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MOMENTS OF PENSION CONTRIBUTIONS AND FUND LEVELS WHEN RATES OF RETURN ARE RANDOM

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1. INTRODUCTION

This paper proposes a simple model for studying the variability of contribution rates and fund levels, when rates of return are random. The funding methods considered are those which (1) produce an actuarial liability (AL) and a normal cost (NC), and which (2) adjust the latter by a constant fraction 'k' of the difference between AL and the actual fund (this discrepancy is known as the 'unfunded liability'; for a description of actuarial cost methods the reader is referred to Trowbridge (1952), Winklevoss (1977) or Turner *et al.* (1984)). Thus at every valuation date

$$\text{total contribution} = \text{normal cost} + k \times (\text{unfunded liability}).$$

If one sets $k = 1/\ddot{a}_{\overline{m}|}$, this may be interpreted as 'spreading' the unfunded liability over a period of m years. It will be assumed that $k \leqslant 1$.

The following assumptions are made.

A1. The earned rates of return ($i(t)$, $t \geqslant 1$) are independent identically distributed ('i.i.d.') random variables, with Prob $(i(t) > -1) = 1$ and $Ei(t)^2 < \infty$. $i(t)$ will designate the rate earned during the period $(t-1, t)$.

A2. All other factors are non-random, e.g. the rate of increase of salaries is assumed known in advance (though it does not have to be constant).

Of these two assumptions, the first one is the most questionable. No suggestion is made that the rates of return actually achieved by pension funds form an i.i.d. sequence. In fact, rates of return are more generally viewed as AR or ARMA processes (e.g. Panjer and Bellhouse (1980), Wilkie (1987)). It is only because it keeps the mathematics tractable that the i.i.d. assumption is imposed here.

The notation used is summarized in §2. In §3 are derived the recursive equations satisfied by the moments of the contributions (C) and fund level (F). These equations are the basis of the analysis set forth in §§4 and 5. Further assumptions are introduced at this point:

A3. All actuarial assumptions are consistently borne out by experience.

A4. The population is stationary (constant membership at every age).

A5. The rate of increase of salaries is nil.

Equivalently, one may imagine that (1) benefits in payment are linked to increases on salaries, and that (2) all monetary amounts relating to time t have been divided by $\Pi_{s=1}^{t}(1+j(s))$, where $j(s)$ is the rate of increase of salaries during the period $(s-1,s)$. In this case, $h(t)=(1+i(t))/(1+j(t))-1$ replaces $i(t)$ in all the formulae; $h(t)$ is seen to be the rate of return on assets above the rate of increase of salaries $(h(t)\doteq i(t)-j(t))$. In §§ 4 and 5 $i(t)$ can therefore be thought of as a 'net' rate of return.

A6. The valuation rate of interest is equal to the mean of the distribution of the earned rates of return, that is to say $i_v=Ei(t)$.

These additional assumptions are not mathematically essential; their use is mainly to simplify the formulae. The interested reader is referred to Dufresne (1986a) and (1986b) for a number of possible generalizations (population only asymptotically stationary, $i_v\neq Ei(t)$, Aggregate method, etc.).

In § 4 the limits, when $t\to\infty$, of the mean and variance of the contributions and fund levels are first calculated. The dependence of these limits on the parameter k is then examined. It turns out that there exists an 'admissible region' for k, outside of which the variance of F and C become unnecessarily high. This region depends solely on the mean and variance of the rates of return.

Section 5 is concerned with the same problem, but for finite durations. Similar conclusions are reached, although an explicit determination of the admissible region now requires numerical methods.

2. NOTATION

a $=(1-k)^2((1+i)^2+\sigma^2)$
AL Actuarial liability
b $=\sigma^2(1+i)^{-2}$
B Benefits paid to members
C Total contribution made $=NC+k(AL-F)$
d $=i/(1+i)$
F Fund
i $=Ei(t)$
i_v Valuation rate of interest
$i(t)$ Rate of return earned on the fund's assets during $(t-1,t)$
k Fraction of unfunded liability $(=AL-F)$ which constitutes the adjustment to the normal cost
NC Normal cost
q $=(1+i)(1-k)$
r $=(1+i)(NC-B+kAL)$
y $=E(1+i(t))^2=(1+i)^2+\sigma^2$
$w(t)$ $=(1+i(t))/(1+i)$
σ^2 $=\mathrm{Var}\ i(t)$

3. MOMENTS OF $F(t)$ AND $C(t)$

We are considering funding methods under which the total contribution at time t is

$$C(t)=NC(t)+k(AL(t)-F(t)). \tag{1}$$

$NC(t)$ and $AL(t)$ depend on both the population and the valuation basis adopted at time t. Observe that the adjustment is the same fraction of the unfunded liability, irrespective of whether the latter is positive or negative. In other words, surpluses and deficiencies are handled in exactly the same fashion; this is not always the case in practice.

