SOCIETE GENERALE
Cross Asset Research

27 November 2009

Fixed Income

Quantitative Special

www.sgresearch.socgen.com

Filtering the interest rate curve

The MENIR framework

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■ In this report we introduce an innovative framework for forecasting the interest rate term structure dynamic. Our approach is based on a multi-factor affine model of the interest rate curve. The model is analogous to the Nelson-Siegel curve specification and is estimated using a standard though improved technique from the engineering and econometric world, namely the Kalman filter and the maximum likelihood estimation. We refer to our model as MENIR, short for Multi-factor Econometric Nelson-Siegel model for Interest Rates.

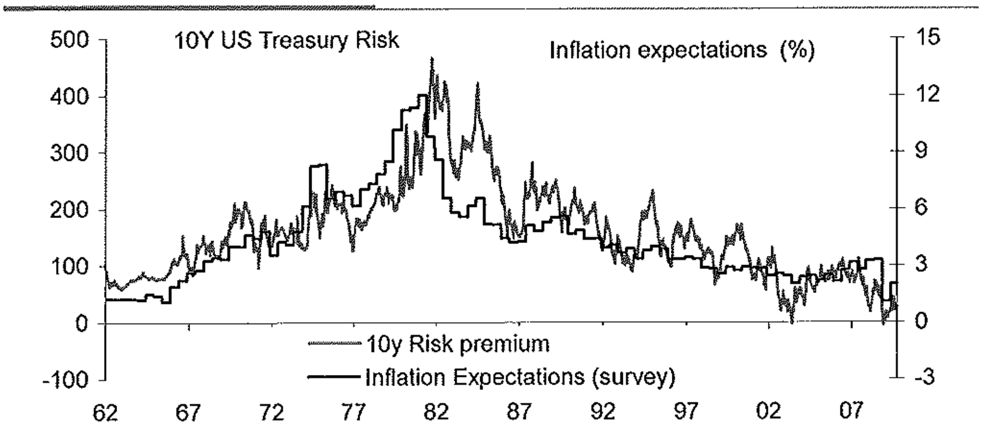
■ This innovative framework is a powerful instrument for a variety of applications in the field of interest rate analysis. As well as quantifying the underlying and optimal dynamics for an interest rate curve, it can also be used as a tool to identify investment opportunities, estimate interest rate risk premia, or extrapolate rates on very long maturities in markets where the instruments do not exist.

■ Such high dimensional models are often known to produce unstable results. We solve the classical stability issue by using an original implementation of the Kalman filter standard engineering technique.

■ Our model shows that the US treasury risk premium is at a 50-year historical low. The 1960s were characterised by low inflation and high productivity, similar to some extent to the situation today. The risk premium in US treasuries seems lower today compared to this period. This can arguably be attributed to foreign central bank purchases of US treasuries.

Contents

US treasury risk premium at 50-year lows



Source: SG Cross Asset Research

Overview

We introduce **MENIR**, our Multi-factor Econometric Nelson-Siegel model for Interest Rates.

This report presents an innovative framework for estimating the interest rate term structure dynamic based on a multi-factor affine model of the interest rate curve. The model is estimated using a standard though improved econometric technique and is analogous to the Nelson-Siegel specification. We refer to our model as **MENIR** (**M**ulti-factor **E**conometric **N**elson-Siegel model for **I**nterest **R**ates).

The Nelson-Siegel model gained popularity in the late 1980s as a flexible way of fitting an interest rate curve. Many practitioners appreciated the model for its simplicity and flexibility: it could be used to fit a wide range of interest rate curve shapes and the numerical procedures involved were easy to implement. However, the choice of the shape of the curve in this model is purely arbitrary. As such, there is no guarantee the model corresponds to a realistic diffusion of the underlying short rate and that the no-arbitrage argument is fulfilled.

An alternative approach would be to use a complicated stochastic model for the short rate and try to calibrate it to the market at a given moment in time. This produces two issues: first, the estimation procedure can be numerically involved and technically more difficult to implement. Second, it can be unstable, leading to swings in the model's implied parameters. This is not a desirable property when the model is to be used for hedging purposes, as the model's hedging performances will be poor.

Combining the Nelson-Siegel specification of the curve and a stochastic model provides a consistent and integrated framework.

In this work, we propose combining the two approaches into a single framework. Following the example of J. Christensen, F. Diebold and G. Rudebusch [2], we define a stochastic model of the short rate consistent with the Nelson-Siegel analytical specification of the interest rate curve. This means that we take the best of both approaches: i.e., we use a flexible analytical shape for the curve and combine it with a stochastic framework to make it arbitrage-free and consistent with the short-rate diffusion.

This still leaves us with the issue of the accuracy and stability of the estimated model parameters. In constructing our framework the risk is to obtain either an over-parameterised model or one which lacks constraints, in which case there will be numerous solutions for the model parameters, leading to unstable estimations. We solve the stability issue by using a smart implementation of a Kalman filter, based on a better estimation of the mean reversion parameters. We provide details on both the Kalman filter technique and our implementation on page 8-9.

This report is organised as follows: in the first part we present the **MENIR** framework, the intuition behind it and the underlying dynamics; in the second part, we detail the estimation procedure, namely the Kalman filter and our choice of implementation; in the last part, we present numerical examples and potential applications in strategy, including the evaluation of risk premia and curve extrapolation.

MENIR in perspective

The Nelson-Siegel model

MENIR is based on a specification of the yield curve described by Nelson-Siegel in the late 80s.

MENIR is based on the standard and classic Nelson-Siegel model for the yield curve [9]. This model gained popularity in the late 80s as a flexible way of fitting an interest rate curve. In its original form, it assumes the zero-coupon rate - with τ years to maturity - is the weighted sum of three factors or **latent variables** (see equation (1) below):

Three latent variables explain the absolute level, the slope and the curvature of interest rates.

- The first latent variable (x_1) is the limit of the zero-coupon rate, as the time-to-maturity τ goes to infinity. As such, this is the long-term level of interest rates. No particular weight is applied to this latent variable. A change in this variable affects all zero-coupon rates with different maturities equally.

- The second latent variable (x_2) represents the slope of the curve. When time-to-maturity τ tends towards 0, it also equates the difference between the short-term zero-coupon rate and the long-term level. As such, it can also be defined as the short to long-term spread. It is weighted by an analytical expression which depends on time to maturity. This analytical expression is designed such that a change in this latent variable will affect the short-term end of the curve more than the long end.

- The third latent variable (x_3) describes the curvature of the curve. It is weighted by another analytical expression, designed such that a change in the third latent variable will affect the body of the curve more than the short or long end of the curve.

Graph 1 below shows the weights or *loading factors* for each latent variable as a function of the time to maturity. The Nelson-Siegel model depends on an additional parameter, denoted by **a** in the equation below. This measures the rate at which the slope and convexity variables decay towards zero: the higher this parameter, the quicker the black and red lines in graph 1 go down to 0.

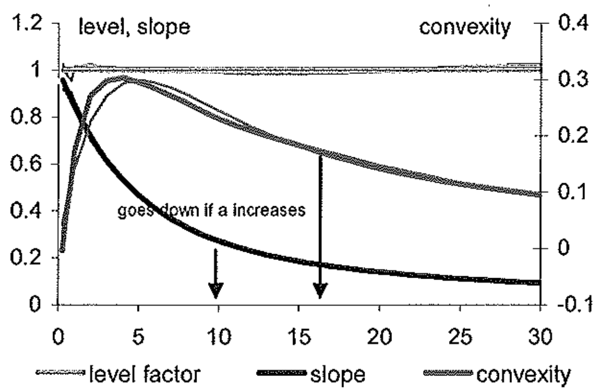
Equation 1: Analytical expression of zero-coupon rate in the Nelson-Siegel model

$$y(t,\tau) = x_1 + x_2 \frac{1-e^{-a\tau}}{a\tau} + x_3\left(\frac{1-e^{-a\tau}}{a\tau} - e^{-a\tau}\right)$$

Source: SG Quantitative Strategy

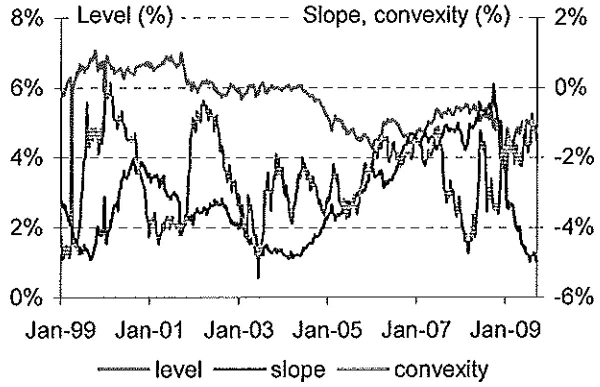
The latent variables **x** are calculated on a daily basis so as to best fit the term structure of the interest rate curve over time. The original procedure makes no further assumptions concerning the distribution of these latent variables; it postulates an ad-hoc value for the mean reversion parameter (a). It then minimises an ordinary least squares model, which finds the best fit for the observed zero-coupon rate in the Nelson-Siegel model (see [5] for an example of implementation). The graph on the right on the next page panel shows the time series for the latent variables obtained using 10-years history of EUR depo and swap rates and with an *apriori* mean reversion parameter of 0.37, corresponding to a half life of a bit less than two years.

Graph 1: Nelson-Siegel loading factors for the three latent variables



Source: SG Quantitative Strategy – Loading factors are calculated either explicitly (bold line), using their analytical expression or statistically, using the regression coefficient of the rates against the time series of the latent variables.

Graph 2: Time series of latent variables in Nelson-Siegel model



The latent variables are estimated in a Nelson-Siegel model where the mean reversion parameter is 0.37. We used 10 years' history of EUR depo and swap rates, from 3-months to 30-year maturity.

Introducing a dynamic model

A dynamic model offers a bridge between the dynamics of the latent variables and the realisation of the interest rates.

