Alternative framework for the fair valuation of participating life insurance contracts*

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Abstract

In this communication, we develop suitable valuation techniques for a with-profit/unitized with profit life insurance policy providing interest rate guarantees, when a jump-diffusion process for the evolution of the underlying reference portfolio is used. Particular attention is given to the mispricing generated by the misspecification of a jump-diffusion process for the underlying asset as a pure diffusion process, and to which extent this mispricing affects the profitability and the solvency of the life insurance company issuing these contracts.

Keywords: Esscher transform, fair value, incomplete markets, Lévy processes, participating contracts.

1 Introduction

Participating life insurance policies are investment/saving plans or contracts (with associated life insurance benefits) which specify a benchmark return, an annual minimum rate of return guarantee and a surplus distribution mechanism, that is a rule for the distribution of the annual investment return in excess of the guaranteed return between the insurer and the customer. Participating life insurance policies represent the most important life insurance products in terms of market size in many countries, for example the UK, US,

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Denmark, Germany, Italy, Japan, Netherlands and Norway. Terminology varies widely, so for example in the UK there are two important classes, viz with-profit and unitized with-profit contracts.

These kinds of contract represent liabilities to the issuers implying that their value and the potential risk to the insurance company's solvency should be properly valued. To the extent that, as a result of the difficulties that un-hedged guarantees embedded in these contracts have caused to the life insurance industry in recent years, the regulatory authorities have increased the monitoring of insurance companies' exposure to market risk, credit risk and persistency risk induced by participating contracts, and the embedded options included in the design of these contracts. For example, in the UK the potential threat to the company' solvency from with-profit policies has been addressed by the Financial Services Authority (FSA) with the introduction, into the regulatory regime for life insurance companies, of the twin peaks approach for the assessment of the financial resources needed for with-profit business. Such an approach (as described in CP195) requires the insurer to set up realistic balance sheets that are designed to capture the cost of guarantees and smoothing on a market consistent basis, so that the firm's provisions are more responsive to changes in the market value of the backing assets for the with-profit funds. This implies the implementation of adequate, consistent and objective models for both the behaviour of the price of the assets backing the policy, and the calculation of realistic liabilities, where by liability it is meant all of the guaranteed elements in the policy plus the projection of future discretionary bonus payments.

In light of the international move towards the market based, fair value accountancy standards mentioned above, in this paper we apply classical contingent claim theory for the valuation of the most common policy design used in the UK for participating contracts. In fact, since the pioneering work of Brennan and Schwartz (1976) on unit-linked policies, there have been several studies on the different typologies of contract design and their features. Thus we would cite Bacinello (2001, 2003), Ballotta et al. (2003), Grosen and Jørgensen (2000, 2002), Guillén et al. (2004), Haberman et al. (2003) and Tansakanen and Lukkarinen (2003), just to mention some of the most recent works.

It is worth pointing out that all these contributions use a Black-Scholes (1973) framework, based on the assumption of a geometric Brownian motion model for the dynamics of the asset fund backing the insurance policy. However, the dramatic changes shown by financial markets over the last 15 years suggest that a better specification of this underlying temporal evolution is needed. In particular, the evidence suggests very strongly that log-stock returns have fatter tails than the normal distribution, meaning that the normal distribution understates the probability of extremes events, especially falls, in the stock prices, thereby inducing biases in the option prices. Extensions to the Black-Scholes model for option pricing began appearing in the finance literature not long after publication of the original paper in 1973. For example, Merton (1973) generalized the Black-Scholes formula to account for a deterministic time-dependent rather than constant volatility later in the same year and, in 1976, he incorporated jump-diffusion models for the price of the underlying asset. From those seminal works, a vast literature on generalizations of the model arose; a state of the art evaluation and comparison of these models is contained, for example, in Bakshi, Cao and Chen (1997).

The purpose of this communication is to consider the valuation problem for one of the smoothing schemes commonly used by insurance companies in the UK and analyzed by Haberman et al. (2003), when a more realistic formulation of the stochastic process driving the reference portfolio is made, than the usual geometric Brownian motion. In particular, we set up a market model based on the use of a Lévy motion as relevant process for the market value of the underlying reference portfolio. In this framework, we consider the problem of determining the fair value of a with profit policy in which the reversionary bonus rate is based on the idea, widely adopted in the UK, of a smoothed "asset share" scheme (Needleman and Roff, 1995).

The rest of the paper proceeds as follows: in section 2 we introduce the participating policy under consideration and the details of the benefits it offers; in section 3 we develop the market set up and the model for the valuation of the contract in a general Lévy process setting, with particular attention to the special case of a geometric Brownian motion driven portfolio. Section 4 is devoted to the pricing in the full jump-diffusion economy; numerical results are presented in section 5 and section 6 concludes.

2 Participating contracts

Let's consider a 100/0 fund, i.e. a fund whose rules provide that 100% of the profits distributed by way of bonuses be allocated to policyholders. At the beginning of the contract, the policyholder pays a single-sum premium, P_0, to purchase from the insurance company a policy expiring after T years, when the account is settled by a single payment from the insurer to the policyholder. At inception of the contract, the insurance company invests the funds received in the financial market, acquiring a portfolio A, and commits itself to crediting interest (the guaranteed benefit plus the reversionary bonus) on the policy's account balance (the policy reserve) until the contract expires, according to some smoothing scheme dependent on each year's market return which aims to reduce the volatility of the company's payouts.

The particular crediting mechanism under initial consideration determines the level of the smoothed policy reserve at time t to be a weighted average of the unsmoothed value of the policy reserve at time t, and the level of the smoothed policy reserve at time $(t-1)$ (Needleman and Roff, 1995). The interest rate credited to the unsmoothed policy account is guaranteed never to fall below the contractually specified guaranteed annual policy interest rate. In this discussion, we ignore lapses and mortality. Hence, the policy reserve is defined as

$$\begin{aligned} P(t) &= \alpha P^1(t) + (1-\alpha) P(t-1), \qquad \alpha \in (0,1), \\ P(0) &= P_{0,} \end{aligned}$$

where $P^1(t)$ is the unsmoothed asset share such that

$$\begin{aligned} P^1(0) &= P_0, \\ P^1(t) &= P^1(t-1)(1+r_P(t)), \\ r_P(t) &= \max\left\{ r_G, \beta \frac{A(t)-A(t-1)}{A(t-1)} \right\}, \end{aligned}$$

and r_G and $\beta \in (0,1)$ are the guaranteed rate and the participation rate respectively. In particular, if

$$r_A(t) = \frac{A(t)-A(t-1)}{A(t-1)}$$

is the annual rate of return on the reference portfolio, then the rate of return credited annually to $P^1(t)$ can be rewritten as

$$r_P(t) = \max\{r_G, \beta r_A(t)\} = r_G + (\beta r_A(t) - r_G)^+ .$$

At maturity, T, the value of the policy reserve is

$$\begin{aligned} P(T) &= \alpha P^1(T) + (1-\alpha) P(T-1) \\ &= \alpha \sum_{k=0}^{T-1} (1-\alpha)^k P^1(T-k) + (1-\alpha)^T P_0 \\ &= P_0 \left[\alpha \sum_{k=0}^{T-1} (1-\alpha)^k \prod_{t=1}^{T-k} (1+r_P(t)) + (1-\alpha)^T \right]. \end{aligned} \tag{1}$$

At the claim date of the contract, a discretionary payment might be made by the insurer on the final surplus earned by the insurance company in addition to the guaranteed amount in the policy reserve. This is the so-called terminal bonus $\gamma R(T)$, where $R(T) = (A(T) - P(T))^+$. As mentioned above, this payment is discretionary as the terminal bonus rate γ, i.e. the participation rate in the company's surplus is not guaranteed but declared only near to the maturity of the contract.

Finally, if at maturity the insurance company is not capable of paying the policy reserve, $P(T)$, the policyholder takes those assets that are available. Hence, the policyholder's overall claim at expiration can be summarised as follows:

$$C(T) = \begin{cases} A(T) & \text{if } A(T) < P(T) \\ P(T) + \gamma R(T) & \text{otherwise,} \end{cases}$$

or, in a more compact way:

$$C(T) = P(T) + \gamma R(T) - D(T), \tag{2}$$

where $D(T) = (P(T) - A(T))^+$. Applying risk-neutral valuation, the market value of the policyholder's claim is:

$$C(0) = V^P(0) + \gamma V^R(0) - V^D(0),$$

with

$$V^P(0) = \hat{\mathbb{E}}\left[e^{-rT}P(T)\right], \; V^R(0) = \hat{\mathbb{E}}\left[e^{-rT}R(T)\right], \; V^D(0) = \hat{\mathbb{E}}\left[e^{-rT}D(T)\right],$$

where $\hat{\mathbb{E}}$ is the expectation taken under the risk-neutral probability measure $\hat{\mathbb{P}}$.

Equation (2) shows that the policy reserve and the terminal bonus are not the only components that affect the valuation of these participating contracts, as we need also to take into account that the insurance company liability is limited by the market value of the reference portfolio. This feature is captured by the quantity D, which represents the payoff of the so-called default option. Ballotta et al. (2003) point out that the value of the default option, V^D, captures information related to the probability and the extent of a shortfall, similarly to the unconditional shortfall expectation risk measure. More precisely, as calculations are carried out under the risk-neutral probability measure, V^D provides an estimate of the market value of the loss that the policyholder incurs if a shortfall occurs and the life insurance company becomes insolvent.