The basic relationship is

$$F(t+1)=(1+i(t+1))(F(t)+C(t)-B(t)), \qquad t=0,1,2,\ldots \tag{2}$$

Using Equation (1), it becomes

$$\begin{aligned} F(t+1) &= ((1+i(t+1))((1-k)F(t)+NC(t)-B(t)+k\,AL(t)) \\ &= w(t+1)(q\,F(t)+r(t)) \end{aligned} \tag{3}$$

where

$$w(t+1)=(1+i(t+1))/(1+i),$$

$$q=(1+i)\,(1-k)$$

and

$$r(t)=(1+i)(NC(t)-B(t)+k\,AL(t)).$$

Equation (3) permits the derivation of recursive equations for the moments of $F(t)$; this is because $F(t)$ depends only on $i(s)$ for $s\leqslant t$, which implies that $w(t+1)$ and $qF(t)+r(t)$ are independent random variables. The moments of $C(t)$ then follow from Equation (1).

For $n=1,2,\ldots$

$$EF(t+1)^n = E\,w(t+1)^n\,E(q\,F(t)+r(t))^n \tag{4}$$

$$= E\,w(t+1)^n[r(t)^n + \sum_{j=1}^{n}\binom{n}{j} q^j\,EF(t)^j\,r(t)^{n-j}] \tag{5}$$

In particular

$$EF(t+1)=q\,EF(t)+r(t) \tag{6}$$

since $E\,w(t+1)=1$. $EF(t+1)$ therefore depends on $r(s)$ and $EF(s)$ for $s\leqslant t$. More generally, the nth moment of $F(t+1)$ depends on $r(s)$ and $EF(s)^j$ for $s\leqslant t$ and $j=1,\ldots,n$.

An example of the application of Equation (5) to a pension fund is the following: compute the mean and variance of $F(t)$ and $C(t)$, $1\leqslant t\leqslant N$, for a range of values of k, and compare the results so as to determine the 'best' values of k. In such a general situation there does not appear to be an explicit way of doing so. With the introduction of assumptions A3 to A6, however, an important simplification of the problem is achieved. This is examined in §§4 and 5.

4. ULTIMATE SITUATION ($t \to \infty$)

We now have $r(t) \equiv r = (1+i)(NC - B + k\ AL)$. From Equation (6)

$$EF(t+1) = q\ EF(t) + r \tag{7}$$

and

$$EF(t) = q^t \cdot F(0) + r(1-q^t)/(1-q), \qquad t \geqslant 0. \tag{8}$$

Clearly $EF(t)$ converges to $EF(\infty) = r/(1-q)$, provided $q = (1+i)(1-k) < 1$, which is equivalent to $k > i/(1+i) = d$.

Since the valuation rate of interest is $i_v = Ei(t)$, the equation of equilibrium

$$AL = (1 + i_v)\,(AL + NC - B) \tag{9}$$

implies

$$B - NC = d\,AL$$

and so

$$r = (1 + i)(k - d)AL.$$

Consequently

$$\begin{aligned} r/(1-q) &= AL(1 + i)(k-d)/[1-(1 + i)(1-k)] \\ &= AL. \end{aligned}$$

This also says that $EC(\infty) = NC + k(AL - EF(\infty)) = NC$. The conclusion is that paying more than interest on the unfunded liability ($k > d$) results in $F(t)$ and $C(t)$ converging in the mean to their target values. This is a direct consequence of A6.

