FINANCIAL RISK MANAGEMENT OF A DEFINED BENEFIT PLAN

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ABSTRACT

The aim of this paper is to demonstrate how to manage the financial risk due to the volatility of the assets with a reserve financed by a majored contribution rate for a defined benefit plan in the accumulation phase. We consider a square-root process to modelize the return of the assets and the contribution rate is assumed to be constant. First, we use the contribution rate which balances the pension scheme. We then show that the mortality risk is negligible compared to the financial risk. To control the financial risk, the insurer has to make a reserve which is financed by a surplus contribution rate. Our findings illustrate the determination of the surplus contribution rate in order to control the ruin probability of the pension scheme. We use simulation techniques to deal with the numerical applications in an actual case for a large French firm.

KEYWORDS: Pension funding, asset liability management, stochastic simulation, Value-at-Risk, ruin probability.

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CONTENTS

1. INTRODUCTION

1.1. CONTEXT

The aim of the present paper is to determine the level of the contribution rate of a defined benefit plan where the return on assets is a random variable. We use the general framework described in Magnin & Planchet [2000] et Planchet [2000].

This type of pension scheme is financed by contributions which are a percentage (the contribution rate) of the salary of working members. These contributions are invested in financial assets. When a member retires, there are two possibilities:

- The mathematical reserve of the pension is transferred to another fund which pays the pensions. This arrangement is examined by Planchet & Thérond [2004a] and is not dealt with in this paper. Or;

- The pensions are paid from the original fund. We will examine this situation in relation to French pension schemes.

If the group is assumed to be closed at the time of the valuation, we can distinguish two phases:

- While there is at least one active member, the fund is financed by contributions and by the financial products of the assets.
- When there are no more active members, the fund is financed only by the return on investment.

We show that, if the rate of investment return is equal to the actualisation rate, the contribution rate which balances the scheme is equal to the ratio of the probabilistic actuarial value of the pensions over the probabilistic actuarial value of the contributions. We can define the balancing contribution rate as the contribution rate which reduces the fund to zero when all the members have died.

We assume that the rate of investment return is a random variable the average of which is the actuarial rate. We may note that, if the fluctuations of the return are symmetrical, the probability of having a negative fund when the fund is closed is about 50%.

In this paper we present a method to control the ruin probability of the scheme by creating a reserve[1] designed to amortize the variations of the investment return. This reserve will be financed by an additional contribution rate. Even if it were possible to change the parameters of the plan each year, it would mean negotiating with the staff representatives. It is therefore particularly important to set the funding of the pension scheme at a constant and reasonable level.

We have illustrated this point in an actual case using simulation methods. In particular, we study the impact of the volatility of the investment return on the level of the balancing contribution rate.

[1] This reserve is the French « provision pour aléa financier ».

1.2. LIABILITY CHARACTERISTICS

The plan we examined is a supplementary defined benefit plan. There are about 15,000 active members at the time of evaluation. When an active member retires, his pension will be 75% of his final salary. If a pensioner dies his or her spouse receives 60% of the pension. The average age of the active members is 43 and the average salary is about 25,000 €.

The following graph shows the evolution of the actuarial value of salaries and pensions.

