Initial and Final Backward and Forward discrete time non-homogeneous semi-Markov Credit Risk Models

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Abstract. In this paper we would show how it is possible to construct efficient Migration models in the study of credit risk problems presented in [Jarrow *et al.*, 1997] with Markov environment. Recently it was introduced the semi-Markov process in the migration models. The introduction of semi-Markov processes permits to overtake some of the Markov constraints given by the dependence of transition probabilities on the duration into a rating category. In this paper, it will be shown how it will be possible taking into account simultaneously backward and forward processes at beginning and at the end of the time in which the credit risk model has to be observed. With such a generalization, it is possible to consider what happen inside the time after the first transition and before the last transition where the problem is studied. This paper generalizes other papers presented before. The model is presented in a discrete time environment. An illustrative example will be shown in the last part of the paper.
Keywords: Backward and forward semi-Markov processes, Credit risk migration model, Reliability.

1 Introduction

Credit risk problem is one of the most important problems that are faced in the financial literature. The banks and other financial intermediaries those most interested in the evaluation of credit risk. To each company issuing a bond it is given a reliability "rating" indicating its capacity to reimburse the debt. The rating level changes over time and one way to follow the time evolution of ratings is by means of Markov processes [Jarrow *et al.*, 1997]. In this environment Markov models are called "migration models".

The problem of the unfitting of Markov processes in the credit risk environment have been outlined in several papers, see for example [Altman, 1999], [Nickell *et al.*, 2000], [Kavvathas, 2001], [Lando and Skodeberg, 2002].

The main problems of non-Markovianity are the following:
i - the duration inside a state. The probability of changing rating depends on the time a company maintains the same rating see for example [Carty and Fons, 1994].
ii - the time dependence of the rating evaluation. This means that in general the rating evaluation depends on when it is carried out and, in particular on the business cycle, see [Nickell *et al.*, 2000].
iii - the dependence of the new rating on all the previous ones and not only on the last [Carty and Fons, 1994], [Nickell *et al.*, 2000].

The three problems have been solved by the authors in some past papers: [D'Amico *et al.*, 2005a], [D'Amico *et al.*, 2005b] using semi-Markov processes (SMP).

The first problem can has been solved by means of semi-Markov processes. In fact, in SMP the transition probabilities are a function of the waiting time spent in a state of the system. Furthermore the introduction of the backward process solves this problem in a complete way giving the opportunity to assign different transition probabilities in function of the duration inside the last visited state.

The second problem can be dealt with in a more general way by means of a non-homogeneous environment or in a more particular way by using different scenarios in the model.

The third effect exists in the case of downward moving ratings but not in the case of upward moving ratings, [Kavvathas, 2001]. More precisely if a company gets a lower rating then there is a higher probability that its subsequent rating will also be lower than the preceding one. In the case of upward movement, this phenomenon doesn't hold. The problem has been solved by enlarging the state space of the process.

In this paper the use of recurrence times processes allow the possibility to construct more efficient migration models that generalize our previous results. It is important to dispose of efficient rating migration models in fact reliable rating prediction is of interest for pricing rating sensitive derivatives (see [**?**]), for the valuation of portfolio of defaulting bonds, for credit risk management and capital allocation.

More precisely the introduction of backward and forward processes at initial and final times permits to have a complete knowledge of the waiting times at beginning and at the end of the observation period of the model. In fact:
• initial backward takes into account the time in which the system went in the state also if the arrival time is before the beginning of the studied time

horizon;
• initial forward considers the time in which the first transition after the beginning of the studied time will happen;
• final backward will take into account the time in which the last transition before the end of the considered time interval is done;
• final forward permits to consider the time in which the system will exit from the state occupied at the final time.

In the credit risk problem a complete knowledge of the duration inside the states is of fundamental importance. The use of the initial and final backward and forward processes gives us the possibility to construct all the waiting time scenarios that could happen in the neighbours of the initial and final observation times. In fact these kind of processes give different transition probabilities depending on the different backward and forward values.

