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Individual claims reserving: a survey

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Abstract

This paper surveys the stochastic modelling of individual claims occurrence and development for reserving purposes in non-life (general) insurance. The paper revisits the continuous time stochastic modelling framework of Norberg (1993) and Hesselager (1994), and provides a consistent presentation of the modelling, inference, and forecasting (with simulation and closed-forms) of individual claims histories as well as aggregate quantities as the overall reserve for both RBNS and IBNR claims. Numerical illustrations are given based on real portfolio datasets, as well as comparisons with classical triangle-based methods.

Keywords: Non-life/General insurance; Stochastic reserving; Individual claims reserving; Poisson point processes; Process and estimation errors; Thinning algorithm.

1 Introduction

Since the work of Mack (1993), stochastic reserving models based on runoff triangles dominate the reserving practice, as surveyed in the book of Wüthrich and Merz (2008) - the triangles are organized by origin period (e.g. accident, underwriting) and development period, and are built as an aggregation of input data made of individual claims paths. At around the same time, other papers proposed to account for a more precise description of the claims development. To our knowledge, Arjas (1989), Jewell (1989), Norberg (1993) and Hesselager (1994) are among the oldest references which introduced a proper probabilistic setting for individual reserving, in the context of increasing awareness of the power of a "point process" description of time patterns in many areas of application. In this setting, the time pattern of

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claims occurrences has systematically been investigated in the form of the Poisson model with time-varying intensity (a stochastic intensity for few exceptions with so-called Cox process), which will be the framework of interest in this paper. In their spirit, individual claims models provide a structural modelling of aggregate 'macro' quantities as a result of the 'micro' components of the individual model - from Norberg (1999), *a parametric specification of the continuous time model, which may be supported by physical reasoning, will automatically induce a parametrization of the discrete time model.* More generally, the virtue of individual-based modelling emerges in a variety of problems from its structural nature, as it is the case of the separate evaluation of IBNR (Incurred But Not Reported) and RBNS (Reported But Not Settled) claims. Although some contributions focused on the development a coherent modelling framework based on runoff triangles providing separate IBNR and RBNS reserves, see e.g. Verrall et al. (2010), this appears in a more natural (and straightforward) way in individual claims modelling, see e.g. the discussion in Antonio and Plat (2014) and references therein, as it is also naturally the case for the valuation of non-proportional reinsurance treaties.

Since then, the individual based modelling of claims has gained recent interest in the literature. Following Norberg (1993), Larsen (2007) revisited the Marked Point Process approach by including claims features to specify the model components. Antonio and Plat (2014) proposed to apply the modelling framework developed in Norberg (1993) and Norberg (1999) to a general liability insurance portfolio of a European insurance company, based on a set of distributional assumptions and a maximum likelihood estimation procedure. In their empirical investigation, and in the context of this specific case study, it is shown that individual reserving provides a better accuracy compared to selected aggregate models. Badescu et al. (2016b) and Badescu et al. (2016a) investigate the modelling of individual claims by means of a Cox process and the related inference; in this work, stochastic intensity is crucial to account for temporal dependence in claims arrival and settlement; the model focuses on number of reported and IBNR claims, without allowing for payments. Among the several authors who focused their attention into inferring the distribution of reporting lags as a first step in the computation of a proper IBNR reserve, one can also refer to Guiahi (1986), Jewell (1989), Zhao and Zhou (2010), and more recently Verrall and Wüthrich (2016) - this latter paper provided additional interesting insights on the analysis and calibration of arrivals and delays distributions based on real data; indeed, it is observed a gain in using individual methods in environments characterized by non-stationarity.

It has also to be noted that other contributions chose another direction for individual models, away from a continuous time point process description: see in particular Pigeon et al. (2013) for a discrete time formulation using stochastic indi-

vidual development factors (see also Taylor et al. (2008)), as well as Godecharle and Antonio (2015) who rely on the *Reserve by Detailed Conditioning* (RDC) method in discrete time from Rosenlund et al. (2012) based on detailed information on individual claims, and also Antonio et al. (2016) for a multi-state framework in discrete time to model the claims development process while allowing for flexible payment distributions.

In this survey, we revisit the original probabilistic formulations of Norberg (1993) and Hesselager (1994) as they provide a continuous-time and structural description of claims arrival and histories, and as we consider additionally that a higher potential emerges from these methods from a pure 'risk analysis' perspective.

Most of the papers investigated the benefit accuracy of individual reserving from a pure empirical point of view, in the form of back-testing efficiency assessment, leading to conclusions that were in each case limited to the data set used. Although this is crucial to get insights on the concrete order of magnitude of the gain of using individual method based on a given dataset, from our point of view, it has not been clearly investigated why do stochastic individual reserving methods outperform aggregate ones from a mathematical perspective. A lot of research work remains to be done in this direction. A first attempt of a theoretical description of the gain in process error has been carried out in Huang et al. (2015), leading to a comparison of individual and aggregate models under a common formalism, although simplified compared to the generic descriptions as provided in Norberg (1993). In this context, we argue that there is a need to revisit the mathematical foundations of individual claims modelling and provide professionals with a proper toolbox allowing for an improved risk assessment.

It is interesting to note that, although for triangle-based models the treatment of estimation error has early become a standard area of interest, dealing with parameter uncertainty in the context of individual reserving has been far less investigated. From an empirical perspective, references as Haastrup and Arjas (1996) (bayesian non-parametric) and Antonio and Plat (2014) (frequentist parametric) are among the few to address this question, although parameter uncertainty is a great area of improvement of accuracy of individual models as data available is "larger" compared to that of an aggregate triangle. Indeed, as noted by Antonio and Plat (2014), *due to our large sample size, confidence intervals for parameters are narrow. This is in contrast with run–off triangles where sample sizes are typically very small and parameter uncertainty is an important point of concern.*

As for simulation issues, Antonio and Plat (2014) rely on the specific form of a piecewise constant intensity by simulating occurrences as Poisson numbers in each (small) interval, or inverting cumulative distribution functions. In this paper, it is used a so-called *thinning* procedure to simulate the (past) occurrences of IBNR

claims, the (future) development of RBNS and IBNR claims, as well as possibly claims which did not occur yet. We argue that such exact simulation method provides a simple and flexible framework which is well suited for the simulation of claims occurrences, development and settlement.

The paper is structured in two parts. In Section 2, we develop the modelling framework for occurrences and reporting delays at the individual claims level, with a special focus on inferring and forecasting IBNR claims (past) occurrences. Section 3 enriches the previous model for claims payments development over time and settlement, allowing to precisely forecast the payment streams of both IBNR and RBNS claims. Each section develops the following components: the modelling framework, the simulation algorithm, the likelihood derivation and inference strategy, a set of macro results and closed-form formulas on the reserve and associated variance, as well as numerical examples. The paper ends with some concluding remarks in Section 4.

2 Occurrence and reporting

2.1 Stochastic individual claims model

Consider claims occurring at times $(T_n)_{n\geq 1}$ given as the jump times of a (non-homogeneous) Poisson process with intensity $\lambda(t)$. Each claim which occurs at time T_n is associated with a delay (waiting time) U_n assumed to have distribution $p_{U|T_n}(\mathrm{d}u)$. In the modelling, this waiting time represents a reporting delay (the difference between time of notification to the insurer and time of occurrence): this is a crucial quantity as it determines the set of claims which are observed in the dataset.

Let us introduce an observation time τ - the set of claims observed is exactly the set of reported claims at time τ, denoted $I^R(\tau)$ and defined as

$$I^R(\tau) = \{(T_n, U_n) \text{ such that } T_n + U_n \leq \tau\}\,. \tag{1}$$

The set of all claims which occurred before time τ, introduced as

$$I^O(\tau) = \{(T_n, U_n) \text{ such that } T_n \leq \tau\}\,,$$

can then be split between reported and so-called *incurred but not reported (IBNR)* claims as

$$I^O(\tau) = I^R(\tau) \cup I^{IBNR}(\tau),$$

where

$$I^{IBNR}(\tau) = \{(T_n, U_n) \text{ such that } T_n \leq \tau \text{ and } T_n + U_n > \tau\}\,.$$

A reserve has to be set to cover the future development of both reported (but not settled) claims, as well as IBNR claims. The prediction of IBNR counts and occurrences is a first step to achieve the latter objective. The overall reserve computation is adressed in the next Section 3 which enriches the individual model for claims payments development.

Poisson point measure representation The Poisson measure formalism is a powerful tool to analyze quantities based on Poisson times and associated marks. We recall below the definition of Poisson point measures.

Definition 1. *A Poisson point measure $Q(\mathrm{d}s, \mathrm{d}y)$ with intensity measure $q(\mathrm{d}s, \mathrm{d}y) = \mathrm{d}s\mu_s(\mathrm{d}y)$ on $\mathbb{R}_+ \times E$ (with μ_s some sigma-finite measure) is a random measure taking values in $\mathbb{N} \cup \{+\infty\}$ satisfying:*
(i) *for all non-overlapping measurable sets $B_1, ..., B_k$ of $\bar{E} = \mathbb{R}_+ \times E$, r.v. $Q(B_1), ..., Q(B_k)$ are independent (infinite r.v. being, as a constant, independent of any other),*
(ii) *for all measurable set $B \subset \bar{E}$ such that $q(B) < +\infty$, $Q(B) \sim \mathcal{P}oisson\,(q(B))$.*
(iii) *$Q(\{0\} \times E) = 0$ (this ensures no jump at time 0). Under this definition, the Poisson point measure is a counting measure on $\mathbb{R}_+^*$, that is a.s. for each $t > 0$, $Q(\{t\} \times E) \in \{0, 1\}$ (see e.g. Çınlar (2011), Theorem 2.17).*

For the purpose of modelling occurrence and reporting in the present section, let us consider $Q(\mathrm{d}t, \mathrm{d}u)$ a Poisson point measure with intensity measure

$$q(\mathrm{d}t, \mathrm{d}u) = \lambda(t)\mathrm{d}t \otimes p_{U|t}(\mathrm{d}u). \tag{2}$$

Then the following representation holds:

$$Q(\mathrm{d}t, \mathrm{d}u) = \sum_{n \geq 1} \delta_{(T_n, U_n)}(\mathrm{d}t, \mathrm{d}u).$$

The number of claims which occurred before τ, whatever their reporting delay, can be simply represented as $Q([0, \tau], [0, \infty)) = \int_0^\tau \int_0^\infty Q(\mathrm{d}t, \mathrm{d}u)$; this is the total sum of numbers in the completed triangle. This provides a straightforward split between reported claims and IBNR claims as

$$Q([0, \tau], [0, \infty)) = \underbrace{\int_{I^R(\tau)} Q(\mathrm{d}t, \mathrm{d}u)}_{\text{Reported}} + \underbrace{\int_{I^{IBNR}(\tau)} Q(\mathrm{d}t, \mathrm{d}u)}_{\text{Incurred but not Reported}} .$$

A key result which follows from i) and ii) of Definition 1 is given below.

Proposition 1. *The number of reported claims $Q\left(I^R(\tau)\right)$ and the number of IBNR claims $Q\left(I^{IBNR}(\tau)\right)$ are independent, and follow Poisson distributions with parameters $q\left(I^R(\tau)\right)$ and $q\left(I^{IBNR}(\tau)\right)$ respectively.*

In particular, this result states that the expected IBNR number writes

$$q\left(I^{IBNR}(\tau)\right) = \int_0^\tau \int_{\tau - t}^\infty \lambda(t) p_{U|t}(\mathrm{d}u)\mathrm{d}t. \tag{3}$$

The frequency of reported claims Proposition 1 provides the distribution of the overall number of claims, both reported and IBNR claims, as a Poisson random variable which can be simulated in a straightforward manner. For our purpose, we are also interested in the *frequency* at which those claims occurred, that is on the distribution of the occurrences (instead of that of the overall number). Let us first compute the probability distribution of reported claims and their associated marks. To do this, we introduce the Poisson point measure of reported claims as

$$Q^R(\mathrm{d}t, \mathrm{d}u) = Q(\mathrm{d}t, \mathrm{d}u)\mathbf{1}_{t+u\leq\tau},$$

whose intensity measure is in fact $q(\mathrm{d}t, \mathrm{d}u)\mathbf{1}_{t+u\leq\tau}$.

Recall that the intensity measure of Q is $q(\mathrm{d}t, \mathrm{d}u) = \lambda(t)p_{U|t}(\mathrm{d}u)\mathrm{d}t$; then the Poisson point measure Q^R, represented as

$$Q^R(\mathrm{d}t, \mathrm{d}u) = \sum_{n\geq 1} \delta_{(T_n^R, U_n^R)}(\mathrm{d}t, \mathrm{d}u),$$

is described as follows:

Proposition 2. (i) *The intensity of reported claims occurrences* $(T_n^R)_{n\geq 1}$ *is*

$$\lambda^R(t) = \lambda(t)p_{U|t}([0, \tau - t]), \tag{4}$$

and at each time of reported claim occurrence T_n^R*, the delay is drawn on* $[0, \tau - T_n^R]$ *with conditional distribution* $p_{U|T_n^R}(\mathrm{d}u)/p_{U|T_n^R}([0, \tau - T_n^R])$.