Turning now to second moments, Equations (4) and (7) give

$$\begin{aligned} EF(t+1)^2 &= E\,w(t+1)^2\,E[q(F(t) - EF(t)) + q(EF(t) + r)]^2 \\ &= E\,w(t+1)^2\,[q^2 \operatorname{Var} F(t) + q(EF(t) + r)^2]; \\ \operatorname{Var} F(t+1) &= E\,F(t+1)^2 - (EF(t+1))^2 \\ &= E\,w(t+1)^2\,q^2 \operatorname{Var} F(t) + (E\,w(t+1)^2 - 1)(EF(t+1))^2. \end{aligned}$$

From

$$\begin{aligned} E\,w(t+1)^2 &= (1 + i)^{-2}\,E(1 + i(t+1))^2 \\ &= (1 + i)^{-2}\,((1 + \mathrm{i})^2 + \sigma^2), \end{aligned}$$

we get

$$\begin{aligned} \operatorname{Var} F(t+1) &= (1-k)^2((1 + i)^2 + \sigma^2) \operatorname{Var} F(t) \\ &\quad + \sigma^2(1 + i)^{-2}\,(EF(t+1))^2 \\ &= a \operatorname{Var} F(t) + b(EF(t+1))^2 \end{aligned} \tag{10}$$

where $a = (1-k)^2\,((1+i)^2 + \sigma^2)$ and $b = \sigma^2(1+i)^{-2}$.

If we let $\operatorname{Var} F(0) = 0$ (meaning that the value of $F(0)$ is known with certainty), we obtain

$$\operatorname{Var} F(t) = b \sum_{j=1}^{t} a^{t-j}\,(EF(j))^2,\ t \geqslant 1. \tag{11}$$

Equation (1) then yields Var $C(t)=k^2$ Var $F(t)$. The limits of Var $F(t)$ and Var $C(t)$ will now be found. Suppose $a<1$. This is equivalent to

$$\begin{aligned} k &> 1-1/[(1+i)^2+\sigma^2]^{1/2} \\ &> 1-1/(1+i)=d. \end{aligned}$$

Thus $EF(t)\to AL$, and Eq. (10) implies

$$\limsup_{t\to\infty} \text{Var } F(t) \leqslant a \limsup_{t\to\infty} \text{Var } F(t) + b\, AL^2$$

or

$$\limsup_{t\to\infty} \text{Var } (F(t) \leqslant b\, AL^2/(1-a).$$

We similarly find

$$\liminf_{t\to\infty} \text{Var } F(t) \geqslant b\, AL^2/(1-a)$$

and thus

$$\lim_{t\to\infty} \text{Var } F(t) = b\, AL^2/(1-a)$$

when $a<1$.

The above results are summarized in Proposition 1.

Proposition 1: Under A1 to A6,
(a) If $k>d$, $EF(\infty)=AL$, $EC(\infty)=NC$;
(b) if $k>1-1/[E(1+i(t)^2]^{1/2}$,

$$\begin{aligned} \text{Var } F(\infty) &= b\ AL^2/(1-a) \\ \text{Var } C(\infty) &= b\ AL^2\ k^2/(1-a) \end{aligned}$$

where $a=(1-k)^2((1+i)^2+\sigma^2)=(1-k)^2\ E(1+i(t))^2$ and $b=\sigma^2(1+i)^{-2}$.

Table 1 shows the standard deviations of $F(\infty)$ and $C(\infty)$, expressed in percentages of AL, for different values of k. The assumptions are $i_v=Ei(t)=\cdot 03$ and $\sigma=(\text{Var } i(t))^{1/2}=\cdot 10$. We see that st. dev. $F(\infty)$ is at its lowest when $k=1$. In this case the whole unfunded liability is paid off at every valuation date, that is to say

$$\begin{aligned} F(t+1) &= (1+i(t+1))\,[F(t)+NC-B+(AL-F(t))] \\ &= (1+i(t+1))\,(AL+NC-B) \\ &= [(1+i(t+1))/(1+i)]AL \end{aligned}$$

(from Equation (9)). Thus st. dev. $F(t)=\sigma(1+i)^{-1}AL$ for all $t\geqslant 1$. When k is decreased, st. dev. $F(\infty)$ increases.

