

# A SYNCHRONOUS BOOTSTRAP TO ACCOUNT FOR DEPENDENCIES BETWEEN LINES OF BUSINESS IN THE ESTIMATION OF LOSS RESERVE PREDICTION ERROR

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## ABSTRACT

In this article we consider the situation in which an insurer requires a loss reserve, together with the estimated prediction error, in respect of a number of stochastically dependent lines of business, individually and in aggregate. We suppose that generalized linear models are used to estimate each of the individual loss reserves, and that bootstrapping is used to estimate prediction errors. Specialized forms of the bootstrap, referred to as *synchronous bootstraps*, are constructed to capture the dependencies. Numerical examples are given in which loss reserve forecasts and their prediction errors are obtained for individual lines of business and in aggregate.

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## 1. INTRODUCTION

Consider an insurance portfolio that consists of  $I$  segments. At the most general level, these may be any subsets of the portfolio, which may or may not be lines of business (LoBs). However, for ease of exposition, they may be conveniently thought of as LoBs.

A *loss reserve* is calculated for each LoB and for the aggregate of them. Naturally the latter is usually taken as the sum of the reserves for the  $I$  LoBs. Each loss reserve is a forecast of the associated liability (call it the *liability forecast*) and subject to prediction error. The liability forecast may therefore be viewed as a random variable with a distribution.

The literature deals with various approaches to the estimation of the distribution for an individual LoB (e.g., Taylor 2000). It is more scant, however, in relation to the distribution of the aggregate, taking account of dependencies between LoBs. If a regression model is applied to each segment, the need to accommodate dependencies between them suggests the framework of *seemingly unrelated regressions* (SURs), introduced by Zellner (1962) and discussed by Srivastava and Giles (1987). However, two difficulties may arise in the application of this sort of framework to loss reserving.

First, while the SUR may be well adapted to general linear models, one will often want to phrase the model of insurance data more generally, such as in terms of generalized linear models (GLMs). Second, if the number of data points involved in the  $I$  LoBs is large, and the structure of the dependencies totally unknown, the problem of modeling this structure by means of an SUR framework would become substantial. This would be so even if it is assumed that observations from within an LoB are stochastically independent.

The *bootstrap* is a well-known approach to the estimation of prediction error for a single LoB, for which standard methodology exists, and it is natural to seek to extend this methodology to the case

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of multiple LoBs with dependencies between them. This article suggests forms of bootstrapping that capture such dependency structures, at least approximately, and even when the structures are unknown.

## 2. SYNCHRONOUS BOOTSTRAPPING

### 2.1 Preliminaries

Consider  $I$  models

$$Y_i = g_i(X_i, \beta_i) + \varepsilon_i, \quad i = 1, 2, \dots, I, \quad (2.1)$$

where

$Y_i$  is a column  $n_i$ -vector of observations  $Y_{ik}$ ,  $k = 1, \dots, n_i$

$X_i$  is an  $n_i \times p_i$  matrix whose rows consist of predictors of the  $Y_{ik}$

$\beta_i$  is a column  $p_i$ -vector of parameters

$g_i$  is a vector-valued function specifying a model structure

$\varepsilon_i$  is a column  $n_i$ -vector of stochastically independent, centered, equidistributed error terms, each with variance  $\sigma_i^2$ , referred to subsequently as *process error*.

For the present analysis, assume, for instance, that  $n_1 = \dots = n_I = n$ .

Assume that there is a dependency between the data sets  $Y_i$ , described by correlation matrices

$$C_{ij} = \text{Corr}(Y_i, Y_j). \quad (2.2)$$

Pearson correlation will be a suitable measure of dependency only if the error terms  $\varepsilon_i$ , while possibly nonnormal, are not too much so. Thus, the procedures developed below may not be effective if the distributions of the  $\varepsilon_i$  are very long tailed.

Note, however, that long-tailed distributions may still be accommodated within the structure. For example, log normal distributions would be accommodated within the GLM structure by the choice  $Y_i(\cdot) = \text{logged observations}$  and  $\varepsilon_i \sim N$  in (2.1). It follows from the above independence assumption that  $C_{ii} = 1$  for each  $i$ .

The  $C_{ij}$  will be supposed unknown, in which case estimation of each  $\beta_i$  will be carried out by reference to just the data set  $Y_i$ . Suppose that there is some unbiased estimation procedure for mapping data  $Y_i$  to estimates  $\hat{\beta}_i$ ,  $\hat{\sigma}_i^2$  of  $\beta_i$ ,  $\sigma_i^2$ . Let  $\hat{Y}_i$  denote the vector of *fitted values*

$$\hat{Y}_i = g_i(X_i, \hat{\beta}_i). \quad (2.3)$$

In practical applications, such as maximum likelihood estimation, unbiasedness may be only approximate, in which case all estimates constructed below will incorporate the same degree of approximation.

Typically  $Y_i$  and  $\hat{Y}_i$  will be correlated, and furthermore the correlations (2.2) will induce correlations between  $\hat{Y}_i$  and  $\hat{Y}_j$ . However, it will be assumed that these are of order  $n^{-1}$ .

Define the *standardized Pearson residual vector*

$$R_i = (Y_i - \hat{Y}_i)/\hat{\sigma}_i. \quad (2.4)$$

Suppose now that one wishes to estimate a vector  $\zeta_i = E[Z_i]$ , where

$$Z_i = \alpha_i(U_i, \beta_i) + \eta_i, \quad (2.5)$$

which is of the same form as (2.1) but with  $Y_i$ ,  $g_i$ ,  $X_i$ ,  $\varepsilon_i$  replaced by  $Z_i$ ,  $\alpha_i$ ,  $U_i$ ,  $\eta_i$ . Dimensions may differ as between (2.1) and (2.5), other than for  $\beta_i$ . Suppose that  $Z_i$  is of dimension  $n' \leq n$ . An estimate of  $\zeta_i$  is given by

$$\hat{\zeta}_i = \alpha_i(U_i, \hat{\beta}_i). \quad (2.6)$$

Define a *resampling matrix* as an  $n' \times n$  ( $n' \leq n$ ) matrix whose rows are distinct natural coordinate vectors, that is, having one unit component and the rest zero. If  $n' = n$ , a resampling matrix is a

*permutation matrix*. In general, if  $M$  is a resampling matrix with  $(k, s)$  element  $M_{ks}$ , then the product  $M_{ks}M_{lt}$  has the following properties:

$$\begin{aligned} \text{For } k = l, M_{ks}M_{lt} &= 1 \text{ for } s = t \\ &= 0 \text{ otherwise;} \\ \text{For } k \neq l, M_{ks}M_{lt} &= 1 \text{ for a single pair } s, t \text{ with } s \neq t \\ &= 0 \text{ otherwise.} \end{aligned}$$

## 2.2 Sources of Correlation

There are two main possible sources of the correlations (2.2). It will be seen below that these different sources of correlation dictate different forms of bootstrapping.

The first source is *correlated noise*, that is, the  $g_i(X_i, \beta_i)$  terms in (2.1) are deterministic and

$$\text{Corr}(\varepsilon_i, \varepsilon_j) = C_{ij}, \quad (2.7)$$

implying (2.2). Alternatively, the source may be *correlated parameters*. This is an apparent form of correlation arising from *model misspecification*.

Suppose that, while (2.1) is the assumed model form, the correct form is, in fact,

$$Y_i = g_i^+(X_i^+, \beta_i, \gamma_i) + \varepsilon_i^+, \quad (2.1a)$$

where  $\gamma_i$  is a vector of parameters unrecognized by the model, and  $g_i^+$  and  $X_i^+$  are modifications of  $g_i$  and  $X_i$  that do recognize these additional parameters.

By (2.1) and (2.1a),

$$\varepsilon_i = \varepsilon_i^+ + b_i,$$

where  $b_i$  denotes the *bias vector*

$$b_i = g_i^+(X_i^+, \beta_i, \gamma_i) - g_i(X_i, \beta_i).$$

Then

$$\text{Cov}(\varepsilon_i, \varepsilon_j) = \text{Cov}(\varepsilon_i^+, \varepsilon_j^+) + b_i b_j^T. \quad (2.8)$$

The left-hand side of this relation is the covariance between  $\varepsilon_i$  and  $\varepsilon_j$ , relative to a model of the form (2.1). It consists of the first term on the right-hand side, which is the correct covariance, plus an adjustment that is a measure of the misspecification of the true model (2.1a).

## 2.3 Bootstrapping a Single Data Set

We next describe the bootstrap (Efron 1979) as it is conventionally applied.