While the Nelson-Siegel model is widely used among practitioners, it suffers from a lack of theoretical background: there is no explicit link between the dynamic of the latent variables and zero-coupon rates. F. Diebold and C. Li [2] propose to bridge this gap: they introduce a three-factor, arbitrage-free, mean-reverting model (AFDNS, Arbitrage-Free affine Dynamic Nelson-Siegel model) which allows the Nelson-Siegel pricing equation for the zero-coupon rate to be recovered. This is the approach we chose for **MENIR**.

The intuition behind the calculations proposed by F. Diebold and C. Li is threefold:

- First, the short-term rate can simply be calculated as the sum of the first two latent variables: as previously demonstrated, the first latent variable is the long-term rate and the second one is the difference between short and long-term rates. These two latent variables sum up to the short-term rate.
- Second, the loading factors/weightings of the second latent variable in the expression of the zero-coupon rate are similar to the ones associated with a mean reverting process, with a mean reversion parameter equal to parameter **a.**
- Finally, the third latent variable intervenes in the zero-coupon rate equation and should therefore be introduced into the short rate via the drift of the first or second latent variable.

Defining the diffusion of the latent variables under the risk-neutral measure leads to an analytical specification of the zero-coupon rate consistent with the Nelson-Siegel model.

Consequently, F Diebold and C Li show that if the short-term rate is defined as the sum of the first two latent variables and if the latent variables follow diffusion equations (2), the zero-coupon rate can be written as in equation (3) (see below and Appendix A for the proof). Using this approach, the valuation equation for the interest rate is the same as equation (1) plus an adjustment term (C).

A closer look at the system of equations (2) can tell us much about the implicit assumptions of the Nelson-Siegel model. But before looking specifically at the diffusions themselves, we emphasize that the passage from the dynamics of the latent variables to the pricing equation for the zero-coupon rate relies on the very important assumption that there is no arbitrage opportunity on the bonds themselves. This is a fundamental point to note as it means the diffusions in the system of equations (2) are defined under the **risk-neutral measure.** This is a theoretical, probabilistic measure, under which the price of any tradable asset (such as bonds) increases according to a drift equal to the short-term rate.

At this point, we note that, although the risk-neutral measure is useful theoretically to relate the pricing equation with the diffusion, it does not allow us to capture any kind of risk premium.

Equation 2: From the dynamic of the three latent variables...

$$dx_t^1 = \sigma_1 dw_t^{1,Q}$$
$$dx_t^2 = -a\left(x_t^2 - x_t^3\right)dt + \sigma_2 dw_t^{2,Q}$$
$$dx_t^3 = -ax_t^3 + \sigma_3 dw_t^{3,Q}$$
$$\left\langle dw_t^{1,Q}, dw_t^{2,Q}\right\rangle = \left\langle dw_t^{1,Q}, dw_t^{3,Q}\right\rangle = \left\langle dw_t^{3,Q}, dw_t^{2,Q}\right\rangle = 0$$

Equation 3: ... to the zero-coupon rate pricing equation

$$y(t,\tau) = x_t^1 + x_t^2\frac{1-e^{-a\tau}}{a\tau} + x_t^3\left(\frac{1-e^{-a\tau}}{a\tau} - e^{-a\tau}\right) + C(\tau)$$

Source: SG Quantitative Strategy

Diffusion of the latent variables is based on uncorrelated Brownian motions.

Returning to the interpretation of the system of equations (2), we note the following points:

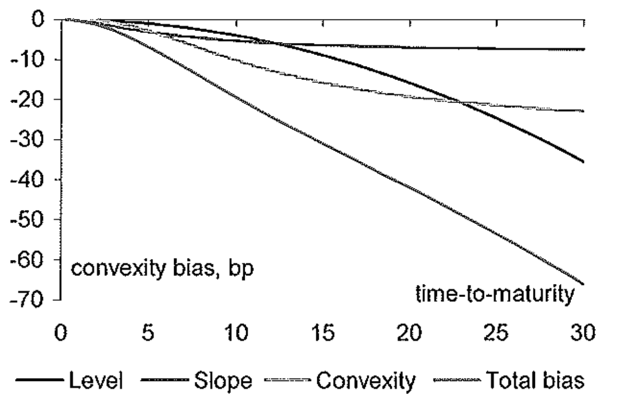
- The first latent variable (long-term rates level) is a simple Brownian motion; the second latent (slope of the curve) mean-reverts to the third latent variable (convexity of the curve) at speed **a**; the third latent variable mean-reverts to 0 at speed **a**.

- The short rate does not depend explicitly on the convexity factor. However, thanks to the specification of the drift term in the equation of the second latent variable, the convexity variable intervenes in the drift-of-the-slope variable. Another way of interpreting this is to say that the slope variable mean-reverts towards a time-varying variable, which is the convexity.

- Lastly, the sources of uncertainty - the Brownian motions - are not correlated. A possible refinement would be to let go of this assumption and to introduce a non-zero correlation between those Brownian motions.

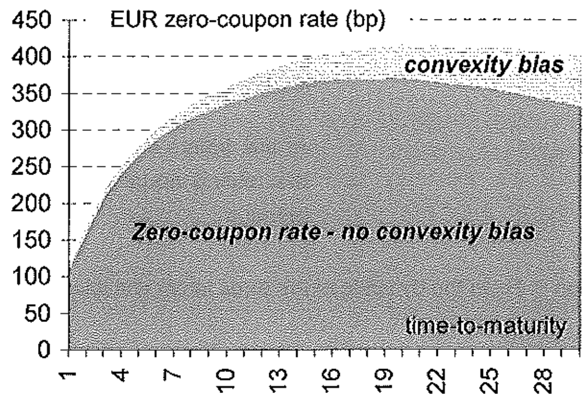
The yield adjustment term (C) in equation (3) depends only on the time-to-maturity and volatility parameters. It is constant over time. It can be split into three components, which correspond to the three volatility parameters in the model. Graph 3 below shows the value of the yield adjustment term for our estimation of the swap EUR curve over a 10-year history (see the following chapter on model estimation, page 8) and divides it into these three components. The yield adjustment term increases along with time to maturity. We estimate that it accounts for 60bp of the total value of the 30-year zero-coupon rate. Graph 4 shows the relationship between the yield adjustment term and the total zero-coupon rate in EUR as of mid-September 2009. Appendix A provides the analytical expression of this yield adjustment term.

Graph 3: Yield adjustment term estimation for EUR swap curve...



Source: SG Quantitative Strategy – Yield adjustment term is estimated on a 10-year history on EUR swap depo and swap rates from 3-months to 30-year maturity. The total bias is calculated as the sum of the level, slope and convexity contributions.

Graph 4: ... accounts for 60bp of long term zero-coupon rates



From theoretical to 'real world' dynamics

Investor preferences are introduced via a risk premium parameter.

We have already defined the diffusion of the latent variables under a theoretical "risk-neutral" measure. In practice, the prices seen in the market are subject to investors' preferences and as such contain a **risk premium**. We have not yet mentioned such a premium in our **MENIR** approach. Measurement of the risk premium in financial assets is an endless subject both for practitioners and academics. We propose including some parameters for the market price of risk so that we can obtain a possible measure of this risk premium in zero-coupon rates.

The market price of risk concept can be explained by taking the example of the stock market. The rate of return on equity is usually the long-term growth rate, which can be estimated. The difference between this long-term growth rate and the short-term rate, divided by the volatility provides a very simplistic measure of the equity market price of risk (see the "*zoom in on the theory*" box below).

Similarly, in bonds and in our MENIR model, we can define a parameter for the market price of risk for each source of uncertainty. Below, we look at how to calculate the risk premium for bonds using this parameter. Please refer to the *SG Cross Asset Research Strategy Special*, March 2006, *Finding value in 50yr sector* [8], for a comprehensive definition of the relationship between this parameter and the risk premium in bond prices in the one-dimensional case.

In the simplest model, the parameters for the market price of risk are assumed to be constant over time. We decided to improve this assumption in our MENIR model: our main intuition is that the market price of risk is not constant over time, as investors' preferences change significantly. We therefore chose these parameters to be a linear function of the latent variables. By doing so, we introduce a large amount of free parameters into our model (12 parameters; see appendix B for more).

The market price of risk parameters are chosen such as the diffusion of the latent variables under the historical measure is tractable.

As a way of restricting the number of parameters to fit, we impose some constraints on the dynamic of the latent variables under the historical probability measure. For reasons of consistency and simplicity, we assume the latent variables under the historical probability measure are independent and follow a Vasicek mean-reverting process. Consequently, the number of additional parameters to fit is restricted to just six: the level and the speed of mean reversion for each latent variable. This means that the total number of parameters to be fitted in MENIR is reduced to ten parameters: this six parameters specific to the distributions under the historical probability measure, the three constant volatilities and the mean reversion **a** introduced in the previous section. Equation 4 below gives the final model for the latent variables in **MENIR** under the historical probability measure, and equation 5 gives the final form for the risk premium associated with each latent variable. We note that specifying the parameters under the historical probability or risk premium parameters is strictly equivalent. We have chosen to specify the parameters under the historical probability measure as it will be easier to interpret in the following:

Equation 4: Diffusion under historical probability measure...

$$dx_t^1 = k_1\left(\theta_1 - x_t^1\right)dt + \sigma_1 dw_t^{1,P}$$
$$dx_t^2 = k_2\left(\theta_2 - x_t^2\right)dt + \sigma_2 dw_t^{2,P}$$
$$dx_t^1 = k_3\left(\theta_3 - x_t^3\right)dt + \sigma_3 dw_t^{3,P}$$
$$\left\langle dw_t^{1,P}, dw_t^{2,P}\right\rangle = \left\langle dw_t^{1,P}, dw_t^{3,P}\right\rangle = \left\langle dw_t^{3,P}, dw_t^{2,P}\right\rangle = 0$$

Equation 5: ... gives value for market prices of risk

$$\lambda_t^1 = k_1/\sigma_1\left(\theta_1 - x_t^1\right)$$
$$\lambda_t^2 = 1/\sigma_2\left(k_2\theta_2 + \left(a - k_2\right)x_t^2 - ax_t^3\right)$$
$$\lambda_t^3 = 1/\sigma_3\left(k_3\theta_3 - \left(a - k_3\right)x_t^3\right)$$

Source: SG Quantitative Strategy

Zoom on the theory: definition of the change of measure

Generally speaking, a change of measure or change of probability is a way of changing the rate of return of a financial asset. More specifically, the risk-neutral measure is defined such that the prices of financial assets under this measure grow according to the short-term interest rate. The change of measure which interests us here is the one which goes from the historical, observed measure to a risk-neutral measure.