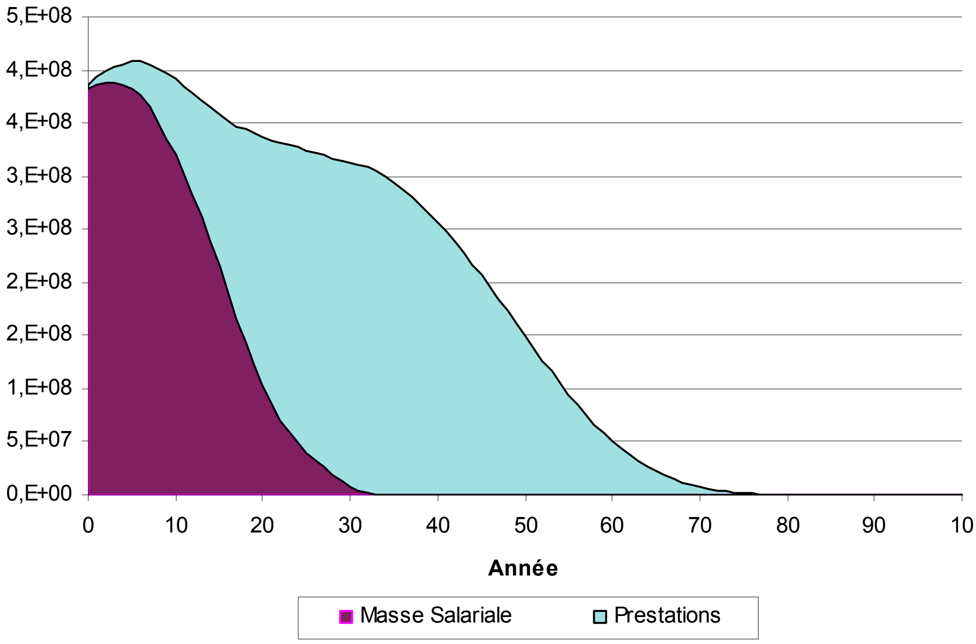

Fig. 1. Evolution of benefits and salaries

We can see the two phases of the plan and note that the probable lifetime of the plan is very long (more than 75 years) due to the reversionary benefits.

Using an actuarial rate of 2.5%, the mathematical reserve is about 5 billion euros. The liability duration[2] is 29 years.

Due to the 3.5 billion euros of the existing fund, the balancing contribution rate is 26% of salaries. Figure 2 shows the evolution of the amount of the fund when the return on investment is equal to the actuarial rate:

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[2] The duration may be interpreted as the average lifespan of a cash flow schedule:

$$\mathbf{D}(f;i) = \frac{1}{V}\sum_{k\geq 1} k * f_k * (1+i)^{-k} \text{ where } V = \sum_{k>1} f_k * (1+i)^{-k}.$$

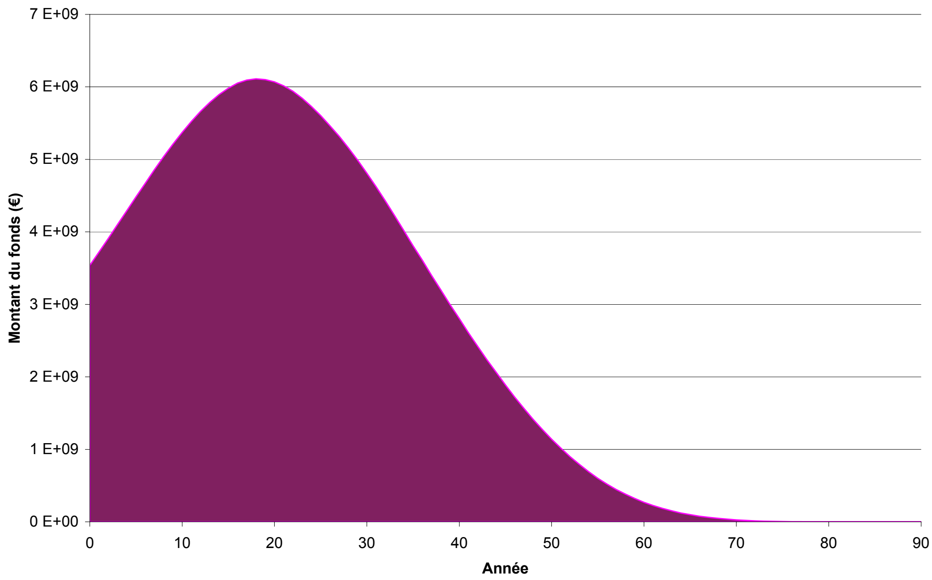

Fig. 2. Average evolution of the fund

We note the continuous increase in the level for approximately 20 years, at which time the level of the fund reaches a maximum before reducing to zero. Indeed, at the end of the first twenty years the benefits are higher than the contributions and investment return.