2 Non-homogeneous semi-Markov Processes

In this section the main results regarding discrete time non-homogeneous semi-Markov processes are reported; the notation adopted follows that of [Çinlar, 1975].
Let $E = \{1, \ldots, m\}$ be the state space and let (Ω, F, P) be a probability space. Let us also define the following random variables:

$$X_n : \Omega \to E, \quad T_n : \Omega \to \mathbb{N},$$

where X_n represents the state at the *n-th* transition and T_n represents the time of the *n-th* transition.

(X, T) is called *"non-homogeneous Markov renewal process"*. The associated *semi-Markov kernel* Q is defined by:

$$Q_{ij}(s,t) = P\left[X_{n+1} = j,\, T_{n+1} \le t \,|\, X_n = i,\, T_n = s\right] \tag{1}$$

We denote with

$$b_{ij}(s,t) = P\left[X_{n+1} = j,\, T_{n+1} = t \,|\, X_n = i,\, T_n = s\right] \tag{2}$$

The following probabilities are of interest:

$$p_{ij}(s) = \lim_{t\to\infty} Q_{ij}(s,t) \quad i, j \in E \tag{3}$$

$$H_i(s,t) = P\left[T_{n+1} \le t \,|\, X_n = i,\, T_n = s\right] = \sum_{j=1}^{m} Q_{ij}(s,t)$$

$$G_{ij}(s,t) = P\left[T_{n+1} \le t \,|\, X_n = i,\, X_{n+1} = j,\, T_n = s\right] = \begin{cases} \frac{Q_{ij}(s,t)}{p_{ij}(s)} & \text{if } p_{ij}(s) \ne 0 \\ U_1(t) & \text{if } p_{ij}(s) = 0. \end{cases}$$

where $U_1(t) = 1 \ \forall t \geq 0$.

The non-homogeneous semi-Markov process Z is defined as $Z(t) = X_{N(t)}$ where $N(t) = max\{n \in \mathbb{N} | \mathrm{T_n} \leq \mathrm{t}\}$.

Its transition probabilities are defined and obtained in the following way:

$$\phi_{ij}(s,t) = P\left[Z(t) = j | Z(s) = i\right],$$

$$\phi_{ij}(s,t) = (1 - H_i(s,t))\delta_{ij} + \sum_{l \in E} \sum_{\tau = s+1}^{t} \phi_{lj}(\tau, t) b_{il}(s, \tau). \tag{4}$$

Given the M.R.P. (X_n, T_n) we define the following processes:

$$B(t) = \begin{cases} t + T_0 & \text{if } t < T_1 \\ t - T_{N(t)} & \text{if } t \geq T_1 \end{cases} \quad ; \quad F(t) = T_{N(t)+1} - t.$$

they are also called auxiliary processes.

The more general distributions of these processes joint with the semi-Markov one is

$${}^{bf}\phi_{ij}^{bf}(l,s,u;l',t,u') = P[Z(t) = j, B(t) = t - l', F(t) = u' - t | Z(s) = i, B(s) = s - l, F(s) = u - s]$$

${}^{bf}\phi_{ij}^{bf}(l,s,u;l',t,u')$ denotes the probability to be at time t in state j with the entrance in this state at time l' and next transition occurring at time u' given that at time s the process was in state i but it entered in that state at time l and it remained there until time u.