Indeed, the pure jump process $N_t^R = Q^R([0, t), \mathbb{R}_+)$ which counts the number of reported claims up to time t can be compensated by the associated intensity measure, so that the following process is a martingale:

$$N_t^R - \int_{(0,t)\times\mathbb{R}_+} q(\mathrm{d}s, \mathrm{d}u)\mathbf{1}_{s+u\leq\tau} = N_t^R - \int_{(0,t)} \lambda(s)p_{U|s}([0, \tau - s])\mathrm{d}s,$$

which proves that the intensity is that given in Equation (4). The associated delay distribution then follows from the basic definition of the Poisson random measure Q^R.

2.2 Simulation algorithm

The previous Proposition 2 characterizes the way the observed data sample can be resimulated: in particular, it states that occurrences of reported claims took place at frequency given in Equation (4). This formula will be useful to derive the likelihood in the next Subsection 2.3 dedicated to parameter inference.

Note that reciprocally, as the overall intensity is $\lambda(t)$, Proposition 2 provides the occurrence frequency of IBNR claims which can be drawn independently from reported claims, as a non-homogeneous Poisson process with intensity

$$\lambda^{IBNR}(t) = \lambda(t) p_{U|t}((\tau - t, \infty)). \tag{5}$$

Once the parameters of the model $\lambda(t)$ and $p_{U|t}(\mathrm{d}u)$ have been computed (see next Subsection 2.3), one is then interested in completing the observed data set based on occurence times of IBNR claims. Due to the stochastic nature of the model, an algorithm has to be set to practically 'draw' the times at which IBNR claims should have occurred in the observation period $[0, \tau]$, which we denote $\left(T_n^{IBNR}\right)$. This can be achieved according to the so-called *thinning* algorithm which provides a very simple way of computing paths of any counting process with time-dependent intensity.

The thinning algorithm, which can be seen as a generalization of the acceptance/rejection method, works as follows:

1. First consider a bound $\bar{\lambda}$ on the intensity - take $\bar{\lambda} = \sup_{t \in [0,\tau]} \lambda^{IBNR}(t)$.
2. Then draw the sequence (S_k) on $[0, \tau]$ as a Poisson process with intensity $\bar{\lambda}$ - for example by drawing $S_{k+1} - S_k$ as iid increments that are exponentially distributed with parameter $\bar{\lambda}$.
3. For each time S_k, draw a Bernoulli random variable with 1-value probability $\lambda^{IBNR}(S_k)/\bar{\lambda}$ - the 1-value corresponds to the acceptance of the time as part of the sequence $\left(T_n^{IBNR}\right)$.

From a mathematical point of view, the thinning algorithm comes from a key martingale result which, again, can be conveniently formulated by means of Poisson random measures - this is detailed at the end of this subsection. Note that this general mathematical formalism for such procedure goes back to Kerstan (1964) and Grigelionis (1971), although the thinning algorithm is most often referred to Lewis and Shedler (1978) and Ogata (1981) who proposed a more operational description of the process. A precise formulation is given in the context of Hawkes and population processes in Boumezoued (2016b). Note also that more advanced thinning procedures can be used, in order e.g. to improve its efficiency, see e.g. Giesecke et al. (2011). In this paper, we restrict our attention to the original thinning approach which is, in its form, sufficient to address efficiently a wide variety of IBNR simulation contexts.

The thinning representation of counting processes Consider a Poisson point measure $\tilde{N}(\mathrm{d}s, \mathrm{d}\theta)$ with intensity measure $\bar{\lambda}\mathrm{d}s \otimes \mathrm{d}\theta$ on the space $\mathbb{R}_+ \times [0, 1]$, and denote by $(\mathcal{F}_t)$ the canonical filtration generated by $\tilde{N}$. Let (λ_t) be a $(\mathcal{F}_t)$-predictable

process such that a.s. for each $t > 0$, $\lambda_t \leq \bar{\lambda}$. Then the following process (N_t) is a counting process with $(\mathcal{F}_t)$-predictable intensity λ_t:

$$N_t = \int_{(0,t]} \int_{\mathbb{R}_+} \mathbf{1}_{[0,\lambda_s/\bar{\lambda}]}(\theta) \tilde{N}(\mathrm{d}s, \mathrm{d}\theta).$$

Indeed, N is clearly a counting process because each atom of $\tilde{N}$ is weighted 1 or 0. Also, since a.s. $\int_0^t \lambda_s \mathrm{d}s < +\infty$, the martingale property for the Poisson point measure $\tilde{N}$ ensures that $N_t - \int_0^t \int_0^1 \mathbf{1}_{[0,\lambda_s/\bar{\lambda}]}(\theta) \mathrm{d}\theta \bar{\lambda} \mathrm{d}s = N_t - \int_0^t \lambda_s \mathrm{d}s$ is a $(\mathcal{F}_t)$-local martingale, which shows that the counting process (N_t) has intensity (λ_t).

2.3 The likelihood

Since claim occurrence is known only if the claim has been reported, it is crucial to be able to write the likelihood of the data sample made of the set $I^R(\tau)$ reported claims, see Equation (1). The partial observation scheme we face here lies in the family of classical censoring and truncation. More precisely, the occurrence T is observed if and only if $T \leq \tau - U$, therefore the occurrence time is right-truncated by the random variable $\tau - U$.

Having established the occurrence intensity for reported claims, see Equation (4), we are now ready to derive the likelihood for the data sample observed between time 0 and time τ. Denote n_τ^R the number of claims reported before τ. In the individual dataset, one observes the occurrences $(t_n^R)_{n=1}^{n_\tau^R}$ and the associated reporting delays $(u_n^R)_{n=1}^{n_\tau^R}$. The likelihood writes, with convention $T_0^R = t_0^R, U_0^R = u_0^R = 0$,

$$\begin{aligned}
&\mathbb{P}\left(\forall 1 \leq n \leq n_\tau^R, T_n^R = t_n^R, U_n^R = u_n^R, \text{ and } T_{n_\tau^R+1}^R > \tau\right) \\
&= \mathbb{P}\left(T_{n_\tau^R+1}^R > \tau \mid T_{n_\tau^R} = t_{n_\tau^R}\right) \prod_{n=1}^{n_\tau^R} \mathbb{P}\left(T_n^R = t_n^R \mid T_{n-1}^R = t_{n-1}^R\right) \mathbb{P}\left(U_n^R = u_n^R \mid T_n^R = t_n^R\right) \\
&= \exp\left(-\int_{t_{n_\tau^R}}^{\tau} \lambda^R(s)\mathrm{d}s\right) \prod_{n=1}^{n_\tau^R} \lambda^R\left(t_n^R\right) \exp\left(-\int_{t_{n-1}^R}^{t_n^R} \lambda^R(s)\mathrm{d}s\right) \frac{p_{U|t_n^R}(u_n^R)}{p_{U|t_n^R}\left([0, \tau - t_n^R]\right)} \\
&= \exp\left(-\int_0^{\tau} \lambda^R(s)\mathrm{d}s\right) \prod_{n=1}^{n_\tau^R} \lambda^R\left(t_n^R\right) \frac{p_{U|t_n^R}(u_n^R)}{p_{U|t_n^R}\left([0, \tau - t_n^R]\right)}.
\end{aligned}$$

From the formula for λ^R given in Equation (4), the likelihood finally writes

$$\exp\left(-\int_0^{\tau} \lambda(s) p_{U|s}([0, \tau - s])\mathrm{d}s\right) \prod_{n=1}^{n_\tau^R} \lambda\left(t_n^R\right) p_{U|t_n^R}(u_n^R),$$

which leads to the following result:

Proposition 3. *The log-likelihood of the observed sample writes*

$$-\int_0^{\tau} \lambda(s) p_{U|s}([0, \tau - s])\mathrm{d}s + \sum_{n=1}^{n_\tau^R} \left\{\ln\left(\lambda\left(t_n^R\right)\right) + \ln\left(p_{U|t_n^R}(u_n^R)\right)\right\}.$$

Remark 1. *(Dependency between occurrence and reporting) An interesting feature lies in the analysis of the case where delays are independent of occurrences, that is* $p_{U|t}(\mathrm{d}u) \equiv p_U(\mathrm{d}u)$. *In this case, the likelihood writes*

$$\exp\left(-\int_0^\tau \lambda(s)p_U([0,\tau-s])\mathrm{d}s\right)\left\{\prod_{n=1}^{n_\tau^R}\lambda\left(t_n^R\right)\right\}\left\{\prod_{n=1}^{n_\tau^R}p_U(u_n^R)\right\},$$

and it is interesting to note that due to the bias of observing reported claims only, a dependence between the occurrence and delay distribution still exists, and is captured in the first term $\exp\left(-\int_0^\tau \lambda(s)p_U([0,\tau-s])\mathrm{d}s\right)$. *That is why in practice the calibration of claims occurrence frequencies and delay distribution has to be performed simultaneously.*

2.4 Some macro results

In the claims reserving practice, reported claims are grouped into years of occurrence i and year of reporting j; in this paper, both indices are integers starting at value 1, and we consider τ to be an integer (the beginning or end of a given year, or month, etc. depending on the time scale considered). The aim of such upper triangle is, again, to forecast IBNR counts (in the lower triangle). Formally, the *aggregate* incremental claim amount for claims which occurred in year i and are reported in their development year j is defined as follows, (see Hesselager (1995) for an extended study on this segmentation), as it is the case in the actuarial practice: for $1 \leq i \leq \tau$ and $1 \leq j \leq \tau$,

$$X_{i,j} = \sum_{n\geq 1}\mathbf{1}_{T_n\in[i-1,i)}\mathbf{1}_{T_n+U_n\in[i+j-2,i+j-1)} = \sum_{n\geq 1}\mathbf{1}_{(T_n,U_n)\in K_{i,j}}, \tag{6}$$

where

$$K_{i,j} = \{(t,u) : t\in[i-1,i), t+u\in[i+j-2,i+j-1)\}.$$

The *aggregate* cumulative claim amount is then defined as

$$C_{i,j} = \sum_{k=1}^{j} X_{i,k}.$$

From Equation (6) the following representation for incremental numbers holds:

$$X_{i,j} = Q(K_{i,j}). \tag{7}$$

The micro-macro result for claims counts is stated below:

Proposition 4. *The individual claims model described in Subsection 2.1 leads to the aggregate Poisson model, that is:*

- *The random numbers* $(X_{i,j})$ *are independent,*
- *For* $1 \leq i \leq \tau$ *and* $1 \leq j \leq \tau$, $X_{i,j}$ *follows a Poisson distribution with parameter*

$$\begin{cases} \alpha_{i,1} = \displaystyle\int_{i-1}^{i} \lambda(t) p_{U|t}\left([0, i-t)\right) \mathrm{d}t \ \textit{if } j = 1, \\ \alpha_{i,j} = \displaystyle\int_{i-1}^{i} \lambda(t) p_{U|t}\left([i+j-t-2, i+j-t-1)\right) \mathrm{d}t \ \textit{for } j \geq 2. \end{cases} \tag{8}$$

Proof of Proposition 4 As the sets $(K_{i,j})$ are disjoint, then by Definition 1 (i), the random variables $(X_{i,j})$ derived in Equation (7) are independent. By Definition 1 (ii), the random variable $Q(K_{i,j})$ has a Poisson distribution with parameter $q(K_{i,j})$, where the intensity measure q is defined in Equation (2). This concludes the proof.

With the previous result, it is clear that for counts, the aggregate model is still Poisson, which largerly justifies why this model is mainly used for claims counts in practical applications. It can be emphasized here that the individual arrival process has a general time-dependent intensity, and that the (time-dependent) distribution of reporting delays is not specified, which shows the remarkable micro-macro consistency feature recovered here. In the following, we detail a particular case of interest.

