Dynamic programming and efficient hedging for unit-linked insurance contracts

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Abstract. This paper considers hedging of the combined insurance and financial risk in a basic unit-linked insurance contract by using dynamic programming. The contracts studied specify a payment to the policy-holders at a given time conditional on survival of the policy-holders. The payment is linked to the value of a traded stock. The insurance company can invest in this stock and in a savings account. Due to the mortality risk involved, the financial market is incomplete such that a perfect hedge is not possible. We determine optimal self-financing investment strategies which minimize the probability of shortfall, i.e. the probability that the capital available at the time of payment is less than the integrated claim. We compare our results with results obtained in the literature via quadratic hedging approaches. In general, only existence results are available for the case of incomplete markets using the shortfall probability criterion. Hence, the dynamic programming approach has been applied. One contribution of the paper is an understanding of solutions for the shortfall probability in the one-period case. In addition, computational procedures (dynamic programming algorithms) for the general discrete time multi-period and multi-policy-holder case are developed. The procedures exploit discrete properties of the problem. Thus, discretization with respect to capital and investment is avoided.

Key words: shortfall probability, expected shortfall, self-financing strategy, unit-linked insurance, incomplete market, discrete time dynamic programming

JEL Classification: G10.

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1 Introduction

Unit-linked life insurance contracts differ from traditional insurance contracts in that benefits (and possibly premiums) are *linked* directly to the value of a *unit* of some investment portfolio. We study unit-linked contracts in their most basic form, which specify a lump sum payment (to the policy-holders) determined as a function of the value of a single stock at a given future time conditional on survival. It is assumed that the insurance company can invest in the same stock and in a savings account with a risk free interest. The problem is then to determine optimal self-financing investment strategies for the company. The integrated financial and insurance claim cannot be hedged perfectly since we are dealing with an incomplete market, see Møller and Steffensen (2007) and references therein.

We study the shortfall probability criterion (and similar criteria) for the optimization problem and compare these results with previous results on quadratic hedging that have been obtained in the literature. Quadratic hedging has been applied in life insurance, see Møller (2001a), where the above problem with unit-linked insurance contracts is considered in a discrete time setting, and Møller (2001b), where more general payment processes in a continuous time setting are treated. For further developments and applications allowing for systematic mortality and longevity effects, see Dahl and Møller (2006) and Dahl, Melchior and Møller (2008). When studying risk-minimizing strategies, a key quantity is the so-called *cost process*, defined as the current value of the strategy reduced by trading gains. A strategy is said to be risk-minimizing if it, at any time during the term of the policy, minimizes the expected squared value of all future (discounted) costs. One advantage of the quadratic hedging approach is that it is analytically tractable and provides solutions that agree well with intuition, e.g. for the above basic problem with unit-linked insurance. Moreover, as the above references show, it has proven applicable for advanced applications. However, one possible drawback of quadratic hedging is that the hedging error is symmetric, i.e. redundant capital is punished in the same manner as lack of capital.

A different approach is to apply so-called *efficient hedging methods*, see Föllmer and Schied (2002) for a detailed account in the discrete time case; continuous time problems have been studied in Föllmer and Leukert (1999, 2000). Within the efficient hedging approach, *quantile hedging* adopts the shortfall probability as the object for the minimization. This amounts to minimizing an expected hedging error, where the hedging error is 1 if the capital is smaller than the claim and 0 otherwise. This criterion may be modified to the case of *expected shortfall*, where the hedging error is defined as the deficit in case of lack of capital and 0 otherwise.