Define *pseudo-data vectors*  $Y_{i(r)}^*$ ,  $r = 1, \dots, R$  associated with  $Y_i$  as

$$Y_{i(r)}^* = \hat{Y}_i + \hat{\sigma}_i M_{i(r)} R_i, \quad (2.9)$$

where  $M_{i(r)}$  is an  $n \times n$  permutation matrix chosen randomly from the set of all such matrices and the  $M_{i(r)}$  are independent.

This resamples the data *without replacement*. The fact that  $Y_{i(r)}^*$  has the same distribution as  $Y_i$  is an essential feature of the bootstrap. Each pseudo-data vector  $Y_{i(r)}^*$  yields *pseudo-estimates*  $\hat{\beta}_{i(r)}^*$ ,  $\hat{\sigma}_{i(r)}^{*2}$  and *pseudo-fitted values*  $\hat{Y}_{i(r)}^*$ . This also yields pseudo-estimates  $\hat{\zeta}_{i(r)}^*$  of  $\zeta_i$ , where

$$\hat{\zeta}_{i(r)}^* = \alpha_i(U_i, \hat{\beta}_{i(r)}^*), \quad i = 1, \dots, I. \quad (2.10)$$

When  $\hat{\zeta}_i$  is in the nature of a *forecast* of  $Z_i$ , one may generate the *pseudo-forecasts*

$$Z_{i(r)}^* = \hat{\zeta}_{i(r)}^* + \eta_{i(r)}^*, \quad (2.11)$$

where  $\eta_{i(r)}^*$  is a component of noise and is sampled just as in (2.9):

$$\eta_{i(r)}^* = \hat{\sigma}_i A_{i(r)} R_i, \quad (2.12)$$

where  $A_{i(r)}$  is an  $n' \times n$  resampling matrix chosen randomly from the set of all such matrices and the  $A_{i(r)}$  are independent.

The collection of  $\hat{\zeta}_{i(r)}^*$ ,  $r = 1, \dots, R$  induces an empirical distribution of  $\hat{\zeta}_i$ , which will be called the *bootstrap distribution* of  $\hat{\zeta}_i$ . Similarly the  $Z_{i(r)}^*$ ,  $r = 1, \dots, R$  induce a bootstrap distribution of  $Z_i$ .

## 2.4 Synchronous Bootstrapping of Multiple Data Sets

### 2.4.1 Motivation

The bootstrap described in Section 2.3 may be validly performed for each value of  $i = 1, \dots, I$ . Suppose, however, that one is interested in forecasting some quantity that combines data sets, for instance,

$$\varepsilon = 1^T Z, \quad (2.13)$$

where  $Z = (Z_1^T, \dots, Z_I^T)^T$  and  $1$  is the vector of the same dimension with all components unity.

Let  $Z_{(r)}^* = (Z_{1(r)}^{*T}, \dots, Z_{I(r)}^{*T})^T$ , but note that, because the  $I$  data sets have been independently bootstrapped, any correlation structure between the  $Z_i$  is likely to have been lost. Hence the bootstrap distribution of  $Z$  will not be representative of the true distribution. Neither will the bootstrap distribution of  $\varepsilon$  be representative of its true distribution.

### 2.4.2 Synchronizing Individual Data Points

Let  $Y_{ik}$  denote the  $k$ th component of  $Y_i$ . Suppose that the correlation structure (2.2) is of the correlated noise form (2.7) and is given by

$$\begin{aligned} C_{ij,kl} &= \rho_{ij1} \quad \text{for } i \neq j, k = l \\ &= \rho_{ij0} \quad \text{for } i \neq j, k \neq l, \end{aligned} \quad (2.14)$$

where  $C_{ij,kl}$  denotes the  $(k, l)$ -element of the matrix  $C_{ij}$ . Recall that  $C_{ii} = 1$ .

Here  $\rho_{ij0}$ , which may be zero, may be regarded as a ‘‘background correlation’’ between distinct data sets  $i$  and  $j$ , while  $\rho_{ij1}$  represents a spike in correlation that occurs for points in corresponding positions in those data sets. The quantities  $\{Z_i\}$  to be forecast are assumed subject to the same correlation structure (2.14) as the data  $\{Y_i\}$  and  $\text{Corr}[Y_{ik}, Z_{jl}] = \rho_{ij0}$ .

Suppose the data to be modeled are of the form (2.1), and apply the bootstrap described in Section 2.3 with the exception that (2.9) and (2.12) are replaced by the following:

$$Y_{i(r)}^* = \hat{Y}_i + \hat{\sigma}_i M_{(r)} R_i, \quad (2.9a)$$

$$\eta_{i(r)}^* = \hat{\sigma}_i A_{(r)} R_i, \quad (2.12a)$$

where  $M_{(r)}$  and  $A_{(r)}$  are  $n \times n$  and  $n' \times n$  randomly chosen resampling matrices.

The difference between this form of bootstrap and the conventional form is that the present form applies the same matrices  $M_{(r)}$  and  $A_{(r)}$  to the bootstrapping of all  $I$  data sets. In this sense the resampling of the residual vectors  $R_1, \dots, R_I$  is *synchronized* for each  $r$ . This form of bootstrapping was suggested by Kirschner, Kerley, and Isaacs (2002) in an insurance loss-reserving context.

It can now be shown that the joint distribution of  $Y_{1(r)}^*, \dots, Y_{I(r)}^*$  is approximately the same as for  $Y_1, \dots, Y_I$ . To see this, note that, from (2.4) and (2.9a),

$$Y_{i(r)}^* = [1 - M_{(r)}]\hat{Y}_i + M_{(r)}Y_i. \quad (2.15)$$

Write  $\mu_i$  to denote  $g_i(X_i, \beta_i)$  so that (2.1) takes the form

$$Y_i = \mu_i + \varepsilon_i \quad (2.1b)$$

and substitute this result into (2.15) to obtain

$$Y_{i(r)}^* - \mu_i = [1 - M_{(r)}](\hat{Y}_i - \mu_i) + M_{(r)}\varepsilon_i. \quad (2.16)$$

From (2.2) and the assumption made immediately after (2.3),

$$\text{Corr}[Y_{i(r)}^*, Y_{j(r)}^*] = M_{(r)}C_{ij}M_{(r)}^T + O(n^{-1}). \quad (2.17)$$

The  $(k, l)$  element of the first member is

$$\sum_{s,t} M_{(r)ks}M_{(r)lt}C_{ij,st}, \quad (2.18)$$

where  $M_{(r)ks}$  denotes the  $(k, s)$  element of  $M_{(r)}$ .

The property of resampling matrices stated just after their definition in Section 2.1, applied to (2.18), reduces it to  $\rho_{ij1}$  for  $k = l$  and  $\rho_{ij0}$  for  $k \neq l$ , in which case (2.17) is just

$$\text{Corr}[Y_{i(r)}^*, Y_{j(r)}^*] = C_{ij} + O(n^{-1}). \quad (2.19)$$

Thus, to within  $O(n^{-1})$ ,  $\{Y_{1(r)}^*, \dots, Y_{I(r)}^*\}$  has the same marginals and correlation structure as  $Y_1, \dots, Y_I$  and therefore has approximately the same joint distribution. The basic requirement for application of the bootstrap, stated just after (2.9), is thus satisfied.

By the same reasoning as above, it follows that the  $Z_{i(r)}^*$  have the same on-diagonal/off-diagonal structure as (2.19). It also follows that the correlation between any component of  $Y_{i(r)}^*$  and any component of  $Z_{j(r)}^*$  is  $\rho_{ij0}$  (subject to an error of order  $n^{-1}$ ), as required. Thus the  $Z_{(r)}^*$  induce a bootstrap distribution that is representative of  $Z$ , including its dependencies.

### 2.4.3 Synchronizing Data Subsets

Let  $N_1, N_2, \dots, N_Q$  be disjoint subsets of  $\{1, \dots, n\}$  whose union is equal to this set. Suppose the correlation structure (2.2) is of the correlated parameter form described in Section 2.2 and is given by

$$\begin{aligned} C_{ij,kl} &= \rho_{ij1} \quad \text{for } i \neq j, \quad k, l \in \text{some } N_q \\ &= \rho_{ij0} \quad \text{for } i \neq j, \quad k, l \text{ from distinct } N_q. \end{aligned} \quad (2.20)$$

Assume that there is no correlated noise, that is, the  $\varepsilon_i, i = 1, \dots, I$  are stochastically independent. As in Section 2.4.2,  $\rho_{ij0}$  may be viewed as a background level of correlation between distinct data sets, while  $\rho_{ij1}$  represents a different level of correlation that occurs for points in corresponding subsets of those data sets.