2. PRESENTATION OF THE MODEL

The purpose of this section is to present the model used and to justify the choices made. The implementation of the previously described example will be outlined in section 2.5

2.1. THE RETURN ON INVESTMENT

Here we have to integrate a stochastic component in the behaviour of the assets; various models are possible: Brownian geometrical, jump processes such as Lévy, standard models of interest rates etc. Le Courtois [2003] reviews traditional modelization on this subject. The purpose of this paper, however, is not to allocate assets, but rather to measure the sensitivity of the balancing operations to the variations of values of the assets around a long-term central value. Time is the determining factor, in our example about 75 years.

This consideration leads us to retain a model in which the return on financial assets fluctuates around a long-term trend, with convergence towards this value in the long term. The level of this value is determined by macro-economic considerations: it cannot be significantly different from the long-term growth rate of the economy.

For these reasons, we retain here the model of Cox, Ingersoll and Ross (CIR)[3] to modelize the return on investment:

$$dr_t = a\left(b - r_t\right)dt + \sigma\sqrt{r_t}\, dB_t \qquad (1)$$

Indeed, this model allows us to integrate this dimension of convergence towards a long-term equilibrium value. This SDE have not any explicit solution so we will retain the discrete version of this process determined by the Euler scheme:

$$\tilde{r}_{t+\delta} = \tilde{r}_t + a\left(b - \tilde{r}_t\right)\delta + \sigma\sqrt{\tilde{r}_t * \delta}\,\varepsilon \qquad (2)$$

In practice we could also use the Milstein scheme which is more precise because it reduces the discretization bias, but this refined version will not be used here; it is reproduced below:

$$\tilde{r}_{t+\delta} = \tilde{r}_t + a\left(b - \tilde{r}_t\right)\delta + \sigma\sqrt{\tilde{r}_t * \delta}\,\varepsilon + \frac{\sigma^2}{4}\delta\left(\varepsilon^2 - 1\right) \qquad (3)$$

[3] This model is normally used to modelize short interest rates which is not the case here.

Questions relating to the simulation of the diffusion process are developed in Planchet & Thérond [2004b].

2.2. THE LIABILITY MODEL

The liability is determined by applying the traditional « individual model ». In practice, if *VA* is the random amount of the liability of the plan, by including the investment return, we obtain the following variance:

$$\mathbf{V}[VA] = \mathbf{E}[\mathbf{V}(VA|X)] + \mathbf{V}[\mathbf{E}(VA|X)] \tag{4}$$

where X is the price of the asset (*i.e.* the synthetic risked asset in which the mathematical reserve is invested). The first term of (4) represents the financial risk and the second term the technical one, *i.e.* the mortality risk. In practice, in a group of this size, the risk of mortality represents a small part of the total risk, since the volatility of the investment return is considerable. This point is developed in Planchet & Thérond [2004a].

From here on, the liability will be assumed to be deterministic, taking into account the size of the group under study. In this context, the liability of the plan is a random variable which only depends on the random investment return:

$$VA = \sum_{k \geq 0} \frac{P_k}{\prod_{l=0}^{k-1} (1 + \tilde{r}_l)} \tag{5}$$

where P_k is the amount of pension paid during the k^{th} year.

2.3. THE EVOLUTION OF THE FUND

By making the assumption that all the movements are carried out in the beginning of the year, the evolution of the collective amount of the funds F is determined by the following dynamic:

$$F_k = \lfloor F_{k-1} - P_k + C_k \rfloor * (1 + r_k) \tag{6}$$

where C_k is the amount of contributions which is a percentage of salaries in the k^{th} year.

We will note:

- ✓ θ the contribution rate,

- α the equilibrium contribution rate,
- β the additional contribution rate: $\theta = \alpha + \beta$.