The methodology used in the literature is typically based on the Neyman-Pearson Lemma from statistical test theory, combined with martingale measures and appropriately chosen knock-out options. Clearly, efficient hedging differentiates between redundant capital and lack of capital as opposed to quadratic hedging. However, the mathematical tractability is typically lost. Methods for obtaining optimal strategies by efficient hedging in complete markets have been derived, whereas typical results only provide existence proofs in case of incomplete markets such that the approach is not readily applicable for our use. Instead we apply dynamic programming methods. Dynamic programming can be applied in virtually any problem with a sequence of decisions to be taken. Numerous applications appear in economics, see e.g. Stokey and Lucas (1989). Life insurance also benefits from dynamic programming, in particular in a continuous time setting through the well-known HJB-equation, see e.g. Björk (2004). A more advanced mathematical treatment can be found in e.g. Yong and Zhou (1998). The HJB-equation re-appears in many stochastic control problems in insurance, e.g. for optimal consumption and investment combined with life insurance, see Schmidli (2008) and references therein.

The paper is organized as follows. In Section 2, the one-period problem for hedging basic unit-linked contracts is stated. Solely from an algebraic point of view, analytical solutions for strategies and shortfall probabilities are presented for the case with a single policy-holder and a single period. These solutions do not really contribute to the overall understanding of the structure of the solutions. However, realizing that hedging the claim for a any number of (surviving) policy-holders imposes a linear relation between initial capital and the number of stocks to be purchased, we can explain the one-period results and a basis for multi-period considerations is established.

Section 3 contains a brief introduction to the finite time horizon dynamic programming algorithm providing a backward recursion procedure for a problem with sequential decisions and costs in each step, specialized to costs at the terminal time only. The presentation draws upon Hernández-Lerma and Lasserre (1995). For a comprehensive review on discrete time dynamic programming, see also Bertsekas and Shreve (1978). In Section 4, computational procedures for the general multi-period and multi-policy-holder case of the basic unit-linked contract are developed using dynamic programming. First, the basis for a naïve discretization approach is outlined. We refer to this method as the *brute force approach*, due to its computational requirements. The brute force approach is mainly relevant for verification of alternatives. Then, with the observations from Section 2 in mind, we exploit that in the one-period case the minimum expected shortfall probability is a piecewise constant function of the initial capital. Moreover, this property propagates in the backward recursion in the dynamic programming algorithm. These observations lead to an algorithm with the shortfall probability criterion which is solely based on discrete properties of the problem and not a discretization of a continuous problem. We refer to the procedure as the *discrete properties approach*. Section 5 contains a number of numerical examples.

2 The model and main problem

We consider a portfolio of unit-linked life insurance contracts in a discrete time setting, where the sum insured, which may depend on the value of some stock, is payable at a fixed time conditional on survival of the policy-holders. We first introduce the financial market and define the basic hedging criterion. In addition, we solve the problem in the one-period case.

2.1 The financial market and trading strategies

Consider a financial market consisting of a stock S and a savings account B. We denote by S_t the value of the stock at time $t = 0, 1, \ldots, T$, where T is a fixed finite time horizon. Formally, the stock price process and all other processes introduced in the following are defined on some probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\{0,1,\ldots,T\}}$. We assume that the underlying price process $S = (S_t)_{t\in\{0,1,\ldots,T\}}$ can be traded in the financial market in addition to a savings account with price process B given by $B_t = (1+r)^t$. Thus, the savings account pays a fixed interest r during each time period. We mainly work with the discounted price processes $S^* = S/B$ and $B^* = B/B$, where we have used the savings account as numeraire.

A trading strategy is a two-dimensional process $\widehat{h} = (h^0, h^1)$, where h^1_t is the number of stocks held at time t and where $h_t^0 B_t$ is the amount deposited in the savings account. More precisely, h_t^1 is the number of stocks chosen at time $t-1$ and held until time t. This means that h_t^1 needs to be determined based on the information available at time $t-1$. The (undiscounted) value at time t of the portfolio $\widehat{h}_t = (h_t^0, h_t^1)$ is given by

$$V_t(\widehat{h}) = h_t^0 B_t + h_t^1 S_t^1, \tag{2.1}$$

and the discounted value is $V_t^*(\widehat{h}) = V_t(\widehat{h})/B_t$. We restrict to self-financing strategies, i.e. strategies $\widehat{h}$, where the value process $V^*(\widehat{h})$ has dynamics given by