Now suppose that the  $Z_i$  extend the correlation structure of the  $Y_i$  in the following sense. Let  $N_1^*, \dots, N_Q^*$  be disjoint subsets of  $\{1, \dots, n'\}$  whose union is equal to this set. Let  $n_q$  and  $n'_q$  be the orders of  $N_q$  and  $N_q^*$ , respectively. Suppose that  $n'_q \leq n_q$  for each  $q$ .

Let  $Y_i^{(q)}$  denote the subvector of  $Y_i$  consisting of just the components  $Y_{ik}, k \in N_q$ . Similarly let  $Z_i^{(q)}$  denote the subvector of  $Z_i$  consisting of just the components  $Z_{il}, l \in N_q^*$ . Define

$$W_i^{(q)} = \begin{bmatrix} Y_i^{(q)} \\ Z_i^{(q)} \end{bmatrix}. \quad (2.21)$$

Let  $W_{ik}^{(q)}$  denote the  $k$ th component of  $W_i^{(q)}$  and let  $D_{ij,kl}^{(q,s)}$  denote  $\text{Corr}(W_{ik}^{(q)}, W_{jl}^{(s)})$ . Suppose that

$$\begin{aligned} D_{ij,kl}^{(q,s)} &= 1 \text{ for } i = j, q = s, k = l \\ &= 0 \text{ for } i = j, q = s, k \neq l \\ &= 0 \text{ for } i = j, q \neq s \\ &= \rho_{ij1} \text{ for } i \neq j, q = s \\ &= \rho_{ij0} \text{ for } i \neq j, q \neq s, \end{aligned} \tag{2.22}$$

which is consistent with (2.20).

Suppose once again that the data to be modeled are of the form (2.1), and apply the bootstrap described in Section 2.3 with the exception that (2.9) and (2.12) are replaced by the following:

$$Y_{i(r)}^{(q)*} = \hat{Y}_i^{(q)} + \hat{\sigma}_i M_{i(r)}^{(q)} R_i^{(q)}, \tag{2.9b}$$

$$\eta_{i(r)}^{(q)*} = \hat{\sigma}_i A_{i(r)}^{(q)} R_i^{(q)}, \tag{2.12b}$$

and

$$Z_{i(r)}^{(q)*} = \hat{\zeta}_i^{(q)*} + \eta_{i(r)}^{(q)*}, \tag{2.23}$$

where  $R_i^{(q)}$  is the subvector of  $R_i$  corresponding to  $Y_i^{(q)}$  as a subvector of  $Y_i$ , and  $M_{i(r)}^{(q)*}$  and  $A_{i(r)}^{(q)*}$  are  $n_q \times n_q$  and  $n'_q \times n_q$  randomly chosen resampling matrices.

Note that *different matrices*  $M_{i(r)}^{(q)}$  and  $A_{i(r)}^{(q)}$  apply to *different subsets within the data sets*. This differs from the situation described by (2.9a) and (2.12a), where common resampling matrices were applied to whole data sets. The resampling of residual vectors is now synchronized within each subset  $N_q$  and its companion  $N_q^*$ .

Corresponding to (2.21), define

$$W_{i(r)}^{(q)*} = \begin{bmatrix} Y_{i(r)}^{(q)*} \\ Z_{i(r)}^{(q)*} \end{bmatrix}, W_{i(r)}^* = \begin{bmatrix} Y_{i(r)}^* \\ Z_{i(r)}^* \end{bmatrix}. \tag{2.24}$$

The argument given in Section 2.4.2 may be adapted to the present situation to demonstrate that  $W_{1(r)}^{(q)}, \dots, W_{I(r)}^{(q)}$  have approximately the same joint distribution as  $W_{1(r)}^{(q)*}, \dots, W_{I(r)}^{(q)*}$ , and that  $W_{1(r)}^*, \dots, W_{I(r)}^*$  have approximately the same joint distribution as  $W_{1(r)}, \dots, W_{I(r)}$  for  $r = 1, \dots, R$ .

Define  $Z$  and  $Z_{(r)}^*$  as in Section 2.4.1. Then the  $Z_{(r)}^*$  induce a bootstrap distribution that is representative of  $Z$ , including its dependencies.

### REMARK 2.1

Synchronized resampling within subsets  $N_q$  is *not* the natural extension of the synchronized point resampling in Section 2.4.2. In that earlier case, if  $R_{ik} \rightarrow R_{il}$  in the resampling of data set  $i$ , then  $R_{jk} \rightarrow R_{jl}$  in the resampling of data set  $j$ . The natural extension of this to the present subsection would be for

- The resampling  $R_{ik} \rightarrow R_{il}$  to involve the mapping of each subset  $N_q$  to a single other subset  $N_s$ , and
- The same mapping of subsets (though not of members of those subsets) in  $R_{jk} \rightarrow R_{jl}$ ; that is, for all  $k \in N_q$ , there is a single  $N_s$  such that

$$R_{ik} \rightarrow R_{il} \text{ with } l \in N_s$$

$$R_{jk} \rightarrow R_{jl} \text{ with } l \in N_s.$$

This, however, creates some difficulty when  $n_q < n_s$ , and the alternative procedure described above will often be more convenient.

**REMARK 2.2**

The resampling scheme represented by the matrices  $M_{i(r)}^{(q)}$  and  $A_{i(r)}^{(q)}$  will encounter difficulties if  $N_q$  is small. There are  $N_q!$  choices of the permutation matrix  $M_{i(r)}^{(q)}$ , and so, for given  $r$ , there are  $n_1!n_2!\cdots n_Q!$  possible resamplings in (2.9b). This needs to be at least as large as  $R$ , and preferably a good deal larger. If not, it may be necessary to aggregate some of the smaller  $N_q$  into larger ones.

**REMARK 2.3**

The correlation structure (2.20) was assumed to involve no correlated noise. Such noise can be added with structure (just for the noise) (2.14), changing the total correlation structure to the following:

$$\begin{aligned} C_{ij,kl} &= \rho_{ij2} \quad \text{for } i \neq j, k = l \\ &= \rho_{ij1} \quad \text{for } i \neq j, k, l \in \text{some } N_q, k \neq l \\ &= \rho_{ij0} \quad \text{for } i \neq j, k, l \text{ from distinct } N_q. \end{aligned} \quad (2.25)$$

This correlation structure will be reproduced by a different form of bootstrap in which (2.9b) and (2.12b) are replaced by the following:

$$Y_{i(r)}^{(q)*} = \hat{Y}_i^{(q)} + \hat{\sigma}_i M_{i(r)}^{(q)} R_i^{(q)}, \quad (2.9c)$$

$$\eta_{i(r)}^{(q)*} = \hat{\sigma}_i A_{i(r)}^{(q)} R_i^{(q)}. \quad (2.12c)$$

That is, the same resampling is applied (at the subset level) to each data set. The modification of (2.9b) and (2.12b) to (2.9c) and (2.12c) is the same as from (2.9) and (2.12) to (2.9a) and (2.12a).

**2.4.4 Synchronizing Geometrically Related Subsets**

Section 2.4.3 was concerned with correlations between subsets whose positions within their respective data sets were fixed in *absolute* terms. The present discussion will be concerned with correlation between data sets that depend on the *relative* positions of members of those sets. This may arise because the data sets, abstractly represented as vectors hitherto, actually have some geometric structure. This is perhaps best explained by way of an example.

Suppose that the data vector  $Y_i$  comprises observations taken over some two-dimensional lattice, so that any component is naturally identified by a *coordinate pair* thus:  $Y_{ik} = Y_{ik_1k_2}$ ,  $k_1 = 1, \dots, K_1$ ;  $k_2 = 1, \dots, K_2$ . An example would be the familiar triangle of property-casualty loss data.

