Hedging Guarantees in Variable Annuities Under Both Equity and Interest Rate Risks

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Abstract

Effective hedging strategies for variable annuities are crucial for insurance companies in preventing potentially large losses. We consider discrete hedging of options embedded in guarantees with ratchet features, under both equity (including jump) risk and interest rate risk. Since discrete hedging and the underlying model considered lead to an incomplete market, we compute hedging strategies using local risk minimization. Our results suggest that risk minimization hedging, under a joint model for the underlying and interest rate, leads to effective risk reduction. Moreover, hedging with standard options is superior to hedging with the underlying when both equity and interest rate risks are appropriately modeled.

Key words: Variable annuity, lookback option, equity risk, interest rate risk, risk minimization

Classification codes: G22, G10, G11; IE43, IM22, IM20

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1. Introduction

Annuities are contracts designed to provide payments to the holder at specified intervals, usually after retirement. Traditionally, insurance companies offered fixed annuities which guarantee a stream of fixed payments over the life of the contract. This type of annuities was attractive to the policy holders in the context of high interest rates and high cost of investment in the equity market. However, bullish markets and low interest rate environments motivate the investors to look for higher returns than those provided by the conventional annuities. Variable annuities, whose future benefits are based on the performance of a portfolio of securities including equities, have proved to be very attractive for investors, since they not only provide participation in the stock market, but they also have some protection against the downside movements in the market. Variable annuities in the U.S. are similar to the unit-linked annuities in the U.K. and the segregated funds in Canada.

Variable annuities are appealing to investors because they are tax-deferred and they offer different types of benefits, such as the guaranteed minimum death benefits (GMDB). Until the beginning of the 1990's, the death benefits were just simple principal guarantees (original investment) or rising floor guarantees (original investment accrued at a minimally guaranteed interest rate, possibly capped at a predetermined level). In the circumstances of the bullish market of the 1990's, insurance companies have started to offer GMDB with more attractive features, such as the ratchet, which guarantees a death benefit based upon the highest anniversary account value. The anniversary dates at which the guarantee is reset are typically annual.

The simultaneous occurrence of death and market downturn seemed unlikely during the strong bullish market of 1990's, however, in the following market crash, insurance companies realized that they may face extremely large losses. Devising good risk management strategies has become of crucial importance. The traditional actuarial methods adopt a passive strategy of holding a sufficient reserve in risk-free instruments in order to meet the liabilities with high probability. Recent research applies methods from finance for computing the fair price of a guaranteed minimum death benefit in a variable annuity and meeting the contract liabilities. The typical risk management strategies in this case consist of holding positions in stocks and bonds and dynamically rebalance these positions in order to cover the guarantees. The financial engineering approach is based on the fact that the guaranteed minimum death benefit can be viewed as a put option with a stochastic maturity date. This put option has a strike equal to the initial investment for a GMDB with principal guarantee, or a strike increasing at the minimum guaranteed rate in the case of a rising floor feature. For a GMDB with ratchet features, the corresponding option is a lookback put for which the strike price is equal to a running maximum of the account value.

Brennan and Schwartz (1976), Boyle and Schwartz (1977), Aase and Persson (1992), Persson (1993), Bacinello and Ortu (1993a) use option theory to price and hedge the embedded options in variable annuities. With the main assumption that the market is complete under both financial and mortality risk, the option price is equal to the expected value of the payoff with respect to a risk-neutral probability measure. Moreover, the option can be exactly replicated using delta hedging. The number of shares of the underlying held in a delta hedging strategy is given by the sensitivity (delta) of the option value to the underlying.

Typically, if the number of policyholders is large enough, it can be assumed that the market is complete under mortality risk. By the Law of Large Numbers, the total liability in this case will be close to its expected value. An insurance company can diversify away its mortality risk by selling enough policies. In this context, the embedded put options can be assumed to have a deterministic maturity. Moller (1998, 2001a,b) investigates pricing and hedging of insurance contracts under mortality risk.

Assuming market completeness under financial risk is, however, more problematic. One issue is that the benefits are sensitive to the tail distributions of the underlying accounts. While empirical market data shows that the distributions of equity returns exhibit fat tails, this behavior cannot be explained by the simple Black-Scholes model for equity prices. Unfortunately, as soon as one allows for stochastic volatility, or if a jump component is added to the model, the market becomes incomplete. Moreover, liquidity constraints and the impossibility of hedging continuously in time, coupled with the need to rebalance as little as possible due to the impact of transaction costs, also lead to an incomplete market. Another problem with modeling the life insurance contracts is that, because of the long maturities of these contracts, stochastic interest rates may be more appropriate than a constant rate.

The main emphasis of the literature has been on pricing the options embedded in the life insurance contracts; however, hedging is also very important for risk management purposes. In this paper we investigate the computation and effectiveness of hedging strategies under both equity and interest rate risks. We assume that the market is complete under mortality risk, but the financial market is incomplete, due to a suitable equity model for fat tails or to discrete hedging. We have analyzed the modeling of implied volatility risk in Coleman, Li and Patron (2004).

We remark that Bacinello and Ortu (1993b, 1994), Nielsen and Sandmann (1995), Miltersen and Persson (1999), Bacinello and Persson (1994) also investigate stochastic interest rates; however, these authors focus on pricing and they assume a complete financial market which leads to the existence of a unique equivalent martingale measure for the equity price. In an incomplete market, however, the equivalent martingale measure is not unique and the expected value of the discounted payoff under any equivalent martingale measure provides a price for the option which excludes arbitrage opportunities. There exists no best choice when trying to consider one of these prices as the fair price of the option. Therefore, options cannot be priced by arbitrage considerations alone. Moreover, since there exists no self-financing strategy that replicates the option payoff, the intrinsic risk of an option cannot be eliminated and there is much uncertainty regarding the choice of an optimal hedging strategy.

Delta hedging is often used by practitioners for hedging the options embedded in a GMDB (Boyle and Hardy, 1997; Hardy, 2000). In theory, delta hedging assumes continuous rebalancing of the hedging portfolio. In practice, a natural problem that occurs is the impossibility of hedging continuously and the necessity to rebalance as little as possible due to transaction costs. Moreover, for evaluating the hedging performance and the riskiness of a hedging strategy, one has to work with the model for the real world underlying price evolution. Since the delta hedging strategy is computed in a risk neutral framework, it is typically not optimal under the real world dynamics when the market is incomplete.

The semi-static hedging proposed by Carr (2002) and Carr and Wu (2002) uses standard options as hedging instruments; however, the existence of a continuum of standard options is assumed. In practice, the availability of only a finite number of standard options leads to incompleteness. Moreover, semi-static hedging, like delta hedging, requires the underlying price dynamics in a risk-neutral framework.

