# Using Aumann-Shapley Values to Allocate Insurance Risk: The Case of Inhomogeneous Losses

Michael R. Powers*

## Abstract

The problem of allocating responsibility for risk among members of a portfolio arises in a variety of financial and risk-management contexts. Examples are particularly prominent in the insurance sector, where actuaries have long sought methods for distributing capital (net worth) across a number of distinct exposure units or accounts according to their relative contributions to the total ''risk'' of an insurer's portfolio. Although substantial work has been done on this problem, no satisfactory solution has yet been presented for the case of inhomogeneous loss distributions—that is, losses $X \sim F_{X|\underline{\lambda}}(x)$ such that $F_{X|t\underline{\lambda}}(x) \neq F_{tX|\underline{\lambda}}(x)$ for some $t > 0$. The purpose of this article is to show that the value-assignment method of nonatomic cooperative games proposed in 1974 by Aumann and Shapley may be used to solve risk-allocation problems involving losses of this type. This technique is illustrated by providing analytical solutions for a useful class of multivariate-normal loss distributions.

## 1. Introduction

The problem of allocating responsibility for risk among members of a group arises in a variety of financial and risk-management contexts. Examples are particularly salient in the insurance sector, where actuaries have long sought methods for distributing capital (net worth) across a number of distinct exposure units, accounts, or lines of business according to their relative contributions to the total ''risk'' of an insurer's portfolio. Although substantial work has been done on this problem (see the surveys of Venter 2004 and Kaye 2005),[1] no satisfactory solution has yet been presented for the case of inhomogeneous loss distributions—that is, losses $X \sim F_{X|\underline{\lambda}}(x)$ such that $F_{X|t\underline{\lambda}}(x) \neq F_{tX|\underline{\lambda}}(x)$ for some $t > 0$. The purpose of this article is to show that the value-assignment method of nonatomic cooperative games proposed by Aumann and Shapley (1974) may be used to solve risk-allocation problems involving losses of this type. This technique is illustrated by providing analytical solutions for a useful class of multivariate-normal loss distributions.

Consider an insurance portfolio composed of $n$ members (exposure units, accounts, or lines of business) and indexed by $i = 1, 2, \ldots, n$. Let each portfolio member's individual loss amount be denoted by the continuous random variable

$$L_i \sim F_{L_i|\Lambda_i}(\ell),$$

* Michael R. Powers, PhD, is Professor of Risk, Insurance, and Healthcare Management at The Fox School, Temple University, Philadelphia, PA, 19122, michael.powers@temple.edu.

[1] The author is indebted to an anonymous reviewer for pointing out that a separate literature has developed in papers by Dhaene, Goovaerts, and Kaas (2003), Goovaerts, Van den Borre, and Laeven (2005), and others. It is hoped that the present article may make researchers in these parallel areas of endeavor more aware of each other's contributions.

where $E[L_i] = \Lambda_i > 0$, and let the portfolio's total loss amount be given by

$$X = \sum_{j=1}^{n} L_j \sim F_{X|\underline{\Lambda}}(x).$$

In this setting the insurer's total capital, K, is to be distributed across the portfolio's members according to their relative contributions to the total risk. For clarity, the term *allocate* will be used to specify how much responsibility for risk is attributable to each member of a portfolio, and the term *distribute* to specify how K is partitioned into the components of a vector $\underline{K} = [K_1, K_2, \ldots, K_n]$, where $\Sigma_{j=1}^{n} K_j = K$. Moreover, $\underline{K}$ will be expressed as $\underline{K} = [\kappa_1\Lambda_1, \kappa_2\Lambda_2, \ldots, \kappa_n\Lambda_n]$ for some fixed vector $\underline{\kappa} = [\kappa_1, \kappa_2, \ldots, \kappa_n]$.

Now let $\mu(X|\underline{\Lambda}, \underline{\kappa})$ denote a fundamental measure of the portfolio's total risk, where $\underline{\Lambda} = [\Lambda_1, \Lambda_2, \ldots, \Lambda_n]$. Examples of commonly used risk measures include the following:[2]

1. *Standard deviation,* $\mu^{SD}(X|\underline{\Lambda}, \underline{\kappa}) = SD_{X|\underline{\Lambda}}[X]$
2. *Variance,* $\mu^{Var}(X|\underline{\Lambda}, \underline{\kappa}) = Var_{X|\underline{\Lambda}}[X]$
3. *Value at risk,* $\mu^{VaR}(X|\underline{\Lambda}, \underline{\kappa}) = F_{X|\underline{\Lambda}}^{-1}(1 - \varepsilon)$
4. *Tail value at risk,* $\mu^{TVaR}(X|\underline{\Lambda}, \underline{\kappa}) = E_{X|\underline{\Lambda}}[X|X > F_{X|\underline{\Lambda}}^{-1}(1 - \varepsilon)]$[3]
5. *Excess tail value at risk,* $\mu^{XTVaR}(X|\underline{\Lambda}, \underline{\kappa}) = E_{X|\underline{\Lambda}}[X|X > E_{X|\underline{\Lambda}}[X]]$
6. *Expected policyholder deficit,* $\mu^{EPD}(X|\underline{\Lambda}, \underline{\kappa}) = E_{X|\underline{\Lambda}}[X - K|X > K]$ and
7. *Default value,* $\mu^{DV}(X|\underline{\Lambda}, \underline{\kappa}) = E_{X|\underline{\Lambda}}[X - (K + \text{premiums}) |X > K + \text{premiums}] \times \Pr\{X > K + \text{premiums}\}$.

Given a selected risk measure $\mu(X|\underline{\Lambda}, \underline{\kappa})$, the two-step allocation/distribution problem proceeds as follows. First, one derives a *value* vector or *contribution* vector

$$\underline{\varphi}(\mu, \underline{\Lambda}, \underline{\kappa}) = [\varphi_1(\mu, \underline{\Lambda}, \underline{\kappa}), \varphi_2(\mu, \underline{\Lambda}, \underline{\kappa}), \ldots, \varphi_n(\mu, \underline{\Lambda}, \underline{\kappa})],$$

in which $\varphi_i(\mu, \underline{\Lambda}, \underline{\kappa})$ reflects the relative responsibility of portfolio member $i$ for the total portfolio risk. Second, one employs the vector $\underline{\varphi}(\mu, \underline{\Lambda}, \underline{\kappa})$ to solve for the optimal distribution vector $\underline{K}^* = [\kappa_1^*\Lambda_1, \kappa_2^*\Lambda_2, \ldots, \kappa_n^*\Lambda_n]$, according to certain specified criteria.

Naturally the selection of an appropriate risk measure should be based upon the characteristics of $F_{X|\underline{\Lambda}}(x)$—for instance, dispersion, asymmetry, long tail—that are considered most relevant in distributing K.[4] Ideally $\underline{\varphi}(\mu, \underline{\Lambda}, \underline{\kappa})$ then would be determined as the unique allocation vector satisfying a list of reasonable properties or axioms. Although the conditions $\varphi_i(\mu, \underline{\Lambda}, \underline{\kappa}) \geq 0$ and $\Sigma_{j=1}^{n} \varphi_j(\mu, \underline{\Lambda}, \underline{\kappa}) = 1$ are attractive, they are not necessary for the purposes at hand. Rather, we can allow each component $\varphi_i(\mu, \underline{\Lambda}, \underline{\kappa})$ to capture member $i$'s responsibility for risk through both its magnitude *and* sign. In this formulation a negative value of $\varphi_i(\mu, \underline{\Lambda}, \underline{\kappa})$ simply means that the presence of member $i$ serves to decrease (i.e., offset some portion of) the total portfolio's risk.

## 2. PREVIOUS RESEARCH

Historically actuaries have employed a number of ad hoc methods to allocate responsibility for risk among members of an insured portfolio. Often a particular member's share would be computed in direct proportion to some measure of the ''risk'' associated with that member. This type of approach is referred to as *proportional spread* (see Venter 2004). For example, given $\mu^{SD}(X,|\underline{\Lambda}, \underline{\kappa}) = SD_{X|\underline{\Lambda}}[X]$, an insurer might compute

---

[2] See Kaye (2005) for brief discussions of items (1) through (6), and Myers and Read (2001) for an application of item (7).

[3] Note that both the value at risk and the tail value at risk require the selection of a particular percentile, $F_{X|\underline{\Lambda}}^{-1}(1 - \varepsilon)$.