3 Market model and the embedded option

Consider a frictionless market with continuous trading. Assume hence that there are no taxes, no transaction costs, no restrictions on borrowing or short sales and all securities are perfectly divisible. Assume further that the risk free security is the money market account $B(t) = e^{rt}$, $r > 0$. Let the reference portfolio be composed only by equity and defined as:

$$\begin{aligned} A(t) &= A(0)\, e^{L(t)}, \\ A(0) &= P_0, \end{aligned}$$

where $\{L(t) : t \geq 0\}$ is a Lévy motion with finite activity under the real probability measure $\mathbb{P}$, i.e.

$$L(t) = at + \sigma W(t) + \int_0^t \int_{\mathbb{R}} x\left(N(ds, dx) - \nu(dx)\, ds\right), \qquad (3)$$

where

- W is a standard $\mathbb{P}$-Wiener process;
- $X_1, X_2, ...,$ is a sequence of i.i.d. random variables with density function $f(dx)$, modelling the size of the jumps in the Lévy process;
- N is an homogeneous Poisson measure of rate λ, with $\mathbb{P}$-compensator $\nu(dx)\, dt = \lambda f(dx)\, dt$, $a = \mathbb{E}[L_1]$ and $\sigma \in \mathbb{R}^+$.

In particular, we assume that the jump size of the Lévy motion is normally distributed, that is $X \sim N(\mu_X, \sigma_X^2)$.

Let $\hat{\mathbb{P}}$ be some risk-neutral probability measure[1]; then the fair value of the policy reserve at inception of the contract is

$$\begin{aligned} V^P(0) &= \hat{\mathbb{E}}\left[e^{-rT} P(T)\right] \\ &= P_0 \left\{ \alpha \sum_{k=0}^{T-1} e^{-rT} (1-\alpha)^k\, \hat{\mathbb{E}}\left[\prod_{t=1}^{T-k} (1 + r_P(t))\right] + e^{-rT} (1-\alpha)^T \right\}. \end{aligned}$$

Note that in this model, the annual rate of return on the reference portfolio is

$$r_A(t) = e^{L(t) - L(t-1)} - 1;$$

therefore, by construction, it generates a sequence of stochastic processes $r_A(t_1), r_A(t_2), ..., r_A(T)$ independent one of the other. Consequently, since

[1] The setup defined by equation (3) is an incomplete market, meaning that there exists at least one contingent claim which cannot be hedged. Alternatively, this means that, under the assumption of no arbitrage, there is a multiplicity of equivalent martingale measures with which to price contingent claims.

$r_P(t) = \max\{r_G, \beta r_A(t)\}$,

$$\begin{aligned} V^P(0) &= P_0 \left\{ \alpha \sum_{k=0}^{T-1} e^{-rk} (1-\alpha)^k \prod_{t=1}^{T-k} \hat{\mathbb{E}}\left[e^{-r}(1+r_P(t))\right] + e^{-rT}(1-\alpha)^T \right\} \\ &= P_0 \left\{ \alpha \sum_{k=0}^{T-1} e^{-rk} (1-\alpha)^k \prod_{t=1}^{T-k} V_t^M(0) + e^{-rT}(1-\alpha)^T \right\}, \end{aligned} \tag{4}$$

with

$$\begin{aligned} V_t^M(0) &= \hat{\mathbb{E}}\left[e^{-r}(1+r_P(t))\right] \\ &= \hat{\mathbb{E}}\left[e^{-r}\left(1+r_G+(\beta r_A(t)-r_G)^+\right)\right] \\ &= e^{-r}(1+r_G) + \hat{\mathbb{E}}\left[e^{-r}\left(\beta e^{L'(1)} - (\beta + r_G)\right)^+\right]. \end{aligned} \tag{5}$$

The term L' denotes an independent copy of the Lévy process L.

Equation (5) shows that the policy reserve can be decomposed into a sequence of one year riskless zero coupon bonds and one year call options on $S_1 = \beta e^{L'(1)}$ with strike price $K = \beta + r_G$.

The same pricing methodology can be applied to the terminal bonus and the default option; however, no similar closed-form expressions are available for their value at inception. In fact, as equation (1) shows, the recursive substitution of P is quite complex; moreover $P(T)$ is obviously highly dependent on the path followed by the reference portfolio A. These facts imply that it is not possible to find analytical expressions for the value of these policy's constituent blocks. Therefore, we have to resort to numerical methods for the analysis of V^R and V^D, which will be described in section 5.

In the following sections of the paper, we consider different specifications of the general Lévy process, L , and the pricing probability measure, in order to derive analytical formulae for the value $V^P(0)$.

3.1 Option pricing in the Black-Scholes framework

Consider the special case in which the underlying Lévy process is a standard Brownian motion. Under the (unique) risk-neutral martingale measure $\hat{\mathbb{P}}$, the reference portfolio A is then described by:

$$A(t) = A(0)\, e^{\left(r - \frac{\sigma_A^2}{2}\right)t + \sigma_A \hat{W}(t)}.$$

Hence

$$r_A(t) = e^{\left(r - \frac{\sigma_A^2}{2}\right) + \sigma_A \hat{W}'(1)} - 1,$$

where $\hat{W}'$ is an independent copy of the standard one-dimensional $\hat{\mathbb{P}}$-Brownian motion.

The one year call option embedded in the policy reserve has value

$$\hat{\mathbb{E}}\left[e^{-r}\left(\beta e^{\left(r-\frac{\sigma_A^2}{2}\right)+\sigma_A \hat{W}'(1)}-(\beta+r_G)\right)^{+}\right]. \tag{6}$$

Applying the Black-Scholes formula to (6) (see also Bacinello, 2001, and Miltersen and Persson, 2003, for similar results), we obtain

$$\hat{\mathbb{E}}\left[e^{-r}\left(\beta e^{\left(r-\frac{\sigma_A^2}{2}\right)+\sigma_A \hat{W}(1)}-(\beta+r_G)\right)^{+}\right]=\beta N\left(d_1\right)-e^{-r}\left(\beta+r_G\right) N\left(d_2\right),$$

where

$$d_1=\frac{\ln \frac{\beta}{\beta+r_G}+\left(r+\frac{\sigma_A^2}{2}\right)}{\sigma_A}; \quad d_2=d_1-\sigma_A.$$

Consequently, the value of the policyholder's account at inception is

$$\begin{aligned} V^P(0) &= P_0\left\{\alpha \sum_{k=0}^{T-1} e^{-rk}\left(1-\alpha\right)^k\left[e^{-r}\left(1+r_G\right)+\beta N\left(d_1\right)-e^{-r}\left(\beta+r_G\right) N\left(d_2\right)\right]^{T-k}\right. \\ &\quad \left.+e^{-rT}\left(1-\alpha\right)^T\right\}. \end{aligned} \tag{7}$$

4 Option pricing in a jump-diffusion economy

Consider now the more general case in which the driving process is described (in the real world) by equation (3). In order to calculate the fair value of the policy, we need to determine the pricing risk-neutral measure $\hat{\mathbb{P}}$. This implies changing the probability measure linked to the driving Lévy process so that asset prices discounted at the risk-free rate are $\hat{\mathbb{P}}$-martingales. Hence, let $\hat{\mathbb{P}}$ be a measure equivalent to $\mathbb{P}$. Then

$$\left.\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathbb{F}_t}=\eta\left(t\right),$$

where

$$\eta\left(t\right)=1-\int_0^t G\left(s\right)\eta\left(s_{-}\right)dW\left(s\right)+\int_0^t\int_{\mathbb{R}}\eta\left(s_{-}\right)\left(H\left(s,x\right)-1\right)\left(N\left(ds,dx\right)-\nu\left(dx\right)ds\right),$$

G is a previsible process and H a previsible and Borel measurable process such that

$$\mathbb{E}\left[\int_0^t G^2(s)\,ds\right] < \infty; \qquad \int_{\mathbb{R}} (H(t,x) - 1)\,\nu(dx) < \infty.$$

As mentioned above, the risk-neutral condition implies that under $\hat{\mathbb{P}}$ asset prices discounted at the risk-free rate are martingales; therefore we need to choose G and H such that this condition is satisfied. In the market described in section 3, discounted asset prices are given by:

$$\begin{aligned}\tilde{A}(t) &= B(t)^{-1} A(t) \\ &= A(0)\, e^{\left(a - r - \int_{\mathbb{R}} x\nu(dx)\right)t + \sigma W(t) + \int_0^t \int_{\mathbb{R}} x N(ds,dx)}.\end{aligned}$$

Consequently, it can be shown that (see equation A1 in the Appendix[2]) for $t < u$

$$\hat{\mathbb{E}}\left[\tilde{A}(u)\,|\mathbb{F}_t\right] = \tilde{A}(t)\, e^{\left(a - r - \int_{\mathbb{R}} x\nu(dx) + \frac{\sigma^2}{2} + \int_{\mathbb{R}} (e^x - 1)\hat{\nu}(dx)\right)(u-t) - \sigma \int_t^u G(s)ds},$$

where $\hat{\nu} = H\nu$.

In order for the process $\tilde{A}$ to be a $\hat{\mathbb{P}}$-martingale, we require

$$\left(a - r - \int_{\mathbb{R}} x\nu(dx) + \frac{\sigma^2}{2} + \int_{\mathbb{R}} (e^x - 1)\,\hat{\nu}(dx)\right)(u - t) - \sigma \int_t^u G(s)\,ds = 0.$$

As the first term of this equation is a linear function of time, in the case under examination the previsible process G is actually expressed by a constant. Therefore, the martingale condition characterizing the equivalent probability measure $\hat{\mathbb{P}}$ can be expressed as

$$a - r - \int_{\mathbb{R}} x\nu(dx) + \frac{\sigma^2}{2} - \sigma G + \int_{\mathbb{R}} (e^x - 1)\, H(t,x)\,\nu(dx) = 0. \tag{8}$$

However, equation (8) shows that the market is incomplete, as in general there are infinitely many ways of choosing G and H so that (8) is satisfied, which means that $\hat{\mathbb{P}}$ is not unique and the market is incomplete. Many different approaches to the problem of identifying a suitable martingale measure for derivative pricing have been proposed, but there is not yet a definitive way of selecting one of them.

The remaining part of this section presents two examples to illustrate the valuation procedure making use of two specific approaches for the selection

[2]Full details of all the calculations presented in this section are offered in the Appendix.

of the risk-neutral martingale measure. In the first example, we assume that the jump component of the assets return represents "non systematic" risk, which is, therefore, uncorrelated with the market. This is the same assumption made by Merton (1976). In the second example, we make use of the Esscher transform technique developed by Gerber and Shiu (1994) to define the Radon-Nikodým derivative η.