Now consider st. dev. $C(\infty)$. It is very high when $k=1$, for in that case

$$\begin{aligned} \text{st. dev. } C(t) &= \text{st. dev. } (AL-F(t)) \\ &= \sigma(1+i)^{-1}\ AL. \end{aligned}$$

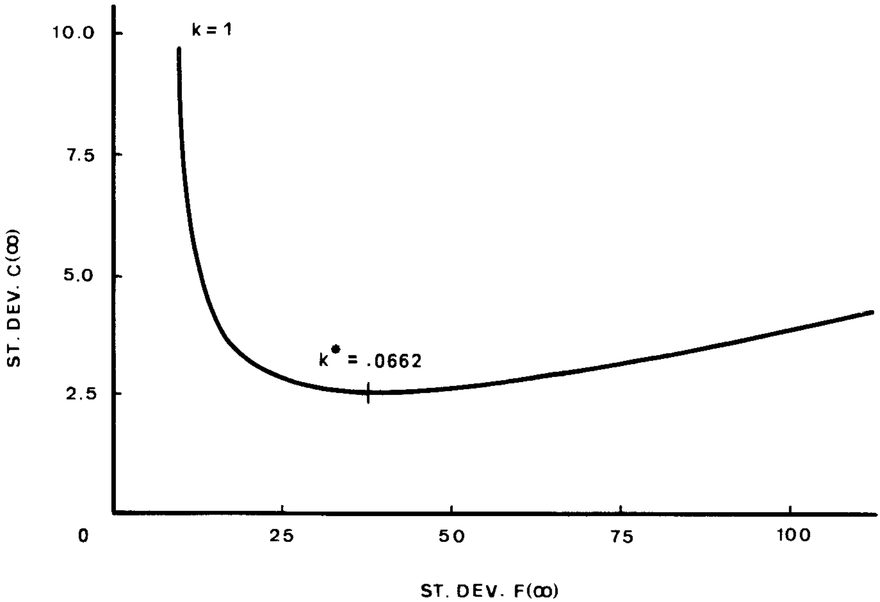

Figure 1. Standard deviations of $F(\infty)$ and $C(\infty)$ (in % of AL) for $k \leqslant 1$. $Ei(t) = \cdot 03$, $(\text{Var } i(t))^{1/2} = \cdot 10$

Table 1. *Standard deviations of* F(∞) *and* C(∞), *in percent of* AL, *for different values of* k. $i = Ei(t) = \cdot 03$, $\sigma = (\text{Var } i(t))^{1/2} = \cdot 10$.

	k	St. Dev. $F(\infty)/AL$	St. Dev. $C(\infty)/AL$
	·04	84·96%	3·40%
	·05	53·03	2·65
	·06	41·88	2·51
$k^* =$	·0662	37·73	2·498
	·07	35·74	2·502
	·08	31·74	2·54
	·09	28·86	2·60
	·10	26·66	2·67
	·20	17·31	3·46
	·30	14·08	4·22
	·40	12·39	4·95
	·50	11·35	5·67
	·60	10·67	6·40
	·70	10·21	7·15
	·80	9·92	7·94
	·90	9·76	8·79
	1·00	9·71	9·71

A noticeable fact is that st. dev. $C(\infty)$ is not a monotonous function of k; it is an increasing function of k, for $k^* = \cdot 0662 < k \leqslant 1$, but a decreasing one, for $k < k^*$.

Figure 1 is a plot of those standard deviations. The declining part of the curve corresponds to $k^* < k \leqslant 1$. Here a 'trade-off' is seen to take place: changing k will decrease one st. dev. but increase the other. The rising part of the curve corresponds to $k < k^*$. There the two st. dev. are decreasing functions of k.

We thus see that if $k < k^*$ then st. dev. $F(\infty)$ and st. dev. $C(\infty)$ are both higher than for $k = k^*$. For this reason these values of k will be called 'inadmissible'. The 'admissible' region is $k^* \leqslant k \leqslant 1$.

The next proposition states that what has just been observed holds in general.

Proposition 2: Let $y = E(1 + i(t))^2$ and $1 - 1/\sqrt{y} < k \leqslant 1$.

(a) If $y \neq 1$, then there exists $k^* < 1$ such that

(1) for $1 - 1/\sqrt{y} < k < k^*$, Var $F(\infty)$ and Var $C(\infty)$ decrease with increasing k;

(2) for $k^* \leqslant k \leqslant 1$, Var $F(\infty)$ decreases and Var $C(\infty)$ increases with increasing k.

Furthermore

$$k^* = \begin{cases} 0 & \text{if } y < 1 \\ 1 - 1/y & \text{if } y > 1. \end{cases} \tag{12}$$

(b) If $y = 1$, then Var $F(\infty)$ decreases and Var $C(\infty)$ increases with increasing k, for all $0 < k \leqslant 1$.