[4] Several authors (e.g., Denault 2001; Kaye 2005) have suggested restricting consideration to *coherent* risk measures, which satisfy the properties of *subadditivity, monotonicity, positive homogeneity,* and *translation invariance.* While I agree that these four properties are certainly reasonable, I recognize that practitioners may sometimes find value in a simple risk measure, like the standard deviation, that is not coherent.

$$\varphi_i^{PS}(\mu^{SD}, \underline{\Lambda}, \underline{\kappa}) = \frac{SD_{X|\Lambda_i; \underline{\Lambda}_{-i}=\underline{0}}[X]}{\sum\limits_{j=1}^{n} SD_{X|\Lambda_j; \underline{\Lambda}_{-j}=\underline{0}}[X]}$$

to allocate risk among members of a portfolio of insured customers, where $\underline{\Lambda}_{-i}$ denotes the vector of $n - 1$ elements formed by removing $\Lambda_i$ from $\underline{\Lambda}$. Then, to distribute the total capital, K, to the various portfolio accounts, the insurer simply would set the capital associated with member $i$ in proportion to the value $\varphi_i^{PS}(\mu^{SD}, \underline{\Lambda}, \underline{\kappa})$, that is,

$$K_i = K\varphi_i^{PS}(\mu^{SD}, \underline{\Lambda}, \underline{\kappa}) = K \frac{SD_{X|\Lambda_i; \underline{\Lambda}_{-i}=\underline{0}}[X]}{\sum\limits_{j=1}^{n} SD_{X|\Lambda_j; \underline{\Lambda}_{-j}=\underline{0}}[X]} .$$

Although easy to implement, proportional-spread methods fail to account specifically for correlations among the losses associated with the different members. To overcome this problem, actuaries have sought more sophisticated methods of *marginal analysis,* based upon the relationship between the marginal changes in the members' contributions to portfolio risk and the marginal demands on K implied by these changes. The first axiomatically derived marginal-analytic approach, proposed by Mango (1998), was to allocate risk in proportion to the *Shapley value,* a concept from cooperative game theory.

For a group of portfolio members, the Shapley value is the simple average—over the set $C_i$ of all subportfolios $C$ containing member $i$—of the marginal contributions to the risk measure of $C$ as member $i$ enters $C$. Mathematically this is given by

$$\varphi_i^{S}(\mu, \underline{\Lambda}, \underline{\kappa}) = \sum_{C \in C_i} \frac{(|C| - 1)(n - |C|)}{n!} [\mu(X|\underline{\Lambda}_C, \underline{\Lambda}_{-C} = \underline{0}, \underline{\kappa}) - \mu(X|\underline{\Lambda}_{C\backslash i}, \underline{\Lambda}_{-C\backslash i} = \underline{0}, \underline{\kappa})],$$

where $|C|$ denotes the number of elements in $C$; $\underline{\Lambda}_C$ denotes the vector of all $\Lambda_j$ for which $j \in C$; $\underline{\Lambda}_{-C}$ denotes the vector of all $\Lambda_j$ for which $j \notin C$; $\underline{\Lambda}_{C\backslash i}$ denotes the vector formed by removing $\Lambda_i$ from $\underline{\Lambda}_C$; and $\underline{\Lambda}_{-C\backslash i}$ denotes the vector formed by adding $\Lambda_i$ to $\underline{\Lambda}_{-C}$.[5]

As shown by Shapley (1953), the above value-assignment method can be derived as the unique quantity satisfying a set of three axiomatic principles, often referred to as *symmetry*, *dummy*, and *additivity*. In applying the Shapley value to the risk-allocation problem, Mango (1998) noted that the first two of these axioms are most relevant in this context. Symmetry means that the order in which portfolio members are added to a subportfolio does not affect that subportfolio's risk measure, whereas dummy means that if a given portfolio member's losses are statistically independent of the losses associated with all other portfolio members, then that portfolio member's contribution to a subportfolio consists only of the member's risk measure.

Mango (1998) found that the risk measure $\mu^{Var}(X|\underline{\Lambda}, \underline{\kappa}) = Var_{X|\underline{\Lambda}}[X]$ was easy to implement, but that $\mu^{SD}(X|\underline{\Lambda}, \underline{\kappa}) = SD_{X|\underline{\Lambda}}[X]$ was analytically intractable. In using the variance, he proposed a modification of the way in which the Shapley value split the effects of covariances between members of the portfolio. This alternative, termed *covariance share,* naturally implies a departure from the standard Shapley axioms (see Mango 1998 and Kaye 2005 for further discussions of covariance-share methods).

Unfortunately there is one critical shortcoming of the Shapley value (with or without the covariance-share modification) that makes it unsuitable as a risk-allocation method: if a member of the portfolio is partitioned into two new members—the aggregation of which remains identical to the original un-

---

[5] In its conventional game-theoretic context, the Shapley value always satisfies the conditions $\varphi_i(\mu, \underline{\Lambda}, \underline{\kappa}) \geq 0$ and $\Sigma_{j=1}^{n} \varphi_j(\mu, \underline{\Lambda}, \underline{\kappa}) = 1$. However, these conditions do not necessarily hold in the risk-allocation context for most risk measures $\mu(X|\underline{\Lambda}, \underline{\kappa})$.

divided member—then the values assigned to the other (nondividing) members typically will change.[6] This defect has led researchers to consider using a generalization of the Shapley value developed by Aumann and Shapley (1974) for the case of *fractional* (or *nonatomic*) cooperative games.

To reconstruct the risk-allocation problem in the fractional setting, we must permit each of the $n$ members of the portfolio to be *infinitely divisible*. This is accomplished by replacing the mean individual loss amount $\Lambda_i$ by $\lambda_i \in (0, \Lambda_i]$ for $i = 1, 2, \ldots, n$, where $\lambda_i$ can be thought of as the level of *participation* or *involvement* of member $i$ in the portfolio. In this case the symbol $\Lambda_i \in (0, \infty)$ is taken to mean *full participation*. Another way of looking at this is to write $\lambda_i = t\Lambda_i$, where $t \in (0, 1]$ denotes the participation or involvement proportion, $\lambda_i/\Lambda_i$. The value assignment is given as follows.

## Definition 1

For the risk measure $\mu(X|\underline{\Lambda}, \underline{\kappa})$, the *Aumann-Shapley value* assigned to member $i$ is given by

$$\varphi_i^{AS}(\mu, \underline{\Lambda}, \underline{\kappa}) = \int_0^1 \frac{\partial \mu(X|\underline{\lambda}, \underline{\kappa})}{\partial \lambda_i}\bigg|_{\underline{\lambda}=t\underline{\Lambda}} \, dt. \tag{2.1}$$

Like the Shapley value, the Aumann-Shapley value can be interpreted as the simple average (hence the integral with respect to the uniform density from 0 to 1) of the marginal changes in the risk measure as the participation level of portfolio member $i$ increases. Also like the Shapley value, the Aumann-Shapley value can be derived from a set of axioms, although this can be done in more than one way (see, e.g., Aubin 1981; Billera and Heath 1982; Mirman and Tauman 1982).

One aspect of the Aumann-Shapley value that appears to deviate qualitatively from the Shapley value is that the partial derivative in the integral on the right-hand side of equation (2.1) is evaluated for the same fixed value of $t$ across all the $\Lambda_j$, whereas the first differences, $\mu(X|\underline{\Lambda}_C, \underline{\Lambda}_{-C} = \underline{0}, \underline{\kappa}) - \mu(X|\underline{\Lambda}_{C \setminus i}, \underline{\Lambda}_{-C \setminus i} = \underline{0}, \underline{\kappa})$, in the Shapley-value summation permit completely arbitrary configurations of the subportfolios $C$. However, as noted by Aumann and Shapley (1974), this "startling" observation is explained by the fact that, in a measure-theoretic sense, *almost all* of the relevant information about $\partial\mu(X|\underline{\lambda}, \underline{\kappa})/\partial\lambda_i|_{\lambda_i=t\Lambda_i;\underline{\Lambda}_{-i}}$ is contained in $\mu(X|t\underline{\Lambda}, \underline{\kappa})$ for $t \in (0, 1]$.

Interestingly, the above concern becomes moot for the special "homogeneous" class of risk measures addressed by the following result.

## Proposition 1

If $\mu(X|t\underline{\Lambda}, \underline{\kappa}) = t\mu(X|\underline{\Lambda}, \underline{\kappa})$ for all $t > 0$, then

$$\varphi_i^{AS}(\mu, \underline{\Lambda}, \underline{\kappa}) = \frac{\partial \mu(X|\underline{\lambda}, \underline{\kappa})}{\partial \lambda_i}\bigg|_{\underline{\lambda}=\underline{\Lambda}}. \tag{2.2}$$

### PROOF

Since $\mu(X|t\underline{\Lambda}, \underline{\kappa})$ is homogeneous of order 1 with respect to each element of the vector $\underline{\Lambda}$, it follows from arguments of elementary calculus that

$$\frac{\partial \mu(X|\underline{\lambda}, \underline{\kappa})}{\partial \lambda_i}\bigg|_{\underline{\lambda}=t\underline{\Lambda}} = \frac{\partial \mu(X|\underline{\lambda}, \underline{\kappa})}{\partial \lambda_i}\bigg|_{\underline{\lambda}=\underline{\Lambda}}$$

for $i = 1, 2, \ldots, n$. The desired result then follows immediately from equation (2.1). ∎

---

[6] As an example, consider what could happen when allocating risk across three different lines of primary insurance (e.g., automobile, homeowners, and business interruption) within an insurer's reinsurance treaty. After computing the Shapley values for the three individual lines, suppose that one decided to split—purely for accounting purposes—the automobile results into their personal automobile and commercial automobile components. Although the homeowners and business-interruption lines would be unaffected by this split, their Shapley values typically would change. For a simple computational example of this type of phenomenon, see Powers (2007).