4.1 Policy fair valuation: the Merton measure

Following Merton (1976), we assume that the jump risk is asset specific, and hence diversifiable (which implies that no premium is paid for such a risk). If we interpret the function G and H included in the Radon-Nikodým derivative η, as indicators of the premia respectively for the risk originated by the Brownian motion component and the risk deriving from the possibility of an "extraordinary" event occurring in the market (the Poisson component), it follows that $H(t,x) = 1$ for the jump risk premium to be zero. In this case, the Radon-Nikodým derivative is

$$\eta(t) = \frac{d\hat{\mathbb{P}}_M}{d\mathbb{P}} = e^{-GW(t) - \frac{G^2}{2}t},$$

where G solves the martingale condition

$$a - r - \int_{\mathbb{R}} x\nu(dx) + \frac{\sigma^2}{2} - \sigma G + \int_{\mathbb{R}} (e^x - 1)\nu(dx) = 0, \tag{9}$$

and $\hat{\mathbb{P}}_M$ denotes the equivalent martingale measure resulting from this approach. Under these assumptions, we obtain that

$$\begin{aligned}
\hat{\mathbb{P}}_M[N(t) = n] &= \mathbb{E}\left[\eta(t)\, 1_{(N(t)=n)}\right] \\
&= \mathbb{E}[\eta(t)]\,\mathbb{E}\left[1_{(N(t)=n)}\right] \\
&= \mathbb{P}[N(t) = n],
\end{aligned}$$

which is expected as we are assuming that investors receive a zero premium for the jump risk. Further, bearing in mind that $X \sim N(\mu_X, \sigma_X^2)$ under $\mathbb{P}$,

$$\hat{\mathbb{E}}_M\left[e^{kL(t)}\,|N(t) = n\right] = e^{k\left(\left(r - \frac{\sigma^2}{2} - \int_{\mathbb{R}}(e^x-1)\nu(dx)\right)t + n\mu_X\right) + \frac{k^2}{2}\left(\sigma^2 t + n\sigma_X^2\right)},$$

(see equation A5 in the Appendix), which implies that conditioning on the number of jumps, the process L follows a Normal distribution with variance

$$\sigma^2 t + n\sigma_X^2,$$

and mean

$$\hat{\mathbb{E}}_M \left[L(t) \left| N(t) = n \right. \right] = \left(r - \frac{\sigma^2}{2} - \int_{\mathbb{R}} (e^x - 1) \nu(dx) \right) t + n\mu_X$$
$$= \left(r - \frac{\sigma^2}{2} - \lambda(\mu - 1) \right) t + n\mu_X$$

where

$$\mu = e^{\mu_X + \frac{\sigma_X^2}{2}}.$$

Hence, if we set

$$r_n = r - \lambda(\mu - 1) + n\left(\mu_X + \frac{\sigma_X^2}{2}\right)$$
$$= r - \lambda(\mu - 1) + n \ln \mu,$$

and

$$v_n^2 = \sigma^2 + n\sigma_X^2,$$

conditioning on the number of jumps occurring in one year, it follows that

$$L(t) - L(t-1) \sim N\left(r_n - \frac{v_n^2}{2}, v_n^2\right).$$

Consider now the one-year European call option in equation (5). Under the framework set out in this section, it follows that

$$\hat{\mathbb{E}}_M \left[e^{-r} \left(\beta e^{L'(1)} - (\beta + r_G) \right)^+ \right]$$
$$= \hat{\mathbb{E}}_M \left\{ \hat{\mathbb{E}}_M \left[e^{-r} \left(\beta e^{L'(1)} - (\beta + r_G) \right)^+ \middle| N'(1) = n \right] \right\}. \quad (10)$$

Consequently, if y is a standardized Normal random variable, the inner expectation in the previous equation can be rewritten as

$$\hat{\mathbb{E}}_M \left[e^{-r} \left(\beta e^{r_n - \frac{v_n^2}{2} + v_n y} - (\beta + r_G) \right)^+ \right] = e^{-\lambda(\mu-1) + n \ln \mu} f(n),$$

with

$$f(n) = \beta N(d_n) - e^{-r_n} (\beta + r_G) N(d_n');$$
$$d_n = \frac{\ln \frac{\beta}{\beta + r_G} + \left(r_n + \frac{v_n^2}{2} \right)}{v_n}, \quad d_n' = d_n - v_n.$$

(See equation A11). Therefore, (10) can be solved and returns[3]

$$\begin{aligned}\hat{\mathbb{E}}_M\left[e^{-r}\left(\beta e^{L'(1)}-(\beta+r_G)\right)^+\right] &= \hat{\mathbb{E}}_M\left[e^{-\lambda(\mu-1)+N'(1)\ln\mu}f\left(N'\left(1\right)\right)\right]\\ &= \sum_{n=0}^{\infty}\frac{e^{-\lambda}\lambda^n}{n!}e^{-\lambda(\mu-1)+n\ln\mu}f\left(n\right)\\ &= \sum_{n=0}^{\infty}\frac{e^{-\lambda\mu}\left(\lambda\mu\right)^n}{n!}f\left(n\right);\end{aligned}$$

whilst the fair value of the policy reserve is

$$\begin{aligned}V^P(0) = P_0 &\left\{\alpha\sum_{k=0}^{T-1}e^{-rk}\left(1-\alpha\right)^k\left[e^{-r}\left(1+r_G\right)+\sum_{n=0}^{\infty}\frac{e^{-\lambda\mu}\left(\lambda\mu\right)^n}{n!}f\left(n\right)\right]^{T-k}\right.\\ &\left.+e^{-rT}\left(1-\alpha\right)^T\right\}. \qquad (11)\end{aligned}$$

4.2 Policy fair valuation: the Esscher measure

In the previous section, we derived a valuation formula under the assumption that the jump risk is not priced, as in Merton (1976). In terms of CAPM assumptions, this means that the jump component of the stock's returns represents "non systematic" risk. Nevertheless, Jarrow and Rosenfeld (1984) provide empirical evidence that the jump component does affect the equilibrium price of contingent claims. In this section, we relax this assumption to allow for a jump risk which is systematic and non-diversifiable. This will lead to a different specification of the risk-neutral martingale measure under which contingent claims are priced.

In particular, the approach we adopt relies on a well established technique in actuarial science, the Esscher transform (Esscher, 1932), which is suitable in the case that the log-returns of the underlying asset are governed by a process with independent and stationary increments, as in our model. The application of this technique to price contingent claims is due to Gerber and Shiu (1994), and it can be described in general terms as follows. Consider the price at time $t \geq 0$ of a non-dividend paying stock $S\left(t\right) = S_0 e^{X(t)}$, where $X\left(t\right)$ is a process with independent and stationary increments. Let $M_X\left(h,t\right)$ be its Laplace transform, i.e.

$$M_X\left(h,t\right) = \mathbb{E}\left(e^{hX(t)}\right).$$

[3]Note the similarity with the option pricing formula derived by Merton (1976) for the same market specification. Merton however priced the call option contract solving the corresponding governing "mixed" partial differential-difference equation.

Because of the independence property of the increments of X,

$$M_X(h,t) = M_X(h,1)^t.$$

Moreover, the process

$$\eta(t) = \left\{ e^{hX(t)} M_X(h,1)^{-t} : t \geq 0 \right\}$$

is a positive $\mathbb{P}$-martingale that can be used to define a change of probability measure, i.e. the Radon-Nikodým derivative of a new equivalent probability measure $\hat{\mathbb{P}}_h$, called the Esscher measure of parameter h. The process $\eta(t)$ is called the Esscher transform of parameter h. In particular, it is possible to select the risk-neutral Esscher measure as the measure $\hat{\mathbb{P}}_h$ such that the discounted price process $e^{-rt}S(t)$ is a $\hat{\mathbb{P}}_h$-martingale. This is obtained by determining the parameter h as solution of

$$e^{-rt}S(t) = \hat{\mathbb{E}}_h \left(e^{-ru} S(u) | \mathbb{F}_t \right), \; t < u,$$

or, equivalently,

$$\begin{aligned} S_0 &= \hat{\mathbb{E}}_h \left(e^{-rt} S(t) \right) \\ &= S_0 e^{-rt} \left(\frac{M_X(1+h,1)}{M_X(h,1)} \right)^t. \end{aligned} \tag{12}$$

The application of this procedure to the market model proposed in section 3, implies that we need to find the parameter h solving

$$r = \ln M_L(1+h,1) - \ln M_L(h,1),$$

or, making use of the Lévy-Khintchine formula,

$$a - r - \int_{\mathbb{R}} x\nu(dx) + \frac{\sigma^2}{2} + \sigma^2 h + \int_{\mathbb{R}} e^{hx} \left(e^x - 1 \right) \nu(dx) = 0. \tag{13}$$

It can be easily checked that this last expression corresponds to the martingale condition (8) for the choices $G = -\sigma h$ and $H(t,x) = e^{hx}$ (see also equation A8 in the Appendix). Consequently,

$$\begin{aligned} \hat{\mathbb{P}}_h [N(t) = n] &= \mathbb{E} \left[\eta(t) 1_{(N(t)=n)} \right] \\ &= \mathbb{P}[N(t) = n] e^{n \ln \mu_h - \lambda t(\mu_h - 1)}, \end{aligned} \tag{14}$$

with

$$\mu_h = e^{h\mu_X + \frac{h^2}{2}\sigma_X^2};$$

and

$$\hat{\mathbb{E}}_h\left[e^{kL(t)}\left|N(t)=n\right.\right]=e^{k\left(\left(r-\frac{\sigma^2}{2}-\int_{\mathbb{R}}e^{hx}(e^x-1)\nu(dx)\right)t+n\mu_X+nh\sigma_X^2\right)+\frac{k^2}{2}\left(\sigma^2t+n\sigma_X^2\right)},$$