Let us first recall that the original Poisson model, which leads to the same reserves as the Chain-Ladder method, is in fact specified in a separable version of Equation (8) such that $\alpha_{i,j} = \beta_i \gamma_j$ with some parameters β_i and γ_j, see Wüthrich and Merz (2008), Section 2.3 - the separable form is the key assumption to recover the same reserves as Chain-Ladder. A question of practical implications is whether this separable assumption is valid - in our setting, this turns out to look for 'micro' specifications which lead to the separable assumption. Interestingly, assuming a time-independent delay distribution $p_{U|t} \equiv p_U$ is not sufficient to recover a separable form. This shows that in general, a micro model with an homogeneous delay distribution does not lead to the separable Poisson assumption. Indeed, in this case the Poisson parameter in Equation (8) can be rewritten as, by a simple change of variable (for $j \geq 2$ here),

$$\alpha_{i,j} = \int_0^1 \lambda(i-1+s) p_U\left([j-1-s, j-s)\right) \mathrm{d}s,$$

which shows that the dependency between occurrence year i and development year j lies in the integral of a product of a i-dependent and a j-dependent function. A general sufficient condition for the separable form to appear (see again Hesselager (1995)), is that the occurrence intensity can be written as an occurrence-year

magnitude times a function on $[0,1]$ driving e.g. the intra-year seasonality, which formally covers the case where there exists some reference intensity $(\lambda_0(s))_{0\leq s\leq 1}$ and constants $(c_i)_{1\leq i\leq \tau}$such that for any $1\leq i\leq \tau$ and $0\leq s\leq 1$, $\lambda(i-1+s)=c_i\lambda_0(s)$. Indeed, the Poisson parameter can then be written in the separable form (for $j\geq 2$),

$$\alpha_{i,j}=c_i\int_0^1 \lambda_0(s)p_U\left([j-1-s,j-s)\right)\mathrm{d}s.$$

In other words, the separable form of the aggregate Poisson model is valid if the intra-year seasonality is similar from one occurrence year to the next.

Parameter uncertainty As stated in the previous result, the number of IBNR claims can be derived, in aggregate, as a Poisson random variable with given parameter. This is the so-called *process error* induced by the individual model. To measure *parameter uncertainty*, or equivalently *estimation error*, one has to address the impact of estimator randomness in the resulting output.

Let us denote generally X the total future claims number (or cash flows) - aim is to estimate the reserve which corresponds to the expectation of X - the estimator of the reserve is denoted $\hat{X}$. To properly decompose the so-called mean square error of prediction defined in the following, one needs to specify the probability measures at stake:

- $\mathbb{P}$ the probability measure related to the uncertainty in the parameters (estimation error) $\lambda(t)$ and $p_{U|t}(\mathrm{d}u)$ - these are estimated based on information on reported claims provided by the random measure $Q^R(\mathrm{d}t,\mathrm{d}u)$,
- $\mathbb{Q}$ the probability measure related to the random measure $Q^{IBNR}(\mathrm{d}t,\mathrm{d}u)$, which drives in particular the random number of IBNR (process variance).

Since by construction of the individual model the number of reported and IBNR claims are independent, this allows us to write the mean square error of prediction under the product measure $\mathbb{P}\otimes\mathbb{Q}$ as

$$\begin{aligned}
&MSEP(\hat{X})\\
&=\mathbb{E}_{\mathbb{P}\otimes\mathbb{Q}}\left[(\hat{X}-X)^2\right],\\
&=\mathbb{E}_{\mathbb{P}\otimes\mathbb{Q}}\left[(\hat{X}-\mathbb{E}_{\mathbb{Q}}[X]+\mathbb{E}_{\mathbb{Q}}[X]-X)^2\right],\\
&=\mathbb{E}_{\mathbb{Q}}\left[(X-\mathbb{E}_{\mathbb{Q}}[X])^2\right]+\mathbb{E}_{\mathbb{P}}\left[\left(\hat{X}-\mathbb{E}_{\mathbb{Q}}[X]\right)^2\right]+2\mathbb{E}_{\mathbb{P}\otimes\mathbb{Q}}\left[\left(\hat{X}-\mathbb{E}_{\mathbb{Q}}[X]\right)\left(\mathbb{E}_{\mathbb{Q}}[X]-X\right)\right],
\end{aligned}$$

where the last term reduces to zero as, again by independence,

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}\otimes\mathbb{Q}}\left[\left(\hat{X}-\mathbb{E}_{\mathbb{Q}}[X]\right)\left(\mathbb{E}_{\mathbb{Q}}[X]-X\right)\right]\\
&=\mathbb{E}_{\mathbb{P}}\left[\hat{X}-\mathbb{E}_{\mathbb{Q}}[X]\right]\mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}[X]-X\right]\ \text{ with } \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}[X]-X\right]=0.
\end{aligned}$$

This shows that the mean square error of prediction writes

$$MSEP(\hat{X}) = Var_{\mathbb{Q}}(X) + \mathbb{E}_{\mathbb{P}}\left[\left(\hat{X} - \mathbb{E}_{\mathbb{Q}}[X]\right)^2\right],$$

and is decomposed into a pure process variance (first term) and a squared bias which measures parameter risk (second term).

As the likelihood has been derived in this model, a convenient way of accounting for parameter uncertainty is to rely on asymptotic properties of maximum likelihood estimators, whose variance-covariance matrix can be computed as the opposite of the Hessian inverse. Such approach is discussed here in the standard parametrization of the individual model, although it can be extended to any piecewise-constant or continuous parametric specification of each model component.

In the standard setting, we consider homogenous Poisson claims arrivals, that is $\lambda(t) = \lambda$, as well as exponential and time-invariant reporting delay distribution, that is $p_U(\mathrm{d}u) = \theta \exp(-\theta u)\mathrm{d}u$. In this framework, the log-likelihood writes (see Proposition 3):

$$\ln \mathcal{L}(\lambda, \theta) = n_\tau^R \ln(\lambda) - \lambda\tau + \frac{\lambda}{\theta}(1 - e^{-\theta\tau}) + n_\tau^R \ln(\theta) - \theta \sum_{n=1}^{n_\tau^R} u_n. \tag{9}$$

As stated in Remark 1, the dependency between the occurrence frequency λ and the delay parameter θ remains, which represents the observation bias, here through the middle term of the form $\frac{\lambda}{\theta}(1 - e^{-\theta\tau})$. The result about the mean-square error of prediction is stated below - it follows from a direct computation of the variance-covariance matrix Σ (opposite of the Hessian inverse) and the application of the Delta method; note that the first term refers to process error whereas the second term relates to estimation error.

Proposition 5. *In the standard parametrization of the individual claims model, the total mean-square error of prediction can be approximated as*

$$\widehat{MSEP}(\hat{X}) = \phi(\hat{\lambda}, \hat{\theta}) + \nabla\phi(\hat{\lambda}, \hat{\theta})\Sigma(\hat{\lambda}, \hat{\theta})\left(\nabla\phi(\hat{\lambda}, \hat{\theta})\right)^T,$$

where $\phi(\lambda, \theta) = \frac{\lambda}{\theta}\left(1 - e^{-\theta\tau}\right)$, *then* $\nabla\phi(\lambda, \theta) = \left(\frac{1}{\lambda}\phi(\lambda, \theta), -\frac{1}{\theta}\phi(\lambda, \theta) + \frac{\lambda\tau}{\theta}e^{-\theta\tau}\right)$, *and the variance-covariance matrix writes*

$$\frac{1}{D(\lambda, \theta)}\begin{bmatrix} \frac{n_\tau^R}{\theta^2} + \frac{\lambda}{\theta^3}\left(e^{-\theta\tau}(\theta^2\tau^2 + 2\theta\tau + 2) - 2\right) & \frac{1}{\theta^2}\left(e^{-\theta\tau}(1 + \theta\tau) - 1\right) \\ \frac{1}{\theta^2}\left(e^{-\theta\tau}(1 + \theta\tau) - 1\right) & \frac{n_\tau^R}{\lambda^2} \end{bmatrix},$$

with the determinant $D(\lambda, \theta)$ *given by*

$$D(\lambda, \theta) = \frac{n_\tau^R}{\lambda^2}\left\{\frac{n_\tau^R}{\theta^2} + \frac{\lambda}{\theta^3}\left(e^{-\theta\tau}(\theta^2\tau^2 + 2\theta\tau + 2) - 2\right)\right\} - \frac{1}{\theta^4}\left\{1 - e^{-\theta\tau}(1 + \theta\tau)\right\}^2.$$

Remark 2. *The result can be adapted for any piecewise constant parameters, in which framework the same structural form can be identified.*

2.5 Numerical illustration

2.5.1 A numerical experiment

Let us start a numerical experiment with parameters $\lambda = 500$ (in average, 500 claims occur each year) and $\theta = 1/3$ (the reporting delay is 3 years in average). This arbitrary setting allows us to easily illustrate key facts about the toy model. In Figure 1, the empirical distribution of observed occurrences and delays is represented. At this stage, the sample bias appears: although the initial occurrences are distributed as an homogeneous Poisson process, the occurrences of reported claims are not Poisson anymore. Therefore, it is not possible to directly fit any distribution on that of occurrences or delays. Instead, by maximizing the log-likelihood (9), we are able to compute MLEs for the frequency parameter λ and the delay parameter θ, see again Figure 1. This example illustrates how the true underlying parameters relate to the observed empirical distributions. In particular, it is shown in Figure 1 that the distance to the true frequency increases with time (due to the existence of reporting delays), this distance representing IBNR claims. In addition, it is shown that the true reporting delay distribution results in transportation of mass from short to longer reporting delays compared to the empirical distribution, due to the fact that larger delays are less likely to be observed - this will be illustrated on real data in the next paragraph.

Note that based on these parameters, the number of IBNR follows a Poisson distribution with a given parameter which takes a simple form in this numerical experiment; it is indeed equal to

$$\int_0^\tau \lambda(s)\left(1 - p_{U|s}([0,\tau - s])\right) \mathrm{d}s = \frac{\lambda}{\theta}\left(1 - e^{-\theta\tau}\right).$$

In order to give a view on estimation error, we repeat the simulation of artificial data and the maximization step, producing a sample of 1000 MLEs; these are depicted in Figure 2.

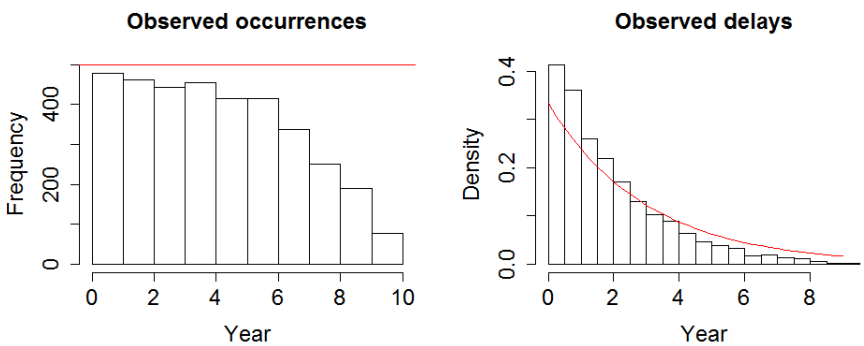

Figure 1: Distribution of observed occurrences (left) and delays (right) - each graph includes the underlying theoretical distribution in red.

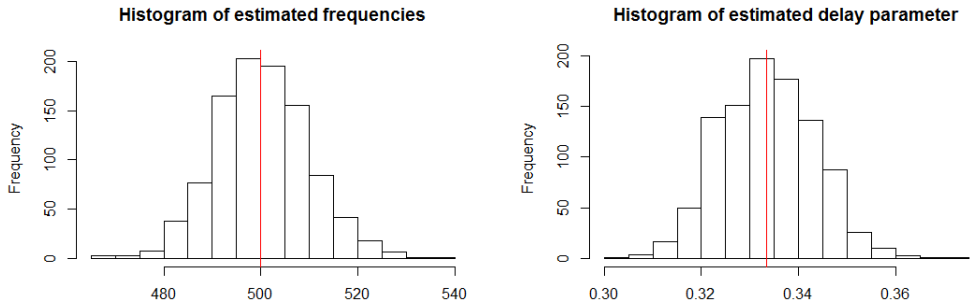

Figure 2: Distribution of maximum likelihood estimators based on 1000 simulations - occurrence frequency λ (left) and delay parameter θ (right)

2.5.2 A practical example on personal injury data

The real dataset 'ausautoBI8999' is obtained from the R package **CASdatasets** (Dutang 2016). It contains information on 22,036 settled personal injury insurance claims in Australia. These claims arose from accidents occurring from July 1989 through to January 1999. Note that claims settled with zero payment are not included. For the purpose of illustration, the model is calibrated on the 3-years observation period from month 50 to month 86, while considering only claims which occurred and were reported in the period. A piecewise constant arrival intensity $\lambda(t)$ on bands of 3 months is chosen to account for non-stationarity, and an empirical specification of the delay distribution is set, augmented with an exponential tail as

$$p_U(\mathrm{d}u) = \sum_{k=1}^{n_0} p_k \mathbf{1}_{[k-1,k)}(u) + \mathbf{1}_{[n_0,\infty)}(u) \left(1 - \sum_{k=1}^{n_0} p_k\right) \alpha \exp\left(-\alpha(u - n_0)\right),$$

where we set $n_0 = 5$ the number of first months for which an empirical distribution is taken. The parameters obtained by the maximization of the log-likelihood established in Proposition 3 are represented in Figure 3. This shows that the model predicts a relatively stable claim occurrence for the last months, although a naïve observation of the reported claims subset would not lead to the same conclusion. As a consequence, it is shown in Figure 3 that the observed delay distribution does not match the fitted one, as the likelihood accounts for unobserved (higher) delays - this is observed in the figure as the density of the smallest delays is reduced in the final distribution and transported in the tail.