In noninsurance financial applications, the premise of Proposition 1 (i.e., $\mu(X|t\underline{\Lambda}, \underline{\kappa}) = t\mu(X|\underline{\Lambda}, \underline{\kappa})$ for all $t > 0$) is frequently satisfied, and so computations may be aided by equation (2.2). However, as will be discussed in the following two sections, this premise typically does not hold for insurance risks, especially those with inhomogeneous total loss distributions. Thus, it is necessary to consider how the general form of the Aumann-Shapley value (i.e., eq. [2.1]) can be applied in the context of inhomogeneous losses.

## 3. THREE TYPES OF HOMOGENEITY

To study the Aumann-Shapley value as a risk-allocation method, it is useful to define three types of mathematical *homogeneity* associated with the total loss random variable $X \sim F_{X|\underline{\Lambda}}(x)$ and its risk measure, $\mu(X|\underline{\Lambda}, \underline{\kappa})$. These definitions, which are insufficiently distinguished and discussed in the relevant literature, provide the basis for understanding the simplification provided by Proposition 1 and how rarely this simplification can be applied in insurance settings.

### Definition 2

A total loss random variable $X \sim F_{X|\underline{\Lambda}}(x)$ is said to be *homogeneous in distribution* (HID) if and only if

$$F_{X|t\underline{\Lambda}}(x) = F_{tX|\underline{\Lambda}}(x) \quad \text{for all } t > 0. \tag{3.1}$$

For the case of $n = 1$ (i.e., for $\underline{\Lambda} = \Lambda$, a scalar parameter), examples of HID random variables are quite common. As noted by Mildenhall (2004), these include appropriate parameterizations of many random variables frequently used to model *individual*—but not necessarily *total*—insurance losses (e.g., the lognormal, gamma, Weibull, and Pareto families). For general $n$, one simple example is the multivariate-normal random vector with mean $\underline{\Lambda}$ and covariance matrix

$$\underline{\Sigma} = \zeta \begin{bmatrix} \Lambda_1^2 & \rho_{1,2}\Lambda_1\Lambda_2 & \cdots & \rho_{1,n}\Lambda_1\Lambda_n \\ \rho_{2,1}\Lambda_2\Lambda_1 & \Lambda_2^2 & \cdots & \rho_{2,n}\Lambda_2\Lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n,1}\Lambda_n\Lambda_1 & \rho_{n,2}\Lambda_n\Lambda_2 & \cdots & \Lambda_n^2 \end{bmatrix},$$

for some positive constant $\zeta$.

### Definition 3

A risk measure $\mu(X|\underline{\Lambda}, \underline{\kappa})$ of the total loss random variable is said to be *homogeneous in scale* (HIS) if and only if

$$\mu(tX|\underline{\Lambda}, \underline{\kappa}) = t\mu(X|\underline{\Lambda}, \underline{\kappa}) \quad \text{for all } t > 0.[7]$$

This property is possessed by many commonly used risk measures, including all of those listed in the first section, with the exception of the variance (item 2). The variance is not HIS because it involves the expectation of a quadratic form, without rescaling units as in the case of the standard deviation. Proposition 3 below provides a useful set of sufficient conditions for a risk measure to be HIS.

### Definition 4

A risk measure $\mu(X|\underline{\Lambda}, \underline{\kappa})$ of the total loss random variable is said to be *homogeneous in mean* (HIM) if and only if

$$\mu(X|t\underline{\Lambda}, \underline{\kappa}) = t\mu(X|\underline{\Lambda}, \underline{\kappa}) \quad \text{for all } t > 0.$$

---

[7] This type of homogeneity is equivalent to the *positive homogeneity* required of a coherent risk measure.

Notice that, using the above terminology, Proposition 1 can be restated by observing that HIM is a sufficient condition for equation (2.2). Essentially, HIM requires that the mean of the loss distribution play a central role in the definition of risk. Thus, if one were to define "risk" as a quantity proportional to the expected loss, $E_{X|\underline{\Lambda}}[X]$, then the risk measure would be HIM. However, such cases are rare in the absence of any special assumptions regarding the loss distribution.

Interestingly, HIM is easily implied by HID because (1) many popular risk measures are HIS and (2) an HIS risk measure becomes HIM when applied to HID losses. These two points are addressed respectively by the following two results.

## Proposition 2

If $\mu(X|\underline{\Lambda}, \underline{\kappa}) = E_{X|\underline{\Lambda}}[\alpha X + \beta u(\underline{\Lambda})|X > \gamma v(\underline{\Lambda})]\Pr\{X > \delta w(\underline{\Lambda})\}$, where $\alpha$, $\beta$, $\gamma$, and $\delta$ are constants and $u(\underline{\Lambda})$, $v(\underline{\Lambda})$, and $w(\underline{\Lambda})$ are either means or percentiles of the distribution of $X \sim F_{X|\underline{\Lambda}}(x)$, then $\mu(X|\underline{\Lambda}, \underline{\kappa})$ is HIS.

### Proof

First, note that

$$\mu(X|\underline{\Lambda}, \underline{\kappa}) = \int_{x > \gamma v(\underline{\Lambda})} [\alpha x + \beta u(\underline{\Lambda})]\, \frac{f_{X|\underline{\Lambda}}(x)}{1 - F_{X|\underline{\Lambda}}(\gamma v(\underline{\Lambda}))}\, dx \cdot [1 - F_{X|\underline{\Lambda}}(\delta w(\underline{\Lambda}))].$$

Now consider that

$$\mu(tX|\underline{\Lambda}, \underline{\kappa}) = \int_{tx > \gamma tv(\underline{\Lambda})} [\alpha tx + \beta tu(\underline{\Lambda})]\, \frac{f_{tX|\underline{\Lambda}}(tx)}{1 - F_{tX|\underline{\Lambda}}(\gamma tv(\underline{\Lambda}))}\, t\, dx \cdot [1 - F_{tX|\underline{\Lambda}}(\delta tw(\underline{\Lambda}))].$$

$$= \int_{x > \gamma v(\underline{\Lambda})} t[\alpha x + \beta u(\underline{\Lambda})]\, \frac{f_{X|\underline{\Lambda}}\left(\frac{tx}{t}\right)\frac{1}{t}}{1 - F_{X|\underline{\Lambda}}\left(\frac{\gamma tv(\underline{\Lambda})}{t}\right)}\, t\, dx \cdot \left[1 - F_{X|\underline{\Lambda}}\left(\frac{\delta tw(\underline{\Lambda})}{t}\right)\right]$$

$$= t \int_{x > \gamma v(\underline{\Lambda})} [\alpha x + \beta u(\underline{\Lambda})]\, \frac{f_{X|\underline{\Lambda}}(x)}{1 - F_{X|\underline{\Lambda}}(\gamma v(\underline{\Lambda}))}\, dx \cdot [1 - F_{X|\underline{\Lambda}}(\delta w(\underline{\Lambda}))]$$

$$= t\mu(X|\underline{\Lambda}, \underline{\kappa}). \qquad \blacksquare$$

## Proposition 3

If $X \sim F_{X|\underline{\Lambda}}(x)$ is HID and $\mu(X|\underline{\Lambda}, \underline{\kappa})$ is HIS, then $\mu(X|\underline{\Lambda}, \underline{\kappa})$ is HIM.[8]

### Proof

First, note that condition (3.1) implies $\mu(X|t\underline{\Lambda}, \underline{\kappa}) = \mu(tX|\underline{\Lambda}, \underline{\kappa})$ for all $t > 0$. The proposition then follows by inspection. $\blacksquare$

Proposition 2 allows us to verify that risk measures (3) through (7) on the list in Section 1 are all HIS. For value at risk, set $\alpha = \gamma = \delta = 0$, $\beta = 1$, and $u(\underline{\Lambda}) = F_{X|\underline{\Lambda}}^{-1}(1 - \varepsilon)$;[9] for tail value at risk, $\alpha = \gamma = 1$, $\beta = \delta = 0$, and $v(\underline{\Lambda}) = F_{X|\underline{\Lambda}}^{-1}(1 - \varepsilon)$; for excess tail value at risk, $\alpha = \gamma = 1$, $\beta = \delta = 0$, and $v(\underline{\Lambda}) = E_{X|\underline{\Lambda}}[X]$; for expected policyholder deficit, $\alpha = 1$, $\beta = -K/E_{X|\underline{\Lambda}}[X]$, $\gamma = K/E_{X|\underline{\Lambda}}[X]$, $\delta = 0$,

---

[8] Although not directly relevant to our argument, it is equally straightforward to show that if $X \sim F_{X|\underline{\Lambda}}(x)$ is HID and $\mu(X|\underline{\Lambda}, \underline{\kappa})$ is HIM, then $\mu(X|\underline{\Lambda}, \underline{\kappa})$ is HIS.