$$V_t^*(\widehat{h}) = V_{t-1}^*(\widehat{h}) + h_t^1 \Delta S_t^*, \tag{2.2}$$

and where $\Delta S_t^* = S_t^* - S_{t-1}^*$. Thus, the value process at time t depends on the strategy via the initial value $V_0(\widehat{h})$ and the number of stocks held $h_1^1, \dots, h_t^1$. We take $S_t = (1+\rho_t)S_{t-1}$, such that

$$S_t^* = S_{t-1}^*(1+\rho_t)/(1+r). \tag{2.3}$$

In the present paper, we work with the so-called binomial model, where the random variables $\rho_1, \dots, \rho_T$ are i.i.d., and where $\rho_1 \in \{a, b\}$ and $0 < P(\rho_1 = b) = p < 1$. In addition, we assume that $a < r < b$. It is convenient to introduce the quantities $\widetilde{\rho}_t$, defined by $1 + \widetilde{\rho}_t = (1+\rho_t)/(1+r)$, which attain the values $\widetilde{a} = \frac{a-r}{1+r}$ and $\widetilde{b} = \frac{b-r}{1+r}$. Thus, $S_t^* = S_{t-1}^*(1+\widetilde{\rho}_t)$. For a thorough treatment of the binomial model, see e.g. Pliska (1997).

It is often relevant to introduce some additional constraints on the strategies. In the following sections, we typically require that the self-financing strategies are chosen such that the value process remains non-negative. In particular, this condition ensures that one cannot use so-called doubling strategies. In Section 2.4, we consider the one-period case without this non-negativity constraint.

2.2 The liability and the choice of criterion

We study a portfolio of n policy-holders and denote by Y_t the number of survivors at time t. The policy-holders' remaining life-times are modeled via some random variables $T_1, \dots, T_n$, which are assumed to be independent of the traded price process S. The discounted liability payable at time T is given by

$$H^* = Y_T\, f(S_T) B_T^{-1} = Y_T \widetilde{f}(S_T^*), \tag{2.4}$$

where f is some measurable function. Thus, the sum payable upon survival to T is assumed to be a function of the terminal value of the stock only. We assume that the insurance company receives some premium at time 0, which is invested in the financial market via a dynamic trading strategy $\widehat{h}$ in order to hedge the risk associated with the liability H.

A liability H payable at time T is said to be *attainable* if there exists a self-financing strategy $\widehat{h}$ such that the terminal value of the investments $V_T(\widehat{h})$ coincides with the liability H almost surely, i.e. if $V_T(\widehat{h}) = H$, P-a.s. In this case, we say that the liability is hedged perfectly. A self-financing strategy $\widehat{h}$ is said to be a *super-hedging* strategy for H if $V_T(\widehat{h}) \geq H$, P-a.s., i.e. if the value process at time T exceeds the liability H with probability 1.

Since the liability (2.4) is assumed to depend on the number of survivors at time T, which is considered to be a non-traded risk, it is in general not possible to hedge the liability perfectly, see Møller (2001a). We therefore study the criterion of minimizing the probability of a shortfall, i.e. the probability of having insufficient capital at time T, where the liability is payable. For a self-financing strategy $\widehat{h}$ and a liability H, the *shortfall probability* is given by

$$P_{sf}(\widehat{h}) = P(V_T(\widehat{h}) < H) = \mathrm{E}\left[1_{\{V_T(\widehat{h})<H\}}\right]. \tag{2.5}$$

In addition, we study the so-called *expected shortfall* given by

$$E_{sf}(\widehat{h}) = \mathrm{E}\left[\left(H - V_T(\widehat{h})\right)^+\right], \tag{2.6}$$

which measures the expected deficit associated with the liability H and the trading strategy $\widehat{h}$. We determine dynamic self-financing strategies $\widehat{h}$ that minimize $P_{sf}(\widehat{h})$ and $E_{sf}(\widehat{h})$ by use of dynamic programming methods in a multi-period setting. First, however, we study the problem in the one-period case, where dynamic programming is not needed. The study of the one-period problem already allows for some important observations.