Given that, in an incomplete market, the intrinsic risk of an option cannot generally be fully hedged, one idea for computing an optimal hedging strategy is to minimize a particular measure of this intrinsic risk. Föllmer and Sondermann (1986), Föllmer and Schweizer (1989), Schäl (1994), Schweizer (1995, 2001), Mercurio and

Vorst (1996), Heath, Platen and Schweizer (2001a,b), Bertsimas, Kogan and Lo (2001) study quadratic criteria for risk minimization. Alternatively, Coleman, Li and Patron (2003, 2006) investigate piecewise linear measures. There are two main criteria: local risk minimization and total risk minimization. Local risk minimization consists in choosing an optimal hedging strategy that exactly matches the option by its final value and minimizes the intermediate cashflows for rebalancing the hedging portfolio. Alternatively, total risk minimization computes an optimal self-financing strategy that best matches the option payoff by its final value. Unfortunately, total risk minimization is a dynamic stochastic programming problem which generally leads to very expensive computations. However, local risk minimizing hedging strategies can be easily computed. Coleman, Li and Patron (2004) illustrate numerically that, for hedging variable annuities with ratchet features, local risk minimization hedging is superior to delta hedging in an incomplete market framework.

Moller (1998, 2001a,b), Lin and Tan (2003) use risk minimization to compute the hedging strategies under mortality risk. Lin and Tan (2003) consider, in addition, a model with stochastic interest rates. However, in the above papers the market is assumed complete from a financial point of view.

The main contribution of this paper is to illustrate the importance of modeling of stochastic interest rates, in addition to the equity risk, in the computation of hedging strategies for the options embedded in a GMDB. Specifically, we evaluate the effectiveness of the discrete local risk minimizing hedging strategy under both equity risk and interest risk with an appropriate equity model for fat tails. The discrete local risk minimizing hedging strategy is computed under the realistic assumption of an incomplete market due to discrete hedging and jump risk. This paper focuses on the hedging of GMDB with a ratchet feature, since this is the most difficult death benefit to hedge. We investigate hedging with the underlying, as well as hedging with standard options. As illustrated in Coleman, Li and Patron (2004), using standard options as hedging instruments can be significantly more effective than using the underlying, especially under jump risk. We demonstrate that the superiority of hedging using standard options, compared with the hedging using the underlying, is greatly compromised if the interest risk is not appropriately modeled. Moreover, by modeling the interest rate risk explicitly, local risk minimization is able to achieve hedging effectiveness comparable to the performance obtained under no interest risk; this is true even when correlation risk is ignored in the risk minimizing hedging computation.

The paper is structured as follows: in Section 2, we introduce the mathematical framework and we review some background notions related to quadratic risk minimization in the context of discretely hedging a lookback option embedded in a GMDB with ratchet feature. In Section 3 we describe the computation of hedging strategies under a Vasicek stochastic interest rate model. We illustrate numerically the hedging effectiveness using the underlying and the sensitivity of the hedging performance to the correlation between the underlying and the stochastic interest rates. The hedging performance of strategies using standard options as hedging instruments is analyzed in Section 4. Section 5 presents the conclusions of the paper.

2. Quadratic risk minimization

The traditional criteria for hedging in an incomplete market are quadratic risk minimizing criteria. This section reviews the main definitions in the context of discretely hedging the option embedded in a GMDB with ratchet feature. Since we assume in this paper that mortality risk can be diversified away, we analyze the hedging of an embedded option with fixed maturity. Moreover, we do not address the problem of basis risk in this paper; consequently, we assume that the underlying account determining the benefit is linked to a market index, such as S&P500.

Let $T > 0$ be the maturity of the embedded option, and

$$0 = \tilde{t}_0 < \tilde{t}_1 < ... < \tilde{t}_{\tilde{M}-1} < \tilde{t}_{\tilde{M}} = T \tag{1}$$

denote the anniversary account step-up dates. A GMDB with ratchet feature guarantees a payoff of $\max(H, S_T)$, where H is the maximum anniversary account value, $H = \max(S_{\tilde{t}_0}, ..., S_{\tilde{t}_{\tilde{M}-1}})$. The GMDB payoff corresponds to the underlying account plus an embedded lookback option with payoff given by:

$$\Pi_T = \max(H - S_T, 0) \tag{2}$$

We suppose that there are only a finite number of hedging dates

$$0 = t_0 < t_1 < ... < t_{M-1} < t_M = T \tag{3}$$

For simplicity, the anniversary step-up dates in (1) are assumed to form a subset of the above trading dates. The financial market is described by a filtered probability space $(\Omega, \mathcal{F}, P)$, with filtration $(\mathcal{F}_k)_{k=0,1,...,M}$, where $\mathcal{F}_k$ corresponds to the hedging time t_k and w.l.o.g. $\mathcal{F}_0 = \{\emptyset, \Omega\}$ is trivial. We suppose that the stock price follows a stochastic process $S = (S_k)_{k=0,1,...,M}$, with S_k being $\mathcal{F}_k$-measurable for all $0 \leq k \leq M$. We assume that the stock price is normalized by the price of the bond with maturity T and thus assume that the price of this bond $B \equiv 1$.

At each hedging time, the instruments available for trading are n risky assets with values $U_k \in \Re^n$ and the bond B. The risky assets can include the underlying, liquid options, and bonds with a shorter maturity (e.g., one year) when hedging interest risk. The values of the risky assets at time t_{k+1}, are given by $U_k(t_{k+1})$. In the literature on hedging in incomplete markets, the hedging instrument is usually the underlying asset ($n = 1, U_n = S_n$). However, with the expansion of option trading, it has become attractive to use liquid standard options for hedging. Hence, we also analyze the case when the n risky instruments available for hedging include standard options. Because of liquidity considerations, we only use standard options with short maturity (e.g., 1 year), thus the hedging instruments may exist for only a sub-period of the hedging horizon. Under these conditions, the hedging portfolio constructed at time t_k is liquidated at time t_{k+1}, when a new portfolio is formed.

A hedging strategy is a sequence of trading positions $\{(\xi_k, \eta_k)|k = 0, 1, ..., M\}$, where ξ_k is a vector of positions in the risky assets at time t_k and η_k is the amount invested in the riskless bond B at t_k. We assume $\xi_M \equiv 0$. This corresponds to liquidating the hedging portfolio at time T in order to cover for the lookback option payoff.