(see equation A9 in the Appendix) which implies that, conditioning on $N(t)$, the process $L(t)$ still follows a Normal distribution with variance[4]

$$\sigma^2 t+n\sigma_X^2;$$

whilst its mean is

$$\begin{aligned}\hat{\mathbb{E}}_h\left[L(t)\left|N(t)=n\right.\right] &= \left(r-\frac{\sigma^2}{2}-\int_{\mathbb{R}}e^{hx}\left(e^x-1\right)\nu(dx)\right)t+n\mu_X+nh\sigma_X^2,\\ &= \left(r-\frac{\sigma^2}{2}-\lambda\left(\mu_{h+1}-\mu_h\right)\right)t+n\mu_X+nh\sigma_X^2,\end{aligned}$$

where the last equality follows from the fact that $X\sim N\left(\mu_X,\sigma_X^2\right)$ under $\mathbb{P}$, and

$$\mu_h=e^{h\mu_X+\frac{h^2}{2}\sigma_X^2},\quad \mu_{h+1}=e^{(1+h)\mu_X+\frac{(1+h)^2}{2}\sigma_X^2}.$$

The one-year European call option contained in equation (5) can be priced following the same steps as in section 4.1. In particular,

$$\begin{aligned}&\hat{\mathbb{E}}_h\left[e^{-r}\left(\beta e^{L'(1)}-(\beta+r_G)\right)^+\right|\left.N'(1)=n\right]\\ =\;& e^{-\lambda(\mu_{h+1}-\mu_h)+n\ln\frac{\mu_{h+1}}{\mu_h}}\left[\beta N\left(d_{n;h}\right)-e^{-r_{n;h}}\left(\beta+r_G\right)N\left(d'_{n;h}\right)\right]\\ =\;& e^{-\lambda(\mu_{h+1}-\mu_h)+n\ln\frac{\mu_{h+1}}{\mu_h}}f\left(n;h\right),\end{aligned}$$

where

$$\begin{aligned}r_{n;h} &= r-\lambda\left(\mu_{h+1}-\mu_h\right)+n\ln\frac{\mu_{h+1}}{\mu_h},\\ d_{n;h} &= \frac{\ln\frac{\beta}{\beta+r_G}+\left(r_{n;h}+\frac{v_n^2}{2}\right)}{v_n},\quad d'_{n;h}=d_{n;h}-v_n,\end{aligned}$$

and

$$v_n^2=\sigma^2+n\sigma_X^2.$$

[4]Note that the conditional variance of the process remains unaffected by each change of measure considered in this paper. This is in line with the spirit of the Girsanov theorem as the density process η shifts the drift of the distribution, i.e. it rescales the mean of the process, without changing its shape.

Equation (14) implies that

$$
\begin{aligned}
\hat{\mathbb{E}}_h\left[e^{-r}\left(\beta e^{L'(1)}-(\beta+r_G)\right)^+\right] &= \hat{\mathbb{E}}_h\left[e^{-\lambda(\mu_{h+1}-\mu_h)+N'(1)\ln\frac{\mu_{h+1}}{\mu_h}}f\left(N'\left(1\right);h\right)\right] \\
&= \sum_{n=0}^{\infty} e^{-\lambda(\mu_{h+1}-\mu_h)+n\ln\frac{\mu_{h+1}}{\mu_h}} f\left(n;h\right)\hat{\mathbb{P}}_h\left[N'\left(1\right)=n\right] \\
&= \sum_{n=0}^{\infty}\frac{e^{-\lambda\mu_{h+1}}\left(\lambda\mu_{h+1}\right)^n}{n!}f\left(n;h\right)
\end{aligned}
$$

Hence, the pricing formula for the fair value of the policy reserve is

$$
\begin{aligned}
V^P(0) = P_0 &\left\{\alpha\sum_{k=0}^{T-1}e^{-rk}\left(1-\alpha\right)^k\left[e^{-r}\left(1+r_G\right)+\sum_{n=0}^{\infty}\frac{e^{-\lambda\mu_{h+1}}\left(\lambda\mu_{h+1}\right)^n}{n!}f\left(n;h\right)\right]^{T-k}\right. \\
&\left. +e^{-rT}\left(1-\alpha\right)^T\right\}. \qquad (15)
\end{aligned}
$$

5 Analysis of the price biases

In this section we use the results obtained above to analyze the differences in the contract value implied by the Lévy process setting proposed, and their implications on the no-arbitrage combinations of contract parameters $(\alpha,\beta,\gamma,r_G)$. By no-arbitrage combinations of parameters, we mean those combinations such that the policy is sold at a price which is determined in a market consistent manner, and in such a way that the contributions from the policyholders are fair with respect to the value of the benefits that they entitle to receive. In particular, since the market value of the policyholder's claim is

$$C\left(0\right)=V^P\left(0\right)+\gamma V^R\left(0\right)-V^D\left(0\right),$$

against the payment of an initial (single) premium P_0, as seen in section 2, then the no-arbitrage combinations of contract parameters must be such that

$$C\left(0\right)=P_0. \qquad (16)$$

Since the market parameters, like the volatility of the reference portfolio or the frequency with which jumps occur in the economy, i.e. λ, are in general not under the control of the life insurance office, we analyze specifically how the design parameters $(\alpha,\beta,\gamma,r_G)$ need to be readjusted for the equilibrium condition (16) to hold when the asset's volatility and λ are allowed to change.

Unless otherwise stated, the base set of parameters[5] is

$$\begin{array}{llllll} P_0 = 100; & r_G = 4\%; & \alpha = 0.6; & \beta = 0.5; & \gamma = 0.7; & T = 20 \text{ years}; \\ a = 10\%; & \sigma_A = 20\%; & \mu_X = -0.0537; & \sigma_X = 0.07; & \lambda = 0.59; & r = 3.5\%. \end{array}$$

Note that, in order to perform a sensible comparison between the prices obtained in the Black-Scholes framework and the ones deriving from the Lévy model considered in this work, we will always consider the value of the total volatility to be constant and equal to σ_A, that is the instantaneous volatility of the log-return on the geometric Brownian motion driven underlying asset. This is done in order to outline the effect of the jump component rather than the effects of changes in the overall volatility of the asset price. This assumption, however, imposes some restrictions on the range of feasible values that σ_A and λ can assume. In fact, from the moment generating function $(A2)$ of the process L, we obtain that

$$\sigma_A^2 = \sigma^2 + \lambda \left(\mu_X^2 + \sigma_X^2 \right).$$

Since $\sigma^2 > 0$, then $\sigma_A^2 - \lambda \left(\mu_X^2 + \sigma_X^2 \right) > 0$, which implies that, for the given base set of parameters, $\sigma_A > 0.07$ and $\lambda < 5.14$.

5.1 Numerical results

Figure 1 shows the sensitivity to the reference portfolio's volatility, σ_A, of the policyholder's overall claim, $C(0)$, together with each contract's component.

As panel (a) highlights, the values of the claim C obtained under the three models considered in this paper are very close (in fact the maximum mispricing generated by the geometric Brownian motion model is a 0.26% overpricing with respect to the $\hat{\mathbb{P}}_M$-measure based model and a 0.65% overpricing with respect to the Esscher model, both in correspondence of $\sigma_A = 10\%$ p.a.).

The breakdown of $C(0)$ into its building blocks, though, shows that this information is misleading. Panels $(b) - (c) - (d)$, in fact, reveal that the geometric Brownian motion-based model overestimates the value of the guaranteed benefits and the probability of default when compared to the $\hat{\mathbb{P}}_M$-measure based model. In other words, the classical Black-Scholes framework leads to a more prudential pricing rule, as it provides an upper bound for the value of the policy reserve and the default option. However, it underprices the value of the terminal bonus, i.e. it underestimates the capacity of the life insurance company for generating enough surplus to distribute to the

[5] The values of the parameters of the jump distribution are taken from Bakshi, Cao and Chen (1997), who estimated them for the S&P500 index using data over the period June 1988-May 1991.

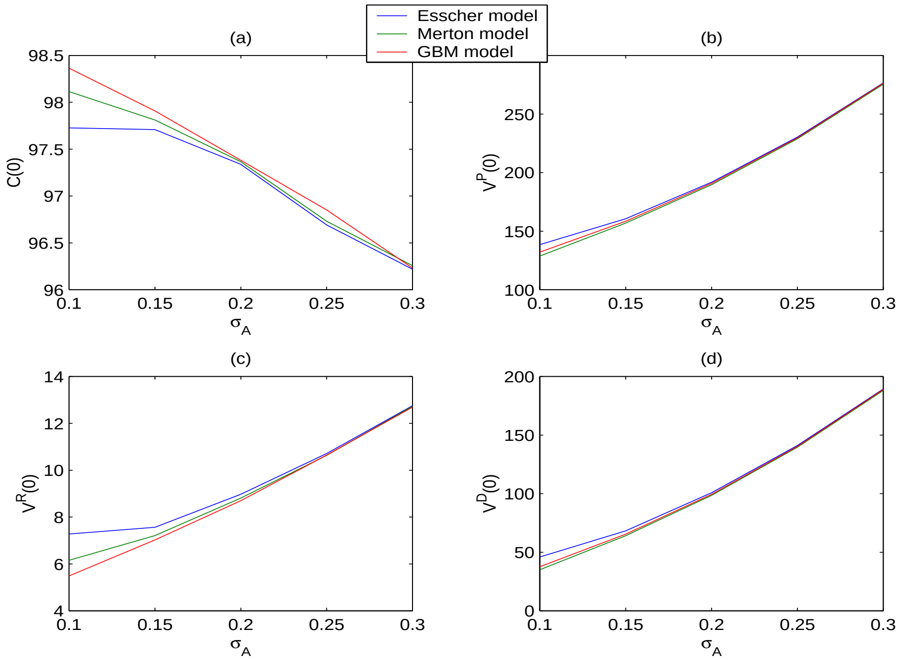

Figure 1: Sensitivity of the overall policyholder's claim and its component to the reference portfolio's volatility.

policyholders. The maximum mispricing is 10.82%, again in correspondence to $\sigma_A = 10\%$ p.a.