It results that

$${}^{bf}\phi_{ij}^{bf}(l,s,u;l',t,u') = \sum_{k \in E} \frac{dQ_{ik}(l,u)}{dH_i(l,u)} \cdot \phi_{ij}^{bf}(u;l',t,u')$$

where $q_{ik}(l,u) = \frac{dQ_{ik}(l,u)}{dH_i(l,u)}$ is the Radon-Nikodym derivatives of Q_{ik} with respect to H_i and

$$\frac{dQ_{ij}(s,x)}{dH_i(s,x)} = P[X_{n+1} = j | X_n = i, T_n = s, T_{n+1} = x] = \frac{b_{ik}(s,x)}{\sum_{k \in E} b_{ik}(s,x)}$$

whereas $\phi_{ij}^{bf}(u;l',t,u') = P[Z(t) = j, B(t) = t - l', F(t) = u' - t | Z(s) = i] =$ satisfy the following system of recursive equations:

$$\phi_{ij}^{bf}(s;l',t,u') = \delta_{ij} \sum_{m \in E} b_{im}(s,u') 1_{\{l'=s\}} + \sum_{m \in E} \sum_{\tau = s+1}^{l'} b_{im}(s,\tau) \phi_{mj}^{bf}(\tau;l',t,u')$$

3 The semi-Markov reliability credit risk model

The credit risk migration problem can be situated in the reliability environment. The rating process, done by the rating agency, gives a reliability's degree of a bond issued by a firm.

In the Standard & Poors case there are 8 different classes of rating that means to have the following set of states:

$$I = \{AAA, AA, A, BBB, BB, B, CCC, D\}.$$

To take into account the downward problem we introduce other 6 states. The set of states becomes the following:

$$I=\{AAA,AA,AA-,A,A-,BBB,BBB-,BB,BB-,B,B-,CCC,CCC-,D\}.$$

The first 13 states are working states (good states) and the last one is the only bad state. The two subsets are the following:

$$U=\{AAA,AA,AA-,A,A-,BBB,BBB-,BB,BB-,B,B-,CCC,CCC-\},\ D=\{D\}.$$

The downward problem (if a firm got a lower rating then has a higher probability that next rating will be lower then the preceding one) is considered using the state space expansion, in fact for example splitting the state B in $B-$ and B the system will be in state B if it is arrived at that rating class from a lower rating, instead it will be in the state $B-$ if it is arrived in that state from a better rating (a downward transition).

The following results are of interest:

• $1 - H_i(s,t) = 1 - \sum_{k \in E} Q_{ik}(s,t)$, that represents the probability that from the time s up to the time t no one new rating evaluation was done for the firm.

$1 - H_i(l, s, t)$ give the same probability under the condition that we entered in the current state i at time l, then $B(s) = s - l$.

• $\varphi_{ij}(s,t) = P[X_{n+1} = j | X_n = i, T_{n+1} > t, T_n = s]$ represents the probability to get the rank j at next rating if the previous state was i and no one rating evaluation was done from the time s up to the time t.

In this way, for example, if the transition to the default state is possible and if the system doesn't move for a time t from the state i, we know the probability that, in the next transition, the system will go to the default state.

It results

$$\varphi_{ij}(s,t) = \frac{p_{ij}(s) - Q_{ij}(s,t)}{1 - H_i(s,t)}$$

note that $\varphi_{ij}(u, s, t) = \varphi_{ij}(s, t)$ being $B(s) = s - s = 0$.

• $\phi_{ij}(s,t)$, that represents the probabilities to be in the state j after a time t starting in the state i at time s. These results take into account the duration problem, the non-homogeneity and the downward. Algorithms to solve the system 4 are well known in the literature, see for example [Janssen and Manca, 2006].

• The dependence of the transition probabilities on the duration inside last visited state can be explicitly considered by means of the probabilities

${}^{b}\phi_{ij}(l,s;t)$.

$$ {}^{b}\phi_{ij}(l,s;t) = \delta_{ij}\frac{1-H_i(s-l,t)}{1-H_i(s-l,s)} + \sum_{k\in E}\sum_{\tau=s+1}^{t}\frac{b_{ik}(s-l,\tau)}{1-H_i(s-l,s)}\phi_{kj}(\tau,t) $$

Very often the rating agencies anticipates the idea to modifies the rating of a specific rated firm, then we can incorporate personal beliefs on the time of next transition and conditionally on that event to compute the transition probabilities. To do so we use the probabilities ${}^{bf}\phi_{ij}(l,s,u;t) = \frac{\sum_{k\in E} b_{ik}(l,u)\phi_{kj}(u,t)}{\sum_{k\in E} b_{ik}(l,u)}$.