The value of the hedging portfolio at any time t_k is given by $P_k = U_k \cdot \xi_k + \eta_k$. The change in value of this portfolio due to changes in the risky asset values at time t_{j+1}, before any rebalancing, is given by $(U_j(t_{j+1}) - U_j) \cdot \xi_j$. Therefore, the *accumulated gain* at time t_k is given by $G_0 = 0$ and

$$G_k = \sum_{j=0}^{k-1} (U_j(t_{j+1}) - U_j) \cdot \xi_j \tag{4}$$

The *cumulative cost* at time t_k, C_k, is defined as $C_k = P_k - G_k$. A strategy is called *self-financing* if the cumulative cost $(C_k)_{k=0,1,\ldots,M}$ is constant over time, i.e. $C_0 = C_1 = ... = C_M$. This is equivalent to $U_{k+1} \cdot \xi_{k+1} + \eta_{k+1} - (U_k(t_{k+1}) \cdot \xi_k + \eta_k) = 0$, for all $0 \leq k \leq M - 1$. In other words, the hedging portfolio can be rebalanced with no inflow or outflow of capital. The value of the portfolio for a self-financing strategy is given by $P_k = P_0 + G_k$ at any time $0 \leq k \leq M$.

In an incomplete market there does not exist, in general, a self-financing strategy that matches exactly the option payoff. Under these conditions, a hedging strategy has to be chosen based on some optimality criterion. One approach is to first impose $P_M = \Pi_T$, then choose the optimal strategy to minimize the incremental cost incurred from adjusting the portfolio at each hedging time. This is the *local risk minimization*. The traditional criterion for local risk minimization is the quadratic criterion, given by minimizing:

$$E((C_{k+1} - C_k)^2|\mathcal{F}_k) = E((P_{k+1} - U_k(t_{k+1}) \cdot \xi_k - \eta_k)^2|\mathcal{F}_k), \tag{5}$$

for all $0 \leq k \leq M - 1$, starting from the final condition $P_M = \Pi_T$. The quadratic local risk minimizing strategy is not self-financing, but it is mean self-financing (i.e., the cost process is a martingale):

$$E(C_{k+1}|\mathcal{F}_k) = C_k \tag{6}$$

Alternative to local risk minimization, one could instead choose to work only with self-financing hedging strategies. Since it is not possible to match exactly the option payoff in this case, an optimal self-financing strategy minimizes the L^2-norm:

$$E((\Pi_T - P_M)^2) = E((\Pi_T - P_0 - \sum_{j=0}^{M-1} (U_j(t_{j+1}) - U_j) \cdot \xi_j)^2). \tag{7}$$

The existence and uniqueness of the optimal strategies for the above criteria have been extensively studied by Schäl (1994) and Schweizer (1995), for the case when the hedging instruments are the underlying assets. Under certain assumptions, the initial values of the local and total risk minimizing hedging portfolios are equal. In the spirit of option pricing in complete markets, Schäl (1994) interprets this value as a “fair value” for the option. However, Mercurio and Vorst (1996) show that this interpretation may not always make sense from an economic point of view. Moreover, the initial value of the hedging portfolio depends, in general, on the subjective criterion for measuring the risk. As Bertsimas, Kogan and Lo (2001) remark, in a dynamically incomplete market an option cannot be priced by arbitrage considerations alone and has to be the result of a market equilibrium based on supply and demand. The focus of our paper is on computing effective hedging strategies; this computation also provides cost information, e.g., initial hedging cost, as well as average hedging cost.

Total risk minimization is a dynamic stochastic programming problem which is, in general, computationally challenging to solve. In this paper we use local risk minimization to compute the hedging strategies. Schweizer (2001) remarks that, since local risk minimization tries to control the riskiness of a hedging strategy as measured by its incremental risk, it is, by its very nature, a hedging approach.

A local risk minimizing hedging strategy is not self-financing, however, we can define a self-financing portfolio related to this strategy. If $\{(\xi_k, \eta_k) | k = 0, 1, ..., M\}$ are the optimal holdings for the local risk minimizing strategy, then the time T value of the associated self-financing portfolio is given by:

$$P_M^{sf} = P_0 + G_M = U_0 \cdot \xi_0 + \eta_0 + \sum_{j=0}^{M-1} (U_j(t_{j+1}) - U_j) \cdot \xi_j \qquad (8)$$

We investigate the hedging efectiveness of the local risk minimizing strategy in terms of the *total risk* and *total cost* computed at the maturity T of the lookback option:

- Total risk, $\Pi_T - P_M^{sf}$, is the amount of money the hedging portfolio is short of meeting the liability.
- Total cost, $\Pi_T - G_M$, is the total amount of money needed for setting up the hedging portfolio and covering the option payoff.

3. Hedging under interest rate risk, using the underlying asset

Hedging the embedded options in a GMDB is difficult due to the long maturity of these options and to their sensitivity to the tail distributions of the underlying assets. Bacinello and Ortu (1993b, 1994), Nielsen and Sandmann (1995), Miltersen and Persson (1999), Bacinello and Persson (1994), Lin and Tan (2003) address the issue of the long maturity of the options by jointly modeling the underlying asset price (which follows an extended Black-Scholes SDE) and the stochastic short interest rate (given by an extended Vasicek or CIR model) or the forward rate model (HJM). However, the assumed underlying asset price model does not explain the fat tails exhibited by equity returns. Moreover, the authors focus on pricing the embedded options in a continuous trading framework using risk-neutral probability measure arguments. Although Lin and Tan (2003) discuss delta hedging in a financially complete market and risk minimization hedging under mortality risk, the paper mainly focuses on pricing instead of hedging. The present paper focuses on hedging the embedded options in GMDB, in an incomplete financial market due to discrete hedging and/or jump risk. This section investigates hedging, under stochastic interest rates, using the underlying asset as hedging instrument. Section 4 illustrates hedging with standard options.

We first analyze the performance of a risk minimizing hedging strategy when there is no interest rate risk, that is, the strategy is computed under the assumption that the term structure of interest rates is deterministic. We assume that the real-world price dynamics of the underlying asset follows

$$\frac{dS_t}{S_t} = (\mu - q - k\lambda)dt + \sigma dW_t + (J-1)d\pi_t \tag{9}$$

where μ is the instantaneous asset return, q is the continuous dividend yield, and σ is the volatility of the asset; λ is the jump intensity, while J models the jump amplitude, and $k = \mathrm{E}(J-1)$. For simplicity, $\log J$ is assumed to be a normal random variable with mean μ_J and standard deviation σ_J. We assume π_t is a Poisson process.

Note that, when $\lambda = 0$, model (9) is the Black-Scholes model. Throughout the paper, a Merton's Jump Model (MJD) refers to a model (9) with a positive jump intensity λ. Even if the Black-Scholes model is not suitable for calibrating the fat tails of equity returns, it is still interesting to investigate hedging performance under this model, since market incompleteness comes only from the assumption of discrete hedging in this case. We will later analyze hedging under a Merton's Jump Model for the underlying price. We emphasize that equation (9) describes the underlying price dynamics in a real-world framework, which is best suited for the computation and hedging performance analysis in an incomplete market. In contrast, a delta hedging strategy, or a semi-static hedging strategy is computed using risk-neutral price dynamics.