In this setting, the joint parameter estimation routine leads to 1,501 IBNR claims in average as an application of formula (3) (whereas the separate fit of the observed occurrence and delay distributions would have led to an expected number of 565). The dispersion of IBNR due to process error can be easily drawn based on the macro consistency, see Proposition 1 - this is shown in Figure 4. This provides an input

of the same 'nature' than an aggregate model would do. More original (and very useful in several practical applications) is the possibility to simulate the times at which those IBNR would have occurred, that is to complete the observed dataset (of reported claims) by that related to IBNR. One stochastic path of IBNR occurrence times performed by the thinning algorithm described in Subsection 2.2 is illustrated in Figure 5. As expected, this scenario shows that most IBNR occurred in the last months, in coherence with the dispersion shape of the IBNR intensity in Equation (5). In addition, it is possible to add the uncertainty on the parameters to the simulation to produce a forecast comparable to aggregate alternatives such as that of Mack (1993) - this has been discussed in Subsection 2.4. The final stochastic forecast including both process and estimation errors is represented in Figure 6.

Finally, we summarize in Table 1 some comparison with Chain-Ladder estimates and the prediction, process and estimation errors by Mack (1993) on triangles with varying granularity - that is we change the aggregation step form 6 to 1 months in the triangle. The numerical computations are performed based on the R package **ChainLadder** (Gesmann et al. 2017). A first result is that the Best Estimate values remain in the same orders of magnitude, although the Chain Ladder method always leads to an overestimation of the expected IBNR compared to the individual model, and moreover that the value depends on the aggregation step (although the underlying data is similar), as it remains to formulate different assumptions on the claims development in each case. In addition, one notices that process error is higher for the Mack method ; the individual model takes here advantage of its pure Poisson formulation. Finally, the estimation error for the Mack model is, as expected, very high for high aggregation step as it leads to fewer data points in the triangle. While decreasing the aggregation step, one increases the number of points which makes estimation error reduces (although the number of parameters increases as well, but not as much). In all cases, the coefficient of variation for the number of IBNR stays at a higher level compared to that of the individual model.

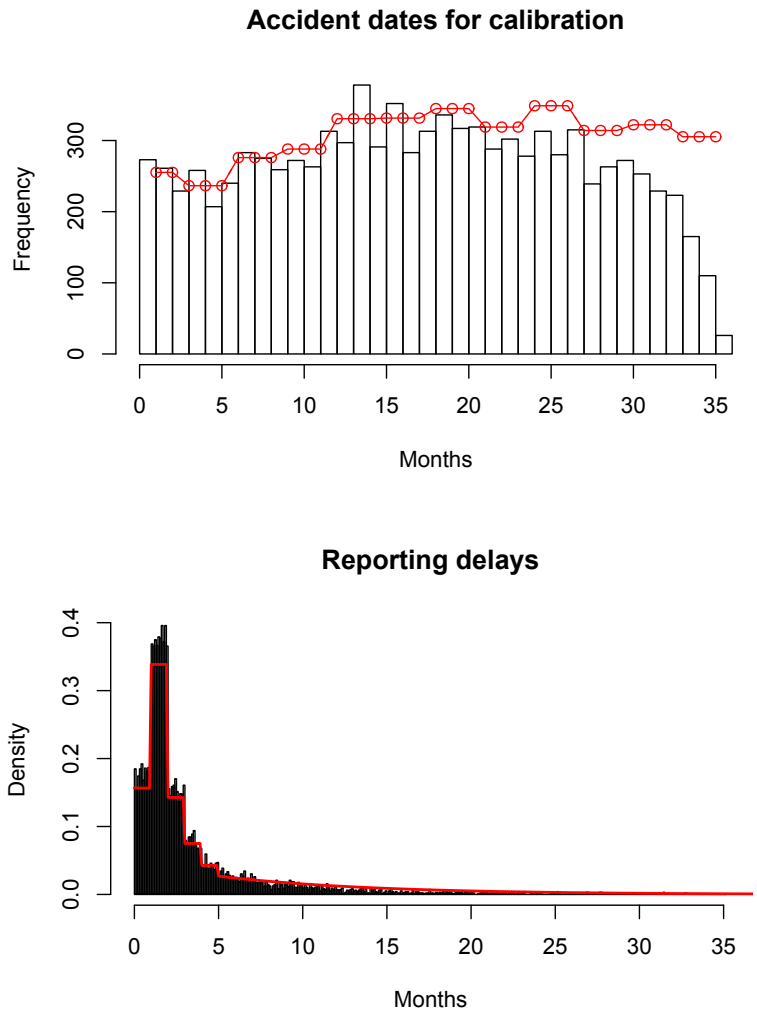

Figure 3: Observed and fitted distribution related to the subsample extracted from the dataset 'ausautoBI8999' is obtained from the R package **CASdatasets**.

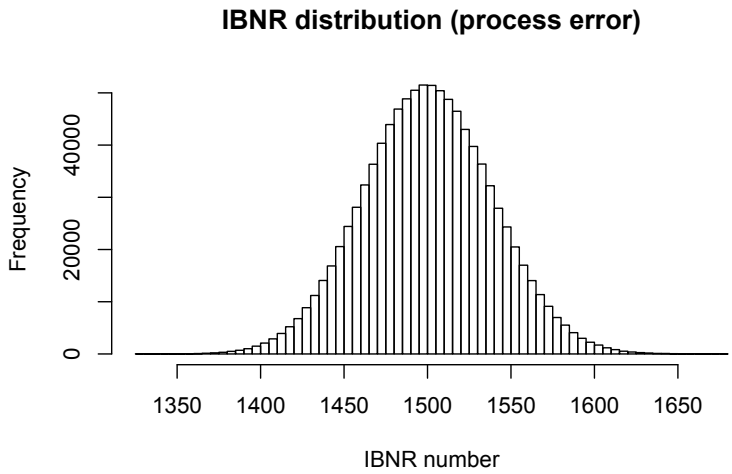

Figure 4: Distribution of IBNR number (process error).

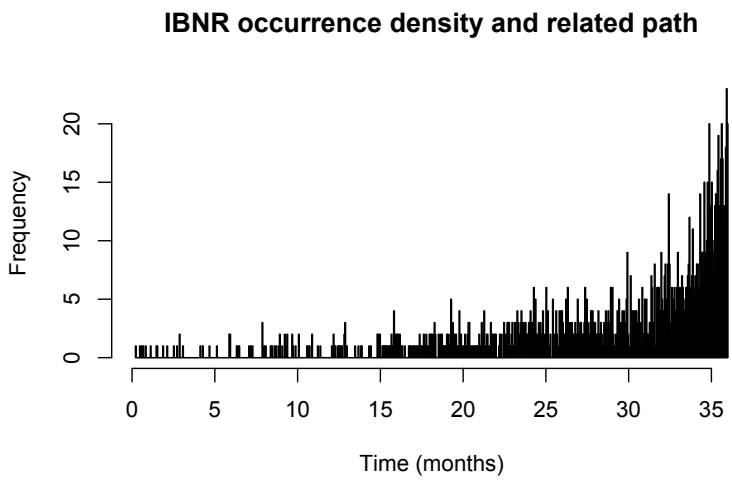

Figure 5: One stochastic path of IBNR occurrence times.

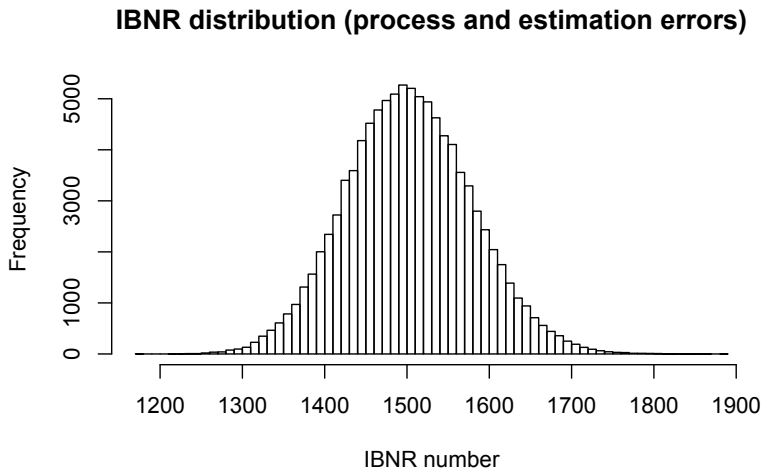

Figure 6: Distribution of IBNR number (process and estimation errors).

Method	Best Estimate	Process error	Estimation error	Prediction error	CV(IBNR)
Individual	1,501	39	66	76	5%
Mack (6)	1,617	335	215	398	25%
Mack (3)	1,618	207	120	240	15%
Mack (2)	1,676	153	91	178	11%
Mack (1)	1,744	122	69	140	8%

Table 1: Best estimate of IBNR number and related process, estimation and prediction errors (standard deviations). The Mack model is applied to a triangle with varying aggregation window, expressed in months in parenthesis.

3 Payments and settlement

3.1 Modelling the claims development

In this part, we develop the modelling framework allowing for claims payments development over time, which enriches the previous occurrence and reporting model developed in Section 2.

In the literature, several studies based on triangles proposed a model in order to account for incurred amounts (in addition to paid) which include the reserve case estimates set by expert judgement, in order to guarantee a practical constraint to match ultimate paid and incurred forecasts - see Mack and Quarg (2004), Merz and Wüthrich (2010), Happ and Wüthrich (2013), as well as Miranda et al. (2012) and Martínez Miranda et al. (2015). From an individual perspective, in the spirit

of the modelling framework developed in Pigeon et al. (2013), Pigeon et al. (2014) proposed an extension to paid-incurred data. In this paper, we rather focus on pure paid claims and leave the extension of the framework surveyed here for further research.

In this section, the time origin 0 corresponds to a given claim's reporting. Our aim is to describe the claims development process from reporting to closing; we rely here on the convenient specification introduced in Antonio and Plat (2014). The claims development process is made of random times (V_k); at each random time, a mark E_k is generated. The mark E_k can take its values in the space $\{1, 2, 3\}$; let us introduce these three states:

- $E_k = 1$ indicates that the claim is settled at time V_k, without any payment at V_k
- $E_k = 2$ indicates that the claim is settled at V_k with a payment,
- $E_k = 3$ indicates that a payment occurs at V_k, without settlement of the claim.

Three intensity functions h_1, h_2 and h_3 are introduced to govern the frequency at which events of type 1, 2 or 3 occur since reporting of the claim. If an event of type 2 or 3 occurs, a payment P_k is generated with distribution which may depend on the time V_k and the type of event, that is on the fact that the claim is closed or not.

3.1.1 Definition with a marked (non-homogenous and absorbed) Poisson process

Inter-arrival times can be seen as a competing risk model; we construct recursively the sequence (V_k) with $V_0 = 0$ and $V_{k+1} = \epsilon_k + V_k$, defined as follows. First introduce

- S_k^1 a random time with rate $h_1(V_k + t)$, associated with event 1,
- S_k^2 a random time with rate $h_2(V_k + t)$, associated with event 2,
- S_k^3 a random time with rate $h_3(V_k + t)$, associated with event 3.

Consider the random times independent from each other for any kind of event and any k. Then construct $\epsilon_k = \min(S_k^1, S_k^2, S_k^3)$ and specify E_k such that $E_k = i$ if and only if $\epsilon_k = S_k^i$. Standard results ('clock lemma') show that the rate of ϵ_k is $\sum_{i=1}^3 h_i(V_k + t)$, and that conditionally on V_k and ϵ_k, the probability that $E_k = i$ is

$$\frac{h_i(V_k + \epsilon_k)}{\sum_{j=1}^3 h_j(V_k + \epsilon_k)}.$$

This then proves that for a given claim, the sequence of the (V_k, E_k) occurs as a marked Poisson process with occurrence rate $\sum_{i=1}^{3} h_i(t)$ and mark probability

$$p_{e|t}(\mathrm{d}e) = \frac{1}{\sum_{k=1}^{3} h_k(t)} \sum_{i=1}^{3} h_i(t)\delta_i(\mathrm{d}e),$$

where $\delta_i(\mathrm{d}e)$ is the Dirac mass at state i.

Finally, the true process is stopped if the generated mark is 1 or 2; one can define S the settlement delay as the stopping time $S = \inf\{V_k : E_k \in \{1, 2\}\}$, and consider the final development process

$$\{(V_k, E_k), V_k \leq S\}.$$

This construction lies in the framework of competing risk models. Although it provides an intuitive representation of inter-arrival times, this framework is not well suited to derive the closed-form formulas at stake. In the next subsection, we propose a continuous time Markov representation of the claims development, which will then allow us to rely on general results by Hesselager (1994).