[9] Note that if one were to extend this analysis to the case of discrete or mixed probability distributions for which $F_{X|\underline{\Lambda}}(0) > 0$, then one would have to adopt the notational conventions that $\gamma = 0 \Rightarrow v(\underline{\Lambda}) = 0$ and $\delta = 0 \Rightarrow w(\underline{\Lambda}) = 0$.

and $u(\underline{\Lambda}) = v(\underline{\Lambda}) = E_{X|\underline{\Lambda}}[X]$; and for default value, $\alpha = 1$, $\beta = -(K + \text{premiums})/E_{X|\underline{\Lambda}}[X]$, $\gamma = \delta = (K + \text{premiums})/E_{X|\underline{\Lambda}}[X]$, and $u(\underline{\Lambda}) = v(\underline{\Lambda}) = w(\underline{\Lambda}) = E_{X|\underline{\Lambda}}[X]$. Given that these measures are HIS, we know from Proposition 3 that any one of them can be combined with an HID loss distribution to produce an HIM risk measure. However, as will be shown in the next section, HID total losses are not common in insurance applications.

## 4. ALLOCATING RISK WITH AUMANN-SHAPLEY VALUES

The first authors to apply Aumann-Shapley values to the problem of allocating insurance risk were Myers and Read (2001), who used their risk allocation to distribute an insurer's capital, K, across several lines of business. Interestingly those authors did not identify the game-theoretic origins of their method, but rather justified the mathematical equivalent of equation (2.2) as a ''marginal contribution'' approach.

Myers and Read (2001) used the default value as their risk measure. This can be expressed as

$$\mu^{DV}(X|\underline{\Lambda}, \underline{\kappa}) = E_{X|\underline{\Lambda}} \left[ X - \sum_{j=1}^{n} (\kappa_j + 1 + \theta)\Lambda_j \middle| X > \sum_{j=1}^{n} (\kappa_j + 1 + \theta)\Lambda_j \right]$$

$$\times \Pr \left\{ X > \sum_{j=1}^{n} (\kappa_j + 1 + \theta)\Lambda_j \right\},$$

where $\kappa_i \Lambda_i$ is the amount of capital distributed to portfolio member $i$, and $\theta$ is the profit loading as a proportion of expected losses (assumed constant for all $i$).[10] Because the default value is HIS (by Proposition 2), but not necessarily HIM,[11] the only way that equation (2.2) can be justified is by assuming losses are HID, so that Proposition 3 applies. Since Myers and Read (2001) did not recognize the critical role of the HID assumption—but rather assumed HID losses while stating that their results applied to *all* loss distributions—it remained for Mildenhall (2004) to provide this important insight.

Mildenhall (2004) also pointed out that HID losses occur in the insurance context if and only if the total loss random variable is determined on a quota-share basis for each portfolio member $i = 1, 2, \ldots, n$. Since quota-share arrangements are more the exception than the rule in insurance and reinsurance markets, one generally cannot use equation (2.2) in conjunction with the default value. Rather, one must use equation (2.1).

Although Denault (2001) noted this general form of the Aumann-Shapley value in his discussion of risk-allocation methods, he neither implemented it nor mentioned its important role in insurance, where losses may not be HID. It is the application of Aumann-Shapley to the general risk-allocation problem with inhomogeneous losses that we now address.

Using the default value as our risk measure, we compute $\underline{\varphi}^{AS}(\mu^{DV}, \underline{\Lambda}, \underline{\kappa})$ from equation (2.1). Then, as in Myers and Read (2001), we find the desired capital-distribution vector $\underline{K}^* = [\kappa_1^* \Lambda_1, \kappa_2^* \Lambda_2, \ldots, \kappa_n^* \Lambda_n]$ by solving the system of $n + 1$ equations:

$$\varphi_i^{AS}(\mu^{DV}, \underline{\Lambda}, \underline{\kappa}^*) = c \text{ (a constant) for } i = 1, 2, \ldots, n \text{ and}$$

$$\sum_{j=1}^{n} \kappa_j^* \Lambda_j = K.$$

---

[10] Since Myers and Read (2001) did not include a profit loading in their analysis, they implicitly set $\theta = 0$.

[11] To see that the default value is not necessarily HIM, it suffices to consider a simple counterexample for the case of $n = 1$. Let $X \sim \text{Normal}(\Lambda, \zeta)$, for some positive constant $\zeta$, and note that $\mu^{DV}(X|\underline{\Lambda}, \underline{\kappa}) = \int_{[(K/\Lambda)+1+\theta]\Lambda}^{\infty} \{x - [(K/\Lambda) + 1 + \theta]\Lambda\}/\sqrt{2\pi\zeta} \exp(-(x - \Lambda)^2/2\zeta) \, dx$, whereas $\mu^{DV}(X|t\underline{\Lambda}, \underline{\kappa}) = \int_{[(K/\Lambda)+1+\theta]t\Lambda}^{\infty} \{x - [(K/\Lambda) + 1 + \theta]t\Lambda\}/\sqrt{2\pi\zeta} \exp(-(x - t\Lambda)^2/2\zeta) \, dx$. It then follows that the second integral must be strictly smaller than the first integral for sufficiently large values of $t$ (because the second integral vanishes as $t \to \infty$).

This distribution vector is optimal in the sense that it distributes the total capital K across the group members in such a way that, once the distribution has occurred, each member's contribution to the portfolio's total risk is exactly the same.[12]

We consider portfolios consisting of jointly distributed normal loss amounts such that

$$\underline{L} = [L_1, L_2, \ldots, L_n] \sim \text{Multivariate Normal } (\underline{\Lambda}, \underline{\Sigma}),$$

where

$$\underline{\Sigma} = \zeta \begin{bmatrix} \Lambda_1^q & \rho_{1,2}\Lambda_1^{q/2}\Lambda_2^{q/2} & \cdots & \rho_{1,n}\Lambda_1^{q/2}\Lambda_n^{q/2} \\ \rho_{2,1}\Lambda_2^{q/2}\Lambda_1^{q/2} & \Lambda_2^q & \cdots & \rho_{2,n}\Lambda_2^{q/2}\Lambda_n^{q/2} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n,1}\Lambda_n^{q/2}\Lambda_1^{q/2} & \rho_{n,2}\Lambda_n^{q/2}\Lambda_2^{q/2} & \cdots & \Lambda_n^q \end{bmatrix}$$

for positive constants $q$ and $\zeta$.[13] Thus,

$$X = \sum_{j=1}^{n} L_j \sim \text{Normal}(\Lambda, \sigma^2(\underline{\Lambda}, q)),$$

where $\Lambda = \sum_{j=1}^{n} \Lambda_j$ and $\sigma^2(\underline{\Lambda}, q) = \zeta(\sum_{j=1}^{n} \Lambda_j^q + \sum_{j=1}^{n} \sum_{k\neq j} \rho_{j,k}\Lambda_j^{q/2}\Lambda_k^{q/2})$.

This family of loss distributions is useful because it gathers three important special cases—$q = 0$, $q = 1$, and $q = 2$—into one continuum. For $q = 0$ we have the situation in which each variance, $Var_{L_i|\underline{\Lambda}}[L_i] = \zeta$, is functionally unrelated to its corresponding mean, $E_{L_i|\underline{\Lambda}}[L_i] = \Lambda_i$. When $q = 1$ the variance, $Var_{L_i|\underline{\Lambda}}[L_i] = \zeta\Lambda_i$, grows in direct proportion to the mean, as in the individual risk model, $L_i = \sum_{h=1}^{m} Y_h$, with i.i.d. components $Y_h$. Finally, the case of $q = 2$ is the only member of the family for which the $L_i$ (and therefore $X$) are HID.

The following result shows that for the family of multivariate-normal loss amounts considered, the amount of capital distributed to portfolio member $i$ is $\kappa_i\Lambda_i$, where $\kappa_i$ is a positive linear function of $\Lambda_i^{q-1} + \sum_{k\neq i} \rho_{i,k}\Lambda_i^{(q/2)-1}\Lambda_k^{q/2}$ with coefficients (i.e., slope and intercept) that are constant over $i$.