We compute the local risk minimizing strategy using a lattice method. Since the embedded lookback option is path dependent, the hedging positions at time t_k depend on both the underlying value S_k and the path dependent value H_k, where $H_k = \max(S_{\tilde{t}_0}, \cdots, S_{\tilde{t}_{k-1}})$ and $H_0 = S_{\tilde{t}_0}$. We note that the lookback payoff is linearly homogeneous in the underlying value and path dependent value, i.e.,

$$\max(\alpha H - \alpha S, 0) = \alpha \max(H - S, 0).$$

Since the log of the price under a model (9) is a process with independent increments, it is shown, in Appendix A, that the local risk minimization holding for the underlying, ξ_k, is homogeneous of degree zero with respect to the current underlying value S and path dependent value H, while the holding for the bond is linearly homogeneous in S and H. Thus the local risk minimizing hedging positions need to be computed only for a fixed path dependent value H.

Consider a discretization $\{S^i\}_{i=0}^{N_s}$, where $S^i = S_0 e^{X^i}$ with $X^i = X^0 + \; i \; \delta X$; for simplicity, assume $N_s > 0$ is an even integer. Starting from $k = M$ and iterating backwards in time, the hedging positions $(\xi_k^{i_0}, \eta_k^{i_0})$ are computed at each grid point S^{i_0} by solving a least squares problem arising from a discrete approximation to (5):

$$\min_{(\xi_k^{i_0}, \eta_k^{i_0})} \sum_{i=-N_s/2}^{N_s/2} q_{i_0}^{i_0+i} \left(P_{k+1}^{i_0+i} - U_k^{i_0+i}(t_{k+1}) \cdot \xi_k^{i_0} - \eta_k^{i_0} \right)^2 \tag{10}$$

where $q_{i_0}^i, i = i_0 - N_s/2, \cdots, i_0 + N_s/2$, denotes the transitional probabilities of the underlying prices $\{S^i = S_0 e^{X_0 + i\delta X}\}$ assuming $S_k = S^{i_0}$, P_{k+1}^i denotes the time t_{k+1} value (corresponding to S^i) of the hedging portfolio formed at t_{k+1}, and $U_k^i(t_{k+1})$ denotes the time t_{k+1} hedging instruments values (corresponding to S^i) of the hedging instruments traded at time t_k. We explicitly compute the transitional probabilities of the underlying asset price since there exists an analytic formula for the transitional density function under a MJD (9). Note that, if the underlying price is not a grid point, hedging positions are interpolated or extrapolated.

The column labeled "ann" and the column "month" in Table 1 illustrates the hedging performance of a risk minimizing strategy in the above framework. The column "ann" corresponds to annual rebalancing, and the column "month" corresponds to monthly rebalancing under no interest rate risk.

As discussed before, the stochastic interest rate evolution needs to be appropriately modeled due to long maturities of options embedded in typical variable annuities. We further assume that the real-world underlying asset value and the short rate are governed by a joint model:

$$
\begin{cases}
\frac{dS_t}{S_t} = (\mu - q - k\lambda)dt + \sigma_1 \, dW_{1,t} + (J-1)d\pi_t \\
\\
dr_t = a(\bar{r}_t - r_t)dt + \sigma_2 \, dW_{2,t}
\end{cases}
\tag{11}
$$

We remark that in (11), the short rate is governed by an extended Vasicek model, with $\bar{r}$ the long term average short rate, a the rate of the mean reversion, and σ_2 the volatility of the short rate.

Even though the real-world underlying price and interest rate are assumed to follow (11), we first ignore the stochastic interest rate model in the computation of the risk minimizing hedging strategy assuming that the term structure of interest rates is deterministic and the underlying asset dynamics are given by the equation (9). Exploring the hedging effectiveness of this risk minimizing strategy in the real-world (11) illustrates the effect of the interest rate risk on the hedging performance. The numerical results for the annual and monthly hedging evaluated under the stochastic interest rate model are presented respectively under the columns labeled "ann-r" and "month-r" in Table 1.

The hedging strategies "ann-r" and "month-r" described in the above paragraph are computed without taking the interest rate risk into account. In fact, it is possible to model this risk in the computation of the hedging strategy by considering the short rate r as a state variable; we denote the corresponding strategies by "ann-$\check{r}$" and "month-$\check{r}$", respectively, for annual and monthly rebalancing. Note that when the interest rate is stochastic, the hedging positions (ξ_{k+1}, η_{k+1}) depend on the values (S_{k+1}, r_{k+1}). Hence the quadratic incremental cost corresponding to a discrete local risk minimizing hedging strategy, cannot be effectively reduced using a single bond with maturity T. To effectively reduce the interest risk under a one factor short rate model, we include an additional bond, with maturity of one year, among the risky hedging instruments U_k, when computing the hedging strategies "ann-$\check{r}$" and "month-$\check{r}$". Without loss of generality, the bond prices are computed under the assumption that the market price of interest rate risk is zero.

When computing the risk minimizing hedging strategies under the joint model (11), we need the joint transitional density function of the underlying and the short rate. Since modeling the correlation between the change of the interest rate and the change of the account value introduces significantly more complex and costly hedging strategy computation, we assume that the Brownian motions W_1 and W_2 are independent when computing the hedging strategies "ann-$\check{r}$" and "month-$\check{r}$". We will later illustrate that the hedging strategies computed by appropriately modeling the interest rate changes, but assuming no correlation, lead to effective hedging under typically observed correlations between the changes in interest rates and the equity values.

The risk minimization strategies "ann-$\check{r}$" and "month-$\check{r}$" can be computed using a lattice method as described before, except that the grid points now include an additional dimension r, accounting for different possible short rate values. As mentioned above, due to homogeneity properties, we can compute the hedging positions only at a fixed path dependent value. The hedging results are illustrated in the columns "ann-$\check{r}$" - annual hedging, and "month-$\check{r}$" - monthly hedging, in Table 1.

Table 1 illustrates the hedging performance over 20000 simulated scenarios for the risk minimizing strategies described above. The numerical results show the initial cost of the hedging portfolio, the average total hedging cost, the first two moments of the total risk, the Value-at-Risk (VaR) and the Conditional Value-at-Risk (CVaR) for a 95% confidence interval. VaR(95%) corresponds to the maximum amount of money the self-financing hedging portfolio, P_M^{sf}, is short of meeting the liability with 95% probability. CVaR(95%) gives information about the right tail of the total risk, it is the expected amount of money the self-financing portfolio is short of meeting the liability, conditional on the total risk being larger than VaR(95%). The column labelled "No hedge" presents information about the unhedged liability.