Thus we have:

$$C_k = \theta S_k \tag{7}$$

The expression of α is given by:

$$\alpha = \frac{\sum_{k \geq 0} \frac{P_k}{\prod_{l=0}^{k-1} (1 + \tilde{r}_l)} - F_0}{\sum_{k \geq 0} \frac{S_k}{\prod_{l=0}^{k-1} (1 + \tilde{r}_l)}} \tag{8}$$

This expression is easily found from the expression of the value of the fund in N directly expressed from the initial amount of the funds F_0 :

$$F_n = F_0 \prod_{k=1}^{n} \left(1 + r_k\right) + \sum_{k=1}^{n} \left(C_k - P_k\right) \prod_{j=k}^{n} \left(1 + r_j\right) \tag{9}$$

Within the framework of the management of the pension scheme, it is important to ensure that the fund is never negative or at least never negative when the fund is closed. This point is dealt with below.

2.4. SIMULATIONS

Close attention is given to the manner of generating the random numbers required for simulations. We use the "tore mélangé" algorithm introduced by Thérond & Planchet [2004b]. We first introduce the torus algorithm: that multi-dimensional generator gives to the n^{th} realization of the d^{th} simulated random variable the value u_n :

$$u_n = n \sqrt{p_d} - \left\lfloor n \sqrt{p_d} \right\rfloor \tag{10}$$

Where :
- p_d is the d^{th} prime number,
- [.] indicates the whole part operator.

The mixed version of that generator is given by:

$$u_m = u_{\varphi(n)} \tag{11}$$

Where :

$$\varphi(n) = [\mu * N * \tilde{u} + 1] \tag{12}$$

with :

- ✓ $\mu \geq 1$,
- ✓ $\tilde{u}$ is a realization of a uniform random variable.

The role of the factor μ in the equation (12) is to reduce the number of selections which would give rise to the same index and thus to the same random number. Indeed, the larger μ is, the lower the probability of twice drawing the same random number. In practice $\mu = 10$ is satisfactory. Moreover, the realization of $\tilde{u}$ could be obtained by a pseudo-number generator.[4]

If N is the number of generated trajectories of the assets and $F^n(\beta)$ the amount of the funds at the expiry time of the plan in the n^{th} simulation with an additional contribution rate β, π is an estimator of the ruin probability:

$$\pi = 1 - \frac{1}{N}\sum_{n=1}^{N} \mathbf{1}_{[0;\infty[}\left(F^n(\beta)\right) \tag{13}$$

We also determine the probability that the funds become negative at least once during the lifetime of the scheme. An estimator of this probability is :

$$\tilde{\pi} = 1 - \frac{1}{N}\sum_{n=1}^{N}\prod_{k\geq 1} \mathbf{1}_{[0;\infty[}\left(F_k^n(\beta)\right) \tag{14}$$

Where $F_k^n(\beta)$ is the amount of the funds at year k.

Of course, this second probability is greater than the ruin probability. These estimators are convergents, without bias and are asymptotically Gaussian.

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[4] The random number generator of Excel/VBA is sufficient for this purpose.

2.5. NUMERICAL RESULTS

For the numerical illustrations, we will use the following parameters:

$$a = 80\% \qquad b = 2{,}5\%$$

$$r_0 = 2{,}5\% \qquad \sigma = 5\%$$

With 10,000 simulations we obtain the following graph that shows the two probabilities according to the additional contribution rate β:

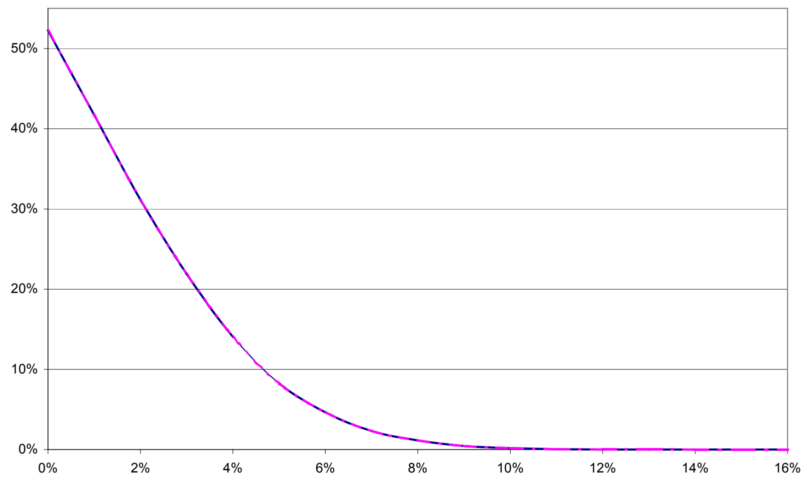

Fig. 3. Ruin probability and probability that the fund becomes negative according to β