(Proof in Appendix.)

This says that the k's larger than $1 - 1/\sqrt{y}$ but smaller than k^* are inadmissible. Only when $y = E(1 + i(t))^2 = 1$ is this region empty.

Using expressions (12) it is possible to calculate the spreading period m^* corresponding to k^*. For $y < 1$ we find $m^* = \infty$, while for $y > 1$

$$k^* = 1/\ddot{a}_{\overline{m^*}|} \langle = \rangle m^* = \begin{cases} -\log(1 - d/k^*)/\log(1 + i), & i \neq 0, \\ 1 + 1/\sigma^2, & i = 0. \end{cases}$$

5. TRANSIENT SITUATION ($t < \infty$)

The previous section has shown that, when examining the ultimate values of the first two moments of F and C, the parameter k should preferably be chosen in the admissible region $k^* \leqslant k \leqslant 1$. Let us now see whether there are inadmissible values of k for finite time periods.

Fix t in Equation (8). Clearly $EF(t)$ and $EC(t)$ are functions of k. In order to remove this dependence, and thus be able to deal with variances only, let us assume that $F(0) = AL$. We obtain

$$EC(t) \equiv AL, \ EC(t) \equiv NC,$$

$$\begin{aligned}\text{Var } F(t) &= b\, AL^2 \sum_{j=1}^{t} a^{t-j} \\ &= b\, AL^2\,(1-a^t)/(1-a), \\ \text{Var } C(t) &= b\, AL^2\, k^2(1-a^t)/(1-a).\end{aligned}$$

The curves (st. dev $F(t)$, st. dev. $C(t)$; $k \leqslant 1$) are shown in Figure 2, for $t = 10, 20, 30$ and ∞, when $i = \cdot 07$ and $\sigma = \cdot 20$. As t increases, the curves get closer and closer to the one corresponding to $t = \infty$. For $t = 10$ or 20 all $0 \leqslant k \leqslant 1$ are admissible. It is only for larger t that some k become inadmissible (rising part of the curve). There then exists k_t^* with the same properties as the $k^* = k_\infty^*$ specified in Proposition 2. In this example we find $k_{30}^* = \cdot 1499$ and $k_\infty^* = \cdot 1560$.

In general, there is no closed form expression for k_t^*. All that can be said is that when it exists $k_t^* \leqslant k_\infty^*$ (see Appendix). Numerical computations indicate that it is only for relatively large values of $y = E(1 + i(s))^2$ that k_t^* is of interest (from the point of view of pension funding). This explains the choice of such extreme assumptions in the above numerical illustration. Setting $i = \cdot 07$ and $\sigma = \cdot 20$ makes $y = 1 \cdot 07^2 + \cdot 20^2 = 1 \cdot 1849$ large enough for k_{30}^* to exist.