We remark from Table 1 that hedging using the underlying asset can significantly reduce risk. Moreover, the effectiveness of the hedging strategies improves as we rebalance more frequently.

To illustrate the sensitivity to interest risk, we compare the effectiveness of the hedging strategy, computed under a model (9) without interest risk, evaluated under a model (9) without interest risk and a model (11) with interest risk; the results correspond to columns ("ann", "month") and ("ann-r", "month-r"), respectively. For annual rebalancing, we notice that hedging using the underlying leads to poor risk reduction and its performance is relatively insensitive to interest rate risk. Hedging with monthly rebalancing, on the other hand, leads to greater risk reduction and the effect of interest risk becomes clearly visible in hedging effectiveness evaluation. For example, the standard deviation of the total risk, VaR and CVaR are larger when the risk minimization hedging strategy computed without modeling interest risk is evaluated under the interest risk, column "month-r", than when the strategy is evaluated under no interest risk, column "month". Specifically, the standard deviation of monthly hedging is increased from 7.97, in the absence of interest rate, to 11.65 under the prescribed interest risk. This suggests that interest risk, if not modeled in the hedging computation, can significantly deteriorate hedging effectiveness. However, when the risk minimizing hedging strategy is computed under the joint model (11), monthly rebalancing ("month-$\check{r}$") effectively hedges the risk introduced by the stochastic interest rates. The standard deviation of the total risk, the VaR(95%) and CVaR(95%) in the column "month-$\check{r}$" are very close to the corresponding values under no interest rate risk ("month"). Strategy "month-$\check{r}$" achieves this reduction in risk,

Table 1

Hedging using the underlying asset in a Black-Scholes framework

	No hedge	Annually			Monthly		
	Π_T	ann	ann-r	ann-$\check{r}$	month	month-r	month-$\check{r}$
C_0	0	13.85	13.85	13.99	14.59	14.59	15.07
$\mathrm{E}(\Pi_T - G_M)$	24.41	26.62	25.59	26.89	27.98	26.86	28.94
$\mathrm{E}(\Pi_T - P_T^{\mathrm{sf}})$	24.41	0.04	-0.98	0.06	-0.007	-1.12	0.02
$\mathrm{std}(\Pi_T - P_T^{\mathrm{sf}})$	36.83	23.44	24.17	23.43	7.95	11.65	7.97
VaR(95%)	96.92	39.65	39.82	39.75	12.54	17.04	12.59
CVaR(95%)	136.60	62.76	63.15	62.82	19.54	25.35	19.66

ann/month: strategy computed by modeling *only* the underlying risk,
effectiveness evaluated under (9) with *no* interest risk

ann/month-r: strategy computed by modeling *only* the underlying risk,
effectiveness evaluated under (11) with *both* equity and interest risks

ann/month-$\check{r}$: strategy computed by modeling *both* the equity and interest risks,
effectiveness evaluated under (11) with *both* equity and interest risks

BS: #scenarios = 20000, $\sigma_1 = 0.2$, $\mu = 0.1$, $S_0 = 100$

Vasicek: $r_0 = 0.05$, $\bar{r} = 0.08$, $a = 0.2$, $\sigma_2 = 0.02$

for a slightly larger average hedging cost than the average hedging cost for "month".

It is important to note that, in order to obtain this effective risk reduction when taking into account the assumed stochastic interest rate model in the computation of the hedging strategies, the hedging portfolio has to contain not one, but two bonds, in addition to the underlying asset. In our computational experience, using a single bond cannot eliminate the additional interest rate risk in the dynamic discrete hedging setting. For subsequent results of the hedging strategies computed by modeling interest risk, we have used a bond with maturity 1 year and a bond with maturity T. Intuitively, the amount invested in one bond at each hedging time is controlled by equation (6) which ensures the strategy is mean self-financing, while the second bond can be used to hedge against the fluctuations of the interest rate.

As mentioned above, in the computation of the risk minimizing strategies "ann-$\check{r}$" and "month-$\check{r}$" under a stochastic interest model, we have assumed, for simplicity, that the underlying asset and the short rate are independent. Let us now investigate the sensitivity of the hedging performance to the correlation between the Brownian motions in (11) in the same Black-Scholes setting as before. We compute the hedging strategies "ann $\check{r}$" and "month-$\check{r}$" as before, assuming the independence of the underlying asset and the interest rates, however, we evaluate the hedging performance on simulated paths for different correlations. For annual hedging with the underlying in the framework given by (11), the hedging error is mainly due to the infrequent rebalancing rather than the interest risk; thus the hedging effectiveness is relatively insensitive to the interest rate risk and the hedging performance is roughly unchanged for different values of the correlation. Table 2 presents the numerical results for monthly hedging, corresponding to the strategies "month-r" which is computed assuming a constant interest rate and "month-$\check{r}$" which is computed under the assumed interest rate model. We have omitted the results for strategy "month", since it corresponds to evaluating hedging performance under a deterministic term structure of interest rates.

We remark that strategy "month-$\check{r}$" leads to an effective reduction in the interest rate risk for all the correlations in Table 2. The values of the standard deviation of the total risk, VaR(95%), CVaR(95%) for strategy "month-$\check{r}$" are close to the corresponding values under no interest rate risk, for strategy "month" (Table 1). However, the average total risk, $\mathrm{E}(\Pi_T - P_T^{\mathrm{sf}})$, suggests that strategy "month-$\check{r}$" tends to overhedge the option as the correlation becomes negative and to underhedge the options for positive correlations.

The results presented above have been obtained in a Black-Scholes framework. We will further investigate risk minimization hedging with the underlying under a Merton's Jump Diffusion (MJD) model. This model is a better choice when trying to calibrate the fat tails exhibited by equity returns. In this framework, market incompleteness is due not only to discrete hedging, but also to the jump component of the underlying asset.

Table 3 presents the numerical results under a MJD model which are obtained from similar hedging experiments to the ones described in the Black-Scholes framework.

Similarly to the Black-Scholes framework, risk minimizing hedging is better than no hedging and the hedging performance improves as rebalancing is more frequent. However, because of the jump risk, this improvement is not as significant as in Table 1. Moreover, we notice that in the MJD framework, both annual and monthly hedging results are less sensitive to the additional interest rate risk. This is because hedging using the underlying asset is ineffective in eliminating the jump risk and thus the presence of interest risk is relatively invisible.