Now suppose that the correlations between  $Y_i$  and  $Y_j$  each take the form

$$\text{Corr}(Y_{ik_1k_2}, Y_{jl_1l_2}) = \text{function}(|k_1 - l_1|, |k_2 - l_2|). \quad (2.26)$$

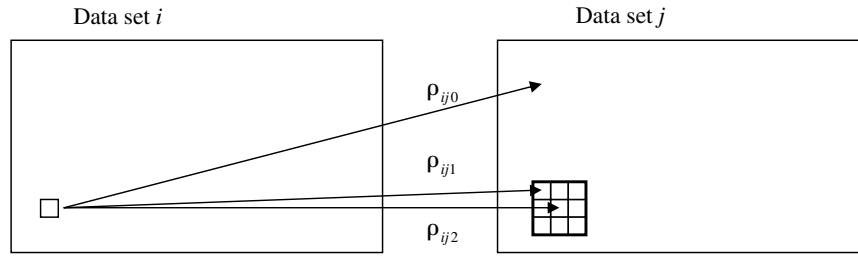
The arguments  $|k_1 - l_1|, |k_2 - l_2|$  measure the relative difference between the spatial positioning of the observations  $Y_{ik}$  and  $Y_{jl}$ . It would be possible to express (2.26) in terms of vector components  $Y_{ik}, Y_{jl}$ , as before, but to do so would seem contrived, and the geometric form appearing in (2.26) would not be retained. Consider the following as an example of (2.26), in which the left-hand side is denoted  $C_{ij,k_1k_2l_1l_2}$ :

$$\begin{aligned} C_{ij,k_1k_2l_1l_2} &= \rho_{ij2} \quad \text{if } i \neq j, (k_1, k_2) = (l_1, l_2) \\ &= \rho_{ij1} \quad \text{if } i \neq j, (k_1, k_2) \neq (l_1, l_2), |k_1 - k_2|, |l_1 - l_2| \leq 1 \\ &= \rho_{ij0} \quad \text{if } i \neq j, \text{ at least one of } |k_1 - k_2|, |l_1 - l_2| > 1. \end{aligned} \quad (2.27)$$

Assume that there is no correlated noise. Geometrically, the situation is as depicted in Figure 1.

Now suppose that the  $Z_i$  extend the lattice on which the  $i$ th data set is defined and that the above correlation structure extends in the natural way. Define

Figure 1  
**Geometric Representation of (2.27)**



$$W_i = \begin{bmatrix} Y_i \\ Z_i \end{bmatrix}.$$

Suppose that the data to be modeled are of the form (2.1), and apply the bootstrap described in Section 2.3 with the exception that (2.9) and (2.12) are replaced as follows.

Just as in (2.9), consider all permutations  $R_i \rightarrow M_{i(r)}R_i$ . However, for each such permutation, allow whole  $3 \times 3$  blocks of cells to be permuted, as illustrated in Figure 2. The permutation acts on the center cell of the block shown in the figure, but induces the mapping shown on the neighboring cells.

The permutation of whole blocks is reminiscent of the *moving block bootstrap* (Carlstein 1986; Kunsch 1989), though that is used for preservation of correlation *within* a data set. Here there is assumed to be no correlation within a data set, and the synchronization of whole blocks is used to preserve correlation *between* data sets. If, however, within-set correlation were present, that would be preserved too. It is also reminiscent of the *historical simulation* procedure used to estimate VaR for a banking portfolio subject to correlation between risk factors (Duffie and Pan 1997).

An exception will arise when the source or target center cell of a permutation lies on the boundary of the data set. In this case permuted neighboring cells that do not make sense are simply ignored, as illustrated in Figure 3.

At this stage it might seem natural to construct pseudo-data by the following replacement of (2.9):

$$Y_{i(r)}^* = \hat{Y}_i + \hat{\sigma}_i R_{i(r)}^Y, \tag{2.9d}$$

where  $R_{i(r)}^Y$  is the array of residuals  $R_{i(r)k_1k_2}^Y$ , with each of these elements calculated as the average, over all permutations, of the residuals mapped to the  $(k_1, k_2)$  position in Figure 2. However, as will be demonstrated below, this form of averaging causes the correlations between some of the  $R_{i(r)k_1k_2}^Y$  to be less than specified by (2.27). Hence, the correct form of the pseudo-data vector is

$$Y_{i(r)}^* = \hat{Y}_i + \hat{\sigma}_i R_{i(r)}^{Y*}, \tag{2.9e}$$

where  $R_{i(r)}^{Y*}$  is an adjusted form of  $R_{i(r)}^Y$  yielding the correct correlation structure.

The correlation structure illustrated in Figure 1 is (partially) preserved by synchronizing (2.9d) across  $i = 1, \dots, I$  in the sense of matched permutations: that is, if a permutation maps  $(k_1, k_2) \rightarrow (l_1, l_2)$

Figure 2  
**Effect of a Single Bootstrap Permutation**

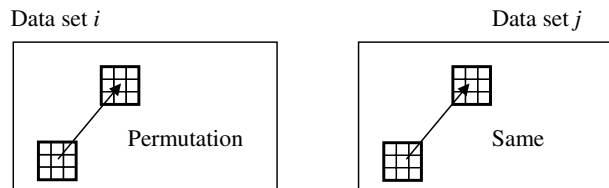
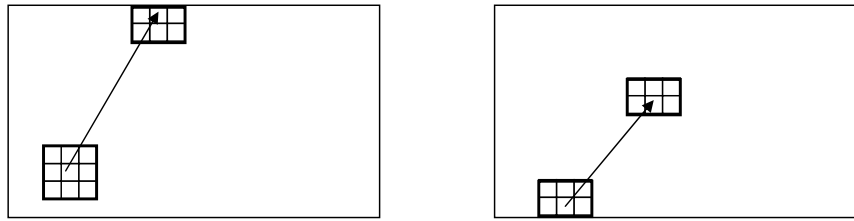




Figure 3  
Examples of Exceptional Permutations



in data set  $i$ , then the same permutation acts on all other data sets  $j$ , as illustrated in Figure 2. The fact that the preservation of correlation is only partial is discussed further below.

In the present geometric context, pseudo-forecasts (2.11) represent a geometric extension  $Z_i$  of resampled residuals from  $Y_i$ . This is illustrated in Figure 4, where the synchronization of data sets is also illustrated. The same boundary issues arise as were illustrated in Figure 3 and are dealt with in the same way.

In sympathy with (2.9e), (2.12) is replaced by

$$\eta_{i(r)}^* = \hat{\sigma}_i R_{i(r)}^{Z_i^*}, \tag{2.12e}$$

where  $R_{i(r)}^{Z_i^*}$  is a correlation-corrected form of  $R_{i(r)}^Z$ , the array of residuals  $R_{i(r)l_1l_2}^Z$  with each of these elements calculated as the average, over all resamplings (within the bootstrap replication  $r$ ), of the residuals mapped to the  $(l_1, l_2)$  position of  $Z_i$  in Figure 4.

The fact that (2.9d) only partially preserves the correlation between  $Y_i$  and  $Y_j$  is illustrated in Figure 5. Suppose the two target blocks there cover coordinates  $(k, l)$  with  $k_1 \leq k \leq k_1 + 5$ ,  $l_1 - 1 \leq l \leq l_1 + 1$ . Then the particular permutation illustrated will have preserved correlation between  $Y_{i,k_1+1,l_1}$  and  $Y_{j,k_1+2,l_1}$  (for example) but not between  $Y_{i,k_1+2,l_1}$  and  $Y_{i,k_1+3,l_1}$ .

In fact, this above resampling scheme will degrade the excess of the covariance between any pair of neighboring cells over the background covariance (corresponding to  $\rho_{ij0}$ ) by a factor of  $1 - p$ , where  $p$  is the proportion of resamplings in which those cells occur within a single  $3 \times 3$  block.

Let  $C_{ij}^Y$  denote  $\text{Corr}(R_{i(r)}^Y, R_{j(r)}^Y)$ , noting that this quantity is independent of  $r$ . Let  $C_{ij}$  denote the correlation matrix defined by (2.27). Note that  $C_{ij}^Y$  will be a distorted version of  $C_{ij}$  in which  $\rho_{ij1}$  is replaced by  $\lambda_{ij,k_1k_2l_1l_2}\rho_{ij1} + (1 - \lambda_{ij,k_1k_2l_1l_2})\rho_{ij0}$ , for some known factor  $\lambda_{ij,k_1k_2l_1l_2}$  between 0 and 1, when it occurs in the  $(k_1, k_2, l_1, l_2)$  position in  $C_{ij}$ .

Let  $\hat{C}_{ij}^Y$  be the estimate of  $C_{ij}^Y$  obtained by estimating  $\rho_{ij0}$ ,  $\rho_{ij2}$  and  $\rho_{ij0} + \lambda_{ij,k_1k_2l_1l_2}(\rho_{ij1} - \rho_{ij0})$  by means of averages of bootstrapped covariances. For example, the third of these quantities would be estimated by averaging the bootstrapped covariances corresponding to  $C_{ij,k_1k_2l_1l_2}$  over the set  $S = \{i, j, k_1, k_2, l_1, l_2: i \neq j, (k_1, k_2) \neq (l_1, l_2), |k_1 - k_2| \leq 1, |l_1 - l_2| \leq 1\}$  (as well as averaging over  $r$ ).