3.1.2 Definition with a Markov process

The difficulty in defining the claims development through a simple Markov process lies in the fact that 'state' 3 (payments without settlement) can occur at consecutive times. To overcome this issue, let us consider (X_t) a Markov process taking values in state space $\mathbb{N}^*$, and introduce the following new specification: if a jump occurs at time V_k, then

- $X_{V_k} = 1$ indicates that the claim is settled at time V_k, without any payment at V_k,
- $X_{V_k} = 2$ indicates that the claim is settled at V_k with a payment,
- $X_{V_k} = j$, for $j \geq 3$, indicates that a payment occurs at V_k, without settlement of the claim.

We consider that $X_0 = 3$ (the construction holds if we choose another integer higher than 3). Let us introduce the following transition intensities: for $j \geq 3$:

- $\lambda_{j,1}(t) = h_1(t)$,
- $\lambda_{j,2}(t) = h_2(t)$,
- $\lambda_{j,j+1}(t) = h_3(t)$.

All other transition intensities are zero. Then one can prove that the construction leads to a similar development process as the one described in 3.1.1.

Remark 3. *(Dealing with reopenings) Once stated in a general Markov setting with countable state space, it is possible to enrich the model to account for other events than occurrence, reporting, payments and settlement. A major issue in many datasets is driven by reopenings, which are observed on several claims path. A specific reopening frequency $h_4(t)$ can be introduced, which allows to go from settlement (state 1 or 2) to new payments (state 3) leading to an other cycle of new payments (states 4 and more) until settlement. In the model, this amounts to consider that transition intensities $\lambda_{1,3}(t)$ and $\lambda_{2,3}(t)$ are non zero, equal to $h_4(t)$. Of course, some constraints may be applied to h_4 to guarantee convergence and avoid infinite payment cycles in the calibration and projection. A basic suggestion is to set $h_4(t) = 0$ for t greater than some time t_0 since reporting, after which the claim can not be reopened.*

3.2 The likelihood

We identify each claim by the index n, given by its observed occurrence T_n ; for a given claim n we denote

- $V_k^{(n)}$ the time since reporting of the kth event in the payments development,
- $S^{(n)}$ the settlement delay for claim n,
- $\tau^{(n)}$ the time during which the claims development is observed, which can be expressed as $\tau^{(n)} = \min\left(S^{(n)}, \tau - T_n - U_n\right)$,
- $E_k^{(n)}$ the associated event type, taking values in $\{1, 2, 3\}$,
- $\delta_k^{(n)}(i)$ the indicator that the kth event for claim n is of type $i \in \{1, 2, 3\}$,
- $X_k^{(n)}$ the payments (if any) related to the kth event,
- $P(.)$ the probability density function of the payments distribution.

Proposition 6. *The likelihood for the claims development component writes*

$$\prod_{n\geq 1} \exp\left(-\int_0^{\tau^{(n)}} (h_1 + h_2 + h_3)(u)\mathrm{d}u\right) \prod_{k\geq 1} h_1(V_k^{(n)})^{\delta_k^{(n)}(1)} h_2(V_k^{(n)})^{\delta_k^{(n)}(2)} h_3(V_k^{(n)})^{\delta_k^{(n)}(3)}$$
$$\times \prod_{n\geq 1}\prod_{k\geq 1} \left\{\delta_k^{(n)}(1) + P(X_k^{(n)})\left(\delta_k^{(n)}(2) + \delta_k^{(n)}(3)\right)\right\}. \tag{10}$$

Remark 4. *The use of the asymptotic normality of maximum likelihood estimators is still possible in this framework, in a parametric or non-parametric (piecewise constant) specification of the intensity parameters. The structure of the log-likelihood shows however that the parameter estimates for each type of event are decorrelated*

(in the precise present specification)- this contrasts with the occurrence and reporting parameters, as discussed in Subsection 2.3, for which we proposed a more detailed discussion.

3.3 Simulation algorithm

The simulation algorithm of each claims path lies, again, in the field of general thinning procedures for marked population processes - we refer the reader to Boumezoued (2016a) for more details on the formalism and the related thinning representation. The algorithm is described here in the spirit of the Marked Poisson process representation of the claims path presented in 3.1.1, although it can also be used to generate general Markov models closer to the construction in 3.1.2. The algorithm works as follows for generating paths from a given time (set at zero) after reporting to any arbitrary time ν:

1. First consider bounds $\bar{h}_i$ for each event-specific intensity function h_i, $i \in \{1, 2, 3\}$, such as $\bar{h}_i = \sup_{t\in[0,\nu]} h_i(t)$.
2. Then draw a sequence (U_k) on $[0, \nu]$ as a Poisson process with intensity $\bar{h}_1+\bar{h}_2+\bar{h}_3$ - for example by drawing $U_{k+1} - U_k$ as iid increments that are exponentially distributed with parameter $\bar{h}_1 + \bar{h}_2 + \bar{h}_3$.
3. For each time U_k, draw a Bernouilli random variable with 1-value probability $(h_1(U_k) + h_2(U_k) + h_3(U_k)) \,/\, \left(\bar{h}_1 + \bar{h}_2 + \bar{h}_3\right)$ - the 1-value corresponds to the acceptance of the U_k as a time of event (1, 2 or 3) for the individual claims path.
4. To determine the type of event, draw another random variable with values in $\{1, 2, 3\}$ with probability for the ith value given by $h_i(U_k)/\left(h_1(U_k) + h_2(U_k) + h_3(U_k)\right)$
5. In case of event related to payment (2 or 3), draw the payment with the appropriate distribution $P(.)$.

3.4 Some macro results

3.4.1 Closed-form formulas for a single claim development

Hesselager (1994) described the path of a given claim as a general continuous time Markov process with state space $\mathbb{N}$, and derived semi-explicit first and second order moments for $X^{(t)}(u, v)$, which denotes the total payments in $(t+u, t+v]$ for a claim which occurred at time t. In the model proposed by Hesselager (1994), the claims path (after reporting) from one state to another is described by a process $S^{(t)}(u)$ with values in $\mathbb{N}^*$ (we removed 0 which represents the IBNR state), with transition

probabilities $p_{mn}^{(t)}(u,v)$ and associated transition intensities $\lambda_{mn}^{(t)}(u)$. Payments are made at the transition times of this dynamics; for a transition $m \to n$ at a time u since reporting, a payment $Y_{mn}^{(t)}(u)$ is drawn, payments being independent from each other, with mean $y_{mn}^{(t)}(u)$ and standard deviation $\sigma_{mn}^{(t)}(u)$. In the following analysis of a single claim, we omit without any restriction the index t. We will rely on the following results by Hesselager (1994).

Proposition 7. *For all $j \in \mathbb{N}^*$, the mean and variance of the payments related to the claim in state j after u times units of development can be expressed as*

$$\mathbb{E}\left[X(u,\infty) \mid S(u)=j\right] = \sum_{m\in\mathbb{N}^*}\sum_{n\in\mathbb{N}^*\setminus\{m\}}\int_u^\infty p_{jm}(u,v)\lambda_{mn}(v)y_{mn}(v)\mathrm{d}v, \tag{11}$$

$$\mathrm{Var}\left(X(u,\infty) \mid S(u)=j\right) = \sum_{m\in\mathbb{N}^*}\sum_{n\in\mathbb{N}^*\setminus\{m\}}\int_u^\infty p_{jm}(u,v)\lambda_{mn}(v)\left\{\sigma_{mn}^2(v)+r_{mn}^2(v)\right\}\mathrm{d}v, \tag{12}$$

where

$$r_{mn}(v) = y_{mn}(v) + \mathbb{E}\left[X(v,\infty) \mid S(v)=n\right] - \mathbb{E}\left[X(v,\infty) \mid S(v)=m\right]. \tag{13}$$

In the next paragraphs, we derive the formulas for the mean and variance (process error) under our particular (non-homogeneous) Markovian setting as described in Subsection 3.1.2. Let us recall that in this setting, intensities can be set at zero, except for the following with $j \geq 3$: $\lambda_{j,1}(t) = h_1(t)$ (settlement without payment), $\lambda_{j,2}(t) = h_2(t)$ (settlement with payment) and $\lambda_{j,j+1}(t) = h_3(t)$ (payment without settlement).

Expected payments along a claim path In our setting, it is clear that $X(u,\infty)$ is zero given that the claim is settled, that is given that $S(u)=1$ or $S(u)=2$. Moreover, without any change in the path distribution, it is sufficient to study $X(u,\infty)$ given that $S(u)=3$. Based on Proposition 7, we then get after some computation:

$$\mathbb{E}\left[X(u,\infty) \mid S(u)=3\right] = \int_u^\infty \Big\{h_2(v)y_2(v)+h_3(v)y_3(v)\Big\}\Big\{\sum_{m\geq 3} p_{3m}(u,v)\Big\}\mathrm{d}v.$$

Note that $\sum_{m\geq 3} p_{3m}(u,v)$ is the probability that the Markov process stays in the set $\{3,4,....\}$, whose transition to states 1 and 2 are h_1 and h_2 respectively ; therefore $\sum_{m\geq 3} p_{3m}(u,v) = \exp\left(-\int_s^u (h_1(v)+h_2(v))\mathrm{d}v\right)$ - this leads to the following result:

Proposition 8. *The expected payments after some time s for a single claims path can be written as*

$$\begin{aligned}\mu(s) &= \mathbb{E}\left[X(s,\infty) \mid S(s)=3\right] \\ &= \int_s^\infty \left\{\big(y_2(u)h_2(u)+y_3(u)h_3(u)\big)\exp\left(-\int_s^u (h_1(v)+h_2(v))\mathrm{d}v\right)\right\}\mathrm{d}u.\end{aligned} \tag{14}$$

Remark 5. *An alternative way of computing the quantities* $\mu_k(u) := \mathbb{E}\left[X(u,\infty) \mid S(u) = k\right]$, *as suggested in Hesselager (1994) (see Remark 3.1), is to rely on Thiele's differential equation given as*

$$\mu_k'(s) = -\sum_{j\in\mathbb{N}^*\setminus\{k\}} \lambda_{kj}(s)(y_{kj}(s) + \mu_j(s) - \mu_k(s)).$$

This equation, stated here in a general Markov framework, provides a convenient numerical scheme for computing transition probabilities in more general individual claims models. This can be proved based on the representation given in Equation (11) - as the proof does not appear in Hesselager (1994), we propose to detail it in Appendix to guarantee self-consistency of the results included in the present survey. In our case, denoting $\mu_3 \equiv \mu$, *Thiele's differential equation reduces to*

$$\mu'(s) = -\Big(y_2(s)h_2(s) + y_3(s)h_3(s)\Big) + \Big(h_1(s) + h_2(s)\Big)\mu(s), \tag{15}$$

whose solution satisfies for $s \leq t$

$$\begin{aligned}\mu(s) = \mu(t)\exp\left(-\int_s^t (h_1(u) + h_2(u))\mathrm{d}u\right) \\ + \int_s^t \left\{\big(y_2(u)h_2(u) + y_3(u)h_3(u)\big)\exp\left(-\int_s^u (h_1(v) + h_2(v))\mathrm{d}v\right)\right\}\mathrm{d}u.\end{aligned} \tag{16}$$

Letting $t \to \infty$, *we recover the closed form in Equation (14) for the expected payments on a single claim path.*

Remark 6. *In the homogeneous Markovian setting, under which* $h_i(s) \equiv h_i$ *and* $y_i(s) \equiv y_i$, *the expected payments along a claims path write*

$$\mu(s) = \mu(0) = \frac{y_2h_2 + y_3h_3}{h_1 + h_2}.$$

Variance of claims payments From Equation (12), and by analogous reasoning as in the previous part on expected payments, one can state the following result:

Proposition 9. *The variance of total payments after some time* s *along a single claims path can be written as*

$$\gamma(s) = \mathrm{Var}\left(X(s,\infty) \mid S(s) = 3\right) = \int_s^\infty H(u)\exp\left(-\int_s^u (h_1(v) + h_2(v))\mathrm{d}v\right)\mathrm{d}u. \tag{17}$$

where the function H, *which depends on time since reporting, is given by*

$$H = h_1\mu^2 + h_2\left(\sigma_2^2 + (y_2 - \mu)^2\right) + h_3\left(\sigma_3^2 + y_3^2\right). \tag{18}$$

Remark 7. *As in Remark 5, one can alternatively use Thiele's differential equation satisfied by the variance of payments; denote* $\gamma_k(u) = \mathrm{Var}\left(X(u,\infty) \mid S(u) = k\right)$*, then*

$$\gamma_k'(s) = -\sum_{j\in\mathbb{N}^*\setminus\{k\}} \lambda_{kj}(s)\left(\sigma_{kj}^2(s) + r_{kj}^2(s) + \gamma_j(s) - \gamma_k(s)\right),$$

where $r_{kj}(s)$ *is defined in Equation (13). Denote* $\gamma(s) \equiv \gamma_3(s)$*; in our case, Thiele's differential equation reduces to*

$$\gamma'(u) = -H(u) + \gamma(u)\left(h_1(u) + h_2(u)\right),$$

which allows to recover the result in Proposition 9.