When compared with the valuation model based on the Esscher measure, instead, the standard Black-Scholes framework leads to the underpricing of the values of all contract's component. In particular, the biggest mispricing is, once again for $\sigma_A = 10\%$ p.a., 5% for the value of the policy reserve, 25% for the value of the terminal bonus and 18% for the value of the default option. Hence, reserving on the basis of the geometric Brownian motion model would lead us to set aside insufficient resources to cover the liabilities. Further, the assumption of a reference portfolio driven by a diffusion process would seriously underestimate the potential threat to the life insurance company solvency represented by the participating contract, as the higher risk of default would not be fully captured.

We also note the differences in the nature of the mispricing generated by the Black-Scholes framework with respect to the $\hat{\mathbb{P}}_M$-measure model and the Esscher measure paradigm, especially for the case of the default option, as $V_M^D < V_{GBM}^D < V_h^D$. This result could be explained by the fact that, differently from the pricing measure $\hat{\mathbb{P}}_M$, the Esscher measure does not preserve the valuation approach's independence of the investors' risk preferences, which is one of the main features of the classical Black-Scholes model. In fact, as equation ($A7$) shows, the $\hat{\mathbb{P}}_h$-dynamic of the process L depends on the parameter h solution to the Esscher martingale condition (13). In order to actually calculate h, we need to make some assumptions regarding the "real" drift of the Lévy process, i.e. the parameter a. Since the drift a represents the expected rate of growth of the reference portfolio, specifying an assumption on its value effectively means specifying the preferences structure of the investors. In this sense the Esscher measure can be seen as the closest probability to the real probability measure $\mathbb{P}$ in terms of information content. Chan (1999), in fact, shows that $\hat{\mathbb{P}}_h$ gives rise to the equivalent martingale measure which has the minimal relative entropy, or Kullback-Leibler index of "information distance" with respect to $\mathbb{P}$. As such the Esscher measure appears to be the most suitable one to capture the additional risk induced by the occurrence of crashes in the market, as shown in Figure 2. In this Figure we represent two possible evolutions for the reference portfolio and the policy reserve. The first scenario is represented in the top three panels of Figure 2 and it is based on the geometric Brownian motion to model the asset backing the participating contract. At the expiration of the contract, the reference portfolio is worth £774 whilst the policy reserve has value £581. In this case the policyholder would be paid the guaranteed benefits in full. The second scenario, represented in the bottom panels of Figure 2, uses the same set of random numbers to generate the diffusion part of the reference portfolio; however, on average λ times per year the asset price jumps discretely of a random amount X. Since σ_A is kept constant, the final result is a higher instability of the rate of returns r_A and r_P to the extent that, at maturity, the portfolio is worth £183 against the £367 liability represented by the policy reserve. In this case the life insurance company would default.

The instability produced in the model by the jump component is also illustrated in Figure 3, in which we show how the probability of default changes under both the Brownian motion and the Lévy process paradigm, when the total volatility σ_A is allowed to change, and in Figure 4, in which, instead, we represent the distributions of the asset and the policy reserve under both market's paradigm. The last Figure, in particular, shows the 99% percentile of the policy reserve distribution; the probability that $A(T)$ is less than this 99% percentile is 73% in the geometric Brownian motion-based model, and 86% in the Lévy process-based model.

The effects on the fair combinations of design parameters, for example the participation rate β, produced by the inclusion of jumps in the market model are represented in Figures 5 and 6. Both Figures show that the additional jump feature restricts the set of optimal choices for β. In particular, in Figure 5, we represent the values that the parameter β is allowed to assume in order

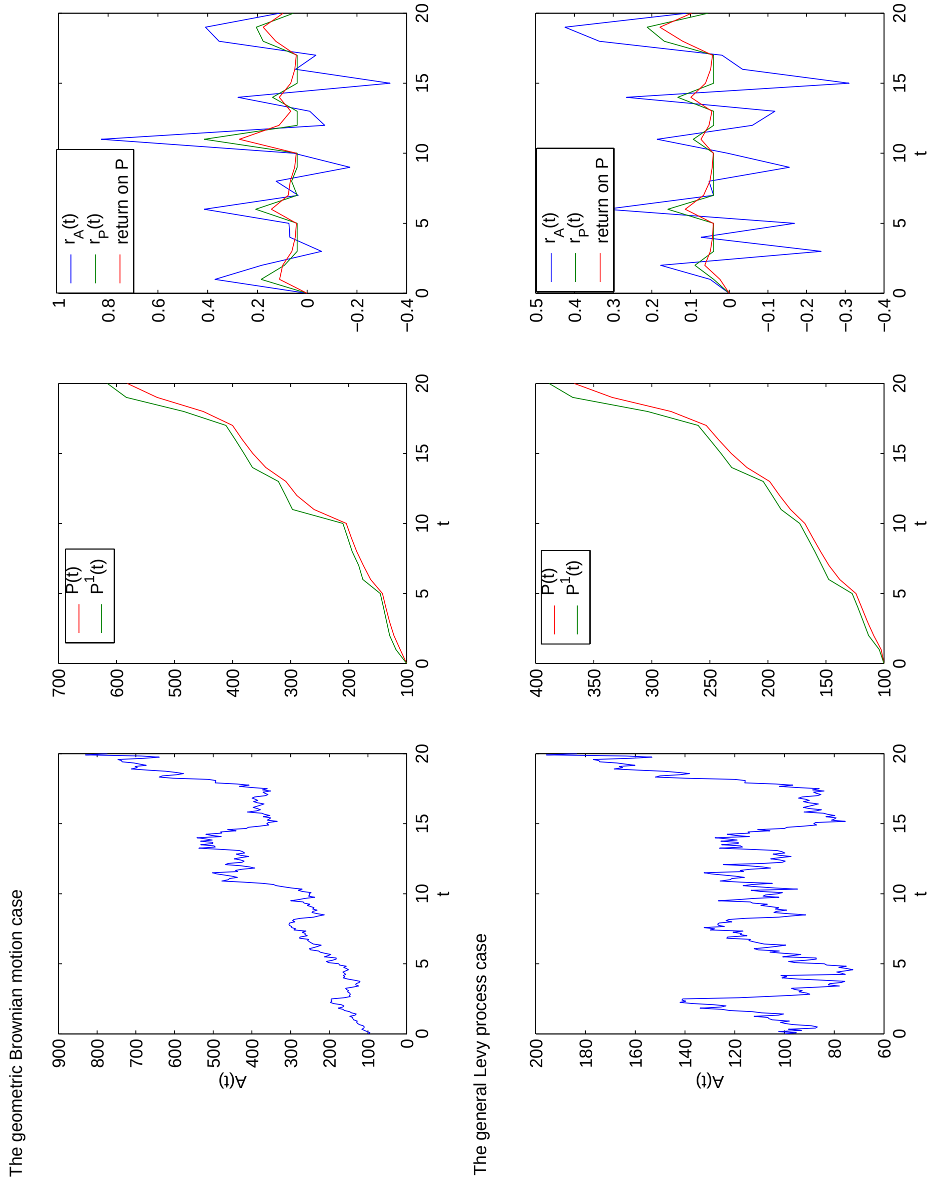

Figure 2: Scenario generation: a possible market scenario under the geometric Brownian motion paradigm and the Lévy process paradigm. The diffusion part of the asset price process has been generated using the same sequence of random numbers. The scenario generation has been obtained for the benchmark set of parameters under the real probability measure $\mathbb{P}$.

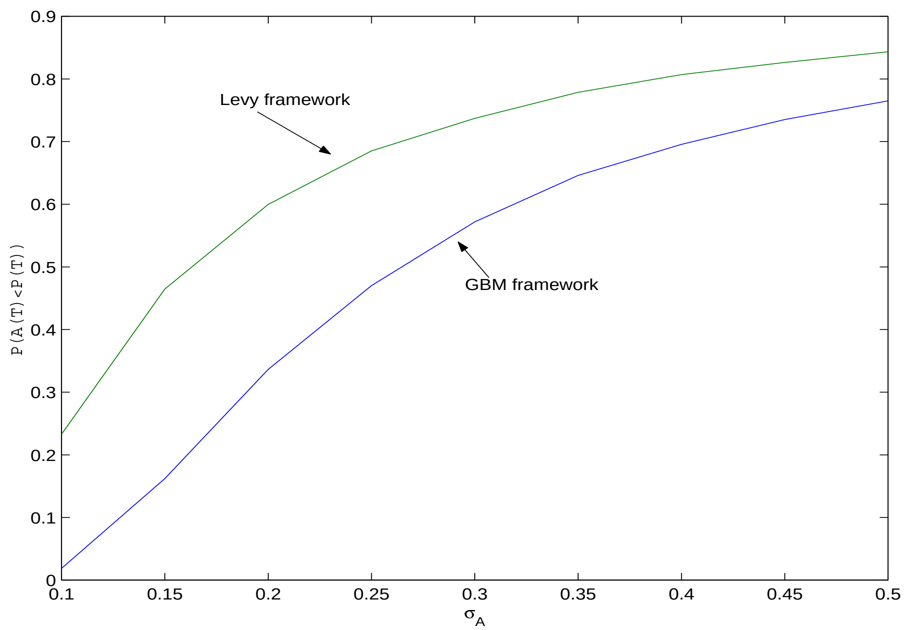

Figure 3: Default probabilities in the two market paradigm presented in the paper and their sensitivity to the reference portfolio volatility.