If we assume that, using the historical measure, a financial asset (a bond or a stock) follows the diffusion:

$$dS_t = \mu\left(t, S_t\right)dt + \sigma\left(t, S_t\right)dW^P$$

Using the risk-neutral measure, this asset will have the following diffusion:

$$dS_t = r_t dt + \sigma\left(t, S_t\right)dW^Q$$

The change of measure corresponds to the following change of stochastic variables in the diffusion:

$$dW^Q = dW^P + \lambda dt$$

Where $\lambda = \dfrac{\mu\left(t, S_t\right) - r_t}{\sigma\left(t, S_t\right)}$

In mathematical literature, λ is identified as the parameter for the market price of risk. λ can be either constant or time-varying. In MENIR, we chose the market price of risk to be a linear function of the three latent variables. As such, in **MENIR**, the market price of risk is a three-dimensional process.

Screening the model

Smart implementation of the MENIR framework leads to a stable and optimal solution for the latent variables and model parameters.

So far in our presentation of the **MENIR** framework, we have examined the different assumptions and theoretical techniques used to construct an arbitrage-free model consistent with a Nelson-Siegel representation of the curve.

We will now deal with three questions that generally need to be answered when estimating a model statistically: **1)** How to construct an efficient measure of the estimation error? **2)** How to efficiently estimate the measurement error function? **3)** How to separate observation noise – in our case liquidity issue, bid/ask spreads or data-entry errors – from real observation?

The first question is resolved by using a maximum likelihood function and is discussed in the first sub-section below. The second and third questions can be addressed via the Kalman filter technique, which we introduce in the second sub-section below. The main challenge in this kind of estimation is to obtain a stable solution. We find an accurate way to account for the mean reversion in our model which allows us to overcome this issue. Lastly, we explain how **MENIR** offers a good framework for estimating the risk premium embedded in zero-coupon rates.

Estimating the quality of the fit

Maximum Likelihood Estimation (MLE), a standard technique in the econometric world, is used to measure the error between rates observed in the market and those calculated in MENIR.

As described in the previous section, **MENIR** is based on ten different model parameters: the parameter controlling mean reversion under the risk neutral measure, the three volatility parameters and the six parameters describing the diffusion of the latent variables under the historical measure and/or describing the risk premium parameters – these are equivalent. We also need to estimate the realised values of the three latent variables at each time step.

The most common technique for estimating the parameters of a distribution is the maximum likelihood estimation (**MLE**). From an intuitive point of view, the MLE gives the most probable model parameters for the wide range of observed interest rate curves.

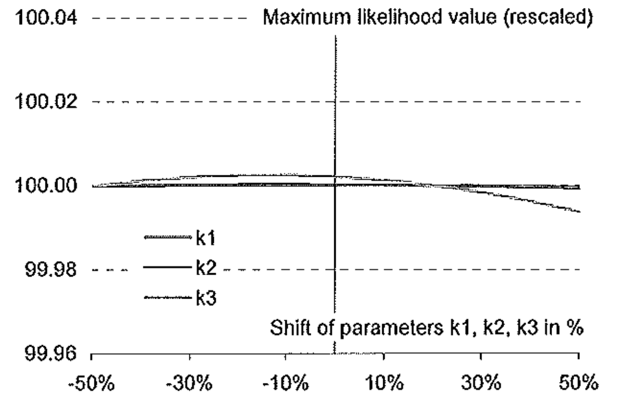
The academic literature says very little about the properties of the maximum likelihood function and its sensitivity to the model parameters. We focus quickly on this point here as it is an important component of our model.

A MLE is usually estimated using only one frequency, typically weekly or monthly data. Intuitively, this will capture the mean reversion corresponding to the sample frequency. But it will be difficult to capture mean reversion corresponding to higher or lower frequencies. We check this intuition empirically by looking at the maximum likelihood function's sensitivity to our three mean reversion parameters when the function is estimated using monthly data. Graph 5 shows how the maximum likelihood function evolves as the different mean reversion parameters change in percentage terms. We present the calculation around the optimal point. It is quite clear from the graph that only the convexity mean reversion has a significant impact on the maximum likelihood function around the optimal point.

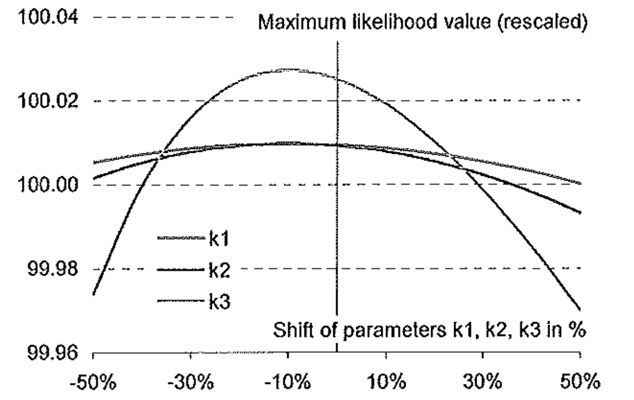
Choice of frequency in the data sampling is fundamental.

In order to solve this issue, we introduce different frequencies into our measure of the maximum likelihood function. Graph 6 shows exactly the same calculation, except that the maximum likelihood function is now calculated using monthly, semi-annual and annual frequencies. The two graphs are on the same scale. It is clear that the optimal point will be much easier to find in the second case. The optimal point is indeed better defined, because the likelihood function is much more concave.

Graph 5: Maximum likelihood functions with only one frequency…



Graph 6: … much flatter than using multiple frequencies



Source: SG Quantitative Strategy

Fast generation of the latent variables: Kalman filter

The maximum likelihood estimation provides a method for finding the most probable parameters for the **MENIR** model. However, because the model depends on non-observable variables, we also need a way to find the most probable realisation for these latent variables. The brute force method would consist in making a global fit and finding the model parameters and the realisation of the latent variable in a single step. However, this is likely to fail to converge, as the dimension of the problem to solve is very high.

The Kalman filter technique is used to generate the optimal path for the latent variables, under the historical measure.

Instead, we used a smart implementation of a standard technique in data analysis called the **Kalman filter**. Kalman filtering is a numerical method based on a state space representation of variables. It is used in a wide range of engineering applications, from computer vision to neural network or control theory. It was first used by the NASA to solve the problem of estimating trajectories for the Apollo programme in the 1960s. We provide a functional description of the algorithm on page 23 (please see appendix C for a technical description). We also refer readers to the *Econometric Analysis of Time Series* by Andrew C. Harvey [6], which contains many insightful and useful details on the implementation.

In **MENIR**, the Kalman filter is used simultaneously to estimate the realisation of the three hidden latent variables, to filter for noise and to calculate the maximum likelihood function. The Kalman filter is in turn used as a black box in an optimisation routine in order to find the optimum parameters for the maximum likelihood function. The numerical routine we use here is a simple gradient descent algorithm.

Most importantly, we use a smart implementation of this algorithm to best fit the mean reversion in our model. As highlighted in the previous section, the choice of the sample frequency is crucial in the calculation of the maximum likelihood function. To take this observation into account, we combine three different frequencies: monthly, semi-annual and annual. This means that instead of using only one Kalman filter to calculate our maximum likelihood function, we use a wide range of different Kalman filters with different frequencies and different starting points, and we then sum up the different maximum likelihood functions.

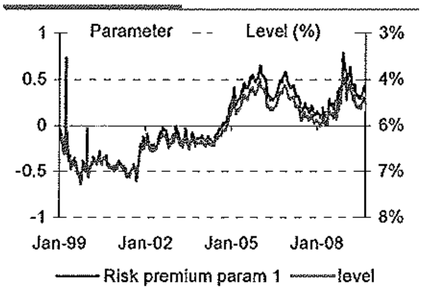
Estimating parameters for market price of risk

The risk premium parameters in our model are linear functions of the latent variables.

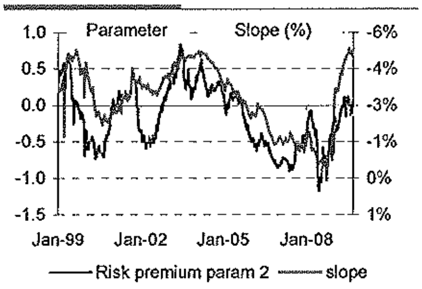
One fundamental point in the estimation process is the difference between risk-neutral and historical probability. In a classic Nelson-Siegel framework, only one parameter, the mean reversion parameter (a) is used to fit the curve term structure. We calculate additional parameters (nine in total), in order to be able to capture the risk premium embedded in zero-coupon rates. The Kalman filter offers a convenient way of distinguishing between the risk-neutral probability used in the measurement equation and the historical probability used in the diffusion of the latent variables.

The graphs below show the variations in the parameters for the market price of risk estimated on EUR swap data. The first and third risk premium parameters are completely linear in function of the slope and convexity latent variables. The second risk premium parameter is well correlated with the slope factor.

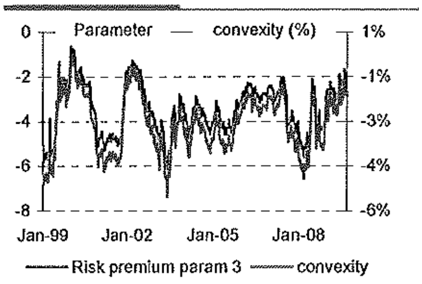
Graph 7: Market price of risk:



Graph 8: a three dimensional process...