The computation of the hedging strategies "ann-$\check{r}$" and "month-$\check{r}$" requires the joint

Table 2

Monthly hedging using the underlying for different correlations

		Correlation				
		-0.4	-0.2	0	0.2	0.4
Initial	month-r	14.59	14.59	14.59	14.59	14.59
	month-$\check{r}$	15.06	15.07	15.03	15.07	15.05
$\mathrm{E}(\Pi_T - G_M)$	month-r	25.36	26.16	27.01	27.63	28.32
	month-$\check{r}$	27.28	28.10	19.00	29.80	30.59
$\mathrm{E}(\Pi_T - P_T^{\mathrm{sf}})$	month-r	-2.62	-1.82	-0.97	-0.35	0.33
	month-$\check{r}$	-1.61	-0.79	0.17	0.88	1.71
$\mathrm{std}(\Pi_T - P_T^{\mathrm{sf}})$	month-r	11.30	11.44	11.44	11.58	12.11
	month-$\check{r}$	7.75	7.95	7.79	7.95	8.23
VaR(95%)	month-r	16.03	16.31	16.53	16.43	17.22
	month-$\check{r}$	10.40	11.40	12.59	13.10	14.42
CVaR(95%)	month-r	26.57	26.06	25.62	24.31	23.94
	month-$\check{r}$	17.42	18.58	19.41	20.32	21.61

BS: #scenarios = 20000, $\sigma_1 = 0.2$, $\mu = 0.1$, $S_0 = 100$

Vasicek: $r_0 = 0.05$, $\bar{r} = 0.08$, $a = 0.2$, $\sigma_2 = 0.02$

Table 3

Hedging using the undelying asset in a MJD framework

	No hedge	Annually			Monthly		
	Π_T	ann	ann-r	ann-$\check{r}$	month	month-r	month-$\check{r}$
C_0	0	19.27	19.27	19.35	22.33	22.33	22.69
$\mathrm{E}(\Pi_T - G_M)$	31.22	37.15	35.76	37.27	42.74	41.11	43.43
$\mathrm{E}(\Pi_T - P_T^{\mathrm{sf}})$	31.22	0.18	-1.20	0.15	-0.08	-1.70	-0.09
$\mathrm{std}(\Pi_T - P_T^{\mathrm{sf}})$	63.88	47.30	47.96	47.18	32.44	34.53	32.36
VaR(95%)	152.84	80.81	79.78	79.79	52.45	52.78	52.30
CVaR(95%)	242.42	136.95	136.25	136.78	96.76	97.72	96.46

ann/month: strategy omputed by modeling *only* the underlying risk,

effectiveness evaluated under (9) with *no* interest risk

ann/month-r: strategy computed by modeling *only* the underlying risk,

effectiveness evaluated under (11) with *both* equity and interest risks

ann/month-$\check{r}$: strategy computed by modeling *both* the equity and interest risks,

effectiveness evaluated under (11) with *both* equity and interest risks

MJD: #scenarios = 20000, $\mu = 0.15$, $\sigma_1 = 0.2$, $\mu_J = -0.344$, $\sigma_J = 0.25$, $\lambda = 0.0916$, $S_0 = 100$

Vasicek: $r_0 = 0.05$, $\bar{r} = 0.08$, $a = 0.2$, $\sigma_2 = 0.02$

density function for the underlying asset and the short rate in equations (11). We have assumed, for simplicity, that the underlying asset and the short rate are independent. However, we have analyzed, as in Section 3, the hedging performance on simulated paths for different values of the correlation between the Brownian motions W_1 and W_2 in (11). The numerical experiments show that, due to the dominant presence of the jump risk, the hedging performance of all the strategies remains almost unchanged for all the different correlations. Thus, we do not include these numerical results.

4. Hedging using standard options, under interest rate risk

Due to the expansion of the option market, hedging with options has become a viable possibility. This section investigates the hedging performance of risk minimizing hedging strategies for a lookback option embedded in a GMDB with ratchet feature, under interest rate risk, when the hedging instruments are options. Due to liquidity constraints, we use only near the money options as hedging instruments. To illustrate, we consider a hedging portfolio rebalanced annually and consisting of 1-year maturity options and risk-free bonds. At each hedging time t_k, the options are: 3 calls with strike prices S_k, $110\%S_k$, $120\%S_k$, and 3 puts with strike prices S_k, $80\%S_k$, $90\%S_k$. The options are priced under the risk neutral measures. Moreover, in order to simplify the computation of the option hedging strategy "opt-$\check{r}$" under the joint model of the underlying asset and the short rate, we assume once again, for both the Black-Scholes and the MJD framework, that the underlying and the interest rate are independent.

In paper (Coleman, Li and Patron, 2004), it has been illustrated that hedging with standard options is significantly more effective than hedging with the underlying, especially under jump risk. While hedging with the underlying is sensitive to instantaneous volatility risk, hedging with options is sensitive to implied volatility risk.

The paper proposes a joint model for the underlying real-world price dynamics and the stochastic at-the-money implied volatility. Computed under this model, the risk minimizing hedging strategies using standard options effectively reduce volatility risk.

In order to investigate the sensitivity to interest rate of a risk minimizing hedging strategy using options, we perform a similar analysis to the case in Section 3, where the hedging instruments are the underlying assets. We first compute the hedging strategy when there is no interest rate risk, in a framework where the term structure of interest rates is deterministic as described by (10), also see Appendix A. The numerical results illustrating the hedging performance in this case are presented in the tables below under the label "opt". We then analyze the performance of the above strategy under a joint model, (11), for the real-world underlying price dynamics and the stochastic short rates. The results are denoted by "opt-r". Finally, we compute the risk minimizing hedging strategy under the joint model by considering the short rate as a state variable. The hedging results in this framework are identified as "opt-$\check{r}$.

Table 4 shows the numerical results in a Black-Scholes framework. The joint model for the underlying asset and the short rate is given by (11) with the jump intensity $\lambda = 0$. Comparing the results for hedging with standard options in Table 4 with the results for monthly hedging with the underlying in Table 1, we remark that option hedging leads to a larger reduction in terms of standard deviation of total risk, VaR and CVaR. Even when the stochastic interest rates are not modeled in the computation of the hedging strategy ("opt-r"), hedging with standard options is slightly better than hedging with the underlying. The average total cost for option hedging is marginally larger than the average total cost for hedging with the underlying.