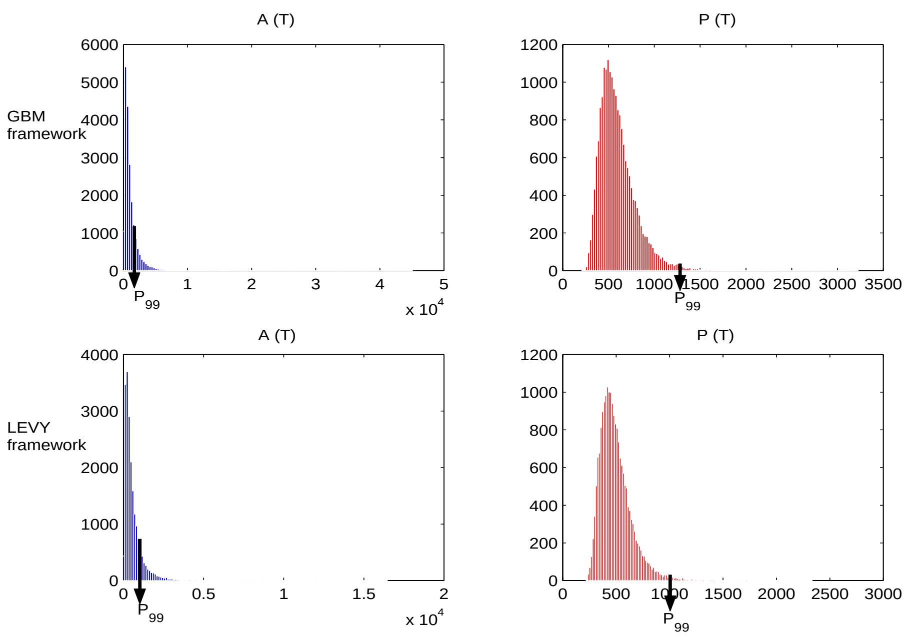

Figure 4: Distribution analysis of $A(T)$ and $P(T)$ in both the geometric Brownian motion and the Lévy process framework. In the panels it is also shown the 99% percentile, P_{99}, of the policy reserve distribution.

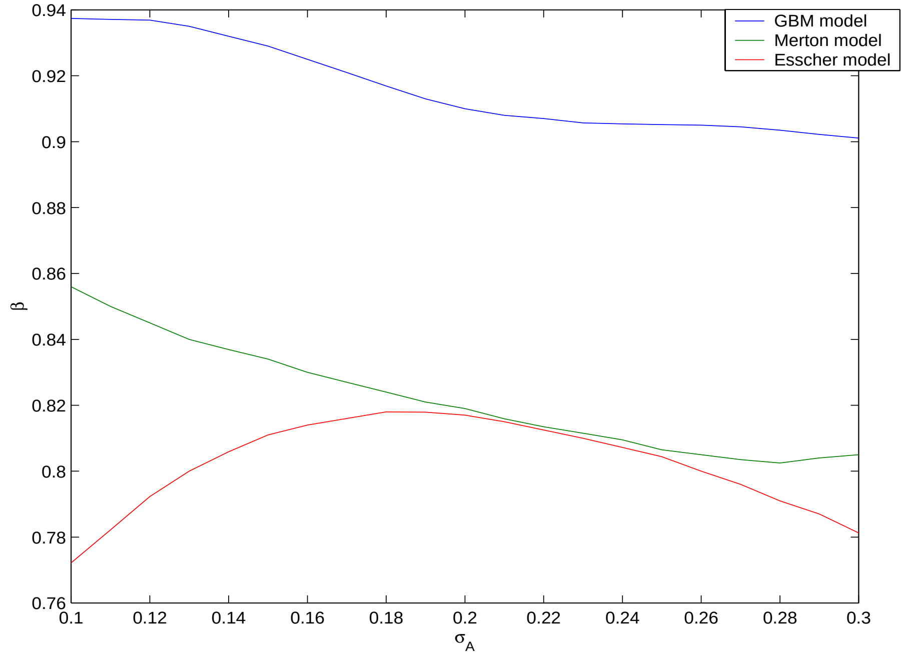

Figure 5: Isopremia curves: the fair set of the participation rate β against the portfolio volatility.

to satisfy equation (16) when σ_A changes. The optimal β set is smaller in the case of the Esscher valuation framework because of the higher default risk that characterizes this model, as already discussed earlier. Although the geometric Brownian motion overprices the value of the claims with respect to the $\mathbb{P}_M$ measure, the participation rate β is set at a lower level in the $\mathbb{P}_M$-based paradigm than in the geometric Brownian motion one. This fact is due to the higher rates of increment of the default option induced by the same increment in σ_A, that characterizes the $\mathbb{P}_M$ model. Figure 6 shows that, the reference portfolio's volatility being equal, the set of fair values for the participation rate β becomes even smaller when the jumps are allowed to occur more often, i.e. when the market becomes more unstable. The reason is again to be sought for in the impact of higher λ on the default risk attached to the participating contract.

6 Concluding remarks

In this paper we have developed a valuation framework for participating life insurance contracts based on a jump-diffusion specification of the asset backing the policy. A market-based pricing methodology has been then ap-

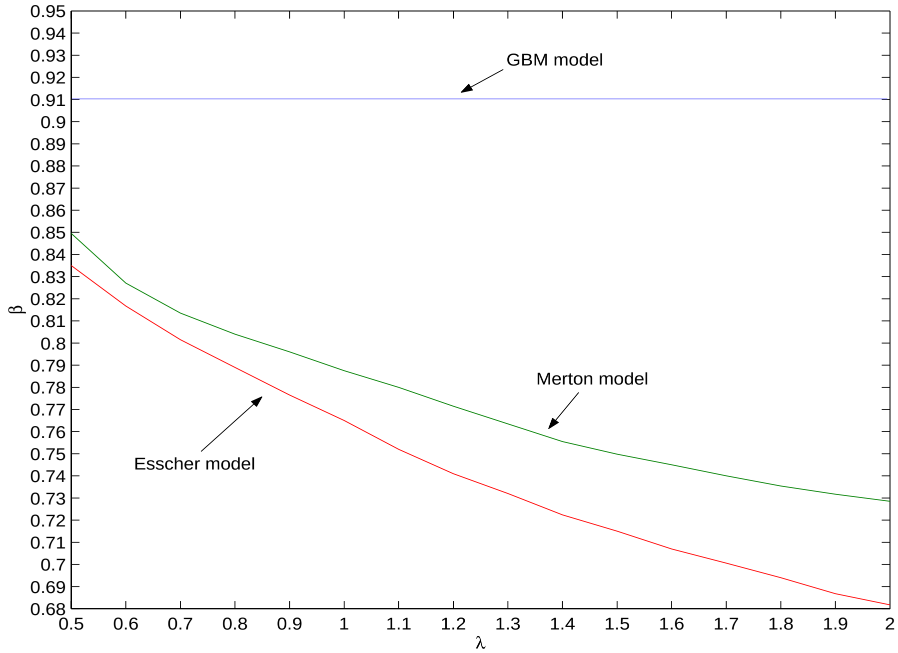

Figure 6: Isopremia curves: β against the jumps' frequency λ.

plied to these contracts and the complex guarantees and option-like features embedded therein.

This study finds its justification in the new recommendations from the IASB and the financial authorities to adopt adequate models for both the dynamic of the asset prices and the calculation of life insurance companies' liabilities. The recent literature has addressed so far only the problem of the implementation of suitable fair valuation techniques for participating contracts. However, the results presented in this paper show the importance of modelling the asset side as well of the company's balance sheet, in order to properly assess market risks, and their impact on the value of these contracts and the company' solvency.

As shown in this paper, an important issue linked to the implementation of valuation schemes in a jump-diffusion context, is the selection of one specific pricing probability measure, which in the end requires the estimation of parameters that are affected by the risk preferences of investors. A possible solution to this problem might rely on possible links between the structure of investors' risk preferences, indices of risk aversion and the expected rate of growth of the underlying asset. We leave this question for future research.

A Equivalent martingale measures and pricing formulae

The purpose of this Appendix is to show the details of the change of measure "machinery" for the case in which the process driving the underlying asset is a Levy process with finite activity, and the derivation of the valuation formulae presented in section 4.

A.1 Equivalent martingale measures in a jump-diffusion economy

As seen in section 4, derivative asset pricing methods rest on converting prices of such assets into martingales. This is done by transforming the underlying probability distribution using the tools provided by the Girsanov's theorem. For the case of the Lévy process introduced in section 3, the change of measure is formalized in the following.

Theorem 1 (Girsanov) *Assume $\mathbb{P}$ and $\hat{\mathbb{P}}$ are two equivalent measure and let η be the density process defined as*

$$\eta(t) = 1 - \int_0^t G(s)\,\eta(s_-)\,dW(s) + \int_0^t \int_{\mathbb{R}} \eta(s_-)\,(H(s,x)-1)\,(N(ds,dx) - \nu(dx)\,ds),$$

where G is a previsible process and H a previsible and Borel measurable process. Suppose further that

$$\mathbb{E}\left[\int_0^t G^2(s)\,ds\right] < \infty$$

and

$$\int_{\mathbb{R}} (H(t,x)-1)\,\nu(dx) < \infty.$$

Then, under $\hat{\mathbb{P}}$, the process

$$\hat{W}(t) = W(t) + \int_0^t G(s)\,ds$$

is a standard Brownian motion, and the process

$$\hat{Z}(t) = \int_0^t \int_{\mathbb{R}} x\,(N(ds,dx) - \hat{\nu}(dx)\,ds) + at + \int_0^t \int_{\mathbb{R}} x\,(H(s,x)-1)\,\nu(dx)\,ds$$

is a quadratic pure jump process with compensator measure

$$\hat{\nu}(dt,dx) = \hat{\nu}(dx)\,dt$$

where

$$\hat{\nu}(dx) = H(t,x)\,\nu(dx).$$

We observe that this result is a version of the more general Girsanov's theorem for semimartingales, modified to fit the features of the process considered in this work. For a detailed treatment of the more general case, we refer to Theorem 3.24 in Chapter III of Jacod and Shiryaev (1987), and Theorem 3.2 in Chan (1999).