Figure 4  
Synchronized Pseudo-Forecasts

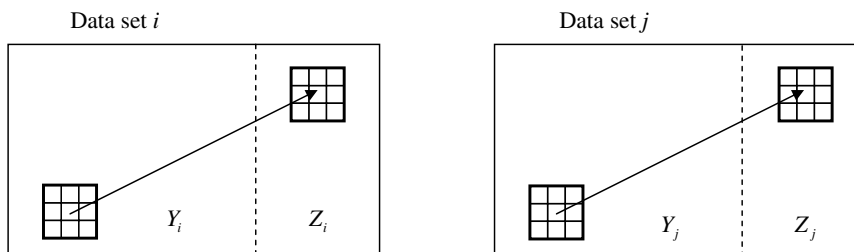
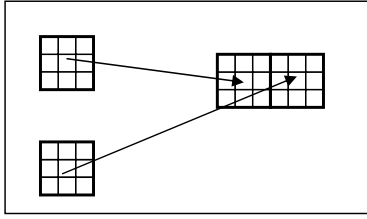


Figure 5  
Source of Partial Correlations



As explained above, the factor  $\lambda_{ij,k_1k_2l_1l_2}$  is just the proportion of bootstrap permutations that place cells  $(k_1, k_2)$  and  $(l_1, l_2)$  within a single permuted  $3 \times 3$  block. These quantities depend only on the permutations, as distinct from the data. They may therefore be calculated ahead of the application of the bootstrap, possibly by a prior application of the bootstrap, in which just *positions* (and not the values) of permuted cells are used.

Let  $\rho'_{ij1,k_1k_2l_1l_2}$  denote the quantity  $\rho_{ij0} + \lambda_{ij,k_1k_2l_1l_2} (\rho_{ij1} - \rho_{ij0})$ . Then  $(\hat{\rho}'_{ij1,k_1k_2l_1l_2} - \hat{\rho}_{ij0})$  estimates  $\lambda_{ij,k_1k_2l_1l_2} (\rho_{ij1} - \rho_{ij0})$ , where  $\hat{\rho}_{ij0}$  and  $\hat{\rho}'_{ij1,k_1k_2l_1l_2}$  are the bootstrap estimates described above. A single estimate of  $(\rho_{ij1} - \rho_{ij0})$  is then given by

$$\sum (\hat{\rho}'_{ij1,k_1k_2l_1l_2} - \hat{\rho}_{ij0}) / \sum \lambda_{ij,k_1k_2l_1l_2},$$

where the summations run over all subscripts in the set  $S$ , and over  $r$ . Combination of this estimate of  $(\rho_{ij1} - \rho_{ij0})$  with  $\hat{\rho}_{ij0}$  then gives a single estimate  $\hat{\rho}_{ij1}$  of  $\rho_{ij1}$ .

Let  $C$  be the  $nI \times nI$  correlation matrix consisting of  $I \times I$  blocks  $C_{ij}$ , and define  $\hat{C}$ ,  $C^Y$  and  $\hat{C}^Y$  similarly. Let  $\Gamma$ ,  $\hat{\Gamma}$ ,  $\Gamma^Y$ ,  $\hat{\Gamma}^Y$  be the covariance matrices corresponding to  $C$ ,  $\hat{C}$ ,  $C^Y$ ,  $\hat{C}^Y$ . Consider the diagonalized forms of  $\hat{\Gamma}$ ,  $\hat{\Gamma}^Y$ :

$$\hat{\Gamma} = PDP^T, \quad \hat{\Gamma}^Y = \mathbf{P}^Y \mathbf{D}^Y (\mathbf{P}^Y)^T, \quad (2.28)$$

where  $D$ ,  $D^Y$  are diagonal and  $P$ ,  $P^Y$  orthogonal. Then

$$\hat{\Gamma} = L^Y \hat{\Gamma}^Y (L^Y)^T, \quad (2.29)$$

where

$$L^Y = PD^{1/2} (D^Y)^{-1/2} (P^Y)^T. \quad (2.30)$$

Now  $\hat{\Gamma}^Y$  is an estimate of  $\Gamma^Y = \text{Var}[R_{(r)}^Y]$ , where  $R_{(r)}^Y$  is the  $n$ -vector obtained by stacking the vectors  $R_{i(r)}^Y$ ,  $i = 1, \dots, I$ . Therefore,  $\hat{\Gamma}$  is an estimate of  $\text{Var}[L^Y R_{(r)}^Y]$ .

But  $\hat{\Gamma}$  is also an estimate of the covariance matrix  $\Gamma$  corresponding to  $C$ , the correlation matrix summarizing (2.27). This means that, under the definition

$$R_{(r)}^{Y*} = L^Y R_{(r)}^Y, \quad (2.31)$$

the correlation matrix of  $R_{(r)}^{Y*}$  will approximate  $C$ , and so this definition will yield the correct  $R_{(r)}^{Y*}$  for substitution in (2.9e).

By similar reasoning, the use of  $R_{i(r)}^Z$  in (2.12e) would only partially preserve the correlation between  $Z_i$  and  $Z_j$ . A process precisely parallel to the above yields a matrix  $L^Z$  such that the residual vector  $R_{i(r)}^{Z*}$  required by (2.12e) is given by

$$R_{(r)}^{Z*} = L^Z R_{(r)}^Z. \quad (2.32)$$

**REMARK 2.4**

It would be necessary to check  $\hat{C}^Y$  and  $\hat{C}$  for positive definiteness. In the event that it is not found, a degenerate choice of (2.30) might be indicated, such as  $\rho_{ij1} = \rho_{ij0}$  (synchronized individual data points (see (2.14)) or  $\rho_{ij2} = \rho_{ij1}$ .

**3. PRACTICAL APPLICATION**

Sections 2.4.2 to 2.4.4 consider specific correlation structures and produce forms of synchronous bootstrap that generate pseudo-data with the same correlations. In these situations the bootstrap will be said to be *adapted* to the given correlation structure.

A poorly adapted bootstrap will tend to destroy correlation between data points. An example of this is given in Section 2.4.1, where poor adaptation consists of bootstrapping correlated data sets independently. For this reason, it is to be expected that measured correlations between bootstrap forecasts  $Z_{i(r)}^*$  and  $Z_{j(r)}^*$  will be maximized when the form of bootstrap is close to adapted to the true correlation structure.

In many practical situations, one may have a range of possible correlation structures in mind as candidates, but little idea of the likelihood of each. Examples are given in Section 4. It is desirable in these situations to allow the data to select the adapted form of bootstrap. This is done by implementing a separate form of the bootstrap adapted to each candidate correlation structure, and selecting that which yields the greatest correlations.

One needs to decide on a criterion for the “greatest correlations” when considering estimates of a whole matrix of correlations  $C_{ij,kl}$ . It may be possible to do this without considering correlations explicitly. For example, if the objective is to estimate  $\varepsilon$  defined by (2.13), then one might choose the bootstrap that maximizes the absolute difference between the estimate of  $\text{Var}[\varepsilon]$  and the estimate of the same quantity obtained by independent bootstrapping of distinct data sets.

**4. EXTENSIONS****4.1 Other Geometrically Related Subsets**

As stated at the beginning of Section 2.4.4, the synchronization of geometrically related subsets was described there only in terms of an example. The example involved a specific geometric relationship. Clearly other relationships are possible.

For example, one might wish to increase the size of the block illustrated in Figure 2.1 from  $3 \times 3$  to  $5 \times 5$  to extend the correlation from “nearest neighbor” to “nearest neighbor but one.” The bootstrap adapted to this structure can be developed along exactly the same lines as Section 2.4.4.

While the correlation structure may be made increasingly complex, and the bootstrap adapted accordingly, this does not come without cost. The greater the complexity, the greater the effort in coordinating the bootstrap’s permutations of whole subsets, the more numerous the boundary problems associated with those permutations, and the more extensive the required correction of correlation biases such as described by the factors  $\lambda$  in Section 2.4.4.

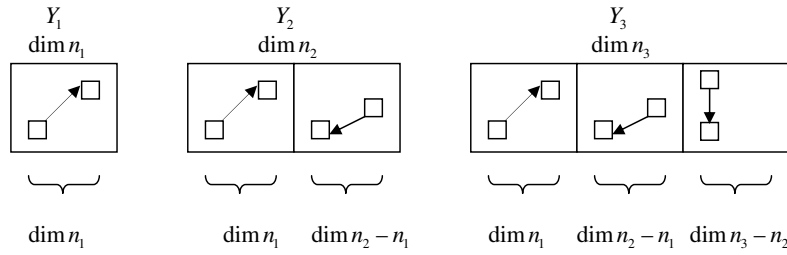
**4.2 Sampling with Replacement Not Permitted**

All bootstrapping in the foregoing sections has been defined in terms of permutations, or sampling without replacement. Sampling with replacement does *not* preserve correlation structures.