We notice that the effectiveness of hedging with options is much more sensitive to interest rate risk than the effectiveness of hedging with the underlying. When the

Table 4

Hedging with standard options in a Black-Scholes framework

	No hedge	Option hedging		
	Π_T	opt	opt-r	opt-$\check{r}$
C_0	0	14.68	14.68	15.10
$\mathrm{E}(\Pi_T - G_M)$	24.41	28.15	27.73	28.98
$\mathrm{E}(\Pi_T - P_T^{\mathrm{sf}})$	24.41	-0.007	-0.42	0.02
$\mathrm{std}(\Pi_T - P_T^{\mathrm{sf}})$	36.83	2.33	7.94	3.17
VaR(95%)	96.92	3.09	11.68	4.76
CVaR(95%)	136.60	5.64	17.66	7.82

opt: strategy computed by modeling *only* the underlying risk,

effectiveness evaluated under (9) with *no* interest risk

opt-r: strategy computed by modeling *only* the underlying risk,

effectiveness evaluated under (11) with *both* equity and interest risks

opt-$\check{r}$: strategy computed by modeling *both* the equity and interest risks,

effectiveness evaluated under (11) with *both* equity and interest risks

BS: #scenarios = 20000, $\sigma_1 = 0.2$, $\mu = 0.1$, $S_0 = 100$

Vasicek: $r_0 = 0.05$, $\bar{r} = 0.08$, $a = 0.2$, $\sigma_2 = 0.02$

market interest rate is stochastic, but this is not accounted for in the computation of the hedging strategy ("opt-r"), the values of the standard deviation of the total risk, VaR and CVaR increase more than 300% compared to the corresponding values when there is no interest rate risk ("opt"). However, when the risk minimization hedging strategy is obtained by modeling the stochastic interest rate as a state variable ("opt-$\check{r}$"), the interest rate risk is greatly reduced. As mentioned in the previous section, an important factor in achieving this interest risk reduction is the inclusion of 2 bonds in the hedging portfolio, one bond being insufficient for maintaining the portfolio self-financing and offsetting the fluctuations of the interest rate.

As in Section 3, we also investigate the sensitivity to interest rate risk in a model with jump risk, which is a more appropriate framework for the price dynamics of the underlying of a lookback option embedded in a GMDB with ratchet feature. Table 5 illustrates the performance of hedging with standard options in a MJD framework. Comparing the results from Tables 3 and Table 5, we remark that hedging with options is more effective in reducing the risk than hedging with the underlying. The remark also applies to the Black-Scholes framework, however the differences between the performances of hedging with options and hedging with the underlying are larger in the MJD framework. This shows that hedging with options is superior to hedging with the underlying in offsetting the jump risk.

As in the case of the Black-Scholes framework, option hedging effectiveness is sensitive to the interest rate risk. When the stochastic interest rates are not modeled in the computation of the hedging strategy ("opt-r"), the standard deviation of the hedging risk, VaR and CVaR are more than double compared to the corresponding values when there is no interest risk ("opt"). However, the hedging strategy calculated under the joint model for the underlying price dynamics and the short rates ("opt-$\check{r}$") substantially reduces the interest rate risk.

Table 5

Hedging with standard options in a MJD framework

	No hedge	Option hedging		
	Π_T	opt	opt-r	opt-$\check{r}$
C_0	0	19.19	19.19	19.58
$\mathrm{E}(\Pi_T - G_M)$	31.22	36.86	36.22	37.60
$\mathrm{E}(\Pi_T - P_T^{\mathrm{sf}})$	31.22	0.03	-0.60	0.03
$\mathrm{std}(\Pi_T - P_T^{\mathrm{sf}})$	63.88	4.30	11.82	5.33
VaR(95%)	152.84	5.64	17.02	7.59
CVaR(95%)	242.42	11.01	26.55	13.34

opt: strategy computed by modeling *only* the underlying risk,

effectiveness evaluated under (9) with *no* interest risk

opt-r: strategy computed by modeling *only* the underlying risk,

effectiveness evaluated under (11) with *both* equity and interest risks

opt-$\check{r}$: strategy computed by modeling *both* the equity and interest risks,

effectiveness evaluated under (11) with *both* equity and interest risks

MJD: #scenarios = 20000, $\mu = 0.15$, $\sigma_1 = 0.2$, $\mu_J = -0.344$, $\sigma_J = 0.25$, $\lambda = 0.0916$, $S_0 = 100$

Vasicek: $r_0 = 0.05$, $\bar{r} = 0.08$, $a = 0.2$, $\sigma_2 = 0.02$

Table 6 illustrates the sensitivity of the option hedging strategies to different correlations, in the MJD framework. The results in the Black-Scholes framework show a similar trend and have been omitted. We remark that "opt" denotes the hedging strategy computed and evaluated under no interest risk; thus no corresponding row appears in Table 6. Comparing the hedging results for strategy "opt-r" in Table 6 with the results in Table 2, it can be observed that hedging with options under a MJD model is more sensitive to the correlation between the changes in the account values and interest rates. However, by modeling the interest rate in the hedging computation, strategy "opt-$\check{r}$" leads to significant risk reduction for all correlations. In addition, the hedging performance of strategy "opt-$\check{r}$", is approximately the same for different correlations. In contrast with hedging using the underlying, the hedging strategy using options, "opt-$\check{r}$", tends to slightly under-hedge the liability for positive correlations and to slightly over-hedge it when the correlation is negative.

5. Conclusions

Hedging the options embedded in guaranteed minimum benefits of variable annuities is a difficult problem due to the sensitivity of these benefits to the tail distributions of the underlying accounts and their long maturity. The popular Black-Scholes model is not adequate for calibrating the fat tails of equity returns. Moreover, delta hedging assumes continuous rebalancing of the hedging portfolio. A more appropriate model for the underlying, such as Merton's jump diffusion model, or discretely rebalancing the hedging portfolio, leads to incomplete markets. In an incomplete market framework, a risk minimization criterion is more suited to compute a hedging strategy. A risk minimization hedging strategy is not only optimal with respect to the particular criterion, but it is also computed under the real-world price dynamics, in contrast to

Table 6

Annually hedging with options for different correlations

		Correlation				
		-0.4	-0.2	0	0.2	0.4
Initial	opt-r	19.19	19.19	19.19	19.19	19.19
	opt-$\check{r}$	19.58	19.58	19.58	19.58	19.58
$\mathrm{E}(\Pi_T - G_M)$	opt-r	37.44	36.83	36.13	35.71	34.97
	opt-$\check{r}$	38.50	38.00	37.50	37.18	36.70
$\mathrm{E}(\Pi_T - P_T^{\mathrm{sf}})$	opt-r	0.62	0.009	-0.69	-1.10	-1.85
	opt-$\check{r}$	0.93	0.42	-0.05	-0.37	-0.86
$\mathrm{std}(\Pi_T - P_T^{\mathrm{sf}})$	opt-r	12.65	12.07	11.83	11.70	12.34
	opt-$\check{r}$	5.79	5.70	5.42	5.30	5.56
VaR(95%)	opt-r	21.07	18.93	16.64	15.67	14.44
	opt-$\check{r}$	9.51	8.27	7.40	6.99	6.92
CVaR(95%)	opt-r	35.04	30.48	26.61	23.82	22.20
	opt-$\check{r}$	15.43	14.37	13.14	13.08	13.08

MJD: #scenarios = 20000, $\mu = 0.15$, $\sigma_1 = 0.2$, $\mu_J = -0.344$, $\sigma_J = 0.25$, $\lambda = 0.0916$, $S_0 = 100$

Vasicek: $r_0 = 0.05$, $\bar{r} = 0.08$, $a = 0.2$, $\sigma_2 = 0.02$

a delta hedging strategy which is computed in a risk-neutral framework. This is important, since the performance of a hedging strategy has to be analyzed with respect to the real-world dynamics. In addition, when hedging the embedded options, due to their long maturities, it is crucial to model the interest rate risk.