## Proposition 4

If $\underline{L} \sim$ Multivariate Normal$(\underline{\Lambda}, \underline{\Sigma})$ and the risk measure is $\mu^{DV}(X|\underline{\Lambda}, \underline{\kappa})$, then (1) the Aumann-Shapley values are given by

$$\varphi_i^{AS}(\mu^{DV}, \underline{\Lambda}, \underline{\kappa}) = \frac{q\zeta \left[ \Lambda_i^{q-1} + \sum_{k\neq i} \rho_{i,k}\Lambda_i^{(q/2)-1}\Lambda_k^{q/2} \right]}{2\sqrt{2\pi}\sigma(\underline{\Lambda}, q)} \int_0^1 t^{q/2-1} \exp\left( -\frac{t^{2-q}(K + \theta\Lambda)^2}{2\sigma^2(\underline{\Lambda}, q)} \right) dt$$

$$- (\kappa_i + \theta) \left[ 1 - \int_0^1 \Phi\left( \frac{t^{1-q/2}(K + \theta\Lambda)}{\sigma(\underline{\Lambda}, q)} \right) dt \right]; \tag{4.1}$$

and (2) the optimal capital distribution is given by $\underline{K}^* = [K_1^*, K_2^*, \ldots, K_n^*] = [\kappa_1^*\Lambda_1, \kappa_2^*\Lambda_2, \ldots, \kappa_n^*\Lambda_n]$, where

---

[12] For this approach to succeed, the risk measure must be a function of $\underline{\kappa}$; consequently none of risk measures (1) through (5) on the list in Section 1 can be used.

[13] Although individual insurance loss amounts are typically characterized by positive skewness, the assumption of normal losses ($L_i$) at the line-of-business level is not unrealistic for noncatastrophe lines with large numbers of individual losses.

$$\kappa_i^* = \frac{K}{\Lambda} + \frac{q}{2\sqrt{2\pi}\sigma(\underline{\Lambda}, q)} \left\{ \zeta \left[ \Lambda_i^{q-1} + \sum_{k \neq i} \rho_{i,k}\Lambda_i^{(q/2)-1}\Lambda_k^{q/2} \right] - \frac{\sigma^2(\underline{\Lambda}, q)}{\Lambda} \right\}$$

$$\times \int_0^1 t^{q/2-1} \exp\left(-\frac{t^{2-q}(K + \theta\Lambda)^2}{2\sigma^2(\underline{\Lambda}, q)}\right) dt \bigg/ \left[ 1 - \int_0^1 \Phi\left(\frac{t^{1-q/2}(K + \theta\Lambda)}{\sigma(\underline{\Lambda}, q)}\right) dt \right]. \quad (4.2)$$

### PROOF

See the Appendix. ■

It is fairly straightforward to extend the analytical results of Proposition 4 to the multivariate-normal family with covariance-matrix elements $\Sigma_{j,k} = \rho_{j,k}\Lambda_j^{q/2}\Lambda_k^{q/2}$, and even to the more general case with $\Sigma_{j,k} = \rho_{j,k}g_j(\Lambda_j)g_k(\Lambda_k)$ for arbitrary positive and differentiable functions $g_j(\cdot)$ and $g_k(\cdot)$. For nonnormal distributions, similar methods may be applied, but clean analytical results are more difficult to achieve. In such cases, as well as those in which loss distributions are given only empirically, the desired Aumann-Shapley values and their associated capital distributions can, in theory, be computed using numerical techniques.

## 5. DISCUSSION

Figures 1 and 2 provide values of $\kappa_i^*$ computed for several values of $q \in [0, 1.2]$ for a fixed insurance portfolio with $n = 4$ lines of business. Here $\Lambda_1 = \$10$ million, $\Lambda_2 = \$20$ million, $\Lambda_3 = \$40$ million, $\Lambda_4 = \$80$ million, $K = \$100$ million, and $\theta = 0.2$. Having experimented with a wide range of values for $\zeta$, this parameter is set equal to K ($100 million) because this order of magnitude affords both representative and reasonable outcomes.

Figure 1
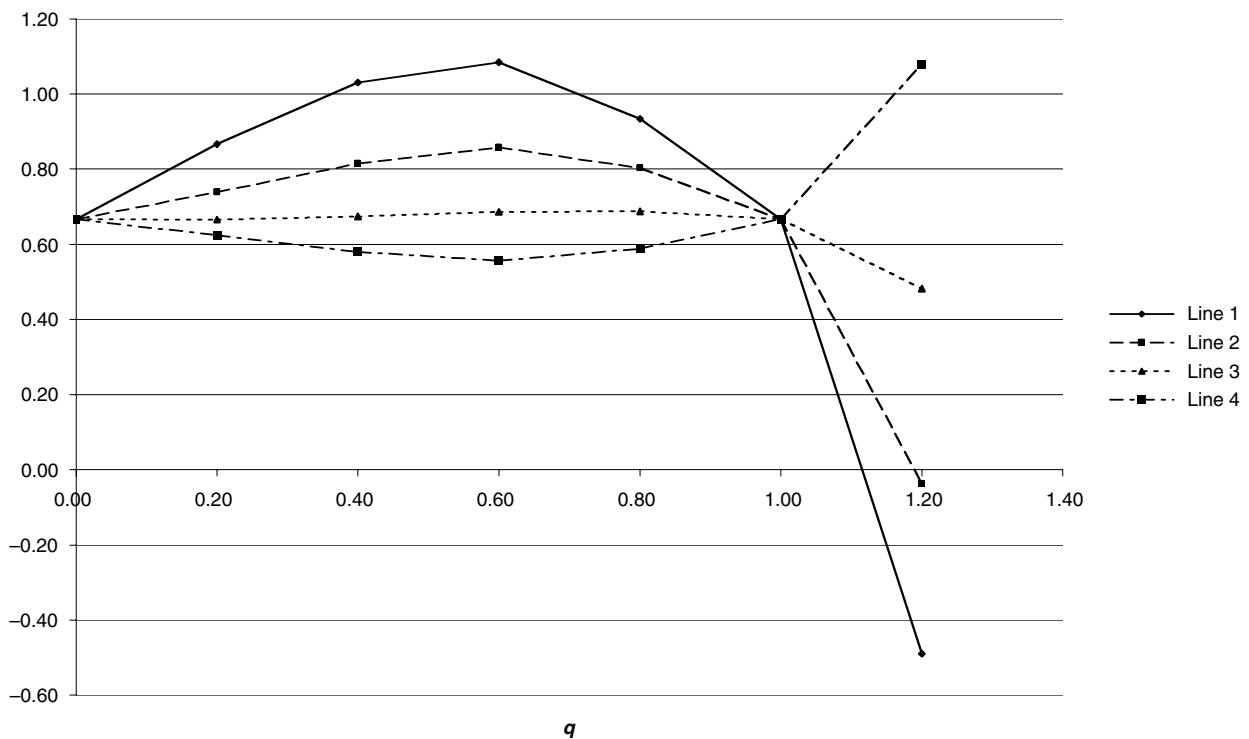**Capital-Distribution Ratios with Uncorrelated Losses**

Figure 2
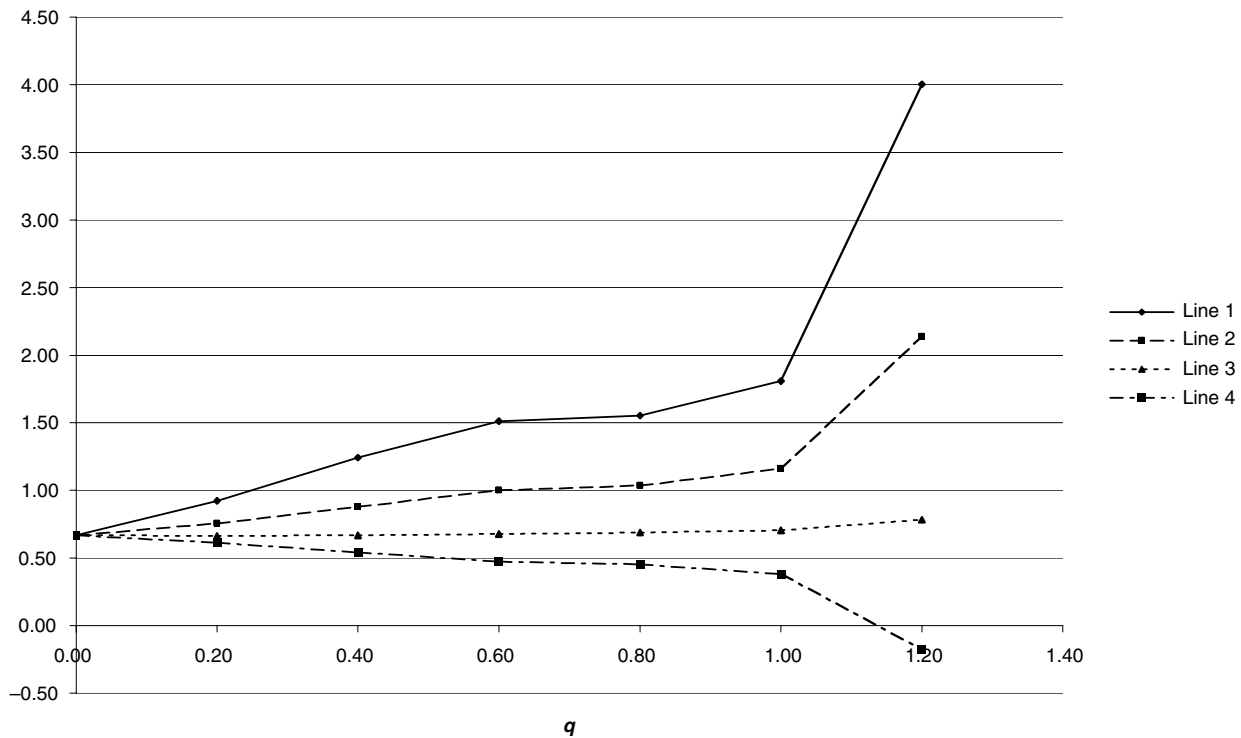**Capital-Distribution Ratios with Loss Correlations = 0.5**



Figure 1 presents the case of uncorrelated loss amounts (i.e., $\rho_{j,k} = 0$ for all pairs $j, k$). We see that the capital-distribution ratio for the smallest line of business ($i = 1$) first increases and then decreases over $q$, reaching a maximum at about 0.6, whereas the capital distribution for the largest line of business ($i = 4$) displays an opposite (decreasing then increasing) pattern of movement.