Graph 9: ...linked to the latent variables time series



Source: SG Quantitative Strategy

MENIR in numbers: all numerical results

So far we have presented our MENIR framework by introducing the equations for the model and we have explained how we used a smart implementation to obtain a stable estimation. In particular, we showed that MENIR allows us to distinguish between the theoretical risk-neutral measure and the "real world" measure. We also showed that choosing an adequate frequency in the data sampling is of critical importance for stabilising the algorithm.

In this section, we present the numerical results we obtained in MENIR using various data sets both in the US and in the eurozone.

We now focus on the various results we obtain from this framework. We first screen the various data sets we tested the model on, in both EUR and USD and look at the quality of the fit. We then examine the long-term rates risk premium estimated in our model. We show that Europe is lagging the US in terms of post-crisis risk premium adjustment. We also briefly illustrate how the model can be used as an extrapolation tool, especially for a very long-term bond newly issued by a sovereign. Finally, we show how our MENIR framework can be improved, notably by introducing a stochastic volatility.

Quality of the fit

We tested MENIR on four sets of data: swap data in EUR, French government bond data (OAT), and swap data in USD and US Treasuries. For the swap data, we use a bootstrapping procedure to transform swap and depo rates into zero-coupon rates.

Table 1: Model parameters for bond and swap data in USD and EUR

	EUR Swap	EUR Bond	USD Swap	USD bonds
Half life - Risk neutral	1.95	1.92	1.47	1.64
Mean rev. level - F1	6.0%	5.6%	6.6%	6.6%
Mean rev. level - F2	-2.8%	-2.9%	-2.9%	-3.5%
Mean rev. level - F3	-2.2%	-1.7%	-2.2%	-2.5%
Half life (year) - F1	3.17	4.25	4.30	1.48
Half life (year) - F2	0.88	1.02	0.95	0.63
Half life (year) - F3	0.29	0.29	0.34	0.57
Volatility - F1	0.5%	0.3%	0.5%	0.6%
Volatility - F2	1.6%	1.5%	2.4%	3.4%
Volatility - F3	2.4%	2.2%	2.8%	3.0%
Maximum likelihood (weekly)	104022	90817	119693	124308

Source: SG Quantitative strategy - Parameters estimated using a 10-year history of EUR or USD swap depo and swap rates from 3-months to 50-year maturity and on OAT or US Treasuries.

Model parameters are consistent with the intuition.

The parameters found in the Kalman filter estimation in all cases are given in the above table. These parameters are consistent with the intuition

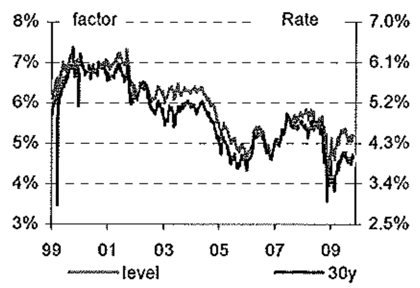
- The first factor represents the level of rates. Its mean reversion level is the long-term level for interest rates. It has the longest half life and the least volatility. This means this factor is very stable over time. Its half life of about 5 years is of the same order of that of an economic cycle. All in all, the first factor is very persistent and stable.

- The second factor represents the slope of the curve and has a half life of just 1 year. Its long-term level is -280bp for swaps compared with -290bp for bonds in EUR and -290bp for swaps versus – 350bp for bonds in USD.

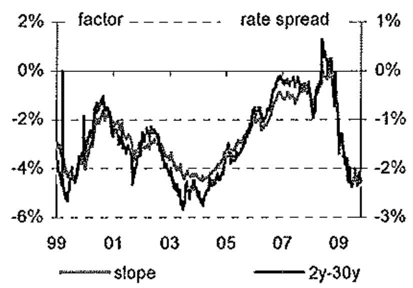
- The third factor represents the convexity of the curve. Its life cycle is very short (under 3 months) and it has the highest volatility. It is the least persistent of the three factors and the quickest to mean-revert. This is interesting from a trading point of view, as one can clearly play the mean reversion of this factor.

The three graphs below show the three latent variables for the EUR swap data compared with the most common rate, spread and butterfly. This visualizes the intuition explained in the first place on the interpretation of the factors.

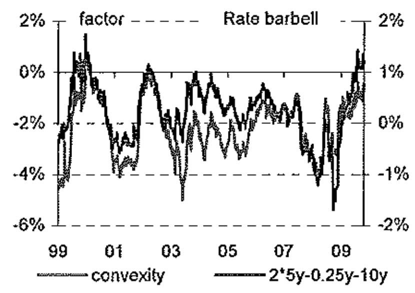
Graph 10: First factor linked to level...



Graph 11: ...Second to slope...



Graph 12: ...And third to convexity



Source: SG Quantitative Strategy – Nelson Siegel model is estimated in MENIR and estimated using 10-year history with depo and swap rates in EUR from 3-month to 30-year tenor.

In-sample results show the data fit is very similar to those reported in the academic literature.

The in-sample fit we obtain for the data is very close to fits reported in the literature. To measure the quality of the in-sample fit, we use the root mean square error (RMSE). This is the square root of the arithmetic average of the squares of the errors. We also show the average of the errors.

Table 2: Summary statistics of in-sample estimation

	Euro Swap		OAT		USD Swap		USD Treasuries	
	Residual mean (bp)	RMSE (bp)	Residual mean (bp)	RMSE (bp)	Residual mean (bp)	RMSE (bp)	Residual mean (bp)	RMSE (bp)
2Y	-0.50	13.46	0.03	6.76	-0.51	17.57	3.15	8.41
5Y	1.45	5.06	0.48	2.00	3.65	5.29	-2.36	4.79
7Y	0.14	3.40	-0.83	3.62	0.08	5.99	-2.43	3.90
10Y	-1.59	4.96	-0.39	3.86	-3.52	7.34	4.36	6.87
12Y	-1.74	6.04	-3.63	6.15	-3.20	6.34		
15Y	-0.20	6.40	-3.51	7.15	-0.63	4.12		
30Y	0.70	9.51	0.85	8.09	3.17	7.02		

Source: SG Quantitative strategy

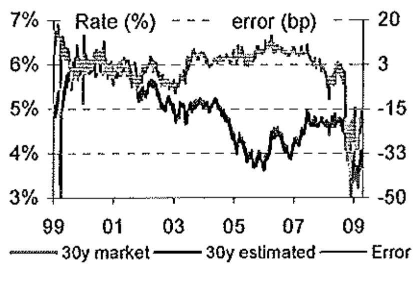
Euro swap data fit over the crisis is not very good, essentially due to structural market flow issues.

Looking at the historical distribution of the errors, as in graphs 13, 14 and 15, is also very insightful: for the EUR swap data, for instance, the fit is very good before the Lehman collapse. However, in the aftermath of Lehman's, the long end of the EUR swap curve came under significant pressure, mainly due to asset managers' unwinding of asset swap packages and hedging flows from the structured desks. The 10y30y segment of the EUR swap curve in particular unexpectedly became inverted, hitting very negative levels (-60bp in year end 2008). MENIR does not work well in such conditions. Over that period, we register the worst in-sample fit, with for example, up to 40bp error on the 30-year swap rate. However, apart from this period, MENIR fits the curve quite well, with estimation errors varying between -6bp and

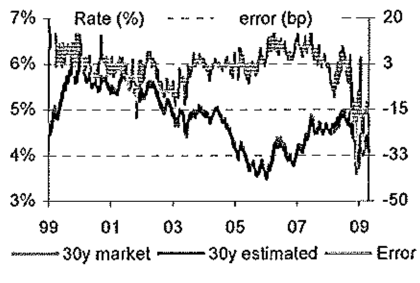
+6bp on the 30-year swap rate, for example. We will see that an adapted volatility process could potentially resolve this issue (page 18).

Conversely, the issue regarding the 30-year rate is less significant for the OAT curve or in USD. For example, the calibration we carried out using 10 years of OAT data shows that the models works better for bonds and dollar swaps, even during the crisis. This highlights that long-term EUR swap rates have suffered from strong, unexpected and unusual market flows, not captured in MENIR.

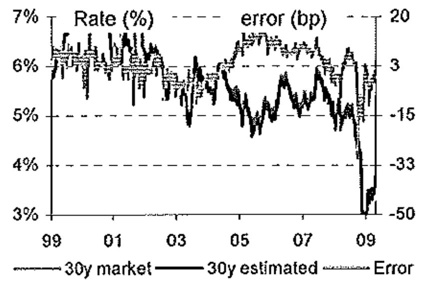
Graph 13: Biased estimation of EUR long-term swap rate during the crisis:



Graph 14: Estimation is better on OAT...



Graph 15: ... and on USD swaps



Source: SG Credit Research

Risk premium estimation in EUR swap data

A risk premium is the extra amount a person requires to be paid in order to agree to take an uncertain bet. This amount is equal to the difference between the expected value and the certain value that this person indifferent to.

Risk premium is defined as the difference between a long-term yield and the average of the short-term yield over the same time period

In the context of interest rate modelling, if there were no risk premium, the long-term interest rate would simply be the average of short rates over the same time window. In other words, investing a given amount in one n-year bond would be equivalent to lending this amount overnight every day over the same time period. In practice, this relationship does not hold and the difference between the two is defined as the risk premium.

In **MENIR**, we define the risk premium as the difference between the long-term interest rate observed in the market and the average of the short rate over the same period. It can also be seen as the difference between the market rate and the rate calculated in our model, assuming that the parameters for the market price of risk are null. Appendix D provides all the details of the calculation.