Sampling with replacement would be effected by redefining the resampling matrix of Section 2.1, no longer requiring that rows be distinct. Such a resampling matrix  $M$  has the following multiplication properties:

$$\begin{aligned} \text{For } k = l, M_{ks} M_{lt} &= 1 \text{ for } s = t \\ &= 0 \text{ otherwise;} \end{aligned}$$

Figure 6  
**Bootstrapping Data Sets of Different Dimensions**



For  $k \neq l$ ,  $M_{ks}M_{lt} = 1$  for a single pair  $s, t$  (no longer  $s \neq t$  necessarily)  
 $= 0$  otherwise.

The allowance here that  $M_{ks}M_{lt}$  may be 1 for  $k \neq l, s = t$ , means that (2.18) can take the value  $C_{ij,ss}$  for  $k \neq l$ , in which case the resampling matrix  $M$  has failed to preserve correlation structure by transferring an on-diagonal correlation to off-diagonal within  $C_{ij}$ .

This can be easily understood by consideration of an example of synchronization of individual data points in which all rows of  $M$  are the same. In this case (2.9a) becomes

$$Y_{i(r)}^* = \hat{Y}_i + \hat{\sigma}_i R_{is} 1, \tag{4.1}$$

where  $R_{is}$  is the  $s$ -th component of  $R_i$  and  $1$  is a vector with all components unity. In other words, the single residual  $R_{is}$  has been resampled for all components of  $Y_{i(r)}^*$ , and so, by synchronization,  $R_{is}$  is sampled for all components of  $Y_{i(r)}^*, i = 2, \dots, I$ . Then  $\text{Corr}[Y_{i(r)}^*, Y_{j(r)}^*]$  will (apart from a correction of  $O(n^{-1})$ ) have all elements the same, and equal to  $\text{Corr}(R_{is}, R_{js}) = C_{ij,ss}$ , as found algebraically above.

It follows that, strictly, only resampling *without* replacement should be used within a synchronous bootstrap. In practice, however, the error introduced by a sampling with replacement often will be relatively small. Provided that the dimensions  $n_i$  of the vectors  $Y_i$  are not too small, the relative incidence of repetitions in sampling with replacement will be low, and the distortion of correlation structure caused by them correspondingly low.

### 4.3 Subsets of Different Dimensions

The observations vectors  $Y_i$  were defined in (2.1) as of dimension  $n_i, i = 1, \dots, I$ . However, for subsequent analysis, it was assumed that  $n_1 = \dots = n_I = n$ . Consider now the implications of allowing the  $n_i$  to differ one from another.

Consider, for example, the synchronization of individual data points (Section 2.4.2) within this extended framework. The correlation structure (2.14) is still meaningful provided that the ‘‘correlation matrix’’  $C_{ij}$  is understood to be of dimension  $n_i \times n_j$ . The required bootstrap design is as follows. Suppose, without loss of generality, that  $I = 3$  and  $n_1 \leq n_2 \leq n_3$ . Then vector  $Y_2$  needs to be decomposed into two subvectors: one of dimension  $n_1$  containing the components that have correlation  $\rho_{211}$  with their counterparts in  $Y_1$ , and the other of dimension  $n_2 - n_1$  containing no such components.

Likewise,  $Y_3$  needs to be decomposed into three subvectors of respective dimensions  $n_1, n_2 - n_1$ , and  $n_3 - n_2$ . Bootstrapping is then synchronized on subvectors of dimension  $n_1$ , and separately on subvectors of dimension  $n_2 - n_1$ . Other subvectors are not synchronized. The synchronization is illustrated schematically in Figure 6.

## 5. NUMERICAL EXAMPLES

### 5.1 Insurance Data

The bootstrap designs discussed in Sections 2.4.2 and 2.4.3 were tested numerically on hypothetical insurance data. The data take the form of *run-off triangles* for different LoBs of an insurer.

Thus each  $Y_i$  denotes a triangular array of data rather than a vector, and the notation  $Y_{ik}$  introduced at the beginning of Section 2.4.2 is conveniently extended to  $Y_{ikl}$ , denoting the  $(k, l)$  element of triangle  $Y_i$ ,  $k = 1, \dots, n_i$ ;  $l = 1, \dots, n_i - k + 1$ . Here  $k$  denotes *accident year* and  $l$  *development year* in insurance parlance (Taylor 2000). Years in which claims experience occurs are represented by diagonals in the triangle. The latest year of observation corresponds to the  $n_i$ -th diagonal. Note that the use of  $n_i$  in this dimensioning is different from that in previous sections. In fact, the dimension of  $Y_i$  in the present framework is  $\frac{1}{2}n_i(n_i + 1)$ .

The vector  $Z_i$  to be forecast consists of the future diagonals, that is, the  $(n_i + 1)$ -th,  $\dots$ ,  $(2n_i - 1)$ -th, as far as they relate to accident years  $1, \dots, n_i$ : that is,  $Z_i$  consists of  $\{Y_{ikl}, k = 1, \dots, n_i; l = n_i - k + 2, \dots, n_i\}$ . If the  $Y_{ikl}$  denote claim payments, then the  $\frac{1}{2}n_i(n_i - 1)$  values subject to forecast represent all future claim costs in respect of the  $n_i$  accident years. The total of these is called the *loss reserve*.

It may be noted that the number of potentially distinct correlations in this situation is large. For the case  $n_i = n$ , it is  $\frac{1}{2}I(I - 1) \times \frac{1}{2}n(n - 1)$ . This can be a large number. For example, in the case  $I = 10, n = 20$ , it is 8,550. This will be too many to estimate from the data, and allowance must be made for them either by introducing some structure to the correlations, such as a SUR framework, or by incorporating them implicitly, such as in a synchronous bootstrap.

The actuarial literature contains a few examples of the inclusion of such correlations in loss reserving procedures, but sometimes in a limited form. For example, Brehm (2002) estimates correlations between LoB-specific loss reserves on the assumption that those correlations arise from dependencies between the rates of superimposed inflation (diagonal effects) influencing the LoBs. Braun (2004) considers the chain ladder model with (in the terminology of earlier sections) pointwise correlations between age-to-age factors and relies on empirical estimation of them. As noted earlier, Kirschner, Kerley, and Isaacs (2002) consider the case of a pointwise bootstrap.

In each example below,  $I = 3, n_i = 20$ , and  $Y_1, Y_2, Y_3$  have identical marginal distributions, but have correlations superimposed. The marginal distributions take the form

$$Y_{ikl} \sim \text{Gamma}, \quad (5.1)$$

$$E[Y_{ikl}] = \exp[\beta_{k0} + \beta_{kl}(l + 1) + \beta_{k2}\log(l + 1)], \quad (5.2)$$

$$\text{Var}[Y_{ikl}] = \varphi E[Y_{ikl}]. \quad (5.3)$$

Note that the distribution of  $Y_{ikl}$  is independent of  $i$ . The gamma error distribution, converging to zero exponentially, meets the criterion of short-tailedness stated in Section 2.1.

The expected value (5.2), as a function of  $l$ , is a Hoerl curve, that is, also gamma shaped. By (5.2) and (5.3), the coefficient of variation (CoV) of  $Y_{ikl}$  is  $\{\varphi/E[Y_{ikl}]\}^{1/2}$ .

### 5.2 Pointwise Bootstrapping

In the model (5.1)–(5.3), the parameter vector  $\beta_k$  was made independent of  $k$ , that is, the same Hoerl curve applied to each accident year as well as to each LoB. The parameter values were as set out in Table 1.

Data triangles for the three LoBs were sampled according to this model, with correlated noise, as defined in Section 2.2, introduced by means of the following steps applied to each pair  $(k, l)$ :

- Sample  $x$  from a trivariate standard random normal.
- Transform  $x$  to  $u = Ax$ , where  $A$  is the matrix

Table 1  
Parameter Values

Parameter	Value
$\beta_{k0}$	+5.022
$\beta_{k1}$	-0.4
$\beta_{k2}$	+2.4
$\varphi$	100

$$\begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.2 & 0.5 & 0.3 \\ 0.3 & 0.2 & 0.5 \end{bmatrix}$$

and the resulting vector  $u$  consists of identically distributed components with all pairwise correlations equal to approximately 80%.

- Transform  $u$  to  $y = \Gamma_{kl}^{-1} [\Phi(u^{(s)})]$ , where  $u^{(s)}$  is the standardized version of  $u$  and  $\Gamma_{kl}$  is the gamma d.f. associated with  $Y_{ikl}$ .
- Let  $y$  be a realization of the vector  $[Y_{1kl}, Y_{2kl}, Y_{3kl}]^T$ .