2.3 The role of the information available

It is of relevance to study the impact of the amount of information that is available to the insurer. We denote by $\mathbb{G}$ the natural filtration associated with the traded price processes and let $\mathbb{H}$ be the natural filtration associated with the process for the number of survivors Y. One example is the natural situation, where the insurance company observes the current number of survivors at each time $t \in \{0, 1, \dots, T\}$. This is the case where the process Y is adapted to the filtration $\mathbb{F}$. For example, we could define $\mathbb{F} = (\mathcal{F}_t)_{t\in\{0,1,\dots,T\}}$ by $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t = \sigma(\mathcal{G}_t \cup \mathcal{H}_t)$, such that we have access to the full information from $\mathcal{G}_t$ and $\mathcal{H}_t$ at time t. Thus, the insurance company can base investments at each time on exact information about the current number of survivors.

Another example is the case, where the insurance company receives information about the financial market but is restricted to information about the number of policy-holders at time 0 and the final number of survivors at time T. At the intermediate times $t = 1, 2, \dots, T-1$, the insurance company does not observe Y_t. This can for example be modeled by working with a filtration $\mathbb{F}^\circ$ defined by $\mathcal{F}_t^\circ = \mathcal{G}_t$ for $t < T$ and $\mathcal{F}_T^\circ = \mathcal{G}_T \vee \mathcal{H}_T$. We compare the minimum obtainable shortfall probability in these two cases and show in an example in Section 5.2 that the optimal strategies based on the filtration $\mathbb{F}$ may indeed lead to lower shortfall probabilities than the ones based on $\mathbb{F}^\circ$ in the general case. This underlines the importance of the choice of the filtration and shows that the insurance company in general will benefit from adapting their investment strategies to the current number of survivors.

2.4 Minimizing the shortfall probability in the one-period case

We study the problem of minimizing (2.5) in the case where $T = 1$ via direct calculations. It follows from (2.2) that $V_1^*(\widehat{h}) = V_0^* + h_1^1 \widetilde{\rho}_1 S_0^*$. Thus, the terminal value V_1^* of the strategy depends on the initial value of the portfolio $v^* = V_0^*$, the initial value of the stock $s^* = S_0^*$, the number h_1^1 of stocks purchased at time 0 and the relative change $\widetilde{\rho}_1$ in the discounted value of the stock. Since we are working with the binomial model, where $\widetilde{\rho}_1 \in \{\widetilde{a}, \widetilde{b}\}$, we see that

$$P(V_1^* = v^* + h_1^1 \widetilde{b} s^*) = p = 1 - P(V_1^* = v^* + h_1^1 \widetilde{a} s^*).$$

We denote by $F_{Y_1}(y) = P(Y_1 \leq y)$ the distribution function for Y_1, and we let $\mathcal{Y} = \{0, 1, \ldots, n\}$. Then, we may write the expected shortfall probability on the form

$$\begin{aligned}
\mathrm{E}\left[1_{\{V_1(\widehat{h})<H\}}\right] &= \mathrm{E}\left[\int_{\mathcal{Y}} P(V_1^*(\widehat{h}) < y\widetilde{f}(S_1^*))F_{Y_1}(\mathrm{d}y)\right] \\
&= \int_{\mathcal{Y}} \left(p1_{\{v^*+h_1^1\widetilde{b}s^*<y\widetilde{f}(s^*+\widetilde{b}s^*)\}} + (1-p)1_{\{v^*+h_1^1\widetilde{a}s^*<y\widetilde{f}(s^*+\widetilde{a}s^*)\}}\right)F_{Y_1}(\mathrm{d}y).
\end{aligned}$$