All calculations being done, the risk premium in our framework is the sum of 4 components:

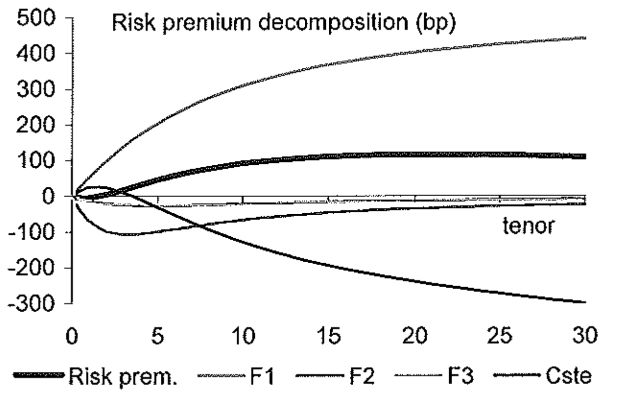
- A term which is constant over time but depends on the tenor of the underlying rate. This is slightly positive for short tenors and strongly negative for longer ones (see graph 16 below);

- A term proportional to the first latent variable. This is the red line on the two graphs below. It monotonously increases with the tenor; it has the highest absolute contribution in the long maturity tenor and is always positive. This is the risk premium due to the overall level of rates: the higher the absolute level of rates, the bigger the premium.

- A term proportional to the second latent variable. This is the risk premium associated with the slope of the curve: the steeper the curve, the higher the premium. It is preponderant in the risk premium in rates in the middle of the curve, i.e. rates from 1- to 10-year maturity.

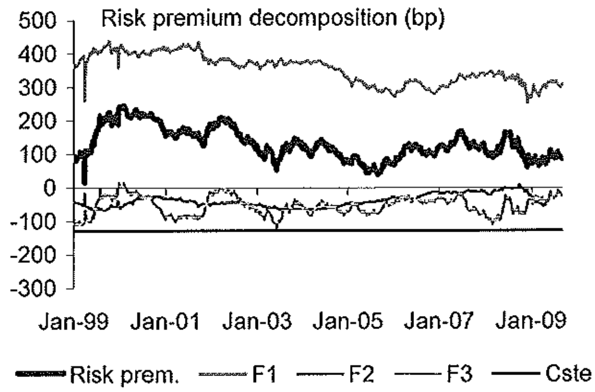
- A term proportional to the third latent variable, namely the convexity factor. This is small across all tenors and volatility is rather low compared with the volatility of the level and slope terms.

Graph 16: Risk premium and various contributions across tenor...



Source: SG Quantitative Strategy . Results are for EUR swap data. Black bold line: total risk premium as of 18 September 2009, across tenors. Red line: first latent variable contribution (F1). Light black line: constant term in the risk premium expression.

Graph 17: ... and over time for the 10y swap rate

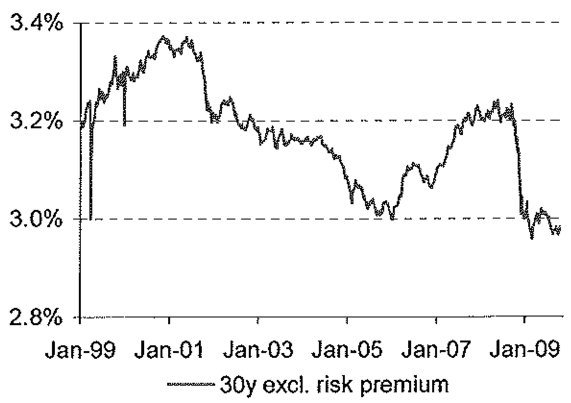


Results are for EUR swap data – Black bold line is the total risk premium for the 10y tenor, since January 2000. Red line is the first latent variable contribution (F1), the light gray line is the third latent variable contribution (F3), the constant term is negligible.

Long term rates dynamics is driven by risk premium dynamics

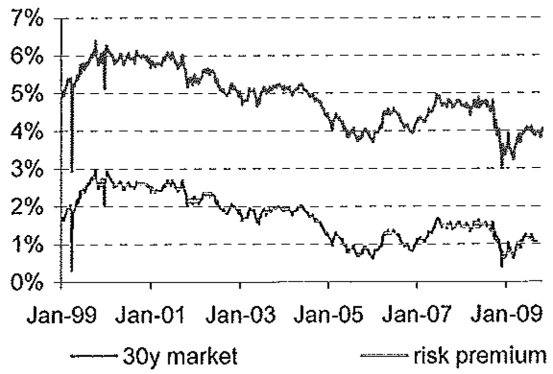
Looking at the total risk premium on long-term zero-coupon rates confirms the standard hypothesis, under which long-term rates are essentially driven by the market-risk premium and long-term expectations are rather stable. In our framework, and for the Euro swap rates for example, we find that 30-year expectations have varied very little since January 2000: they remained bounded between 2.9% and 3.4%, while the 30-year swap rate moved in a 6%-2.8% range. The Lehman collapse had very little impact on the long-term expectation (20bp from end of August 2008 to the lows of December 2009), while the risk premium plunged by over three times by 90bp over the same time period. Our analysis also shows that long-term expectations still haven't quite recovered while the risk premium has started to recover. 30y expectations were anchored at 3.2% prior to the crisis, but they fell to 3% in autumn 2008. They are still oscillating around this level.

Graph 18: Long-term expectations stable...



Source: SG Quantitative Strategy – Results are for EUR swap. The graph shows the 30-year long term, risk-free expectations

Graph 19: ... Long-term rates essentially driven by risk premium

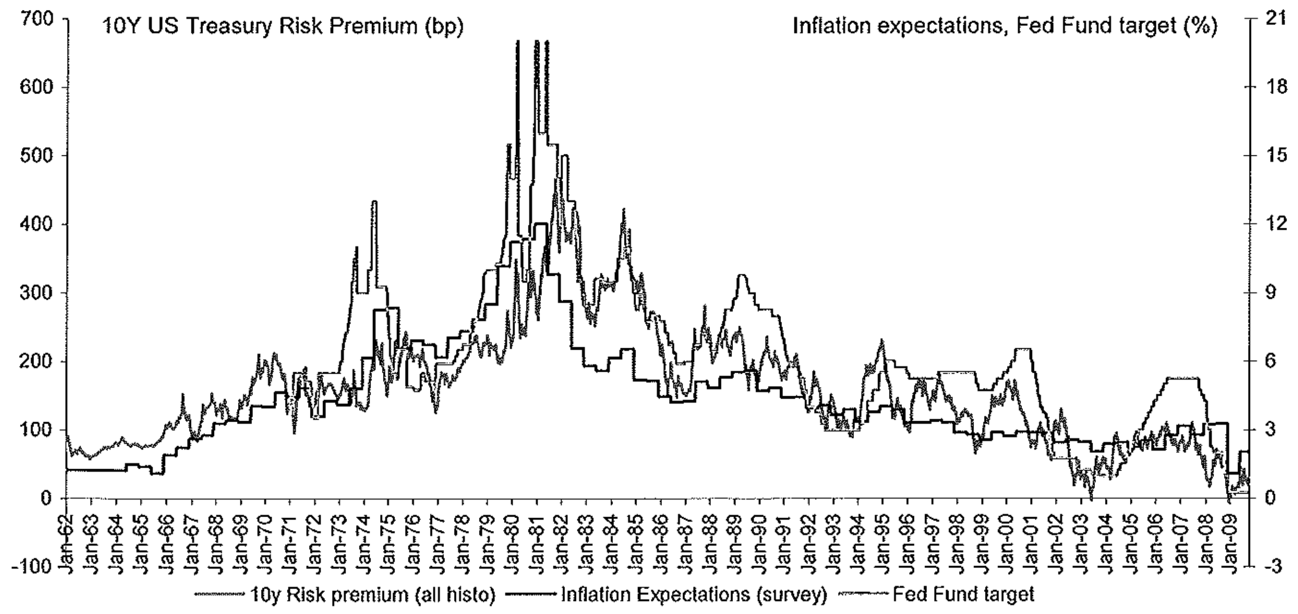


Risk premium estimation in USD bond data

Risk premium in the US have been in a secular downtrend

The fall of term risk premia in US Treasuries has been discussed at length in the literature. Risk premia in US Treasuries have been on a downtrend over the past 20 years. We estimate the 10-year risk premium was 360bps in the early 1990s, while it is now estimated to be 26bp – when the model is estimated on 50 years of history. Following the Lehman bankruptcy, investors sought the safety of treasury notes so much that the 10-year risk premium fell below 0bp. This long-term trend has been arguably attributed to several potential factors, including the better anchoring of inflation expectations, the decrease in real volatility and increased foreign investment in US Treasuries. With our MENIR framework, we look at a longer timescale to see if we can identify the potential driving factor for the long-term risk premium in US treasuries. In our model, the risk premium is correlated with inflation expectations and the Fed funds target. In addition, the following charts shows that the current risk premium levels are among the lowest since half a century. The 1960s was a period of low inflation and high productivity, similar to some extent to the situation today, but without foreign central bank purchases of US Treasuries. The risk premium in US Treasuries seems too low today compared to this period.

Graph 20: US Treasury Risk Premium from 1962 until today



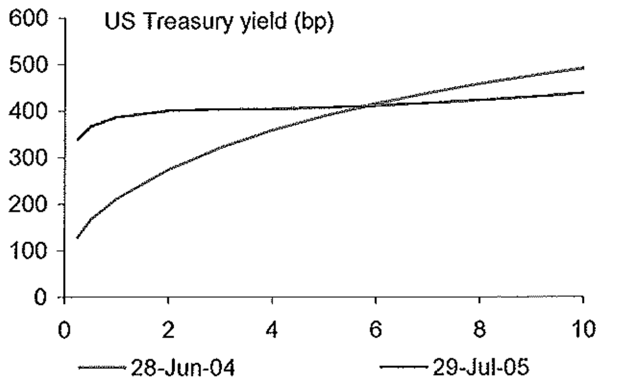
Source: SG Quantitative Strategy

Do we understand the conundrum in MENIR ?