We compute local risk minimizing hedging strategies under a joint model for the real-world underlying price dynamics and the short rate. We analyze the hedging performance under both equity and interest rate risks. The hedging instruments considered are either the underlying or liquid standard options.

Hedging with standard options leads to a considerably better performance than annual or monthly hedging with the underlying, especially under jump risk. Since hedging with options effectively reduces equity risks, including jump risk, the sensitivity of the hedging performance to interest rate risk becomes significant and can be more visibly observed. Computing the hedging strategies by ignoring the stochastic interest rate risk may lead to large hedging errors, possibly losing the advantage of hedging with options over hedging with the underlying. It is possible to reduce the interest rate risk by modeling the stochastic interest rates in the computation of the hedging strategies. In fact the hedging errors for the strategies computed under the joint model of the underlying asset and the short rates are close in values to the hedging errors under no interest risk.

For simplicity, when computing the local risk minimizing hedging strategies under the joint model we assume the independence of the underlying and the interest rates. We analyze, however, the sensitivity of the hedging strategies to different correlations. The numerical results illustrate that the relative risk reduction achieved by the hedging strategy which takes into account the stochastic interest rates is approximately the same for different values of the correlation.

In addition to the jump risk and interest rate risk, hedging the embedded options in GMDB is susceptible to other risk factors such as volatility risk, mortality risk, lapsation risk, or basis risk. We have investigated the modeling of volatility risk in a previous paper (Coleman, Li and Patron, 2004). It will be interesting analyze the sensitivity of the hedging performance to basis risk.

Appendix A: Dimension reduction in the risk minimization hedging of lookback options

We consider the local risk minimizing hedging of a lookback option using bonds and the underlying or standard options; the options traded at time t_k have strikes proportional to the underlying value S_k. We show that, under the model assumption:

$$\begin{cases} \frac{dS_t}{S_t} = (\mu - q - k\lambda)dt + \sigma_1 \; dW_{1,t} + (J-1)d\pi_t \\ \\ dr_t = a(\bar{r}_t - r_t)dt + \sigma_2 \; dW_{2,t} \end{cases} \tag{12}$$

with $dW_{1,t}$ and $dW_{2,t}$ independent, the local risk minimizing hedging holdings satisfy:

$$\xi_k^i(\alpha S, \alpha H) = \begin{cases} \xi_k^i(S,H), & \text{if } U_k^i \text{ is the underlying or a standard option} \\ \\ \alpha \cdot \xi_k^i(S,H), & \text{if } U_k^i \text{ is a bond} \end{cases} \tag{13}$$

$$\eta_k(\alpha S, \alpha H) = \alpha \cdot \eta_k(S,H) \quad \text{for the bond with maturity } T.$$

where, for notational simplicity, we have suppressed the dependence of the various terms on the interest rate.

We note that the Black-Scholes model is a special case of (12). Under the assumption that $dW_{1,t}$ and $dW_{2,t}$ are independent, we have $S_{t_{k+1}} = S_{t_k} e^{\delta X_k}$, where $\delta X_k = X_{t_{k+1}} - X_{t_k}$ and X_t is a Lévy process with stationary independent increments. The analysis below assumes the hedging instruments are standard options and bonds, the proof in the case where the hedging instruments are the underlying and bonds being similar.

In order to prove the result we use the following key facts: the log of the underlying value is a process with independent increments, the lookback option payoff is linearly homogeneous in the underlying value and the path dependent value, the standard options are linearly homogeneous in the underlying value and strikes, and the changes in the interest rate are independent of the changes in the underlying value.

Clearly (13) holds at $T = t_M$, since $\xi_M = 0$, $\eta_M = \Pi_T$ and the lookback payoff is linearly homogeneous with respect to S and H. Assume (13) holds at t_{k+1}. We show that it also holds at t_k.

The optimal local risk minimization holding $(\xi_k(\alpha S, \alpha H), \eta_k(\alpha S, \alpha H))$ at time t_k, is the solution of the minimization problem:

$$\begin{aligned} \min_{x,y} \; E_k^{S,r} \{ [\; & U_{k+1} \; (\alpha S e^{\delta X_k}, t_{k+1}) \cdot \xi_{k+1}(\alpha S e^{\delta X_k}, \alpha H) + \eta_{k+1}(\alpha S e^{\delta X_k}, \alpha H) \\ & - U_k \; (\alpha S e^{\delta X_k}, t_{k+1}) \cdot x - y \;]^2 \, | S_k = \alpha S, r_k = r \; \} \end{aligned} \tag{14}$$

where $E_k^{S,r}(\cdot)$ denotes the conditional expectation of the joint distribution of the underlying value and short rate.

Using the induction hypothesis and the key facts mentioned above, (14) becomes:

$$\begin{aligned} \min_{x,y} \; E_k^{S,r} \{ [\; & \alpha \, U_{k+1}(S e^{\delta X_k}, t_{k+1}) \cdot \xi_{k+1}(S e^{\delta X_k}, H) + \alpha \eta_{k+1}(S e^{\delta X_k}, H) \\ & - \alpha U_k(S e^{\delta X_k}, t_{k+1}) \cdot x - y \;]^2 \, | S_k = S, r_k = r \; \} \end{aligned} \tag{15}$$

On the other hand, $(\xi_k(S,H), \eta_k(S,H))$ is the minimizer of:

$$
\begin{aligned}
\min_{x',y'} \; E_k^{S,r}\{\,[\;\; & U_{k+1}\;(Se^{\delta X_k}, t_{k+1}) \cdot \xi_{k+1}(Se^{\delta X_k}, H) + \eta_{k+1}(Se^{\delta X_k}, H) \\
& - U_k\;(Se^{\delta X_k}, t_{k+1}) \cdot x' - y'\;]^2\,|S_k = S, r_k = r\;\}
\end{aligned}
\tag{16}
$$

Comparing problems (15) and (16), we see that (13) holds at t_k.

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