Here, the second equality follows by conditioning on the two possible outcomes at time 1 for the stock price process. If we in addition assume that the remaining life-times are i.i.d. with one-period survival probability ${}_1p_x = e^{-\mu_x}$, the number of survivors at time 1 is binomially distributed with parameters $(n, {}_1p_x)$. Thus, the minimum shortfall probability $P_{sf,min}$ is given by

$$\begin{aligned}
P_{sf,min} &= \min_{h_1^1}\Bigg[\sum_{y=0}^{n}\Big(p1_{\{v^*+h_1^1\widetilde{b}s^*<y\widetilde{f}(s^*+\widetilde{b}s^*)\}} \\
&\quad +(1-p)1_{\{v^*+h_1^1\widetilde{a}s^*<y\widetilde{f}(s^*+\widetilde{a}s^*)\}}\Big)\frac{n!}{(n-y)!y!}{}_1p_x{}^y(1-{}_1p_x)^{n-y}\Bigg]. \qquad (2.7)
\end{aligned}$$

Finding $P_{sf,min}$ amounts to minimizing a sum of $2(n+1)$ terms, where each term involves the indicator function of the event $v^* + h\widetilde{\rho}s^* < \widetilde{f}(s^* + \widetilde{\rho}s^*)y$, $\widetilde{\rho} \in \{\widetilde{a}, \widetilde{b}\}$, and $y = 0, 1, \ldots, n$.

One policy-holder

In the case of a single policy-holder with $y = 1$ and with $P_{Y_1}(1) = {}_1p_x = 1 - P_{Y_1}(0)$, the equation (2.7) reduces to:

$$\begin{aligned}
P_{sf,min} &= \min_{h_1^1}\Big[(p1_{\{v^*+h_1^1\widetilde{b}s^*<0\}} + (1-p)1_{\{v^*+h_1^1\widetilde{a}s^*<0\}})(1-{}_1p_x) \qquad (2.8)\\
&+ \quad (p1_{\{v^*+h_1^1\widetilde{b}s^*<\widetilde{f}(s^*+\widetilde{b}s^*)\}} + (1-p)1_{\{v^*+h_1^1\widetilde{a}s^*<\widetilde{f}(s^*+\widetilde{a}s^*)\}}){}_1p_x\Big].
\end{aligned}$$

If we further impose the natural assumptions $\widetilde{a} < 0$, $\widetilde{b} > 0$, and $s^* > 0$, we see from (2.8) that $P_{sf,min} = 0$, if the following four inequalities are met by the optimal value h^{opt} of h_1^1:

$$\begin{aligned}
h^{opt} &\geq -\frac{v^*}{\widetilde{b}s^*} && \text{(b0)},\\
h^{opt} &\leq -\frac{v^*}{\widetilde{a}s^*} && \text{(a0)},\\
h^{opt} &\geq \frac{\widetilde{f}(s^*+\widetilde{b}s^*)-v^*}{\widetilde{b}s^*} && \text{(b1)},\\
h^{opt} &\leq \frac{\widetilde{f}(s^*+\widetilde{a}s^*)-v^*}{\widetilde{a}s^*} && \text{(a1)}.
\end{aligned} \qquad (2.9)$$

Each of these criteria represents a possible combined outcome of the stock price process and the number of survivors. One can interpret the 4 criteria (2.9) as follows. The first one, referred to as criterion b0, is related to the first term in (2.8) and represents the situation where the policy-holder does not survive and the stock jumps upwards. In this case, the inequality implies that the capital requirement 0 is hedged, i.e. the capital is non-negative at time 1, and this is sufficient to super-hedge the liability in this scenario. Similarly, the second criterion (criterion a0) is the case where the stock jumps downwards and the policy-holder does not survive. The last two criteria (criterion b1 and a1, respectively) represent the case where the policy-holder survives and the stock jumps either upwards or downwards. In these outcomes, the conditions on h_1^1 are sufficient to require that the value process at time 1 exceeds the liability.