Theorem 1 implies in particular that

$$\eta(t) = e^{-\int_0^t\left(\frac{G^2(s)}{2}+\int_{\mathbb{R}}(H(s,x)-1)\nu(dx)\right)ds-\int_0^t G(s)dW(s)+\int_0^t\int_{\mathbb{R}}\ln H(s,x)N(ds,dx)}.$$

Therefore, for $t < u$

$$\begin{aligned} & \mathbb{E}\left[\tilde{A}(u)\eta(u)|\mathbb{F}_t\right] \\ = \; & \tilde{A}(t)\eta(t)\mathbb{E}\left[e^{\left(a-r-\int_{\mathbb{R}}(H(s,x)-1+x)\nu(dx)\right)(u-t)-\int_t^u\frac{G^2(s)}{2}ds+\int_t^u(\sigma-G(s))dW(s)+\int_t^u\int_{\mathbb{R}}(x+\ln H(s,x))N(ds,dx)}\right]. \end{aligned}$$

Since W, N and X are independent of each other, it follows that

$$\begin{aligned} & \mathbb{E}\left[\tilde{A}(u)\eta(u)|\mathbb{F}_t\right] \\ = \; & \tilde{A}(t)\eta(t)e^{\left(a-r-\int_{\mathbb{R}}(H(s,x)-1+x)\nu(dx)\right)(u-t)-\int_t^u\frac{G^2(s)}{2}ds+\int_t^u\frac{(\sigma-G(s))^2}{2}ds+\int_t^u\int_{\mathbb{R}}(e^xH(s,x)-1)\nu(dx)ds} \\ = \; & \tilde{A}(t)\eta(t)e^{\left(a-r+\frac{\sigma^2}{2}-\int_{\mathbb{R}}x\nu(dx)+\int_{\mathbb{R}}(e^x-1)H(s,x)\nu(dx)ds\right)(u-t)-\sigma\int_t^uG(s)ds} \\ = \; & \tilde{A}(t)\eta(t)e^{\left(a-r+\frac{\sigma^2}{2}-\int_{\mathbb{R}}x\nu(dx)+\int_{\mathbb{R}}(e^x-1)\hat{\nu}(dx)ds\right)(u-t)-\sigma\int_t^uG(s)ds}, \end{aligned}$$

where the last equality follows from Theorem 1. The Bayes rule implies that

$$\hat{\mathbb{E}}\left[\tilde{A}(u)|\mathbb{F}_t\right] = \frac{\mathbb{E}\left[\tilde{A}(u)\eta(u)|\mathbb{F}_t\right]}{\mathbb{E}\left[\eta(u)|\mathbb{F}_t\right]},$$

therefore

$$\hat{\mathbb{E}}\left[\tilde{A}(u)|\mathbb{F}_t\right] = \tilde{A}(t)e^{\left(a-r+\frac{\sigma^2}{2}-\int_{\mathbb{R}}x\nu(dx)+\int_{\mathbb{R}}(e^x-1)\hat{\nu}(dx)ds\right)(u-t)-\sigma\int_t^uG(s)ds} \tag{A1}$$

A.2 Distribution properties of $L(t)$ under $\hat{\mathbb{P}}$

In section 3, we specified the Lévy decomposition to be:

$$L(t) = at + \sigma W(t) + \int_0^t\int_{\mathbb{R}} x\left(N(ds,dx)-\nu(dx)ds\right).$$

Hence, the Lévy-Khintchine formula implies that the moment generating function of the process L can be written as

$$\mathbb{E}\left[e^{kL(t)}\right] = e^{t\varphi(k)}$$

where

$$\varphi(k) = ak + \frac{\sigma^2}{2}k^2 + \int_{\mathbb{R}} \left(e^{kx} - 1 - kx\right)\nu(dx). \tag{A2}$$

In particular, under the risk-neutral martingale measure $\hat{\mathbb{P}}$, the moment generating function will take the form

$$\begin{aligned}
\hat{\mathbb{E}}\left[e^{kL(t)}\right] &= e^{t\hat{\varphi}(k)}, \\
\hat{\varphi}(k) &= Ak + \frac{\Gamma^2}{2}k^2 + \int_{\mathbb{R}} \left(e^{kx} - 1 - kx\right)\hat{\nu}(dx).
\end{aligned}$$

The aim of this section is to determine the functions A, Γ and $\hat{\nu}$, i.e. the characteristic triplet of the semimartingale L, under the two alternative martingale measures considered in this paper, using the fact that

$$\hat{\mathbb{E}}\left[e^{kL(t)}\right] = \mathbb{E}\left[\eta(t)\, e^{kL(t)}\right],$$

where $\eta(t)$ is the density process defined in Theorem 1.

Let's consider the case of the Merton measure first. As discussed in section 4.1, the Radon-Nikodým derivative for the probability $\hat{\mathbb{P}}_M$ is

$$\eta(t) = e^{-GW(t) - \frac{G^2}{2}t}.$$

Hence

$$\begin{aligned}
\hat{\mathbb{E}}\left[e^{kL(t)}\right] &= \mathbb{E}\left[e^{-GW(t)-\frac{G^2}{2}t} e^{k\left(at+\sigma W(t)+\int_0^t\int_{\mathbb{R}} x(N(ds,dx)-\nu(dx)ds)\right)}\right] \\
&= e^{kat-\frac{G^2}{2}t+\frac{(\sigma k-G)^2}{2}t+t\int_{\mathbb{R}}\left(e^{kx}-1-kx\right)\nu(dx)ds} \\
&= e^{t\hat{\varphi}_M(k)},
\end{aligned}$$

with

$$\hat{\varphi}_M(k) = (a - \sigma G)k + \frac{\sigma^2}{2}k^2 + \int_{\mathbb{R}} \left(e^{kx} - 1 - kx\right)\nu(dx). \tag{A3}$$

Therefore, the characteristic triplet is:

$$A = a - \sigma G; \quad \Gamma = \sigma; \quad \hat{\nu}(dx) = \nu(dx).$$

This implies that under $\hat{\mathbb{P}}_M$ the decomposition of the process L is

$$L(t) = (a - \sigma G)t + \sigma \hat{W}_M(t) + \int_0^t \int_{\mathbb{R}} x\left(N(ds, dx) - \nu(dx)\, ds\right).$$

The martingale condition (9) implies

$$L(t) = \left(r - \frac{\sigma^2}{2} - \int_{\mathbb{R}} (e^x - 1)\nu(dx)\right)t + \sigma \hat{W}_M(t) + \int_0^t \int_{\mathbb{R}} xN(ds, dx). \tag{A4}$$

Equations $(A3)$ and $(A4)$ and Theorem 1 imply that $\hat{W}_M$ is a standard one-dimensional $\hat{\mathbb{P}}_M$-Brownian motion, whilst the $\hat{\mathbb{P}}_M$-law of the compound Poisson process is the same as the one under the real probability measure $\mathbb{P}$.

Moreover,

$$\begin{aligned}\hat{\mathbb{E}}_M\left[e^{kL(t)}\left|N(t)=n\right.\right] &= e^{k\left(r-\frac{\sigma^2}{2}-\int_{\mathbb{R}}(e^x-1)\nu(dx)\right)t+\frac{\sigma^2}{2}k^2 t}\hat{\mathbb{E}}_M\left[e^{k\int_0^t\int_{\mathbb{R}}xN(ds,dx)}\right| N(t)=n\Big] \\ &= e^{k\left(r-\frac{\sigma^2}{2}-\int_{\mathbb{R}}(e^x-1)\nu(dx)\right)t+\frac{\sigma^2}{2}k^2 t}\hat{\mathbb{E}}_M\left(e^{kx}\right)^n;\end{aligned}$$

since $X\sim N\left(\mu_X,\sigma_X^2\right)$, then

$$\hat{\mathbb{E}}_M\left[e^{kL(t)}\left|N(t)=n\right.\right]=e^{k\left(\left(r-\frac{\sigma^2}{2}-\int_{\mathbb{R}}(e^x-1)\nu(dx)\right)t+n\mu_X\right)+\frac{k^2}{2}\left(\sigma^2 t+n\sigma_X^2\right)}. \tag{A5}$$

Equation $(A4)$ is used in section 5 to implement the Monte Carlo procedure for the valuation of terminal bonus and the default option. Equation $(A5)$ is instead used in section 4.1 to calculate the value of the policy reserve.

Analogous calculations can be carried out for the case of the Esscher measure $\hat{\mathbb{P}}_h$. In this case, the Radon-Nikodým derivative is defined as

$$\eta(t)=e^{hL(t)}M_L(h,1)^{-t}.$$

Note that, using Itô's lemma for semimartingales,

$$\begin{aligned}d\eta(t) &= \eta'(t_-)\,dL(t)+\frac{1}{2}\eta''(t_-)\,d[L,L]_t^C+\eta'(t)-\eta'(t_-)-\eta'(t_-)\,\Delta L(t) \\ &= \sigma h\eta(t_-)\,dW(t)-\eta(t_-)\int_{\mathbb{R}}\left(e^{hx}-1\right)\left(N(dt,dx)\,\nu(dx)\,dt\right).\end{aligned}$$

From Theorem 1 we obtain that

$$d\eta(t)=-G(t)\,\eta(t_-)\,dW(t)-\eta(t_-)\int_{\mathbb{R}}\left(H(t,x)-1\right)\left(N(dt,dx)\,\nu(dx)\,dt\right),$$

which implies that $G(t)=-\sigma h$ and $H(t,x)=e^{hx}$. The moment generating function of the Lévy process is given by

$$\begin{aligned}\hat{\mathbb{E}}_h\left[e^{kL(t)}\right] &= M_L(h,1)^{-t}\,\mathbb{E}\left[e^{(h+k)L(t)}\right] \\ &= e^{k\left(a+\sigma^2 h\right)t+\frac{\sigma^2}{2}k^2+t\int_{\mathbb{R}}\left(e^{hx}\left(e^{kx}-1\right)-kx\right)\nu(dx)} \\ &= e^{k\left(a+\sigma^2 h-\int_{\mathbb{R}}x\nu(dx)+\int_{\mathbb{R}}xe^{hx}\nu(dx)\right)t+\frac{\sigma^2}{2}k^2+t\int_{\mathbb{R}}\left(e^{kx}-1-kx\right)e^{hx}\nu(dx)} \\ &= e^{t\hat{\varphi}_h(k)}\end{aligned}$$

with

$$\hat{\varphi}_h(k)=\left(a+\sigma^2 h-\int_{\mathbb{R}}x\nu(dx)+\int_{\mathbb{R}}xe^{hx}\nu(dx)\right)k+\frac{\sigma^2}{2}k^2+\int_{\mathbb{R}}\left(e^{kx}-1-kx\right)\hat{\nu}(dx). \tag{A6}$$