To analyze this behavior, consider what happens when line $i = 1$ is added to the three larger lines to form the total portfolio. Regardless of the value of $q$, the new line must be accompanied by premiums of $(1 + \theta)\Lambda_1$ just to place it on the same footing as the other lines in terms of expected profitability. Then a capital allotment of $\kappa_1^* \Lambda_1$ must be made to account for the new line's contributions to the variance of profitability through its *diversification effect*—that is, from adding the new line's uncorrelated losses to the rest of the portfolio. This effect, which may be expressed mathematically as

$$\frac{\zeta(\Lambda_2^q + \Lambda_3^q + \Lambda_4^q)}{\Lambda - \Lambda_1} - \frac{\zeta(\Lambda_1^q + \Lambda_2^q + \Lambda_3^q + \Lambda_4^q)}{\Lambda} = \frac{\zeta}{\Lambda}\left[\left(\frac{\Lambda_1}{\Lambda - \Lambda_1}\right)(\Lambda_2^q + \Lambda_3^q + \Lambda_4^q) - \Lambda_1^q\right], \quad (5.1)$$

makes two distinct contributions: (1) it reduces the average variance of the three original lines, and (2) it reduces the variance of the new line.

Heuristically one might express the first contribution as

$$\frac{\zeta}{\Lambda}\left[\left(\frac{\Lambda_1}{\Lambda - \Lambda_1}\right)(\Lambda_2^q + \Lambda_3^q + \Lambda_4^q) - \Lambda_1^q\right] \times \frac{\zeta(\Lambda_2^q + \Lambda_3^q + \Lambda_4^q)}{3}, \quad (5.2)$$

and the second as

$$\frac{\zeta}{\Lambda}\left[\left(\frac{\Lambda_1}{\Lambda - \Lambda_1}\right)(\Lambda_2^q + \Lambda_3^q + \Lambda_4^q) - \Lambda_1^q\right] \times \zeta\Lambda_1^q. \quad (5.3)$$

Since the first contribution reduces the capital needs of the three original lines, its impact on $\kappa_1^* \Lambda_1$ is positive. In contrast, the second contribution reduces the capital needs of the new line, and so its impact on $\kappa_1^* \Lambda_1$ is negative. Subtracting expression (5.3) from expression (5.2) then yields a net contribution of

$$\frac{\zeta}{\Lambda} \left[ \left( \frac{\Lambda_1}{\Lambda - \Lambda_1} \right) (\Lambda_2^q + \Lambda_3^q + \Lambda_1^q) - \Lambda_1^q \right] \times \zeta \left[ \frac{(\Lambda_2^q + \Lambda_3^q + \Lambda_4^q)}{3} - \Lambda_1^q \right]. \tag{5.4}$$

Given that $\Lambda_1$ is the smallest of the $\Lambda_j$, it follows that the first factor in expression (5.4)—that is, the diversification effect—is positive at $q = 0$, decreases over $q$ for $q \geq 0$, and crosses the horizontal axis at $q = 1$, while the second factor, $\zeta \lfloor (\Lambda_2^q + \Lambda_3^q + \Lambda_4^q)/3 - \Lambda_1^q \rfloor$, equals 0 at $q = 0$, and increases over $q$ for $q \geq 0$. It is the sign of the product of these two factors that indicates how the new line's capital must be adjusted; specifically the positive product in the range $q \in (0, 1)$ means that $\kappa_1^* \Lambda_1$ should be increased disproportionately (i.e., $\kappa_1^*$ becomes the largest of the $\kappa_j^*$), whereas the negative product in the range $q > 1$ means that $\kappa_1^* \Lambda_1$ should be decreased disproportionately (i.e., $\kappa_1^*$ becomes the smallest of the $\kappa_j^*$). When the product is 0 at $q = 0$ and $q = 1$, the new line's capital is simply set in the same proportion as those of the other three lines (i.e., $\kappa_1^* = \kappa_j^*$ for all $j$).

Note that for values of $q > 1$ (including the HID case of $q = 2$), the capital may be decreased to such an extent that it becomes negative.[14]

Figure 2 addresses a case of positively correlated losses, with $\rho_{j,k} = 0.5$ for all pairs $j, k$. For the values of $q$ shown, the behavior of the capital-distribution ratios is the same as in Figure 1 for smaller values of $q$—that is, $\kappa_1^*$ increases (and $\kappa_4^*$ decreases) over $q$. In this case $\kappa_1^*$ does not begin to decrease (and $\kappa_4^*$ increase) until about $q = 1.6$ (as shown in Fig. 3 with $q \in [0, 1.8]$). The reason for the delay in the characteristic behavior is simply that the positive correlations retard the diversification effect (given by a suitable generalization of eq. [5.1]), so that while this effect is positive at $q = 0$, it does not cross the horizontal axis until $q > 1$.

## 6. CONCLUSIONS

This article has demonstrated that the value-assignment method of Aumann and Shapley (1974) may be used to solve insurance risk-allocation problems with inhomogeneous, as well as homogeneous, loss distributions. This result permits actuaries and others to allocate responsibility for risk and distribute capital in the most common insurance contexts.

One natural extension of the present research is to consider how Aumann-Shapley values may be used to distribute both capital and profit, simultaneously, across the members of an insurance portfolio. In the analysis of Section 4, only capital was distributed across lines while the profit per line was fixed at $\theta \Lambda_i$. However, a close inspection of equation (4.1) reveals that, in computing the Aumann-Shapley values, one alternatively could replace $\theta$ with $\theta_i$ (a variable profit parameter for line $i$), and thereby replace $\theta \Lambda = \Sigma_{j=1}^n \theta \Lambda_j$ with $\Theta = \Sigma_{j=1}^n \theta_j \Lambda_j$. This adjustment would lead immediately to

$$\kappa_i^* + \theta_i^* = \frac{K + \Theta}{\Lambda} + \frac{q}{2\sqrt{2\pi}\sigma(\underline{\Lambda}, q)} \left\{ \zeta \left[ \Lambda_i^{q-1} + \sum_{k \neq i} \rho_{i,k} \Lambda_i^{(q/2)-1} \Lambda_k^{q/2} \right] - \frac{\sigma^2(\underline{\Lambda}, q)}{\Lambda} \right\}$$

$$\times \int_0^1 t^{q/2-1} \exp \left( -\frac{t^{2-q}(K + \Theta)^2}{2\sigma^2(\underline{\Lambda}, q)} \right) dt \bigg/ \left[ 1 - \int_0^1 \Phi \left( \frac{t^{1-q/2}(K + \Theta)}{\sigma(\underline{\Lambda}, q)} \right) dt \right], \tag{6.1}$$

which provides values for the sums $\kappa_i^* + \theta_i^*$, but not for the individual $\kappa_i^*$ and $\theta_i^*$.

---

[14] A negative capital ratio for line of business *i* simply means that line *i* must borrow funds (e.g., by issuing bonds or selling equities short) to subsidize other lines of business. Although possible in practice, the presence of negative capital distributions generally precludes standard rate-of-return analyses.