Further, we take $\widetilde{f}(s^*) = f(s)/((1+r)^T) > 0$, where $s^* = s(1+r)^{-T}$. One example is the case $f(s) = \max(K, s)$, $K > 0$. If $\widetilde{f}$ is strictly positive, we see that

$$\frac{\widetilde{f}(s^* + \widetilde{a}s^*) - v^*}{\widetilde{a}s^*} < -\frac{v^*}{\widetilde{a}s^*},$$

and

$$\frac{\widetilde{f}(s^* + \widetilde{b}s^*) - v^*}{\widetilde{b}s^*} > -\frac{v^*}{\widetilde{b}s^*}.$$

This shows that the probability of insufficient capital is zero for

$$h^{opt} \in \left[\frac{\widetilde{f}(s^* + \widetilde{b}s^*) - v^*}{\widetilde{b}s^*} ; \frac{\widetilde{f}(s^* + \widetilde{a}s^*) - v^*}{\widetilde{a}s^*} \right], \quad (2.10)$$

if $\frac{\widetilde{f}(s^*+\widetilde{b}s^*)-v^*}{\widetilde{b}s^*} < \frac{\widetilde{f}(s^*+\widetilde{a}s^*)-v^*}{\widetilde{a}s^*}$. If the condition $\frac{\widetilde{f}(s^*+\widetilde{b}s^*)-v^*}{\widetilde{b}s^*} < \frac{\widetilde{f}(s^*+\widetilde{a}s^*)-v^*}{\widetilde{a}s^*}$ is not satisfied, there are several cases, depending on the ordering of the r.h.s. in the four inequalities above, to be considered. It is noted that if the initial capital v^* is such that the condition $\frac{\widetilde{f}(s^*+\widetilde{b}s^*)-v^*}{\widetilde{b}s^*} < \frac{\widetilde{f}(s^*+\widetilde{a}s^*)-v^*}{\widetilde{a}s^*}$ is not satisfied, it is not possible to obtain a zero shortfall probability.

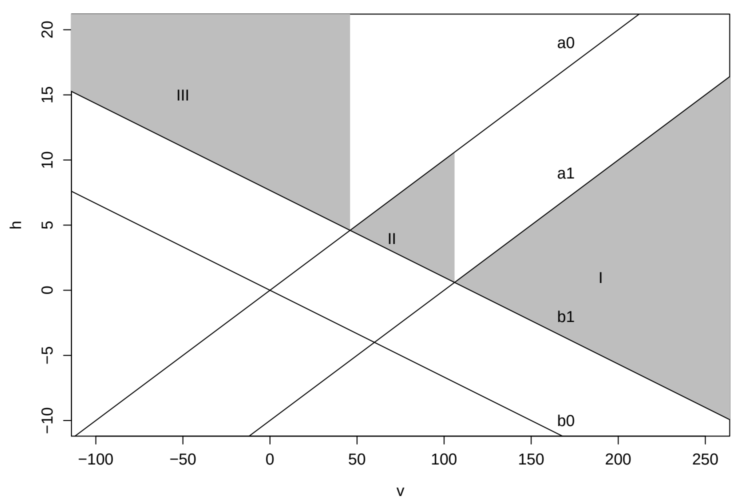

Figure 1: *Optimal amount of stock h versus start capital v shown as grey areas for $a = -0.10$, $b = 0.15$, $r = 0$, $p = 0.7$, $f(s) = \max(s, K)$, $K = 100$, $S_0 = 100$, and $\mu = 1$.*

The approach outlined above provides complete results regarding shortfall probabilities and the corresponding (non-unique) optimal strategies. However, the considerations do not really contribute much to the general understanding of the overall structure of solutions. Moreover, they do not appear to be constructive when considering multi-period problems. The four inequalities representing criteria a0, a1, b0, and b1 in (2.9) can be translated into lines which characterize the optimal solution. Indeed, the first inequality leads to combinations of h and v^* such that $h \geq -\frac{v^*}{\widetilde{b}s^*}$, which gives a lower bound on the optimal number of stocks h. Analogously, the 2nd, 3rd, and 4th inequalities give upper, lower, and upper bounds, respectively, represented by lines.