The two curves merge perfectly whereas the probability that the funds become negative at least once is higher or equal to the ruin probability. The observation of the trajectories shows us that these probabilities are very strongly dependent on the first return on assets: if it is lower than 2.5% the fund will be negative at the end and conversely.

In addition, when the additional contribution rate is greater than 10%, the ruin probability is practically reduced to zero. The additional contribution rate which brings the ruin probability to zero grows with the volatility of the investment return. Figure 4 illustrates this:

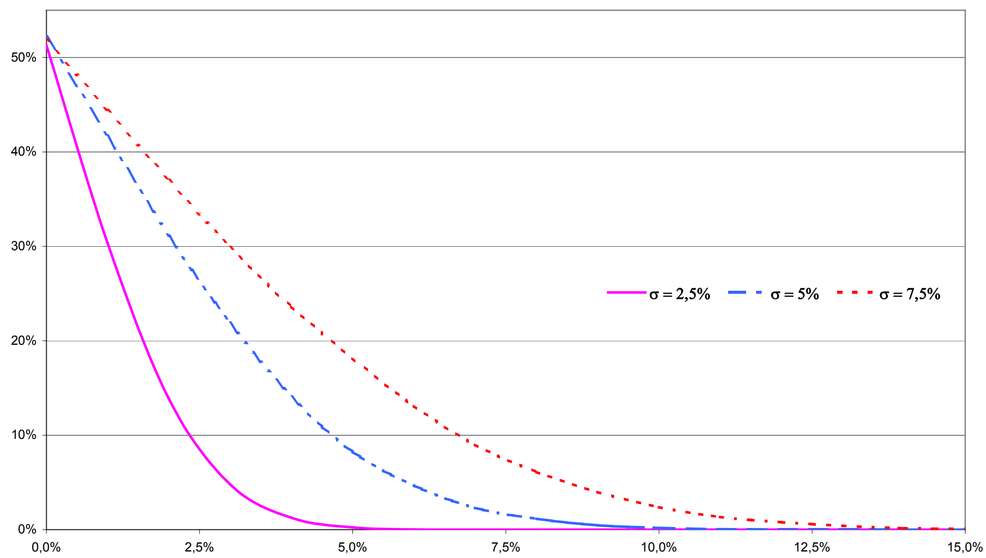

Fig. 4. Ruin probability according to β

In our case it will therefore be essential for the manager to have a good knowledge of the volatility of the return on assets: with an additional rate of contribution of 2% the ruin probability is between 0% to 38% for volatility ranging from 0% to 8%:

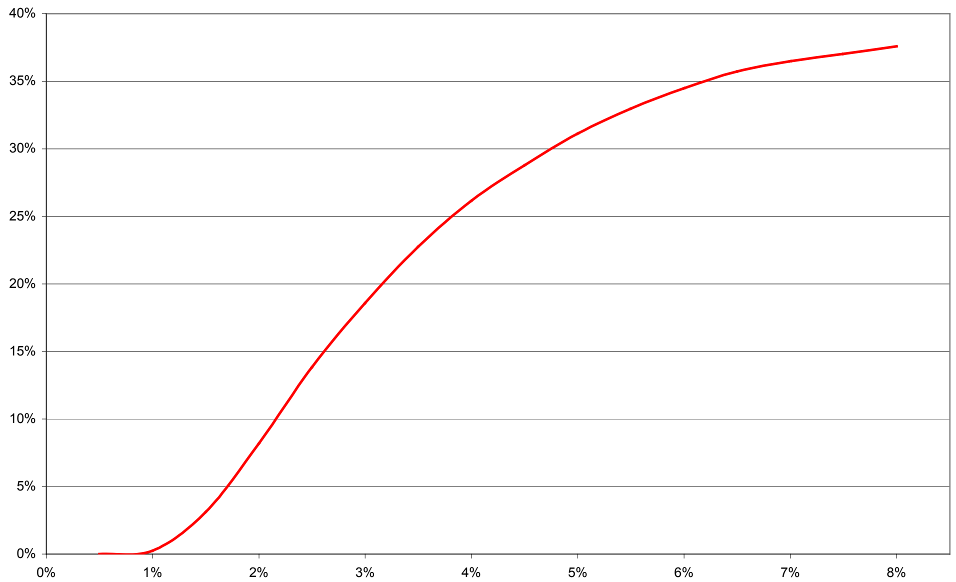

Fig. 5. Ruin probability for β=2% according to σ

For a fixed level of contribution, the long term equilibrium of the funds is thus extremely sensitive to the volatility of the investment return.

3. CONTINUOUS-TIME ANALYSIS

Taking into account the significant horizon of projection used and the relative "macroscopic" regularity of the variables, it is interesting to seek to express the problem in continuous time: this allows us to integrate the continuous model of rate, in a natural way and avoiding the question of its discretization.

3.1. MODEL

In continuous-time, we will note:

- ✓ p_u the instantaneous flow of pensions at time *u*,
- ✓ s_u the instantaneous amount of salaries at time *u*,
- ✓ $c_u = \theta\, s_u$ the instantaneous flow of contributions at time *u*,
- ✓ r_u the instantaneous financial return at time *u*,
- ✓ Φ the filtration associated to the historical probability measure **P**.

Then the value of the funds at time *t* is give by:

$$F_t = F_0\, B_t + \int_0^t \frac{\left(\theta s_u - p_u\right) B_t}{B_u}\, du \tag{15}$$

Where $B_t = \mathbf{exp} \int_0^t r_u\, du$ is the price in *t* of 1 € invested in the same assets as the reserves in 0.

This expression can be seen as the continuous version of (9) or as the integral of the differential equation governing the variation of the funds at each instant:

$$\frac{dF_t}{dt} = F_t\, r_t + \theta s_t - p_t \tag{16}$$

This equation can be obtained when $h \to 0$ in the discrete-time equation governing the evolution of the funds between *t* and *t*+*h*.

The equation (15) enables us to find the rate of contribution α which, in hope, balances the plan in its term *T*:

$$\mathbf{E}\left[F_T\right]=0 \quad \Leftrightarrow \quad \alpha = \frac{\mathbf{E}\left[\int_0^T \frac{p_u}{B_u}\,du\right]-F_0}{\mathbf{E}\left[\int_0^T \frac{s_u}{B_u}\,du\right]} \tag{17}$$

This expression is similar to the equation (8) but in a continuous-time version.

$\mathbf{E}\left[\frac{1}{B_u}\right]$ is well known as it is the price of a zero-coupon which pays 1 at time u when the short-term interest rate is modelized by a square-root process:

$$\mathbf{E}\left[\frac{1}{B_t}\right]=\mathbf{exp}\left\{-a\varphi(t)-r_0\psi(t)\right\} \tag{18}$$

Where :

- ✓ $\varphi(t)=-\frac{2}{\sigma^2}\,\mathbf{ln}\left\{\frac{2\gamma\, e^{\frac{t(\gamma+b)}{2}}}{\gamma-b+(\gamma+b)e^{\gamma t}}\right\}$
- ✓ $\psi(t)=\frac{2\left(e^{\gamma t}-1\right)}{\gamma-b+(\gamma+b)e^{\gamma t}}$
- ✓ $\gamma=\sqrt{b^2+2\sigma^2}$

Moreover, under the assumption that the functions of wages and instantaneous benefits are known, this rate can be calculated. An application with a polynomial adjustment of these 2 functions is proposed hereafter.