Remark 8. *In the homogeneous Markovian setting, under which* $h_i(s) \equiv h_i$ *and* $y_i(s) \equiv y_i$*, the variance of the total payments along a claims path writes*

$$\gamma(s) \equiv \gamma = \frac{h_1\mu^2 + h_2\left(\sigma_2^2 + (y_2-\mu)^2\right) + h_3(\sigma_3^2 + y_3^2)}{h_1 + h_2},$$

where

$$\mu = \frac{y_2h_2 + y_3h_3}{h_1 + h_2}.$$

3.4.2 Closed-form formulas for the claims population

IBNR reserve Recall that times and reporting delays are given as a Poisson point measure $Q(\mathrm{d}t, \mathrm{d}u)$ with intensity measure $q(\mathrm{d}t, \mathrm{d}u) = \lambda(t)\mathrm{d}t \otimes p_{U|t}(\mathrm{d}u)$. Therefore the ultimate payments for IBNR claims at observation time τ can be expressed as

$$X_\tau^{IBNR} := \int_0^\tau \int_{\tau-s}^\infty X^{(s)}(0,\infty)Q(\mathrm{d}s, \mathrm{d}u). \tag{19}$$

The expected payments are therefore given as

$$\mathbb{E}\left[X_\tau^{IBNR}\right] = \int_0^\tau \int_{\tau-s}^\infty \mu^{(s)}(0)\lambda(s)p_{U|s}(\mathrm{d}u)\mathrm{d}s,$$

where $\mu^{(s)}(0) \equiv \mu(0)$ is expressed in Equation (14). This leads to the following result:

Proposition 10. *The total IBNR reserve writes*

$$\begin{aligned}\mathbb{E}\left[X_\tau^{IBNR}\right] &= \left\{\int_0^\tau \mathrm{d}s\lambda(s)\int_{\tau-s}^\infty p_{U|s}(\mathrm{d}u)\right\} \\ &\times \int_0^\infty \left\{\left(y_2(u)h_2(u) + y_3(u)h_3(u)\right)\exp\left(-\int_0^u (h_1(v)+h_2(v))\mathrm{d}v\right)\right\}\mathrm{d}u.\end{aligned} \tag{20}$$

Based on the representation (19), we can also derive the variance of the IBNR payments as

$$\mathrm{Var}\left(X_\tau^{IBNR}\right) = \int_0^\tau \int_{\tau-s}^\infty \mathbb{E}\left[X^{(s)}(0,\infty)^2\right]\lambda(s)p_{U|s}(\mathrm{d}u)\mathrm{d}s. \tag{21}$$

Remark 9. *Note that this is not a trivial result; it holds according to the fact that the family of processes* $\left(X^{(s)}(u,\infty)\right)$ *for occurrence times* s *is independent from the Poisson point measure* Q*. To prove it, let us use this independence and construct an extended Poisson measure* $M(\mathrm{d}s,\mathrm{d}u,\mathrm{d}\bar{X})$ *with intensity measure* $m(\mathrm{d}s,\mathrm{d}u,\mathrm{d}\bar{X}) = \lambda(s)\mathrm{d}s \otimes p_{U|s}(\mathrm{d}u) \otimes p_{\bar{X}|s,u}(\mathrm{d}\bar{X})$*, where we use a general notation* $p_{\bar{X}|s,u}(\mathrm{d}\bar{X})$ *for the density distribution of the* $\mathbb{R}$*-valued random variable* $X^{(s)}(0,\infty)$*. Based on this Poisson point measure, the following representation holds:*

$$X_\tau^{IBNR} = \int_0^\tau \int_{\tau-s}^\infty \int_{\mathbb{R}} \bar{X} M(\mathrm{d}s,\mathrm{d}u,\mathrm{d}\bar{X}), \tag{22}$$

and from the classical properties of Poisson point measure, one gets

$$\mathrm{Var}\left(X_\tau^{IBNR}\right) = \int_0^\tau \int_{\tau-s}^\infty \int_{\mathbb{R}} \bar{X}^2 m(\mathrm{d}s,\mathrm{d}u,\mathrm{d}\bar{X}),$$

which then reduces to Equation (21).

We therefore get the following result:

Proposition 11. *The variance of the IBNR reserve (process error) writes*

$$\mathrm{Var}\left(X_\tau^{IBNR}\right) = (\gamma(0)+\mu(0)^2)\int_0^\tau \int_{\tau-s}^\infty \lambda(s) p_{U|s}(\mathrm{d}u)\mathrm{d}s, \tag{23}$$

where $\mu(0)$ *and* $\gamma(0)$ *can be computed from Equations (14) and (17) respectively.*

Based on the representation (22), it is also possible to compute the Laplace transform of the total IBNR payments, which involves the distribution $p_{\bar{X}|s,u}(\mathrm{d}\bar{X})$ as a whole. Indeed, from Poisson point measures properties as well, one gets

Proposition 12. *The Laplace transform of the IBNR reserve writes*

$$\mathbb{E}\left[\exp\left(-\theta X_\tau^{IBNR}\right)\right] = \int_0^\tau \mathrm{d}s \int_{\tau-s}^\infty p_{U|s}(\mathrm{d}u) \int_{\mathbb{R}} p_{\bar{X}|s,u}(\mathrm{d}\bar{X}) \exp\left(1-\exp\left(-\theta\bar{X}\right)\right).$$

RBNS reserve The future payments for the RBNS claims write

$$X_\tau^{RBNS} = \int_0^\tau \int_0^{\tau-s} X^{(s)}(\tau-s-u,\infty)\mathbf{1}_{S^{(s)}(\tau-s-u)\notin\{1,2\}} Q(\mathrm{d}s,\mathrm{d}u).$$

Let us denote by $\mathcal{F}_\tau$ the information about reported claims (including RBNS) up to time τ ; then the RBNS reserve writes

$$\mathbb{E}\left[X_\tau^{RBNS} \mid \mathcal{F}_\tau\right] = \int_0^\tau \int_0^{\tau-s} \mathbb{E}\left[X^{(s)}(\tau-s-u,\infty)\mathbf{1}_{S^{(s)}(\tau-s-u)\notin\{1,2\}} \mid \mathcal{F}_\tau\right] Q(\mathrm{d}s,\mathrm{d}u).$$

Using the independence between claims paths and the Markov property of each single path, we get the following result

Proposition 13. *The RBNS reserve writes*

$$\mathbb{E}\left[X_\tau^{RBNS} \mid \mathcal{F}_\tau\right] = \int_0^\tau \int_0^{\tau-s} \mu(\tau - s - u)\mathbf{1}_{S^{(s)}(\tau-s-u)\notin\{1,2\}} Q(\mathrm{d}s, \mathrm{d}u), \tag{24}$$

where $\mu(.)$ is given in Equation (14).

Finally, the variance of RBNS future payments is given in the following proposition, as that of a sum (integral) of independent trajectories:

Proposition 14.

$$\mathrm{Var}\left(X_\tau^{RBNS} \mid \mathcal{F}_\tau\right) = \int_0^\tau \int_0^{\tau-s} \gamma(\tau - s - u)\mathbf{1}_{S^{(s)}(\tau-s-u)\notin\{1,2\}} Q(\mathrm{d}s, \mathrm{d}u).$$

3.5 Numerical illustration

The numerical illustration is based on a real portfolio, in which 8 groups of claims can naturally be identified by means of an observed covariate. For purpose of confidentiality, the covariates possible values are labeled from A to H. We represent in Figure 7 the observed occurrence times and reporting delays used for calibration (all groups merged in the illustration). The sample is made of 19,870 observed claims, split into 6,121 settled claims and 13,749 RBNS claims.

The occurrence and reporting component of the model is taken as the standard parametrization in each group - that is, a constant occurrence frequency is chosen (which depends on claims covariate) as well as a constant reporting parameter (which again depends on claims group) for the exponentially distributed delay. The estimated occurrence and reporting parameters are illustrated in Figure 8 and compared to the 'naive' estimates based on a separate fit of frequency and reporting parameters. Note that this is the average reporting delay (in years) which is represented here (the inverse of each claims group reporting parameter). As expected, the joint estimation of occurrence and reporting parameters leads to a higher occurrence frequency and a higher average delay compared to a separate estimation.

Based on these parameters, it is possible to derive the distribution of the number of IBNR - this is given in Figure 9 (with the associated prediction error) and compared with the Mack model (with Gaussian distribution assumption). Again here, similar conclusions hold as in 2.5.2: the Mack model provides a higher IBNR (unitary) reserve (4,485) compared to the individual claims model (4,334), while providing a larger total confidence interval. Note that here the Mack method is applied to the full triangle (all 8 groups merged) in order to guarantee reasonable forecasts - note also that the joint application of the Mack method to both 8 triangles would require an extended framework to account for correlations, see Braun (2004) and Merz and Wüthrich (2007).

Let us now focus on the claims payment streams, and recall that the overall events frequency in the claims path (whatever their type) is driven by $h_1 + h_2 + h_3$ and that the weight of each parameter relates to the event probability, as described in 3.1.1. We depict the frequency parameters h_1, h_2 and h_3 in Figure 10, which are assumed to be constant in a first step of the analysis. This first indicates that both settlement frequencies and payments without settlement are of the same order of magnitude, although one can identify in more details that group C is characterized by both a high settlement frequency (shorter claims history) and a low payment frequency (more time between two payments). This contrasts with group B, with a larger time to settlement, and more frequent payments during the claims paths. In between, claims such as that of group F are characterized by a medium average time to settlement and times between payments, compared to the other groups. Note that as almost no payments at settlement occurred, the estimates for parameter h_2 are close or equal to zero, depending on the group considered.

To refine the analysis, we propose to fit yearly piecewise constant parameters h_1, h_2 and h_3, to better account for the timing of the kind of events which are observed here. Note that maximum likelihood estimators in this piecewise constant setting can be derived in closed form based on the likelihood representation (10) - the calibration results are depicted in Figure 11. This setting appears quite informative (except for h_2 whose values are negligible) as a proper structure can be exhibited among the several claims groups considered, namely:

- The settlement frequency h_1 is low for the two first years, then increases for years 3 and 4, then a slight decrease is observed and finally an increase or decrease in the latest years up to 10, depending on the group considered. In particular, the overall increase of the settlement intensity from earliest years of development to the latest ones means that the probability for an IBNR claim to be settled in its first year of development is lower than that of an RBNS claim with already 2 years of development, which in comparison is more likely to be settled. Therefore such conclusions contradict a simplified homogeneous Markov framework, which would lead in particular to consider that expected future payments are the same for RBNS and IBNR claims, see Remarks 6 and 8. Note however that the illustrative computations for the Markov case can serve as a building block for the piecewise constant intensity model (worth mentioning that it is not homogeneous Markov).

- The payment (without settlement) frequency h_3 shows a different pattern, as it is high for the first year, then deceases to the second year, and finally increases (overall) up to the latest year. This means in particular that, the settlement intensity being given (let us think about it as constant for the interpretation), payments are more frequent in the first and the last year, and increases from

the second year to the last year. Also, this means that e.g. payments are more likely to occur (still in this interpretation framework) in the first year of an IBNR claim compared to the second year of a one-year old RBNS claim.

As for the observed payments, we also account for the covariate information and fit a separate distribution for each claim group. Moreover, due to the nature of the payments distribution, we use a mixture of a log-normal distribution and an exponential in the tail - the density is written as

$$P(x) = p\frac{1}{x\sigma\sqrt{2\pi}}\exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right)\mathbf{1}_{x\leq x_0} + (1-p)\gamma\exp\left(-\gamma(x-x_0)\right)\mathbf{1}_{x>x_0},$$

with x_0 a fixed threshold separating attritional and large claims, which depends on the group considered - the results are depicted in Figure 12.

The frequency and payments parameters being fitted, this allows us to compute by closed-form $\mu(s)$ and $\gamma(s)$, the expected payments and related variance along a single claim path conditional on development length, see Equations (14) and (17) in Subsection 3.4. The results are depicted in Figure 13 - they can be described as follows: as a function of claim development, the expected payments decrease in the 3 first years, due to the combination of a decrease of the payment frequency and an increase (spike) in the settlement frequency h_1 (see again Figure 11); then the expectation increases again, due to the reduction in the settlement frequency after the third year barrier is attained. This illustrates the benefit in analyzing the timing of claims development, and in particular the bias when considering simple frequency $\times$ cost models.

On this topic, the computation of the IBNR and RBNS reserves is informative, which can be performed in closed-form based on Equations (20) and (24). Indeed, although the expected number of IBNR represents around 20% of the claims numbers to be paid (both RBNS and IBNR), which would be the ratio in terms of reserve in a homogeneous Markov setting, in terms of amounts this represents here around 50%, which illustrates the importance of quantifying IBNR claims accurately for lines of business with extended reporting delays - this is depicted in Figure 14.