MENIR also illustrates the so-called conundrum in 2004-2006. Over this period, the Fed increased short term rates in an effort to control and to help raise the global level of long term rates. In reality, the inverse happened. As Fed Chairman Alan Greenspan pushed the Fed refi rate further up, long-term rates went down. For example, between end June 2004 and July 2005, the Fed increased the refi rate by 2.25%, from 1% to 3.25%. Meanwhile, long-term US treasury yields decreased by slightly over 50bp, from 4.91% to 4.37%. Graphs 21 and 22 below illustrate this phenomenon: the 10y rate decreased by about 50bp over the period, while the risk premium decreased by 100bp. This means 10y rate expectations increased in reality by 60bp, consistent with the hike in short-term rates.

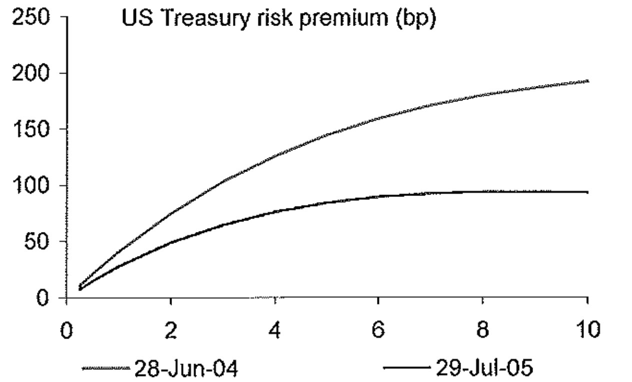
We compared the results of our study with those from the Fed [7]. The results are very similar, albeit different. The main difference between our study and the study by Kim and Wright is the definition of short-term rates: we take the treasury spot yield while they take forward rates from investor survey data. A natural development of our framework in the US would be the inclusion of such survey data.

Graph 21: 10y rates in the US decreased by 54bp



Source: SG Quantitative Strategy – Results are for EUR swap. The graph shows the 30-year long term, risk-free expectations

Graph 22: While 10y risk premium decreased by 90bp



Is volatility really constant over time?

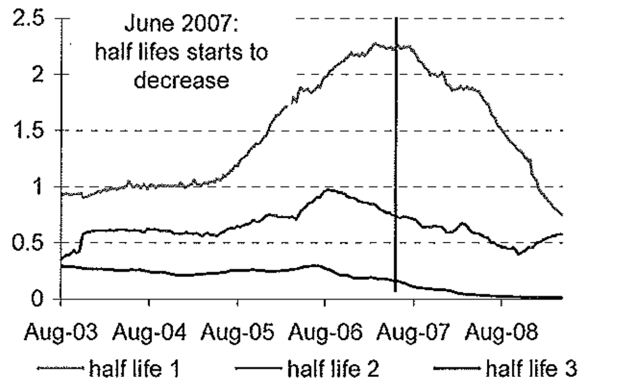
In the previous section, we showed that the in-sample fit we obtain in MENIR for the data is very close to fits reported in the literature. We also noticed that the fit deteriorated over the financial crisis for some data sets.

Constant volatility assumption breaks down in a crisis scenario

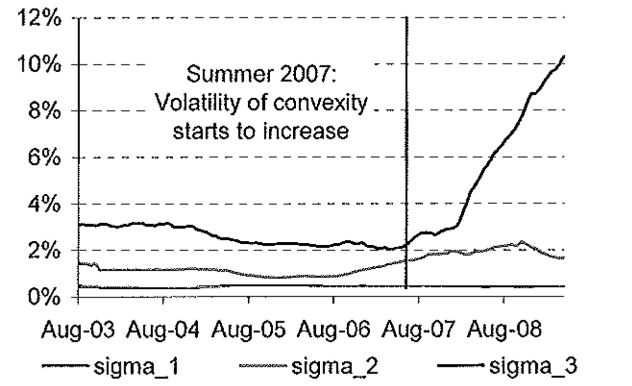
One possible interpretation is that one of the assumptions in the MENIR model is flawed or at least not good enough to efficiently capture the EUR swap curve over the last month of 2008.

The main assumption that springs to mind is the assumption of constant volatility/parameters. To measure the validity of this assumption, we restricted the calibration of MENIR to a smaller sample and calibrated it on a rolling horizon. We used 5-year data and calibrated from August 2003 until August 2009. Results are depicted in the two graphs below: graph 25 shows the evolution of the half life. As the crisis period starts to be taken into account in the calibration, the first factor half life starts to decrease significantly. Similarly, graph 26 shows the change in the volatility of the three latent variables.

Graph 25: Biased estimation of EUR long term swap rate in the crisis...



Graph 26: ... Stability of the volatility parameter



Source: SG Quantitative Strategy

of taking into account the regime shift in volatility would be to include some stochastic volatility in the structure of the factors. This is left for further development.

A brief comment on curve extrapolation

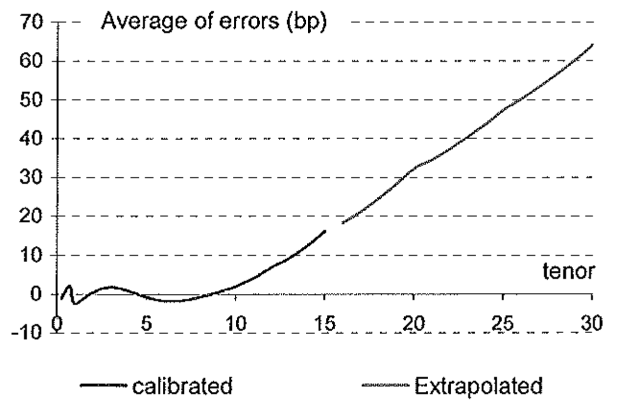
An application to new markets: calculating long-term yields from existing yield term structure

The issue in some markets (in particular emerging markets) is that only the short end of the swap curve is liquid, while the longer points of the curve (10 years and above, for example) may be inexistent. Another issue is if a sovereign wishes to issue a bond with a longer maturity than the other standard benchmarks. The extrapolation is then left to some uncertain ad-hoc assumptions. The model we developed here provides us with a consistent arbitrage-free framework to come up with an extrapolation method. In order to gauge the quality of the extrapolation, we study the case of the EUR swap curve, the EUR bond curve and the USD swap curve. In the three cases, we assume that data is known only up to 10 years. We subsequently look at the quality of the extrapolation for the remaining rates.

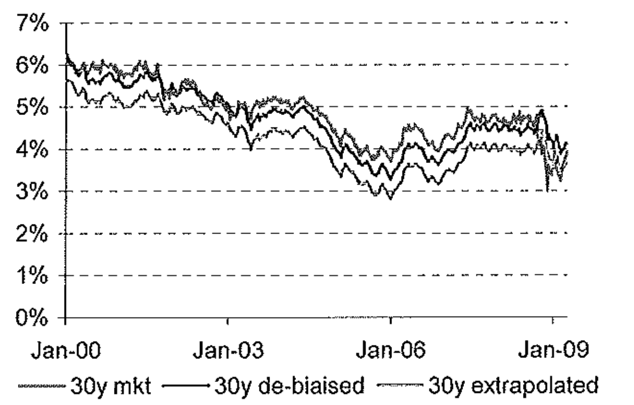
Correcting for a systematic bias improves the yield calculation

In EUR, the calculation exhibits a systematic bias: as an example, we calibrated the EUR swap curve using the first 15 years of the curve. We then use the calibrated model to compute all the rates of maturity above 15 years. We find that the model systematically undervalues long-term rates. Correcting for this bias leads to a much more accurate estimate for long-term rates (see graph 28).

Graph 27: The extrapolated points exhibit a systematic bias



Graph 28: Correcting for this bias gives a much better proxy for long-term rates



Source: SG Quantitative Strategy

Bibliography

[1] David J. Bolder, 2001, *Affine Term-structure models: theory and implementation* – Bank of Canada, working paper

[2] J. Christensen, F. Diebold and G. Rudebusch, *The Affine Arbitrage-Free Class of Nelson-Siegel Term Structure Models*

[3] John Cochrane, Monika Piazzesi, March 2008, *Decomposing the yield curve*

[4] F. Diebold and C. Li, 2006, *Forecasting the term structure of government bond yields*, Journal of Econometrics, Vol. 130, 337-364

[5] Franck J. Fabozzi, Lionel Martellini, Philippe Priaulet, 2007, *Exploiting Predictability in the Time-Varying Shape of the Term Structure of Interest Rates*, Edhec Risk and Asset management Research Center

[6] Andrew C. Harvey, *The Econometric Analysis of Time Series*, the MIT Press March 1990

[7] Don H. Kim, Jonathan H. Wright, *An Arbitrage-Free Three-Factor Term Structure Model and the Recent Behavior of Long-Term Yields and Distant-Horizon Forward Rate*, 2005, Finance and Economics Discussion Series Divisions of Research & Statistics and Monetary Affairs Federal Reserve Board, Washington, D.C.

[8] Adam Kurpiel, March 2006, *Finding value in 50yr sector*, SG Cross Asset Research

[9] Charles R. Nelson, Andrew F. Siegel, 1987, *Parsimonious modeling of yield curves*, Journal of Business, Vol. 60, 473-489

Appendix A – Zero-coupon rate analytics in the AFDNS model

Let's assume that the short rate can be written as $r_t = x_t^1 + x_t^2$ and that the factors x_1 and x_2 are defined by the following diffusion:

$$\begin{pmatrix} dx_t^1 \\ dx_t^2 \\ dx_t^3 \end{pmatrix} = -\begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda & -\lambda \\ 0 & 0 & \lambda \end{pmatrix}\begin{pmatrix} x_t^1 \\ x_t^2 \\ x_t^3 \end{pmatrix} dt + \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}\begin{pmatrix} dw_t^{1,Q} \\ dw_t^{2,Q} \\ dw_t^{3,Q} \end{pmatrix} \tag{A.1}$$

Where $dw_t^{1,Q}, dw_t^{2,Q}, dw_t^{3,Q}$ are independent Brownian motions.