By means of ${}^{bf}\phi_{ij}^{bf}(l,s,u;l',t,u')$ all waiting time scenarios in the neighbours of initial and final times.

• Let denote by $\tau_i(s) = inf\{h > s : Z(h) = D\}$. The reliability function is defined as the probability to be always in up states that is $R_i(s,t) = P\{Z(h) \in U, \forall h \in \{s+1, s+2, ..., t\} | Z(s) = i\} = \sum_{j\in U}\phi_{ij}(s,t)$, that represents the probability that the system never goes in the default state from the time s up to the time t.

The Reliability function changes because of the duration inside the state occupied. In this light it is possible to define and obtain

$$ {}^{b}R_i(l,s,t) = P\{Z(h) \in U, \forall h \in \{s+1, s+2, ..., t\} | Z(s) = i, B(s) = l\} = \sum_{j\in U} {}^{b}\phi_{ij}(l,s,t) $$

The news of a possible new rating assessment changes the reliability values too. Then we define and obtain

$$ {}^{bf}R_i(l,s,u;t) = P[Z(h) \in U, \forall h \in \{s+1, ..., t\} | Z(s) = i, B(s) = s-l, F(s) = u-s] = $$

$$ \sum_{j\in U} {}^{bf}\phi_{ij}(l,s,u;t) $$

• It is possible to generalize the reliability function in another direction too. Let $\{D(t)\}$ be a time varying barrier denoting the set of down states in function of time. We define by $\tau_i(s;D) = inf\{h > s : Z(h) = D_h\}$. The generalized reliability function is defined as the probability to be always in up states that is ${}^{g}R_i(s,t) = P\{Z(h) \in U(h), \forall h \in \{s+1, s+2, ..., t\} | Z(s) = i\}$. That represents the probability that the system never goes below the time varying barrier from the time s up to the time t. This probability can have some interest in the migration problem because very often there are financial institutions and investors that have some constraints expressed by rating values on the possibility to invest in bond and other financial objects. As a consequence these investors can be interested in bonds that for example for the first x years are very reliable (we consider all classes below BBB as defaulting), for other y years the bonds have never to fall below rating B and so on.

The generalized reliability can be evaluated as follows:
let consider a time varying barrier $D(t)$ and a semi-Markov process $Z(t)$ with kernel $b(s,t)$. Its generalized reliability is given by

$$ {}^{g}R(o,\gamma) = [\underline{\alpha}, 0] * P^{w}_{EE}(0,\gamma) * \mathbf{1_E} \tag{5} $$

where $P^{w}_{EE}(0,\gamma)$ are the transition probabilities of a DTNHSMP $W(t)$ with state space $\overline{E} = E \bigcup \{\Delta\}$ with kernel

$$ b^{w}(s,t) = b^{y}(s,t) * A(s,t) + diag\{1 - H_i(s,t)\} * \overline{C}(s,t) \tag{6} $$

with $b^{y}(s,t) = \begin{pmatrix} b(s,t) & \underline{0} \\ \underline{0} & 1 \end{pmatrix}$ $A(s,t) = \prod_{h=s}^{t} a(h)$ with $a(h) = \begin{cases} 1 \text{ if } i \in U(t),\ j = i \\ 1 \text{ if } i \in D(t),\ j = \Delta \\ 0 \text{ otherwise} \end{cases}$

$$ \overline{C}(s,t) = \begin{pmatrix} \mathbf{0} & C(s,t) \\ \underline{0} & 0 \end{pmatrix} \quad C(s,t) = A(s,t-1) * [a_{1\Delta}(t), a_{2\Delta}(t), ..., a_{\Delta\Delta}(t)]^{t} \tag{7} $$

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