3.2. VALUE-AT-RISK

The VaR approach used in the time-discrete framework with simulations can be made analytically explicit in the continuous-time framework. Indeed, by considering that:

$$\mathbf{P}\left[F_T<0\,\middle|\,\Phi_0\right]\le\omega \quad \Leftrightarrow \quad \mathbf{P}\left[F_0+\int_0^T \frac{\theta s_u-p_u}{B_u}\,du<0\,\middle|\,\Phi_0\right]\le\omega \tag{19}$$

We have :

$$\mathbf{P}\left[F_T < 0 \middle| \Phi_0\right] \leq \omega \quad \Leftrightarrow \quad \mathbf{P}\left[\theta < \frac{\int_0^T \frac{p_u}{B_u}du - F_0}{\int_0^T \frac{s_u}{B_u}du} \middle| \Phi_0\right] \leq \omega \tag{20}$$

If we note $\Theta_T = \dfrac{\int_0^T \frac{p_u}{B_u}du - F_0}{\int_0^T \frac{s_u}{B_u}du}$

Θ_T is a random variable whose density function is not easy to obtain; however a simulation approach enables us to obtain an empirical distribution useful in determining the level of contribution which controls the probability of ruin at the level desired by the means of the quantile of level 1-ω of the empirical function of distribution of Θ_T :

$$\mathbf{P}\left[F_T < 0 \middle| \Phi_0\right] \leq \omega \quad \Leftrightarrow \quad \theta \geq F_\Theta^{-1}(1-\omega) \tag{21}$$

By interpolating benefit flows and evolution of salaries by polynomials, *i.e.* by writing:

$$\begin{aligned} p_u &= \sum_{k \geq 0} p^{(k)} u^k \\ s_u &= \sum_{k \geq 0} s^{(k)} u^k \end{aligned} \tag{22}$$

the simulation of realizations of Θ_T is easy by simulating a trajectory of the curve of the rates.

$$\Theta = \frac{\sum_{k \geq 0} p^{(k)} \int_0^T \frac{u^k}{B_u} - F_0}{\sum_{k \geq 0} s^{(k)} \int_0^T \frac{u^k}{B_u}} \tag{23}$$

3.3. NUMERICAL RESULTS

In our example, benefits and salaries can, relatively precisely[5], be modelled by polynomials of degree 3:

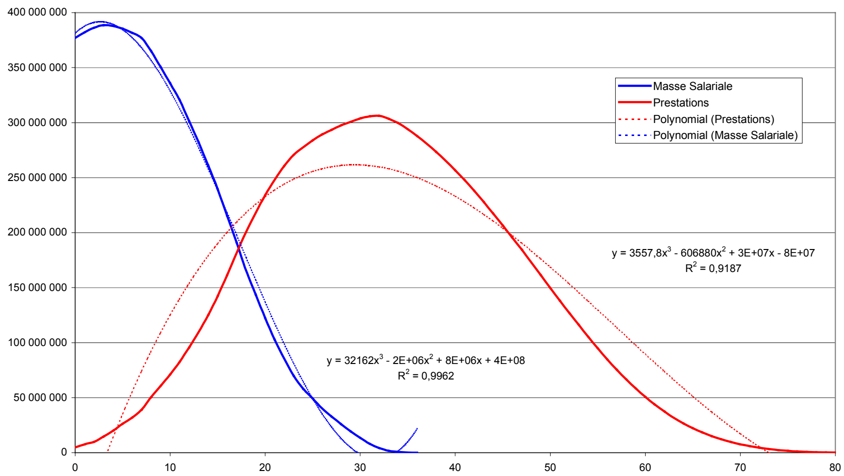

Fig. 6. Polynomial adjustment of the curves of benefits and salaries

The generation of curves of the rates enables us to build an empirical distribution of Θ_T. The following graph uses the distributions obtained starting from 10,000 variables simulated for various volatilities of the investment return.