Figure 3
**Capital-Distribution Ratios with Loss Correlations = 0.5 (Expanded Domain)**
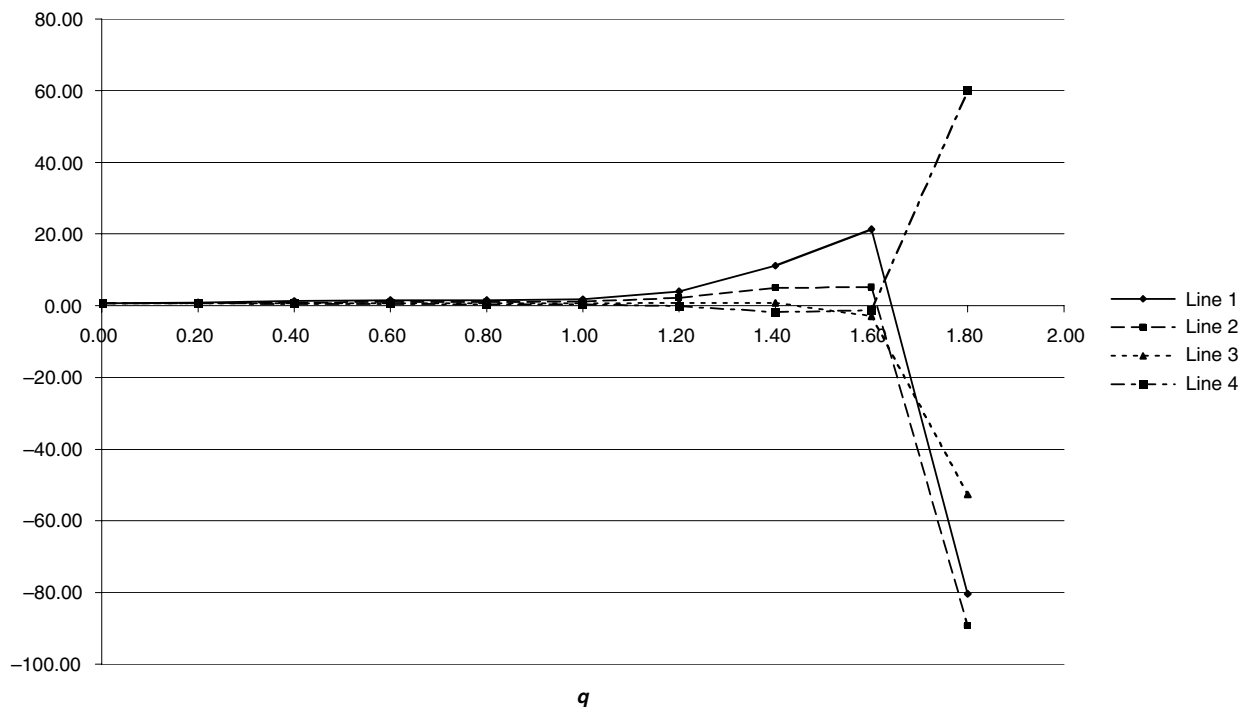


To overcome this problem, one simply could assume that the rate-of-return parameter for line of business $i$—that is, $r_i^* = \theta_i^* \Lambda_1 / \kappa_i^* \Lambda_i = \theta_i^* / \kappa_i^*$—is specified exogenously (e.g., by a financial model such as the CAPM). Then the above equation would simplify to

$$
\kappa_i^* = \frac{1}{1 + r_i^*} \left\{ \frac{K + \Theta}{\Lambda} + \frac{q}{2\sqrt{2\pi}\sigma(\underline{\Lambda}, q)} \left\{ \zeta \left[ \Lambda_i^{q-1} + \sum_{k \neq i} \rho_{i,k} \Lambda_i^{(q/2)-1} \Lambda_k^{q/2} \right] - \frac{\sigma^2(\underline{\Lambda}, q)}{\Lambda} \right\} \right.
$$
$$
\left. \times \int_0^1 t^{q/2-1} \exp\left( -\frac{t^{2-q}(K + \Theta)^2}{2\sigma^2(\underline{\Lambda}, q)} \right) dt \bigg/ \left[ 1 - \int_0^1 \Phi\left( \frac{t^{1-q/2}(K + \Theta)}{\sigma(\underline{\Lambda}, q)} \right) dt \right] \right\}. \qquad (6.2)
$$

To endogenize the $\theta_i^*$ would require adding further constraints to the model.

## Appendix

## Proof of Proposition 4
As previously noted, for $q \neq 2$ the total loss amount $X$ is non-HID. Given that the default value with inhomogeneous losses is not HIM, we can use equation (1) to find

$$
\varphi_i^{AS}(\mu^{DV}, \underline{\Lambda}, \underline{\kappa}) = \int_0^1 \left. \frac{\partial \mu^{DV}(X|\underline{\lambda}, \underline{\kappa})}{\partial \lambda_i} \right|_{\underline{\lambda}=t\underline{\Lambda}} dt,
$$

where

$$\frac{\partial \mu^{DV}(X|\underline{\lambda}, \underline{\kappa})}{\partial \lambda_i}\Bigg|_{\underline{\lambda}=t\underline{\Lambda}} = \frac{\partial}{\partial \lambda_i} \int_{x > \sum_{j=1}^{n} \lambda_j(\kappa_j+1+\theta)} \left[ x - \sum_{j=1}^{n} \lambda_j(\kappa_j + 1 + \theta) \right] f_{X|\underline{\lambda}}(x)\ dx \Bigg|_{\underline{\lambda}=t\underline{\Lambda}}$$

$$= \frac{\partial}{\partial \lambda_i} \left\{ \int_{x > \sum_{j=1}^{n} \lambda_j(\kappa_j+1+\theta)} x f_{X|\underline{\lambda}}(x)\ dx - \left[ \sum_{j=1}^{n} \lambda_j(\kappa_j + 1 + \theta) \right] \int_{x > \sum_{j=1}^{n} \lambda_j(\kappa_j+1+\theta)} f_{X|\underline{\lambda}}(x)\ dx \right\} \Bigg|_{\underline{\lambda}=t\underline{\Lambda}}$$

$$= \frac{\partial}{\partial \lambda_i} \left\{ \frac{\sigma(\underline{\lambda}, q)}{\sqrt{2\pi}} \exp\left( -\frac{\left[ \sum_{j=1}^{n} \lambda_j(\kappa_j + \theta) \right]^2}{2\sigma^2(\underline{\lambda}, q)} \right) + \left( \sum_{j=1}^{n} \lambda_j \right) \left[ 1 - \Phi\left( \frac{\sum_{j=1}^{n} \lambda_j(\kappa_j + \theta)}{\sigma(\underline{\lambda}, q)} \right) \right] \right.$$

$$\left. - \left[ \sum_{j=1}^{n} \lambda_j(\kappa_j + 1 + \theta) \right] \left[ 1 - \Phi\left( \frac{\sum_{j=1}^{n} \lambda_j(\kappa_j + \theta)}{\sigma(\underline{\lambda}, q)} \right) \right] \right\} \Bigg|_{\underline{\lambda}=t\underline{\Lambda}}$$

$$= \frac{\partial}{\partial \lambda_i} \left\{ \frac{\sigma(\underline{\lambda}, q)}{\sqrt{2\pi}} \exp\left( -\frac{\left[ \sum_{j=1}^{n} \lambda_j(\kappa_j + \theta) \right]^2}{2\sigma^2(\underline{\lambda}, q)} \right) \right.$$

$$\left. - \left[ \sum_{j=1}^{n} \lambda_j(\kappa_j + \theta) \right] \left[ 1 - \Phi\left( \frac{\sum_{j=1}^{n} \lambda_j(\kappa_j + \theta)}{\sigma(\underline{\lambda}, q)} \right) \right] \right\} \Bigg|_{\underline{\lambda}=t\underline{\Lambda}}$$

$$= \left\{ \exp\left( -\frac{\left[ \sum_{j=1}^{n} \lambda_j(\kappa_j + \theta) \right]^2}{2\sigma^2(\underline{\lambda}, q)} \right) \left\{ \frac{1}{\sqrt{2\pi}} \frac{\partial \sigma(\underline{\lambda}, q)}{\partial \lambda_i} \right. \right.$$

$$- \frac{\sigma(\underline{\lambda}, q)}{\sqrt{2\pi}} \left\{ \frac{4\sigma^2(\underline{\lambda}, q) \left[ \sum_{j=1}^{n} \lambda_j(\kappa_j + \theta) \right] (\kappa_i + \theta) - 4\left[ \sum_{j=1}^{n} \lambda_j(\kappa_j + \theta) \right]^2 \sigma(\underline{\lambda}, q) \frac{\partial \sigma(\underline{\lambda}, q)}{\partial \lambda_i}}{[2\sigma^2(\underline{\lambda}, q)]^2} \right\}$$

$$+ \frac{\left[ \sum_{j=1}^{n} \lambda_j(\kappa_j + \theta) \right]}{\sqrt{\pi}} \left\{ \frac{\sqrt{2}\sigma(\underline{\lambda}, q)(\kappa_i + \theta) - \sqrt{2}\left[ \sum_{j=1}^{n} \lambda_j(\kappa_j + \theta) \right] \frac{\partial \sigma(\underline{\lambda}, q)}{\partial \lambda_i}}{[\sqrt{2}\sigma(\underline{\lambda}, q)]^2} \right\} \right\}$$

$$\left. - (\kappa_i + \theta) \left[ 1 - \Phi\left( \frac{\sum_{j=1}^{n} \lambda_j(\kappa_j + \theta)}{\sigma(\underline{\lambda}, q)} \right) \right] \right\} \Bigg|_{\underline{\lambda}=t\underline{\Lambda}}$$