Also, the comparison with the Chain-Ladder prediction is given in Figure 15. As for the IBNR unitary reserve (see 2.5.2), the Chain-Ladder method provides higher reserves for all (except one) groups.

Finally, we propose in Figure 16 a detail of the coefficients of variation (CV) for estimation and process errors for each claim group, and we compare the results of the individual model with that of the Mack model. First, the graph shows a reduction in process variance in the individual model, whose stochastic formulation is different (and apparently less overdispersed) compared to the original discrete time Markov model of Mack (1993). More importantly, it appears that the use of the individual claims model provides a great reduction in estimation error when

payments are modelled, which strengthens the results obtained in 2.5.2 for IBNR claim number. This key conclusion, which has also been observed in other studies as referenced in this survey, can be interpreted based on two facts: first, the more natural formulation seems to allow for a better fit of the several individual model building block compared to the Mack model, and second, the larger amount of data available for each part of claims history combined with this smart parametrization makes estimators have a low variance. On this basis, we argue that such modelling framework represents a promising (and complementary) method to assess future claims development with improved accuracy.

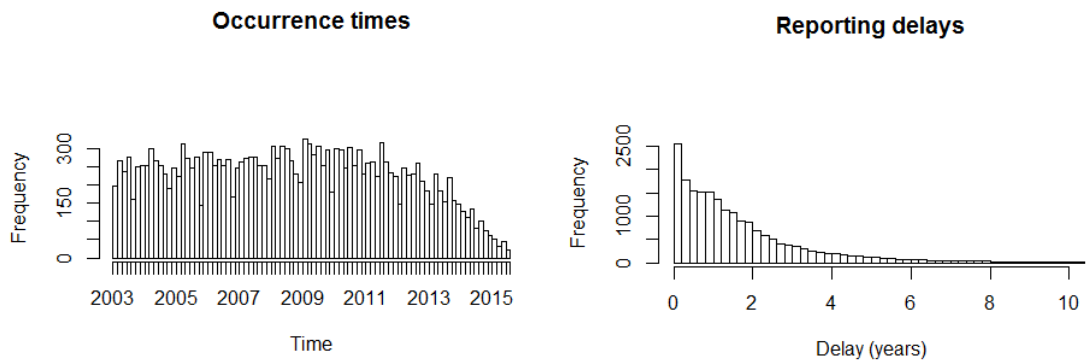

Figure 7: Occurrence times and reporting delays used for calibration

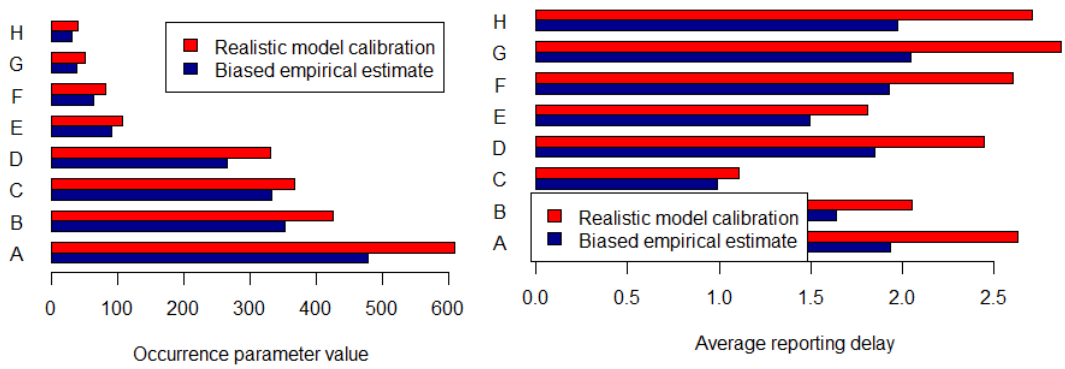

Figure 8: Occurrence and reporting parameters estimated for each claims group

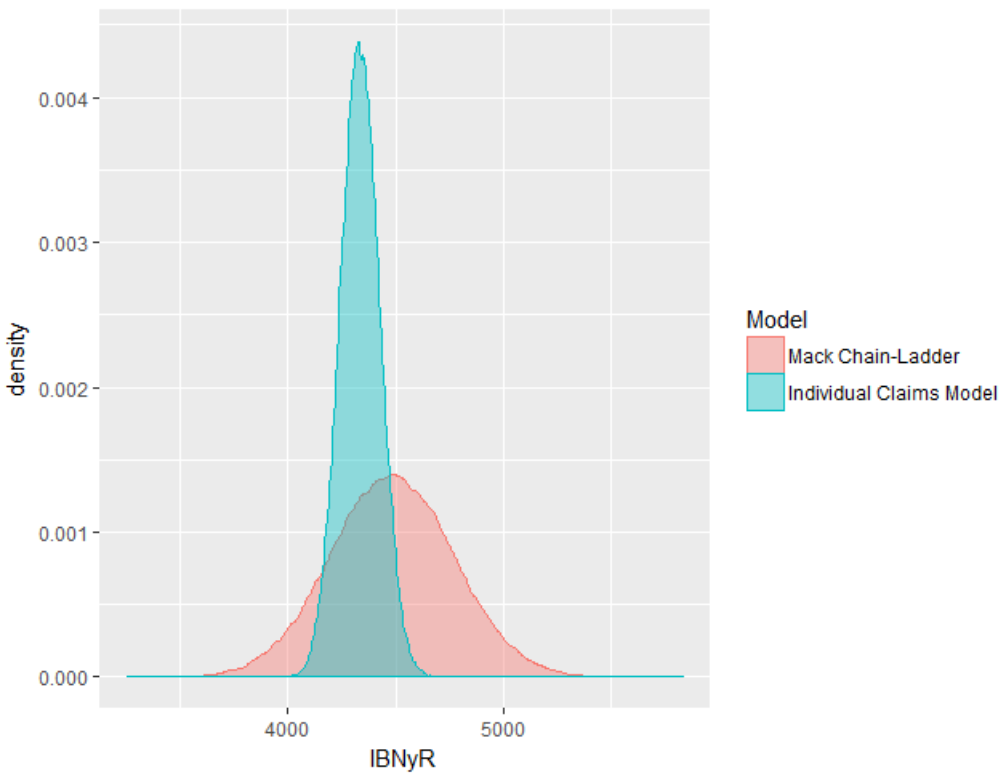

Figure 9: Prediction of the number of IBNR - comparison between the individual claims model and the Mack method

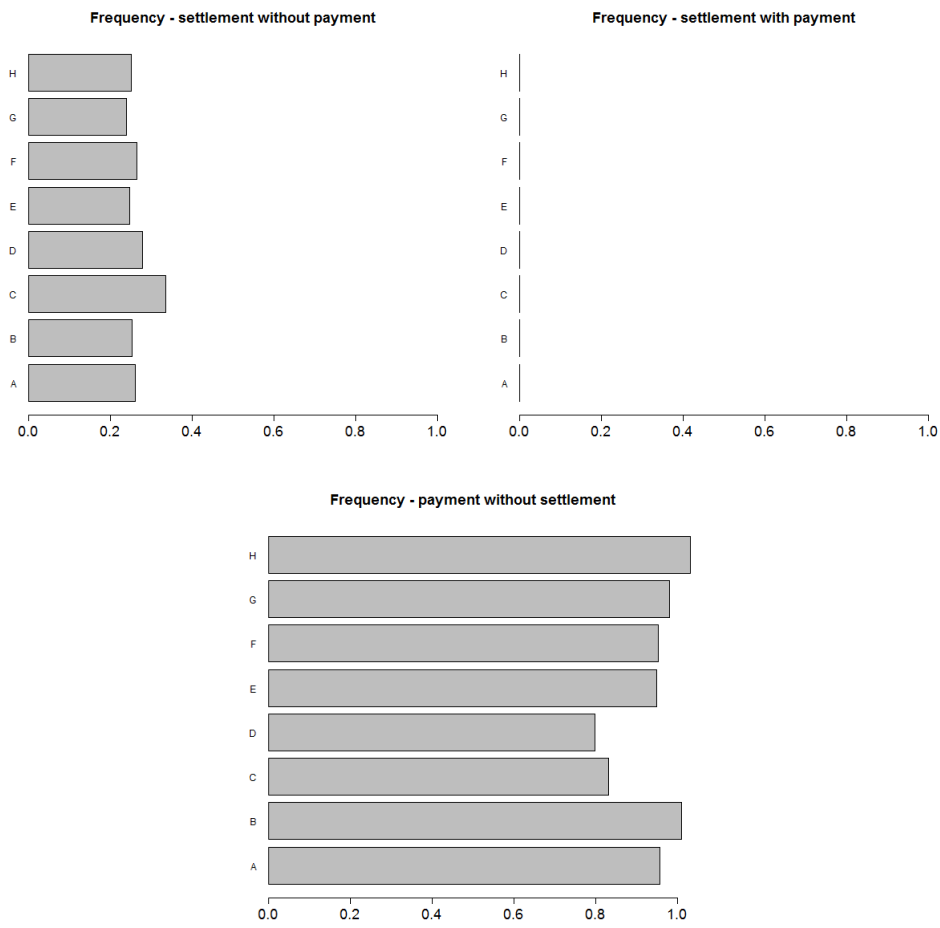

Figure 10: Computation of the frequency parameters h_1, h_2 and h_3 related to payments and settlement.

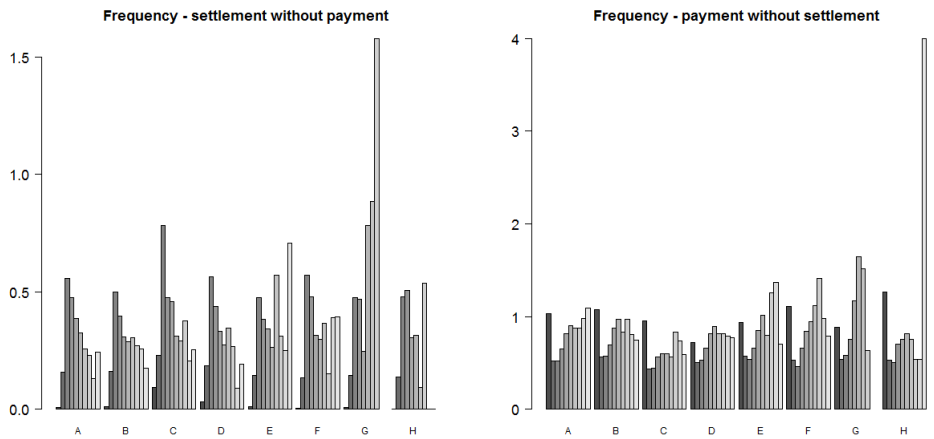

Figure 11: Computation of the frequency parameters h_1 and h_3 driving settlement and payments respectively, in a time dependent setting.

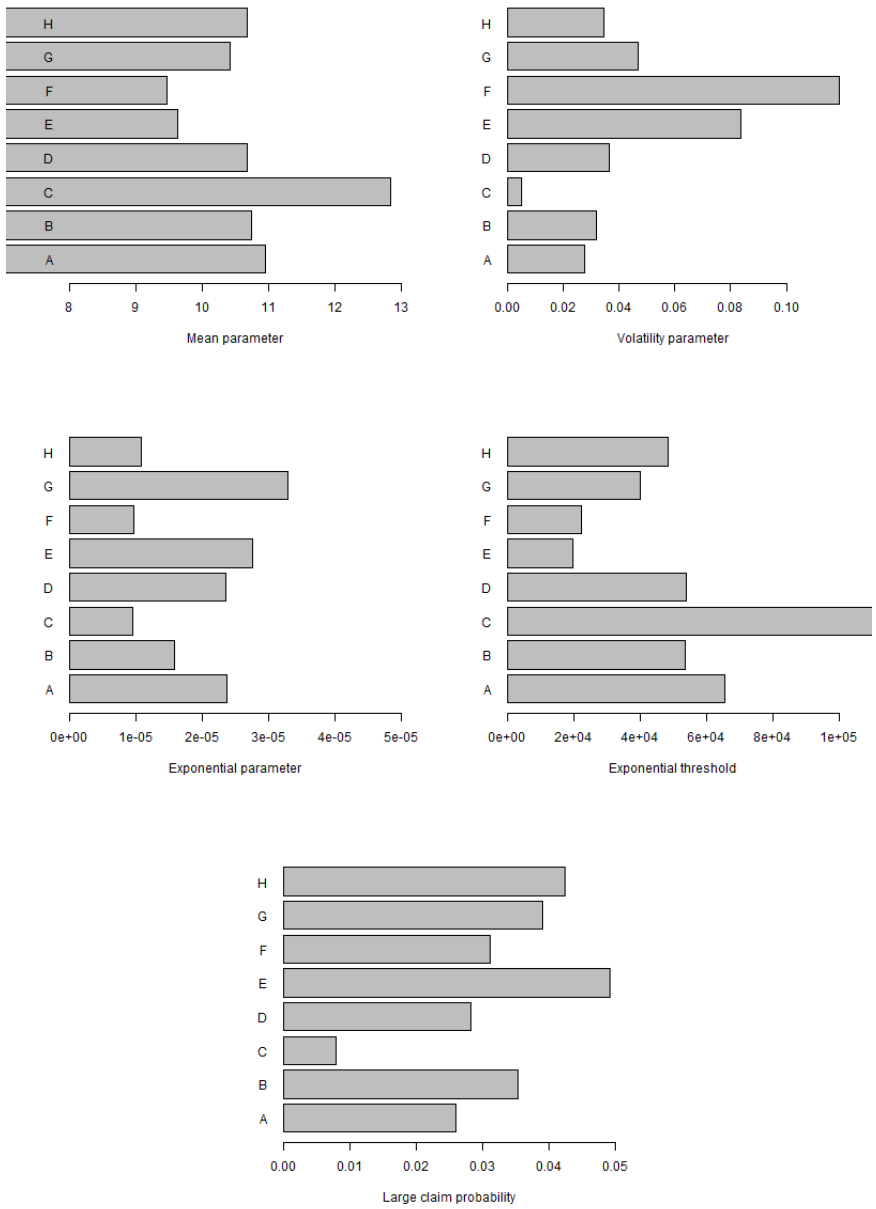

Figure 12: Fitted for each claims group: mean and volatility parameters of the log-normal distribution, as well as the exponential parameter, exponential threshold and large claim probability.