Because the model is assumed to be arbitrage-free, zero-coupon bond prices are constrained to verify the following diffusion:

$$\frac{dP(t,T)}{P(t,T)} = r_t dt + \sigma_P dw^Q \tag{A.2}$$

We look for a solution to this differential equation and express the zero-coupon price as an exponential affine function of the latent variables:

$$P(t,T) = e^{C(t,T) - B_1(t,T)x_t^1 - B_2(t,T)x_t^2 - B_3(t,T)x_t^3} = e^{-(T-t)\times y(t,T)} \tag{A.3}$$

If we denote the time-to-maturity as: $\tau = T - t$, and introduce (A.3) into the (A.2) diffusion, we find that the unknown functions B_1, B_2, B_3 and C in (A.3) should verify the following system of ordinary differential equation:

$$\begin{cases} \dfrac{dB_1}{dt} = -1, \\ \dfrac{dB_2}{dt} = -1 + \lambda B_2, \\ \dfrac{dB_3}{dt} = -\lambda B_2 + \lambda B_3, \\ \dfrac{dC}{dt} = -\dfrac{1}{2} B_1^2 \sigma_1^2 - \dfrac{1}{2} B_2^2 \sigma_2^2 - \dfrac{1}{2} B_3^2 \sigma_3^2 \end{cases} \tag{A.4}$$

with final conditions: $c(T,T) = B_1(T,T) = B_2(T,T) = B_3(T,T) = 0$

The unique solutions to this system are:

$$\begin{cases} B_1(t,T) = T - t \\ B_2(t,T) = \dfrac{1 - e^{-\lambda(T-t)}}{\lambda} \\ B_3(t,T) = \dfrac{1 - e^{-\lambda(T-t)}}{\lambda} - (T-t)e^{-\lambda(T-t)} \end{cases} \tag{A.5}$$

And the expression of the yield adjustment term:

$$\begin{cases} C(t,T) = p_1(T-t) + p_2(T-t) + p_3(T-t) \\ p_1(\tau) = \tau \dfrac{\sigma_1^2 \tau^2}{6} \\ p_2(\tau) = \tau\sigma_2^2 (\dfrac{1}{2\lambda^2} - \dfrac{1 - e^{-\lambda\tau}}{\lambda^3 \tau} + \dfrac{1 - e^{-2\lambda\tau}}{4\lambda^3 \tau}) \\ p_3(\tau) = \tau\sigma_3^2 (\dfrac{1}{2\lambda^2} + \dfrac{e^{-\lambda\tau}}{\lambda^2} - \dfrac{\tau_t e^{-2\lambda\tau}}{4\lambda} - \dfrac{3e^{-2\lambda\tau}}{4\lambda^2} - \dfrac{2(1 - e^{-\lambda\tau})}{\lambda^3 \tau} + \dfrac{5(1 - e^{-2\lambda\tau})}{8\lambda^3 \tau}) \end{cases} \tag{A.6}$$

We recognise the three components of the yield adjustment term, as presented in the first part of this document and as illustrated in graph 3. p_1 depends on the first factor (level) volatility, p_2 depends on the second factor (slope) volatility and p_3 depends on the third factor (convexity) volatility.

Appendix B – Linking the risk-neutral and the historical model with the market price of risk

We have not yet mentioned historical probability. We now introduce the Brownian motion under historical probability. By definition, it relates linearly to the Brownian motion under the risk neutral probability: $dW^Q = dW^P + \lambda dt$

The risk premium parameter can be either constant or time dependant. For a more general set-up, we chose the risk premium parameter to be a linear function of the latent variables:

$$\begin{pmatrix} dw_t^{1,Q} \\ dw_t^{2,Q} \\ dw_t^{3,Q} \end{pmatrix} = \begin{pmatrix} dw_t^{1,P} \\ dw_t^{2,P} \\ dw_t^{3,P} \end{pmatrix} + \left(\begin{pmatrix} \lambda_1^0 \\ \lambda_2^0 \\ \lambda_3^0 \end{pmatrix} + \begin{pmatrix} \lambda_1^1 & \lambda_1^2 & \lambda_1^3 \\ \lambda_2^1 & \lambda_2^2 & \lambda_2^3 \\ \lambda_3^1 & \lambda_3^2 & \lambda_3^3 \end{pmatrix} \begin{pmatrix} x_t^1 \\ x_t^2 \\ x_t^3 \end{pmatrix} \right) dt \tag{B.1}$$

Introducing the expression (B.1) into the diffusion (A.1), we obtain the following expression:

$$\begin{pmatrix} dx_t^1 \\ dx_t^2 \\ dx_t^3 \end{pmatrix} = \begin{bmatrix} \sigma_1\lambda_1^0 \\ \sigma_2\lambda_2^0 \\ \sigma_3\lambda_3^0 \end{bmatrix} dt - \begin{pmatrix} \sigma_1\lambda_1^1 & \sigma_1\lambda_1^2 & \sigma_1\lambda_1^3 \\ \sigma_2\lambda_2^1 & \sigma_2\lambda_2^2 - a & \sigma_2\lambda_2^3 + a \\ \sigma_3\lambda_3^1 & \sigma_3\lambda_3^2 & \sigma_3\lambda_3^3 - a \end{pmatrix} \begin{pmatrix} x_t^1 \\ x_t^2 \\ x_t^3 \end{pmatrix} dt + \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} \begin{pmatrix} dw_t^{1,P} \\ dw_t^{2,P} \\ dw_t^{3,P} \end{pmatrix}$$

This means that, in a very generic definition of the risk premium parameter, we can obtain a wide range of mean-reverting models. For the sake of simplicity, and for consistency with the observed time series of the latent variables (see Diebold and Li [2]), we chose the latent variables under the "real world" measure to be independent and mean-reverting according to the following:

$$\begin{pmatrix} dx_t^1 \\ dx_t^2 \\ dx_t^3 \end{pmatrix} = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix} \begin{pmatrix} \theta_1 - x_t^1 \\ \theta_2 - x_t^2 \\ \theta_3 - x_t^3 \end{pmatrix} dt + \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} \begin{pmatrix} dw_t^{1,P} \\ dw_t^{2,P} \\ dw_t^{3,P} \end{pmatrix} \qquad \text{(B.2)}$$

This completely defines our risk premium parameters:

$$\begin{bmatrix} \lambda_1^0 \\ \lambda_2^0 \\ \lambda_3^0 \end{bmatrix} = \begin{bmatrix} k_1\theta_1 / \sigma_1 \\ k_2\theta_2 / \sigma_2 \\ k_3\theta_3 / \sigma_3 \end{bmatrix} \qquad \text{(B.3)}$$

$$\begin{pmatrix} \lambda_1^1 & \lambda_1^2 & \lambda_1^3 \\ \lambda_2^1 & \lambda_2^2 & \lambda_2^3 \\ \lambda_3^1 & \lambda_3^2 & \lambda_3^3 \end{pmatrix} = \begin{pmatrix} -k_1/\sigma_1 & 0 & 0 \\ 0 & (a-k_2)/\sigma_2 & -a/\sigma_2 \\ 0 & 0 & (a-k_3)/\sigma_3 \end{pmatrix} \qquad \text{(B.4)}$$

Appendix C – Kalman filter in equations

Our Kalman filter implementation is a five-step iterative procedure:

lien paramètres

1. It starts with the initialisation of the latent variables. The state variables are initialised to their expected values in the "real world". Integrating equation (B.2) and calculating the expected values and the variance of the state variables leads to the following result:

$$E[x_0^1 \quad x_0^2 \quad x_0^3] = [\theta_1 \quad \theta_2 \quad \theta_3] \qquad \text{(C.1)}$$

$$Var\left[x_0^1\right] = \frac{\sigma_1^2}{2k_1}, Var\left[x_0^2\right] = \frac{\sigma_2^2}{2k_2}, Var\left[x_0^3\right] = \frac{\sigma_3^2}{2k_3} \qquad \text{(C.2)}$$

2. Step 2 consists in forecasting the state variables. According to equation (B.2), the state variables can be calculated in discrete time according to the following equation (the **state equation**):

$$x_{t+dt}^i = \theta_i(1 - e^{-k_i dt}) + e^{-k_i dt} x_t^i + \varepsilon_{dt}^i, i = 1,2,3 \qquad \text{(C.3)}$$

discrétisation exacte modèle Vasicek

With $E\left[\varepsilon_{dt}^i\right] = 0, Var\left(\varepsilon_{dt}^i\right) = \frac{\sigma_i^2}{2k_i}(1 - e^{-2k_i dt})$

According to equation (4), the expectation of zero-coupon rates is given by the following equation (the **measurement equation**):

$$E\begin{bmatrix} y(t,T_1) \\ y(t,T_2) \\ \vdots \\ y(t,T_n) \end{bmatrix} | F_t \Bigg] = \begin{bmatrix} C(T_1-t) \\ C(T_2-t) \\ \vdots \\ C(T_n-t) \end{bmatrix} + \begin{bmatrix} 1 & \frac{1-e^{-a(T_1-t)}}{a(T_1-t)} & \frac{1-e^{-a(T_1-t)}}{a(T_1-t)} - e^{-a(T_1-t)} \\ 1 & \frac{1-e^{-a(T_2-t)}}{a(T_2-t)} & \frac{1-e^{-a(T_2-t)}}{a(T_2-t)} - e^{-a(T_2-t)} \\ \vdots & \vdots & \vdots \\ 1 & \frac{1-e^{-a(T_n-t)}}{a(T_n-t)} & \frac{1-e^{-a(T_n-t)}}{a(T_n-t)} - e^{-a(T_n-t)} \end{bmatrix} \begin{bmatrix} x_t^1 \\ x_t^2 \\ x_t^3 \end{bmatrix} + \eta_t \quad \text{(C.4)}$$

$$R = Var(\eta) = \begin{bmatrix} r^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & r^2 \end{bmatrix}$$

The expression of $C(T-t)$ given in equation (A.6). η_t is the vector of measurement error with average 0 and matrix of variance R. The value of r is usually taken as equal to 0.1% and corresponds to the noise linked to the measurement. In our case, this would typically be the occurrence of bad data due to database issues, the issue of timing when measuring zero-coupon rates, and the assumptions made when some data are missing.