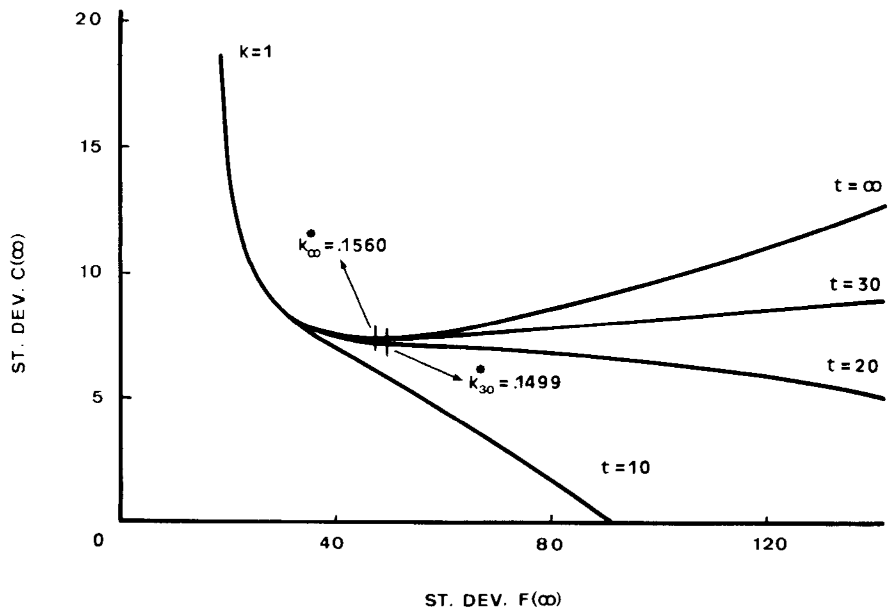

Fig. 2. Standard deviations of $F(t)$ and $C(t)$ (in % of AL) for $k \leqslant 1$ and $t = 10, 20, 30$ and ∞. $Ei(t) = \cdot 07$, $(\text{Var } i(t))^{1/2} = \cdot 20$

6. CONCLUSION

The formulae of § 3 permit the calculation of all the moments of $F(t)$ and $C(t)$, under the assumption that rates of return are i.i.d. random variables.

Section 4 examined Var $F(t)$ and Var $C(t)$ when the process F has become stationary. It appeared that the better values of k are in the interval $[k^*,1]$. Observe that when $i \geqslant 0$

$$\begin{aligned} k^* &= 1 - 1/[(1+i)^2 + \sigma^2] \\ &> 1 - 1/(1+i)^2 \\ &= d(1+v). \end{aligned}$$

Hence, in order to minimize variances, the adjustment to the normal cost should be greater than approximately twice the interest on the unfunded liability.

In § 5, it was seen that for finite time periods (and when $F(0) = AL$) there may or may not be inadmissible values of k, depending on the length of the period and the magnitude of $E(1+i(s))^2 = (1+i)^2 + \sigma^2$.

Of course in general $F(0) \neq AL$, and the population is not stationary. An appropriate value of k can still be obtained, by comparing the sequences of moments $(EF(t)^n, EC(t)^n, n = 1,2, t \geqslant 1)$ produced by different possible values of k.

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APPENDIX

Proof of Proposition 2

Let $k \in (1-\sqrt{y}, 1]$. From Prop. 1 we have Var $F(\infty)=\sigma^2 AL^2 f(k)$ and Var $C(\infty)=\sigma^2 AL^2\, c(k)$, where

$$f(k)=\ 1/(1-(1-k)^2 y)$$
$$c(k)=k^2/(1-(1-k)^2 y).$$

These functions are finite for $k>1-1/\sqrt{y}$. Clearly $f'(k)<0$ for all $1-1/\sqrt{y}<k\leqslant 1$. But

$$\begin{aligned} c'(k) &= [2k(1-(1-k)^2y)-k^2\cdot 2(1-k)y]\,(1-(1-k)^2y)^{-2} \\ &= 2k[1-y+yk]\,(1-(1-k)^2y)^{-2} \\ &= 2ky[k-(1-1/y)]\,(1-(1-k)^2y)^{-2}. \end{aligned}$$

If $y<1$, then $1-1/y<1-1/\sqrt{y}<0$, and so the only k in $(1-1/\sqrt{y},\ 1]$ such that $c'(k)=0$ is $k^*=0$. If $y=1$, then $1-1/\sqrt{y}=1-1/y=0$ and $c'(k)>0$ for all $k\in(0,1]$. If $y>1$, then $0<1-1/\sqrt{y}<1-1/y<1$ and $c'(k)$ only vanishes at $k^*=1-1/y$.

About k_t^*, $t<\infty$.

We have

$$\begin{aligned} \text{Var } C(t) &= \sigma^2\, AL^2\,(1-a^t)/(1-a) \\ &= \sigma^2\, AL^2\, c_t(k) \end{aligned}$$

where

$$c_t(k) = c(k)(1-a^t).$$

Thus

$$c_t'(k)=c'(k)\,(1-a^t)+2ty(1-k)a^{t-1}\,c(k).$$

Let us restrict our attention to $k>1-1\sqrt{y}=>a<1$. Let $y<1$. For $k>0$ $c_t'(k)>0$, while $c_t'(0)=0$. Thus $k_t^*=0$ for all t. If $y=1$ then $c_t'(k)>0$ for all $k>0$, so k_t^* does not exist. If $y>1$, then $c_t'(k_\infty^*)>0$. The second member of the equation above is always positive, while the first one is negative for $k<k_\infty^*$ only. Thus if k_t^* exists $k_t^*<k_\infty^*$.