$$= \left\{ \frac{1}{\sqrt{2\pi}} \frac{\partial \sigma(\underline{\lambda}, q)}{\partial \lambda_i} \exp\left( -\frac{\left[ \sum_{j=1}^{n} \lambda_j(\kappa_j + \theta) \right]^2}{2\sigma^2(\underline{\lambda}, q)} \right) - (\kappa_i + \theta) \left[ 1 - \Phi\left( \frac{\sum_{j=1}^{n} \lambda_j(\kappa_j + \theta)}{\sigma(\underline{\lambda}, q)} \right) \right] \right\} \Bigg|_{\underline{\lambda}=t\underline{\Lambda}}$$

$$
= \left\{ \frac{1}{\sqrt{2\pi}} \frac{q\zeta \left[ \lambda_i^{q-1} + \sum_{k\neq i} \rho_{i,k} \lambda_i^{(q/2)-1} \lambda_k^{q/2} \right]}{2\sigma(\underline{\lambda}, q)} \exp\left( -\frac{\left[ \sum_{j=1}^{n} \lambda_j(\kappa_j + \theta) \right]^2}{2\sigma^2(\underline{\lambda}, q)} \right) \right.
$$

$$
\left. - (\kappa_i + \theta) \left[ 1 - \Phi\left( \frac{\sum_{j=1}^{n} \lambda_j(\kappa_j + \theta)}{\sigma(\underline{\lambda}, q)} \right) \right] \right\} \Bigg|_{\underline{\lambda}=t\underline{\Lambda}}
$$

$$
= \frac{q\zeta \left[ \Lambda_i^{q-1} + \sum_{k\neq i} \rho_{i,k} \Lambda_i^{(q/2)-1} \Lambda_k^{q/2} \right]}{2\sqrt{2\pi}\sigma(\underline{\Lambda}, q)} t^{q/2-1} \exp\left( -\frac{t^{2-q}(K + \theta\Lambda)^2}{2\sigma^2(\underline{\Lambda}, q)} \right)
$$

$$
- (\kappa_i + \theta) \left[ 1 - \Phi\left( \frac{t^{1-q/2}(K + \theta\Lambda)}{\sigma(\underline{\Lambda}, q)} \right) \right].
$$

We then obtain

$$
\varphi_i^{AS}(\mu^{DV}, \underline{\Lambda}, \underline{\kappa}) = \frac{q\zeta \left[ \Lambda_i^{q-1} + \sum_{k\neq i} \rho_{i,k} \Lambda_i^{(q/2)-1} \Lambda_k^{q/2} \right]}{2\sqrt{2\pi}\sigma(\underline{\Lambda}, q)} \int_0^1 t^{q/2-1} \exp\left( -\frac{t^{2-q}(K + \theta\Lambda)^2}{2\sigma^2(\underline{\Lambda}, q)} \right) dt
$$

$$
-(\kappa_i + \theta) \left[ 1 - \int_0^1 \Phi\left( \frac{t^{1-q/2}(K + \theta\Lambda)}{\sigma(\underline{\Lambda}, q)} \right) dt \right]. \tag{A.3}
$$

Setting $\varphi_i^{AS}(\mu^{DV}, \underline{\Lambda}, \underline{\kappa}) = c$ for $i = 1, 2, \ldots, n$, we can rewrite (A.3) as

$$
\kappa_i^* = \frac{-c + \dfrac{q\zeta \left[ \Lambda_i^{q-1} + \sum_{k\neq i} \rho_{i,k} \Lambda_i^{(q/2)-1} \Lambda_k^{q/2} \right]}{2\sqrt{2\pi}\sigma(\underline{\Lambda}, q)} \displaystyle\int_0^1 t^{q/2-1} \exp\left( -\dfrac{t^{2-q}(K + \theta\Lambda)^2}{2\sigma^2(\underline{\Lambda}, q)} \right) dt}{1 - \displaystyle\int_0^1 \Phi\left( \dfrac{t^{1-q/2}(K + \theta\Lambda)}{\sigma(\underline{\Lambda}, q)} \right) dt} - \theta. \tag{A.4}
$$

Then, since $\sum_{j=1}^{n} \kappa_j^* \Lambda_j = K$, it follows that

$$
K = \frac{-c\Lambda + \dfrac{q\sigma^2(\underline{\Lambda}, q)}{2\sqrt{2\pi}\sigma(\underline{\Lambda}, q)} \displaystyle\int_0^1 t^{q/2-1} \exp\left( -\dfrac{t^{2-q}(K + \theta\Lambda)^2}{2\sigma^2(\underline{\Lambda}, q)} \right) dt}{1 - \displaystyle\int_0^1 \Phi\left( \dfrac{t^{1-q/2}(K + \theta\Lambda)}{\sigma(\underline{\Lambda}, q)} \right) dt} - \theta\Lambda,
$$

and so

$$
c = \frac{q\sigma^2(\underline{\Lambda}, q)}{2\Lambda\sqrt{2\pi}\sigma(\underline{\Lambda}, q)} \int_0^1 t^{q/2-1} \exp\left( -\frac{t^{2-q}(K + \theta\Lambda)^2}{2\sigma^2(\underline{\Lambda}, q)} \right) dt - \left( \frac{K}{\Lambda} + \theta \right) \left[ 1 - \int_0^1 \Phi\left( \frac{t^{1-q/2}(K + \theta\Lambda)}{\sigma(\underline{\Lambda}, q)} \right) dt \right].
$$

Substituting this expression into (A.4) yields

$$\kappa_i^* = \frac{K}{\Lambda} + \frac{q}{2\sqrt{2\pi}\sigma(\underline{\Lambda}, q)} \left\{ \zeta \left[ \Lambda_i^{q-1} + \sum_{k \neq i} \rho_{i,k}\Lambda_i^{(q/2)-1}\Lambda_k^{q/2} \right] - \frac{\sigma^2(\underline{\Lambda}, q)}{\Lambda} \right\}$$

$$\times \int_0^1 t^{q/2-1} \exp\left( -\frac{t^{2-q}(K + \theta\Lambda)^2}{2\sigma^2(\underline{\Lambda}, q)} \right) dt \left/ \left[ 1 - \int_0^1 \Phi\left( \frac{t^{1-q/2}(K + \theta\Lambda)}{\sigma(\underline{\Lambda}, q)} \right) dt \right], \right.$$

for $i = 1, 2, \ldots, n$. ∎

## ACKNOWLEDGMENTS

## REFERENCES

AUBIN, JEAN-PIERRE. 1981. Cooperative Fuzzy Games. *Mathematics of Operations Research* 6(1): 1–13.

AUMANN, ROBERT J., AND LLOYD S. SHAPLEY. 1974. *Values of Non-Atomic Games*. Princeton: Princeton University Press.

BILLERA, LOUIS J., AND DAVID C. HEATH. 1982. Allocation of Shared Costs: A Set of Axioms Yielding a Unique Procedure. *Mathematics of Operations Research* 7(1): 32–39.

DENAULT, MICHEL. 2001. Coherent Allocation of Risk Capital. *Journal of Risk* 4(1): 1–34.

DHAENE, JAN, MARK J. GOOVAERTS, AND ROB KAAS. 2003. Economic Capital Allocation Derived from Risk Measures. *North American Actuarial Journal* 7(2): 44–56.

GOOVAERTS, MARK J., EDDY VAN DEN BORRE, AND ROGER J. A. LAEVEN. 2005. Managing Economic and Virtual Economic Capital within Financial Conglomerates. *North American Actuarial Journal* 9(3): 77–89.

KAYE, PAUL. 2005. Risk Measurement in Insurance: A Guide to Risk Measurement, Capital Allocation, and Related Decision Support Issues. Casualty Actuarial Society Discussion Paper.

MANGO, DONALD F. 1998. An Application of Game Theory: Property Catastrophe Risk Load. *Proceedings of the Casualty Actuarial Society* 85: 157–86.

MILDENHALL, STEPHEN J. 2004. A Note on the Myers and Read Capital Allocation Formula. *North American Actuarial Journal* 8(2): 32–44.

MIRMAN, LEONARD J., AND YAIR TAUMAN. 1982. Demand Compatible Equitable Cost Sharing Prices. *Mathematics of Operations Research* 7(1): 40–56.

MYERS, STEWART C., AND JAMES A. READ, JR. 2001. Capital Allocation for Insurance Companies. *Journal of Risk and Insurance* 68(4): 545–80.

POWERS, MICHAEL R. 2007. Sharing Responsibility: What They Didn't Teach You in Kindergarten. *Journal of Risk Finance* 8(2): 93–96.

SHAPLEY, LLOYD S. 1953. A Value for *n*-Person Games. *Contributions to the Theory of Games*. Vol. II. Princeton: Princeton University Press.

VENTER, GARY G. 2004. Capital Allocation Survey with Commentary. *North American Actuarial Journal* 8(2): 96–107.