The above lower and upper bounds on the number of stocks h as a function of the initial capital v given by lines and corresponding optimal combinations of v and h (grey areas) are shown in Figure 1. The three areas, which represent optimal combinations of h and v, are labeled I, II and III, respectively. In addition, we have shown the 4 optimality lines derived from the criteria (2.9). Area I represents combinations of initial capital v and number of stocks h, where, irrespective of the outcome of the stock and the number of survivors, the integrated claim is super-hedged. In area II, the case of 1 (and 0) survivors is hedged, provided that the stock jumps upwards. If the stock jumps downwards, no survivor is hedged (criterion a0). Finally, in area III, 1 (and 0) survivors are hedged (criteria b1 and b0) if the stock jumps upwards. If the stock jumps downwards, the capital becomes negative.

3 Dynamic programming with finite time horizon

In this section, we reformulate the main problem in a dynamic programming framework with a finite time horizon. For a general treatment of discrete time dynamic programming, see Hernández-Lerma and Lasserre (1995). Since we restrict to self-financing strategies, we may focus on the component h_1^1 of the strategy (h^0, h^1) and refer to this component as $\widetilde{h} = (h_1, \ldots, h_{T-1})$. Further, denote by $X_t(\widetilde{h}) = (V_t^*(\widetilde{h}), S_t^*, Y_t)^{tr}$ the vector-valued process consisting of the current (discounted) value $V_t^*(\widetilde{h})$ of the investment strategy, the (discounted) value of the stock S_t^* and the current number of survivors Y_t. In the general case, this process can be observed by the insurance company. Moreover, the company is able to control the process $V_t^*(\widetilde{h})$ via the initial value V_0^* and the number of risky assets held. We assume that $\widetilde{h}$ is a Markov control, i.e. $h_{t+1} = h_{t+1}(V_t^*(\widetilde{h}), S_t^*, Y_t)$. Thus, the process $X_t(\widetilde{h})$ is a Markov process.

The goal is to minimize the finite horizon performance criterion given by

$$J^{\widetilde{h}}(t,x) = \mathrm{E}^{t,x}\left[c_T\left(X_T(\widetilde{h})\right)\right] = \mathrm{E}\left[c_T\left(X_T(\widetilde{h})\right)\Big| X_t(\widetilde{h}) = x\right], \tag{3.1}$$

for all $t = 0, 1, \ldots, T$, for $x = (v, s, y)^{tr}$. As above, we study shortfall probability minimization with $c_T(v,s,y) = 1_{\{v < y\widetilde{f}(s)\}}$ and the expected shortfall, where $c_T(v,s,y) = (y\widetilde{f}(s) - v)^+$.

We denote by J^{opt} the optimal value function

$$J^{opt}(x) = \inf_{\widetilde{h}\in\widetilde{\mathcal{H}}} J^{\widetilde{h}}(0,x), \tag{3.2}$$

where $\widetilde{\mathcal{H}}$ is the set of all admissible strategies. This problem can be solved by a simplified version of the general dynamic programming theorem, see Hernández-Lerma and Lasserre (1995, Theorem 3.2.1), which provides an algorithm for finding both the value function J^{opt} and an optimal strategy h^{opt}. In order to formulate this result we introduce some additional notation. We denote by

$$Q(dy|x,h) = P(X_{t+1}(\widetilde{h}) \in dy | X_t(\widetilde{h}) = x, h_{t+1} = h),$$

the distribution of $X_{t+1}(\widetilde{h})$ given that $X_t(\widetilde{h}) = x$ and given the control $h_{t+1} = h$. In addition, we introduce the sets $\mathcal{H}(x)$ of possible values for the control h_{t+1} given the present state $X_t(\widetilde{h}) = x$. For example, if we are working with the condition that the process $V_t(\widetilde{h})$ may not attain negative values, this leads to a condition on the admissible values for the number of stocks held h. Finally, we denote by $\mathcal{X}$ the space of possible values for the process $X_t(\widetilde{h})$. We can now formulate the dynamic programming theorem.