3. In turn, in step 3, we can compute the error made on the zero-coupon rates as expressed in equation (C.4). These errors are computed as the difference between the observed zero-coupon rates and the zero-coupon rates calculated in the previous step:

$$\xi_{j,t+dt} = y(t+dt,T_j) - E(y(t+dt,T_j) | F_t),\ j = 1..n \quad \text{(C.4)}$$

According to equation (C.4), these variables are normally distributed, with mean 0 and variance given by:

$$Var(\begin{bmatrix} \xi_{1,t+dt} \\ \vdots \\ \xi_{n,t+dt} \end{bmatrix} | F_t) = Var\begin{pmatrix} y(t+dt,T_1) \\ \vdots \\ y(t+dt,T_n) \end{pmatrix} | F_t \Bigg] = NVar(\begin{bmatrix}\begin{bmatrix} x_{t+dt}^1 \\ x_{t+dt}^2 \\ x_{t+dt}^3 \end{bmatrix}\end{bmatrix} | F_t)N^T + R \quad \text{(C.5)}$$

With $N = \begin{bmatrix} 1 & \frac{1-e^{-a(\tau_{1,t+dt})}}{a(\tau_{1,t+dt})} & \frac{1-e^{-a(\tau_{1,t+dt})}}{a(\tau_{1,t+dt})} - e^{-a(\tau_{1,t+dt})} \\ 1 & \frac{1-e^{-a(\tau_{2,t+dt})}}{a(\tau_{2,t+dt})} & \frac{1-e^{-a(\tau_{2,t+dt})}}{a(\tau_{2,t+dt})} - e^{-a(\tau_{2,t+dt})} \\ \vdots & \vdots & \vdots \\ 1 & \frac{1-e^{-a(\tau_{n,t+dt})}}{a(\tau_{n,t+dt})} & \frac{1-e^{-a(\tau_{n,t+dt})}}{a(\tau_{n,t+dt})} - e^{-a(\tau_{n,t+dt})} \end{bmatrix}$, and $R = \begin{bmatrix} r^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & r^2 \end{bmatrix}$

In the previous equations, N is the projection matrix of the three factors onto the zero-coupon rates. In other words, it gives the linear combination of rates which recovers the value for the three factors.

4. Finally, in **step 4**, the prediction errors (C.5) are used to update the unobservable state variables. The errors are split onto the state variables, following a weighting scheme, corresponding to the inverse of the projection matrix N, including some of the measurement noises:

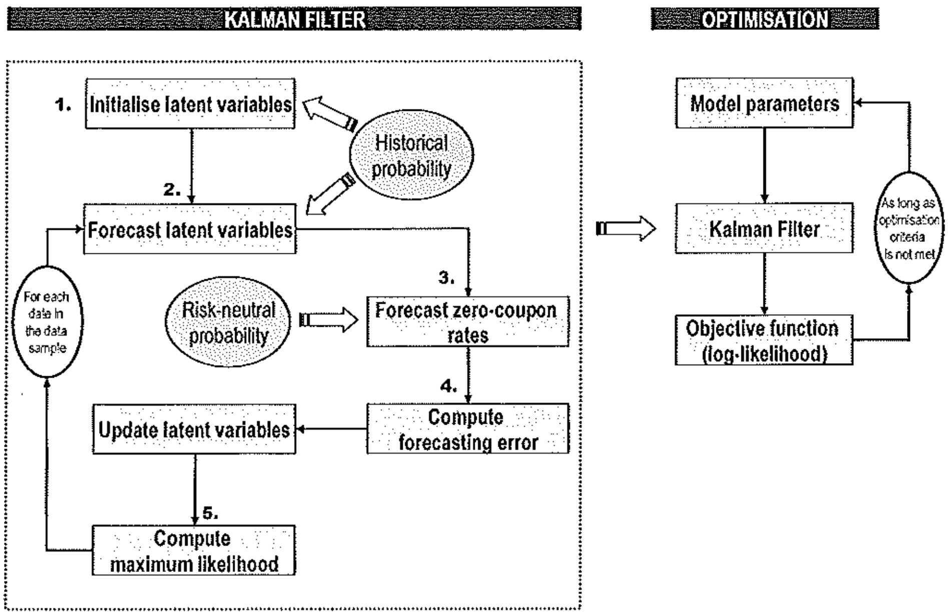
$$E(\begin{bmatrix} x^1_{t+dt} \\ x^2_{t+dt} \\ x^3_{t+dt} \end{bmatrix} \mid F_{t+dt}) = E(\begin{bmatrix} x^1_{t+dt} \\ x^2_{t+dt} \\ x^3_{t+dt} \end{bmatrix} \mid F_t) + K_{t+dt} \begin{bmatrix} \xi_{1,t+dt} \\ \vdots \\ \xi_{n,t+dt} \end{bmatrix} \quad \text{(C.6)}$$

$$K_{t+dt} = Var(\begin{bmatrix} x^1_{t+dt} \\ x^2_{t+dt} \\ x^3_{t+dt} \end{bmatrix} \mid F_t) N^T Var^{-1}(\begin{bmatrix} zcr_1(t+dt, T_1) \\ \vdots \\ zcr_n(t+dt, T_n) \end{bmatrix} \mid F_t)$$

5. We finally compute the maximum likelihood for this date. It is constructed from the conditional variance of the measurement errors and assumes that the errors are Gaussian variables. The final objective function is the sum of objective functions calculated at all dates:

$$\log\_likelihood = \sum_t \ln[(2\pi)^{-n/2} \det{}^{-1/2}(Var(\xi_{t+dt} \mid F_t)) e^{-\frac{1}{2}\xi^T_{t+dt} Var^{-1}(\xi_{t+dt} \mid F_t)\xi_{t+dt}}] \quad \text{(C.7)}$$

MENIR: a five-step iterative process



Source: SG Quantitative strategy

Appendix D – Measuring the risk premium in long term rates: rates under the P-measure

If we consider that there is no risk premium parameter, the model for the three factors is the same under the risk neutral and the historical measure, and we have $dw_t^Q = dw_t^P$. This means the model under the risk-neutral measure becomes similar to the one under the historical measure model (B.2).

$$\begin{pmatrix} dx_t^1 \\ dx_t^2 \\ dx_t^3 \end{pmatrix} = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix} \begin{pmatrix} \theta_1 - x_t^1 \\ \theta_2 - x_t^2 \\ \theta_3 - x_t^3 \end{pmatrix} dt + \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} \begin{pmatrix} dw_t^{1,Q} \\ dw_t^{2,Q} \\ dw_t^{3,Q} \end{pmatrix} \quad \text{(D.1)}$$

Integrating this equation and using the expression of the short rate $r_t = x_t^1 + x_t^2$, we can compute the expression of the zero-coupon rate in this model.

As in appendix A, we assume that the zero-coupon bond price is an affine function of the state variables, and we try to identify the parameters:

$$P(t,T) = e^{C(t,T) - B_1(t,T)x_t^1 - B_2(t,T)x_t^2 - B_3(t,T)x_t^3} = e^{-(T-t)y^*(t,T)} \quad \text{(D.2)}$$

We find that C, B_1, B_2 and B_3 should be solutions of the following system:

$$\begin{cases} \dfrac{dB_1}{dt} = -1 + k_1 B_1 \\ \dfrac{dB_2}{dt} = -1 + k_2 B_2 \\ \dfrac{dB_3}{dt} = k_3 B_3 \\ \dfrac{dC}{dt} = \sum_{i=1}^{3} B_i k_i \theta_i - \sum_{i=1}^{3} \dfrac{1}{2} B_i^2 \sigma_i^2 \end{cases} \quad \text{(D.3)}$$

With the boundary conditions: $c(T,T) = B_1(T,T) = B_2(T,T) = B_3(T,T) = 0$.

The solution is given by:

$$\begin{aligned} B_1(t,T) &= \frac{1 - e^{-k_1(T-t)}}{k_1} \\ B_2(t,T) &= \frac{1 - e^{-k_2(T-t)}}{k_2} \\ B_3(t,T) &= 0 \end{aligned} \quad \text{(D.4)}$$

And:

$$C(t,T)-C(T,T)=\sum_{i=1}^{2}(\theta_i-\frac{\sigma_i^2}{2k_i^2})(B_i(t,T)-(T-t))-\frac{\sigma_i^2}{4k_i}B_i^2(t,T)$$

The risk premium is defined as the difference between the zero-coupon rate as calculated under the risk-neutral probability, as in (A.3), and the zero-coupon rate as calculated under the historical probability, as in (D.4). As a result, the risk premium, RP, is the sum of 4 different terms:

$$RP=y(t,T)-y^*(t,T)=RP_{f1}+RP_{f2}+RP_{f3}+RP_c \tag{D.4}$$

$$RP_{f1}=\left(1-\frac{1-e^{-k_1\tau}}{k_1\tau}\right)x_t^1$$

$$RP_{f2}=\left(\frac{1-e^{-a\tau}}{a\tau}-\frac{1-e^{-k_2\tau}}{k_2\tau}\right)x_t^2$$

$$RP_{f3}=\left(\frac{1-e^{-a\tau}}{a\tau}-e^{-a\tau}\right)x_t^3$$

$$RP_C=\left(C(\tau)+\frac{\tilde{C}(t,T)}{\tau}\right)$$

Note that the values of the 3 factors we use to compute the rates under the historical measure" are the same as the ones used to compute the rate under the risk neutral measure.