Each of the resulting  $Y_{ikl}$  has d.f.  $\Gamma_{kl}$  and,  $\text{Corr}[Y_{ikl}, Y_{jkl}]$  is independent of  $i, j$  for  $i \neq j$  and is equal to about 80%.

This procedure is carried out independently for each pair  $(k, l)$ , that is, the values of  $x$  are sampled independently for distinct  $(k, l)$ . As a result,

$$\text{Corr}[Y_{ikl}, Y_{jst}] = 0 \quad \text{for } (k, l) \neq (s, t). \tag{5.4}$$

This creates a correlation structure of the form (2.14), to which the synchronized bootstrap of individual points described in Section 2.4.2 is adapted. This will be referred to as a *pointwise bootstrap*. It is applied to the data triangles  $\{Y_{ikl}\}$  generated as above, with the results of 10,000 replications set out in Table 2.

Here, and in the subsequent numerical examples, sampling with replacement is considerably easier to implement than sampling without replacement and has been used even though theoretically less accurate. This is justified by the final remark of Section 4.2.

The true values in the table are obtained by simulation. Values of  $E[Y_{ikl}]$  are generated according to (5.2), applying equally to the LoBs  $i = 1, 2, 3$ . For each  $(k, l)$ , correlated gamma noise terms, generated as described above, are added. This is done independently for different  $(k, l)$ .

The true loss reserve for LoB  $i$  is calculated as the sum of the simulated values of  $Y_{ikl}$  over future cells  $(k, l)$ . The correlated noise of about 80% translates into a similar correlation between loss reserves of pairs of LoBs.

Naturally, conventional independent bootstrapping produces no correlation between the loss reserves of different LoBs. The CoV of loss reserve for a single LoB is 5.4% and is reduced by a factor of  $\sqrt{3}$  to 3.0% when the loss reserve is aggregated over all three LoBs.

Table 2  
Results of Pointwise Bootstrap

Basis of Estimation	Pairwise Correlation of LoB Loss Reserves	CoV of Aggregate Loss Reserve across 3 LoBs
True (simulated)	0.81	5.4%
Independent bootstrap	-0.00	3.0
Synchronous pointwise bootstrap	0.79	5.0

However, the correlation between LoBs prevents this degree of reduction, and the true CoV of the aggregate is 5.4%. The pointwise synchronous bootstrap captures the majority of the correlation between LoB-specific loss reserves and therefore produces a substantially more accurate estimate of CoV of the aggregate.

### 5.3 Rowwise Bootstrapping

#### 5.3.1 Data

Model (5.1)–(5.3) was applied once again, with the following variation. As in Section 5.2, the parameters  $\beta_{k1}$  and  $\beta_{k2}$  were made independent of  $k$ , that is, the same Hoerl curve applied to each LoB, and also, up to a multiplier, to each accident year. These two parameters, and  $\varphi$ , were assigned the same values as in Table 1.

However,  $\beta_{k0}$  followed a random walk over  $k$ , with the same initial value  $\beta_{10}$  as in Table 1, and single-step variance of 0.4. The model specification was therefore

$$Y_{ikl} \sim \text{Gamma}, \quad (5.1)$$

$$E[Y_{ikl}] = \exp[\beta_{k0} + \beta_1(l + 1) + \beta_2 \log(l + 1)], \quad (5.2a)$$

$$\text{Var}[Y_{ikl}] = \varphi E[Y_{ikl}]. \quad (5.3)$$

Data triangles for the three LoBs were sampled according to this model, but this time with independent process error for the different LoBs. The model of each data set is now deliberately misspecified to omit the accident year multiplier. That is, instead of the correct specification (5.2a), the following model was fitted:

$$E[Y_{ikl}] = \exp[\beta'_0 + \beta'_1(l + 1) + \beta'_2 \log(l + 1)], \quad (5.2b)$$

where the parameters in the misspecified model are distinguished by a prime, and the notation indicates that the same set of parameters is estimated for each of the three data triangles. This creates a bias vector  $b_i$ , as in Section 2.2, whose  $(k, l)$  component is of the form

$$b_i = \exp[\beta_{k0} + \beta_1(l + 1) + \beta_2 \log(l + 1)] - \exp[\beta'_0 + \beta'_1(l + 1) + \beta'_2 \log(l + 1)] \quad (5.5)$$

and that may be observed to be independent of  $i$ .

The misspecification with respect to rows of the triangles creates parameter correlation of the form (2.20), where the subsets  $N_q$  are the rows. A bootstrap with synchronized rows, as described in Section 2.4.3, is therefore applied. This will be referred to subsequently as a *rowwise bootstrap*.

#### 5.3.2 Performance Measurement

As in Section 2, let  $Y_i$ ,  $Z_i$  denote the past and future data respectively in LoB  $i$ . The measurement of bootstrap performance will be concerned with  $Z_i$ .

Let

$$\zeta_i = E[Z_i], \quad (5.6)$$

which is the *true* mean, given by an equation like (5.2a), and let  $\zeta'_i$  denote the corresponding mean in the misspecified model (5.2b).

Let  $\hat{\zeta}'_i$  denote the estimate of  $\zeta'_i$  derived from the data set  $Y_i$  and let  $Z_{i(r)}^*$  denote the rowwise bootstrap forecast of  $Z_i$ . The *prediction error* in the bootstrap forecast is

$$\begin{aligned}
Z_i - Z_{i(r)}^* &= (Z_i - \zeta_i) && \text{[process error]} \\
&+ (\zeta_i - \zeta'_i) && \text{[specification error]} \\
&+ (\zeta'_i - \hat{\zeta}'_i) && \text{[parameter error]} \\
&+ (\hat{\zeta}'_i - Z_{i(r)}^*) && \text{[internal bootstrap error]}.
\end{aligned} \tag{5.7}$$

It will be convenient to write this in the notation

$$Z_i - Z_{i(r)}^* = \varepsilon_i + \xi_i + \chi_{i(r)}, \tag{5.8}$$

where  $\varepsilon_i$ ,  $\chi_{i(r)}$  denote the process and internal bootstrap errors, respectively, and  $\xi_i$  denotes the sum of the specification and parameter errors.

Note that

- $\xi_i$  is a deterministic quantity
- $\varepsilon_i$  is stochastically independent of  $\varepsilon_j$  for  $j \neq i$
- $\varepsilon_i$  is stochastically independent of  $\xi_j$  for any  $i, j$

and typically

$$E[\varepsilon_i] = 0, \tag{5.9}$$

$$\xi_i \neq 0, \tag{5.10}$$

$$E[\chi_{i(r)}] \neq 0. \tag{5.11}$$

Relation (5.10) relates mainly to the existence of specification error. Relation (5.11) reflects the fact that forecasts from a synchronous bootstrap will contain a bias that tends to correct for specification error.

Now consider the *predictive covariance*

$$\begin{aligned}
E[(Z_i - Z_{i(r)}^*)(Z_j - Z_{j(r)}^*)] &= E[(\varepsilon_i + \xi_i + \chi_{i(r)})(\varepsilon_j + \xi_j + \chi_{j(r)})] \\
&= \delta_{ij} \text{Var}[\varepsilon_i] + \text{Cov}[\chi_{i(r)}, \chi_{j(r)}] + E[\xi_i + \chi_{i(r)}]E[\xi_j + \chi_{j(r)}] \\
&= \delta_{ij} \text{Var}[\varepsilon_i] + \text{Cov}[\chi_{i(r)}, \chi_{j(r)}] + \{\xi_i + E[\chi_{i(r)}]\}\{\xi_j + E[\chi_{j(r)}]\}.
\end{aligned} \tag{5.12}$$

It will now be convenient to write

$$\xi_i = s_i + p_i \tag{5.13}$$

recognizing  $\xi_i$  as the sum of specification and parameter errors, respectively.

Substitution of this in (5.12) gives

$$\begin{aligned}
E[(Z_i - Z_{i(r)}^*)(Z_j - Z_{j(r)}^*)] &= \delta_{ij} \text{Var}[\varepsilon_i] + \text{Cov}[\chi_{i(r)}, \chi_{j(r)}] + \{p_i + E[\chi_{i(r)}]\}\{p_j + E[\chi_{j(r)}]\} \\
&+ s_i s_j + s_i \{p_j + E[\chi_{j(r)}]\} + s_j \{p_i + E[\chi_{i(r)}]\}.
\end{aligned} \tag{5.14}$$

Express this relation in the notation

$$(\text{PC})_{ij} = (\text{PC})_{ij}^{ns} + (\text{PC})_{ij}^s, \tag{5.15}$$

where  $(\text{PC})_{ij}$  denotes predictive covariance,  $(\text{PC})_{ij}^s$  denotes the last three members of (5.14), involving specification error, and  $(\text{PC})_{ij}^{ns}$  denotes the remainder of the right-hand side of (5.14), not involving specification error.