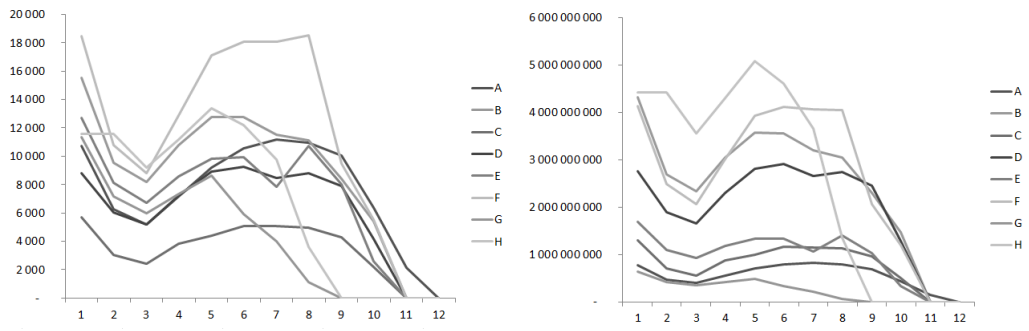

Figure 13: Expectation (left) and variance (right) of future payments on a single claim path, conditional on claim group and development length.

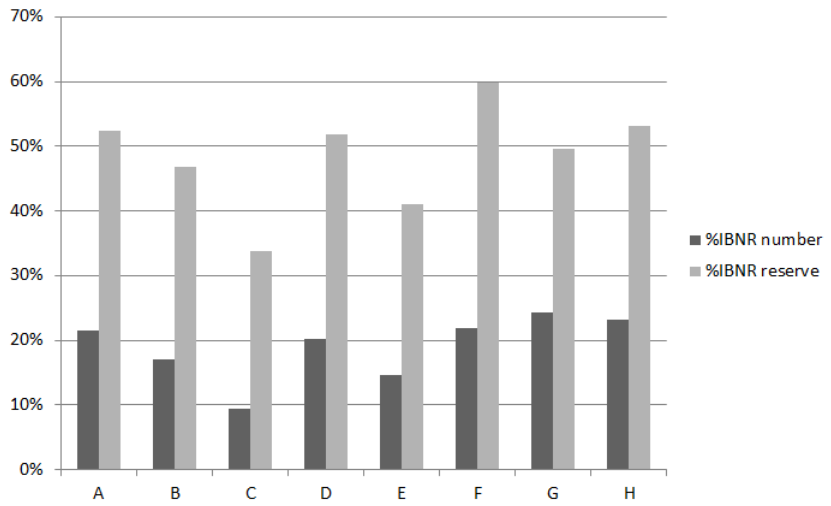

Figure 14: Proportion of IBNR claims in all claims to be payed (both IBNR and RBNS): numbers and amounts.

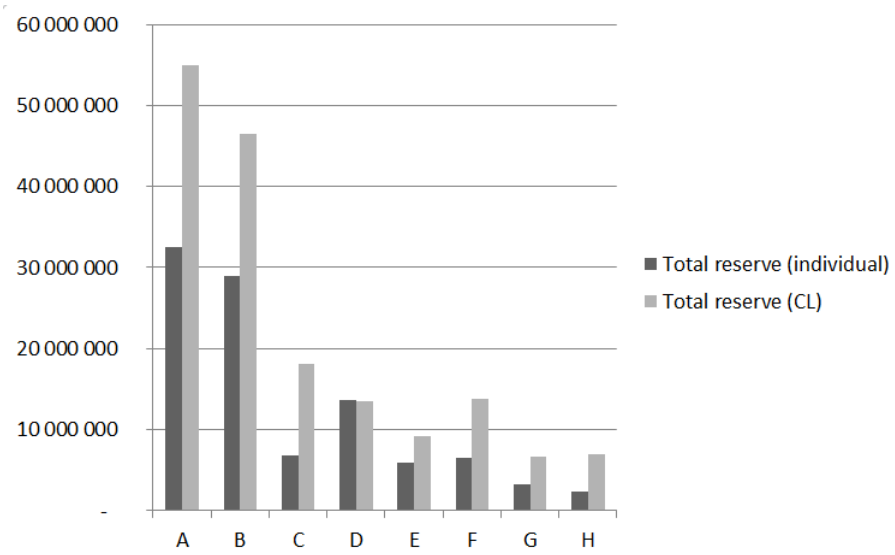

Figure 15: Comparison with the Chain-Ladder method

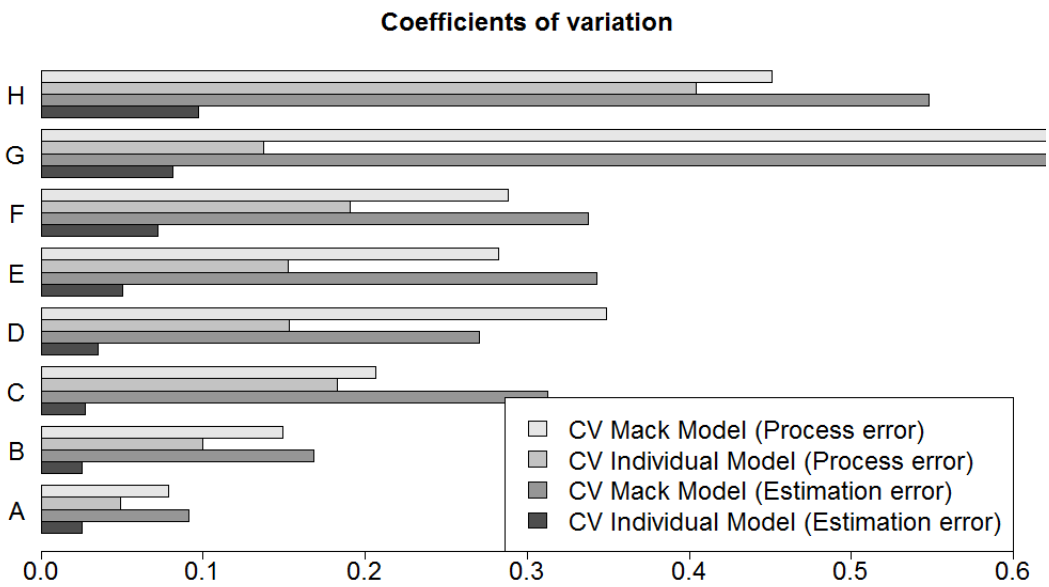

Figure 16: Coefficients of variation related to process and estimation errors, for both the Individual model and the Mack model.

4 Concluding remarks

In this survey, we proposed a unified framework to address the stochastic modelling of individual claims for reserving purposes, based on the earliest developments of Norberg (1993) and Hesselager (1994) in this direction. The model is built upon two major components, driving the occurrence and reporting process, as well as the claims payments path until settlement. It is shown in this paper how such framework can be used for calibration, risk analysis and simulation in a variety of contexts based on a coherent modeling framework. We argue that individual-based methods, which allow to analyze, understand and forecast risk in a more precise way compared to triangle-based approaches while taking advantage of detailed individual information, is worth implementing in current risk management practices.

Further work remains to be done on the exploration of the numerous potentialities of individual models, in particular by calibrating more flexible models involving advanced dependence structures between occurrence, reporting and payments (timing and amount). Also, the comparison with classical methods based on run-off triangles suggests mathematical perspectives which are out of scope of the present survey, but needed to continue improving our understanding of the link with more classical models.

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Appendix

In this appendix, we propose the proof of Thiele's differential equation (discussed in 3.4) satisfied by any prospective reserve on a (general) individual Markov model, as introduced by Hesselager (1994).

Proposition 15. *For any set of continuous maps $f_{mn}(.)$ for all $n \neq m$, the prospective reserve conditioned on state j at time s,*

$$R(s \mid j) = \sum_{m \in \mathbb{N}^*} \sum_{n \in \mathbb{N}^* \setminus \{m\}} \int_s^\infty p_{jm}(s,t) \lambda_{mn}(t) f_{mn}(t) \mathrm{d}t, \tag{25}$$

satisfies Thiele's differential equation

$$\frac{\mathrm{d}R(s \mid j)}{\mathrm{d}s} = - \sum_{k \in \mathbb{N}^* \setminus \{j\}} \lambda_{jk}(s) \left(f_{jk}(s) + R(s \mid k) - R(s \mid j) \right). \tag{26}$$

Proof of Proposition 15 Let us perform a standard differentiation of Equation (25), leading to

$$\frac{\mathrm{d}R(s \mid j)}{\mathrm{d}s} = - \sum_{m \in \mathbb{N}^*} \sum_{n \in \mathbb{N}^* \setminus \{m\}} p_{jm}(s,s) \lambda_{mn}(s) f_{mn}(s) + \sum_{m \in \mathbb{N}^*} \sum_{n \in \mathbb{N}^* \setminus \{m\}} \int_s^\infty \frac{\partial p_{jm}(s,t)}{\partial s} \lambda_{mn}(t) f_{mn}(t) \mathrm{d}t,$$

where the first term reduces to $-\sum_{k \in \mathbb{N}^* \setminus \{j\}} \lambda_{jk}(s) f_{jk}(s)$, since $p_{jm}(s,s) = \mathbf{1}_{m=j}$.

As for the second term, let us focus on that in Equation (26) which can be rewritten as

$$
\begin{aligned}
&-\sum_{k\in\mathbb{N}^*\setminus\{j\}} \lambda_{jk}(s)(R(s\mid k)-R(s\mid j)),\\
&=-\sum_{k\in\mathbb{N}^*\setminus\{j\}} \lambda_{jk}(s)\sum_{m\in\mathbb{N}^*}\sum_{n\in\mathbb{N}^*\setminus\{m\}}\int_s^\infty \left(p_{km}(s,t)-p_{jm}(s,t)\right)\lambda_{mn}(t)f_{mn}(t)\mathrm{d}t,\\
&=-\sum_{m\in\mathbb{N}^*}\sum_{n\in\mathbb{N}^*\setminus\{m\}}\int_s^\infty \lambda_{mn}(t)f_{mn}(t)\sum_{k\in\mathbb{N}^*\setminus\{j\}}\lambda_{jk}(s)\left(p_{km}(s,t)-p_{jm}(s,t)\right)\mathrm{d}t.
\end{aligned}
$$

Now, let us compute the following differential:

$$
\begin{aligned}
&\frac{\partial p_{jm}(s,t)}{\partial s}\\
&=\lim_{h\to 0}\frac{p_{jm}(s+h,t)-p_{jm}(s,t)}{h}\\
&=\lim_{h\to 0}\frac{p_{jm}(s+h,t)-\sum_{k\in\mathbb{N}^*\setminus\{j\}}p_{jk}(s,s+h)p_{km}(s+h,t)}{h}\\
&=\sum_{k\in\mathbb{N}^*\setminus\{j\}}\lim_{h\to 0}\frac{p_{jk}(s,s+h)}{h}\left(p_{jm}(s+h,t)-p_{km}(s+h,t)\right)\\
&=\sum_{k\in\mathbb{N}^*\setminus\{j\}}\lambda_{jk}(s)\left(p_{jm}(s,t)-p_{km}(s,t)\right).
\end{aligned}
$$

This concludes the proof.