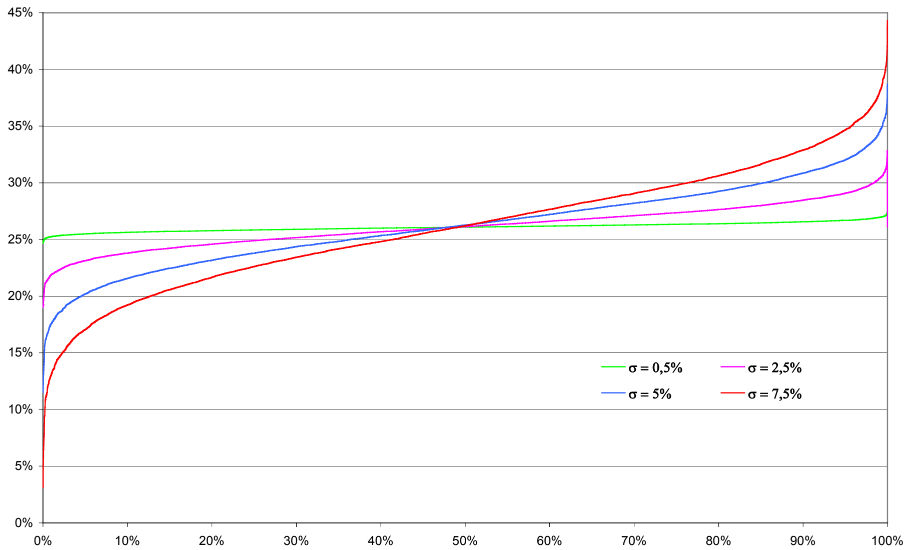

Fig. 7. Distribution of Θ for several volatilities

[5] Correlation coefficients are greater than 95%.

If the various distributions have the same value, as we would expect, their dispersion increases with the volatility of the return on assets. The relation is quasi-linear for values of $\sigma < 10\%$:

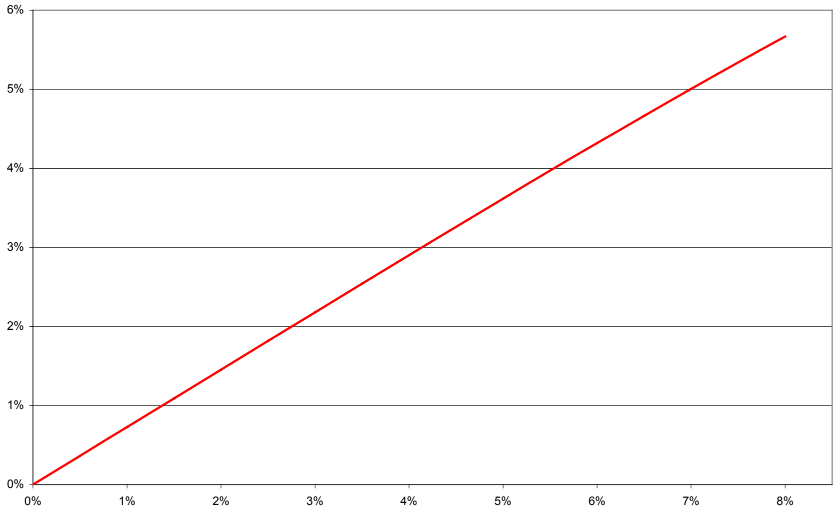

Fig. 8. Standard deviation of Θ according to σ

Figure 7 enables us to determine the level of additional rate of contribution β which will balance the scheme with a probability of *x*%. Thus for $\sigma = 5\%$, $\beta = 5\%$ will allow us to guard against ruin in 95% of the cases.

4. Conclusion

The rate of contribution is a decisive parameter for a defined benefit plan. The traditional approach which consists in determining the level which will balance the plan is valid if that rate can be regularly changed during the lifetime of the plan. But social constraints limit this faculty.

In a model which integrates stochastic return on financial assets, the insurer may use a Value-at-Risk method to fix a contribution rate which controls the ruin probability of the fund.

This approach, in phase with management of very long term liabilities, can be implemented in a relatively simple manner. It leads to a contribution rate which is coherent with the asset allocation policy. Indeed, a level of contribution rate is associated with an acceptable level of volatility of investment return which places a constraint on the asset management of the fund.

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