Define  $\varepsilon$  as in (2.13). Then the *mean square error of prediction* of  $\varepsilon$  is

$$\begin{aligned}
\text{MSEP}[\varepsilon] &= E[(\varepsilon - \varepsilon_{(r)}^*)^2] = \sum_{ij} 1^T (\text{PC})_{ij} 1 \\
&= \text{MSEP}^{ns}[\varepsilon] + \text{MSEP}^s[\varepsilon],
\end{aligned} \tag{5.16}$$



where

$$\text{MSEP}^{ns}[\boldsymbol{z}] = \sum_{ij} \mathbf{1}^T (\text{PC})_{ij}^{ns} \mathbf{1}, \quad \text{MSEP}^s[\boldsymbol{z}] = \sum_{ij} \mathbf{1}^T (\text{PC})_{ij}^s \mathbf{1}.$$

Also define

$$\text{MSEP}_{ind}[\boldsymbol{z}] = \sum_i \mathbf{1}^T (\text{PC})_{ii} \mathbf{1}, \tag{5.17}$$

which is the MSEP in the case of independence between all LoBs, and define  $\text{MSEP}_{ind}^s[\boldsymbol{z}]$  and  $\text{MSEP}_{ind}^{ns}[\boldsymbol{z}]$  similarly. Then  $\text{MSEP}[\boldsymbol{z}] - \text{MSEP}_{ind}[\boldsymbol{z}]$  is the inter-LoB covariance contribution to  $\text{MSEP}[\boldsymbol{z}]$ .

The true MSEP is  $\text{MSEP}[\boldsymbol{z}]$ . However, in practice, one attempts to eliminate specification error from a model and so assumes that  $\text{MSEP}^s[\boldsymbol{z}] = 0$ , so that the observable MSEP is  $\text{MSEP}^{ns}[\boldsymbol{z}]$ . Therefore, a measure of the efficiency of the synchronous bootstrap in capturing the inter-LoB covariances is

$$\text{Efficiency} = \frac{\text{MSEP}^{ns}[\boldsymbol{z}] - \text{MSEP}_{ind}^{ns}[\boldsymbol{z}]}{\text{MSEP}[\boldsymbol{z}] - \text{MSEP}_{ind}[\boldsymbol{z}]}. \tag{5.18}$$

### 5.3.3 Numerical Results

The properties of the rowwise bootstrap depend on the degree of misspecification that it needs to correct. If there is little variation in the row parameters  $\beta_{k0}$ , there will be correspondingly little parameter correlation, and little need for the bootstrap to estimate it efficiently.

Five different examples were considered, each consisting of a triple of data sets for the three LoBs. The random walk of parameters  $\beta_{k0}$  differed between the examples. Figure 7 plots (on a logarithmic scale) the sampled values of  $\exp(\beta_{k0} - \beta_{10})$ , which is the multiplier that scales the expected values of accident year  $k$  relative to year 1.

In Example 1 claims experience drifts to very high levels in the later accident years. Example 5 provides more or less the mirror image, with claims experience drifting to low levels. Examples 2, 3, and 4 are more moderate.

The effects of these parameters are illustrated in Table 3, which displays the mean and prediction error of the loss reserve for a single LoB, as estimated by a simple (unsynchronized) bootstrap with 1000 replications. The prediction error takes the form of the root mean square error of prediction

Figure 7  
Accident Year Multipliers

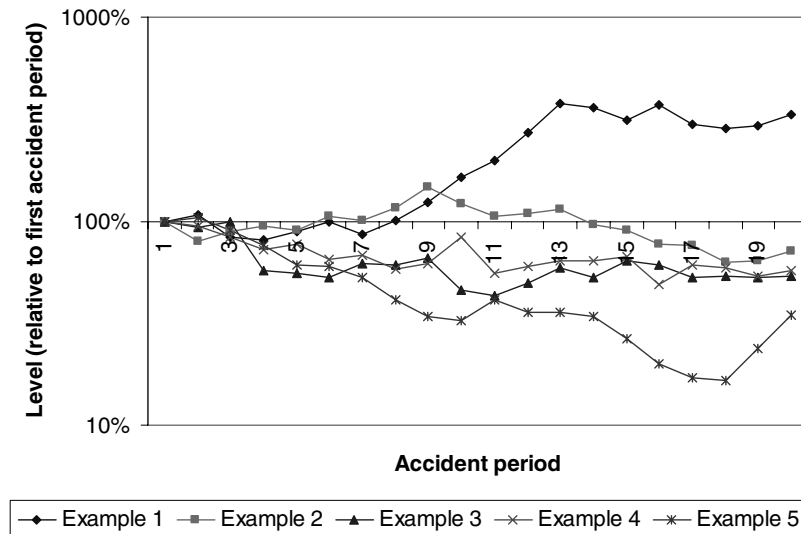


Table 3  
**Mean and Prediction Error of Loss Reserve for  
 a Single LoB**

Example	Mean Loss Reserve	RMSEP (% of Mean)
1	\$211,325	60
2	47,572	22
3	39,610	32
4	42,831	33
5	27,840	160

(RMSEP), including specification error, and is estimated by simulation. The effect of the parameter drift on prediction error is evident in Examples 1 and 5.

Table 4 displays values of the efficiency statistic (5.18) for the rowwise bootstrap applied to the five examples, with 10,000 replications in each case. The table also displays values of  $\{1 - \text{MSEP}_{ind}[\varepsilon]/\text{MSEP}[\varepsilon]\}$ , representing the relative contribution of the inter-LoB correlations to  $\text{MSEP}[\varepsilon]$ . It has a maximum value of  $\frac{2}{3}$ . The efficiency of the rowwise bootstrap is displayed in the column headed “rowwise.” The column headed “pointwise” displays the corresponding efficiency measure for the pointwise bootstrap.

Just the presence of noise in the data will prevent any forecast procedure from capturing full parameter correlation. However, the rowwise bootstrap is seen to perform well in cases of high correlation and moderately well in cases of lower correlation. Somewhat surprisingly, the pointwise bootstrap also performs moderately well, though it does not match the rowwise version in cases of high correlation.

It should be recognized that, despite the semi-effectiveness of the synchronous bootstrap in capturing the contribution of parameter correlation to RMSEP, the existence of specification error nevertheless causes a substantial part of RMSEP to remain unrecognized. Table 5 illustrates.

The first column of results in Table 5 is simply reproduced from Table 3. The synchronized bootstrap RMSEP including specification error is based on  $(\text{PC})_{ij}$  from (5.15), excluding specification error based on  $(\text{PC})_{ij}^{ns}$ . The table illustrates how the RMSEP to be estimated (i.e., including specification error) is reduced by the introduction of the synchronous bootstrap. This occurs because this form of bootstrap assigns part of what is otherwise interpreted as noise-to-row effects.

The final column of the table contains the estimate of RMSEP obtained in practice where specification cannot be measured and is assumed zero. It is seen that 80% of the RMSEP is lost in one of the cases of high parameter correlation, and of the order of half in the cases of milder correlation.

Such losses are, of course, serious. However, two points should be made:

- The examples, particularly those involving high parameter correlation, are rather exaggerated. In practice, specification errors of the magnitude contained in these examples would hardly go unnoticed.

Table 4  
**Efficiency of Rowwise and Pointwise Bootstrap Estimation of Prediction Error**

Example	Contribution of Inter-LoB Correlations to Prediction Error	Efficiency	
		Rowwise Bootstrap	Pointwise Bootstrap
1	66.4%	73%	32%
2	59.7	60	9
3	63.9	24	39
4	63.1	26	25
5	65.0	103	51

Table 5  
**Prediction Error Unrecognized Due to Specification Error**

Example	RMSEP (as % of Mean)		
	Including Specification Error		Excluding Specification Error According to Rowwise Synchronized Bootstrap
	According To Unsynchronized Bootstrap	According to Rowwise Synchronized Bootstrap	
1	60%	50%	10%
2	22	15	10
3	32	24	9
4	33	24	10
5	160	62	80

- The failure to correct for specification error is not a failure of the synchronous bootstrap as such. Indeed, Table 5 demonstrates that this form of bootstrap makes a significant contribution to correcting for such errors. Specification error will bedevil any form of forecast. This accepted, the synchronous bootstrap performs reasonably well in recognizing inter-LoB dependencies.

## 6. ACKNOWLEDGMENTS

We are grateful to Hong Ooi of Insurance Australia Group and Mark Johnston of PricewaterhouseCoopers who alerted us to some connections between our work and the statistical and economics literatures.

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