This implies that the $\hat{\mathbb{P}}_h$-characteristic triplet is:

$$A = a + \sigma^2 h - \int_{\mathbb{R}} x\nu(dx) + \int_{\mathbb{R}} xe^{hx}\nu(dx); \quad \Gamma = \sigma; \quad \hat{\nu}(dx) = e^{hx}\nu(dx).$$

The corresponding decomposition of the process L is then:

$$L(t) = \left(a + \sigma^2 h - \int_{\mathbb{R}} x\nu(dx) + \int_{\mathbb{R}} xe^{hx}\nu(dx)\right) t + \sigma \hat{W}_h(t) + \int_0^t \int_{\mathbb{R}} x\left(N(ds, dx) - \hat{\nu}(dx)\, ds\right),$$

or, bearing in mind that h solves the Esscher martingale condition (13)

$$L(t) = \left(r - \frac{\sigma^2}{2} - \int_{\mathbb{R}} e^{hx}(e^x - 1)\nu(dx)\right) t + \sigma \hat{W}_h(t) + \int_0^t \int_{\mathbb{R}} xN(ds, dx). \tag{A7}$$

Theorem 1 and equations $(A6)$ and $(A7)$ imply that, under $\hat{\mathbb{P}}_h$, $\hat{W}_h$ is a standard one-dimensional Brownian motion, and the compound Poisson process $\int_0^t \int_{\mathbb{R}} xN(ds, dx)$ has compensator measure $\hat{\nu}(dx) = e^{hx}\nu(dx)$. Therefore, the $\hat{\mathbb{P}}_h$-rate of the Poisson process N is

$$\lambda_h = \lambda e^{h\mu_X + h^2 \frac{\sigma_X^2}{2}},$$

whilst the $\hat{\mathbb{P}}_h$-distribution of the jump random size X is $N(\mu_X + h\sigma_X^2, \sigma_X^2)$. In fact:

$$\begin{aligned}
\hat{\mathbb{P}}_h[N(t) = n] &= \hat{\mathbb{E}}_h\left[1_{(N(t)=n)}\right] = \mathbb{E}\left[\gamma(t)\, 1_{(N(t)=n)}\right] \\
&= M_L(h, 1)^{-t}\, \mathbb{E}\left[e^{hL(t)} \,|N(t) = n\right] \mathbb{P}[N(t) = n] \\
&= \frac{\mathbb{P}[N(t) = n]}{M_L(h, 1)^t} e^{h\left(a - \int_{\mathbb{R}} x\nu(dx) + \frac{\sigma^2}{2} h\right) t + nh\mu_X + nh^2 \frac{\sigma_X^2}{2}}.
\end{aligned}$$

Let

$$\mu_h = e^{h\mu_X + h^2 \frac{\sigma_X^2}{2}},$$

then

$$\hat{\mathbb{P}}_h[N(t) = n] = \mathbb{P}[N(t) = n]\, e^{n \ln \mu_h - \lambda t(\mu_h - 1)}. \tag{A8}$$

Therefore

$$\begin{aligned}
\hat{\mathbb{P}}_h[N(t) = n] &= \frac{e^{-\lambda t}(\lambda t)^n}{n!} e^{n \ln \mu_h - \lambda t(\mu_h - 1)} \\
&= \frac{e^{-\lambda_h t}(\lambda_h t)^n}{n!}.
\end{aligned}$$

Moreover

$$\begin{aligned}\hat{\mathbb{E}}_h\left[e^{k\int_0^t\int_{\mathbb{R}}xN(ds,dx)}\right] &= \frac{e^{h\left(a-\int_{\mathbb{R}}x\nu(dx)+\frac{\sigma^2}{2}h\right)t}}{M_L(h,1)^t}\mathbb{E}\left[e^{(h+k)\int_0^t\int_{\mathbb{R}}xN(ds,dx)}\right]\\ &= e^{t\int_{\mathbb{R}}\left(e^{kx}-1\right)e^{hx}\nu(dx)}.\end{aligned}$$

Since, under the real probability measure $\mathbb{P}$, $X\sim N\left(\mu_X,\sigma_X^2\right)$ and $\nu(dx)=\lambda f(dx)$, then

$$\hat{\mathbb{E}}_h\left[e^{k\int_0^t\int_{\mathbb{R}}xN(ds,dx)}\right]=e^{\lambda_h t\left(e^{k\left(\mu_X+h\sigma_X^2\right)+\frac{k^2\sigma_X^2}{2}}-1\right)}.$$

On the other hand, the moment generating function of a compound Poisson process has form:

$$\hat{\mathbb{E}}_h\left[e^{k\int_0^t\int_{\mathbb{R}}xN(ds,dx)}\right]=e^{\lambda_h t\left(\hat{\mathbb{E}}_h\left(e^{kx}\right)-1\right)},$$

which implies that

$$\hat{\mathbb{E}}_h\left(e^{kx}\right)=e^{k\left(\mu_X+h\sigma_X^2\right)+\frac{k^2\sigma_X^2}{2}}.$$

Finally, we can also calculate the conditional moment generating function of the process L, which returns

$$\hat{\mathbb{E}}_h\left[e^{kL(t)}\left|N(t)=n\right.\right]=e^{k\left(r-\frac{\sigma^2}{2}-\int_{\mathbb{R}}e^{hx}(e^x-1)\nu(dx)\right)t+\frac{\sigma^2}{2}k^2t}\hat{\mathbb{E}}_h\left(e^{kx}\right)^n.$$

Hence

$$\hat{\mathbb{E}}_h\left[e^{kL(t)}\left|N(t)=n\right.\right]=e^{k\left(\left(r-\frac{\sigma^2}{2}-\int_{\mathbb{R}}e^{hx}(e^x-1)\nu(dx)\right)t+n\mu_X+nh\sigma_X^2\right)+\frac{k^2}{2}\left(\sigma^2t+n\sigma_X^2\right)}. \tag{A9}$$

A.3 Valuation using the Merton measure

Equation (10) in section 4.1 shows that the one-year call option embedded in the policy reserve has value

$$\begin{aligned}&\hat{\mathbb{E}}_M\left[e^{-r}\left(\beta e^{L'(1)}-(\beta+r_G)\right)^+\right]\\ =\ &\hat{\mathbb{E}}_M\left\{\hat{\mathbb{E}}_M\left[e^{-r}\left(\beta e^{L'(1)}-(\beta+r_G)\right)^+\middle| N'(1)=n\right]\right\}.\end{aligned}$$

Since, as shown in the previous section, conditioning on the number of jumps occurring in one year

$$L\left(t\right)-L\left(t-1\right)\sim N\left(r_n-\frac{v_n^2}{2},v_n^2\right),$$

then the inner expectation can be written as

$$\hat{\mathbb{E}}_M\left[e^{-r}\left(\beta e^{L'(1)}-(\beta+r_G)\right)^+\middle|N'\left(1\right)=n\right]=\hat{\mathbb{E}}_M\left[e^{-r}\left(\beta e^{r_n-\frac{v_n^2}{2}+v_n y}-(\beta+r_G)\right)^+\right],$$

where $y\sim N\left(0,1\right)$. Therefore

$$\begin{aligned}
&\hat{\mathbb{E}}_M\left[e^{-r}\left(\beta e^{r_n-\frac{v_n^2}{2}+v_n y}-(\beta+r_G)\right)^+\right]\\
=\;&\hat{\mathbb{E}}_M\left[\beta e^{-r+r_n-\frac{v_n^2}{2}+v_n y}1_{\left(\beta e^{r_n-\frac{v_n^2}{2}+v_n y}>\beta+r_G\right)}\right]-e^{-r}\left(\beta+r_G\right)\hat{\mathbb{P}}_M\left(\beta e^{r_n-\frac{v_n^2}{2}+v_n y}>\beta+r_G\right)\\
=\;&\beta e^{-r+r_n}\int_a^\infty\frac{1}{\sqrt{2\pi}}e^{-\frac{(y-v_n)2}{2}}dy-e^{-r}\left(\beta+r_G\right)\hat{\mathbb{P}}_M\left(y>a\right),
\end{aligned}$$

with

$$a=\frac{\ln\frac{\beta+r_G}{\beta}-\left(r_n-\frac{v_n^2}{2}\right)}{v_n}.$$

Hence

$$\hat{\mathbb{E}}_M\left[e^{-r}\left(\beta e^{r_n-\frac{v_n^2}{2}+v_n y}-(\beta+r_G)\right)^+\right]=\beta e^{-r+r_n}N\left(d_n\right)-e^{-r}\left(\beta+r_G\right)N\left(d_n'\right),\tag{A10}$$

with

$$\begin{aligned}
d_n&=\frac{\ln\frac{\beta}{\beta+r_G}+\left(r_n+\frac{v_n^2}{2}\right)}{v_n};\\
d_n'&=d_n-v_n.
\end{aligned}$$

Since

$$r_n=r-\lambda\left(\mu-1\right)+n\ln\mu,$$

we can rewrite equation $(A10)$ as

$$\begin{aligned}
&\beta e^{-\lambda(\mu-1)+n\ln\mu}N\left(d_n\right)-e^{-r}\left(\beta+r_G\right)N\left(d_n'\right)\\
=\;&e^{-\lambda(\mu-1)+n\ln\mu}\left[\beta N\left(d_n\right)-e^{-r+\lambda(\mu-1)-n\ln\mu}\left(\beta+r_G\right)N\left(d_n'\right)\right]\\
=\;&e^{-\lambda(\mu-1)+n\ln\mu}\left[\beta N\left(d_n\right)-e^{-r_n}\left(\beta+r_G\right)N\left(d_n'\right)\right]\\
=\;&e^{-\lambda(\mu-1)+n\ln\mu}f\left(n\right),
\end{aligned}$$

where

$$f(n) = \beta N(d_n) - e^{-r_n}(\beta + r_G) N(d_n') . \tag{A11}$$

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