Theorem 3.1 *Let $J_0, J_1, ..., J_T$ be functions defined by*

$$J_T(x) = c_T(x), \tag{3.3}$$

and

$$J_t(x) = \min_{h \in \mathcal{H}(x)} \left[\int_{\mathcal{X}} J_{t+1}(y) Q(dy|x,h) \right], \tag{3.4}$$

for $t = T-1, T-2, ..., 0$. Suppose that these functions are measurable and that there exists a strategy $\widetilde{h} = \{h_1, ..., h_T\}$ such that $h_{t+1} = h_{t+1}(x) \in \mathcal{H}(x)$, $t = 1, \dots, T$, and $h_{t+1}(x)$ attains the minimum in (3.4) for all $x \in \mathcal{X}$ and $t = 1, ..., T$, i.e.,

$$J_t(x) = \int_{\mathcal{X}} J_{t+1}(y) Q(dy|x, h_{t+1}). \tag{3.5}$$

Then the strategy $\widetilde{h} = \{h_1, ..., h_T\}$ is optimal and the value function J^{opt} equals J_0, i.e.,

$$J^{opt}(x) = J_0(x) \ \ \forall x \in \mathcal{X}. \tag{3.6}$$

Proof. See Hernández-Lerma and Lasserre (1995).

The main result in the theorem is that the value function $J^{opt}(x)$ which is defined in (3.2) as the infimum with respect to all strategies of the performance criterion in (3.1), can be calculated as $J_0(x)$ from the backward recursion (3.4). A further assumption, the so-called *measurable selection condition*, ensures that we obtain a minimum, and not just an infimum in (3.4). The measurable selection condition, see Hernández-Lerma and Lasserre (1995), adapted to the current presentation, can be stated as in the following assumption.

Assumption 3.2 *The model and a given measurable function $w : \mathcal{X} \to I\!R$ are such that*

$$\hat{w}(x) = \inf_{h \in \mathcal{H}(x)} \left[\int_{\mathcal{X}} w(y) Q(dy|x,h) \right], \ \ x \in \mathcal{X}, \tag{3.7}$$

is measurable, and there exists a measurable function $g : \mathcal{X} \to \mathcal{H}$ satisfying $g(x) \in \mathcal{H}(x)$ for all $x \in \mathcal{X}$, such that the function within brackets attains its minimum at $g(x)$ for all x, i.e.,

$$\hat{w}(x) = \int_{\mathcal{X}} w(y) Q(dy|x,g) \ \ \forall x \in \mathcal{X}, \tag{3.8}$$

It is noted that we have confined ourselves to *stationary Markov control models* in that $\mathcal{X}$, $\mathcal{H}$, $\mathcal{H}(x)$, Q, and c_t, $t < T$, are time-invariant ($c_t = 0$ for $t < T$).

3.1 Reduced information on the number of survivors

In order to investigate the role of the amount of information available, we also study the situation where the insurance company is restricted to information concerning the number of survivors in the insurance portfolio. This situation may be described by the process $X_t^\circ(\widetilde{h}) = (V_t^*(\widetilde{h}), S_t, Y_t^\circ)^{tr}$, where $Y_t^\circ := Y_0$ for $t < T$ and $Y_T^\circ = Y_T$. This means that the company does not observe the current number of survivors during $t = 1, \dots, T-1$. An alternative approach is discussed below. We consider the case from Section 2.3, where the insurance company only receives information about the number of policy-holders at time 0 and the final number of survivors Y_T at time T. Such a state component for which the value is unknown at intermediate time points cannot be dealt with directly by the dynamic programming algorithm, since it assumes full information on all state components entering the control. Generally, methods for imperfect state information in dynamic programming may be employed, see e.g. Bertsekas and Shreve (1978). However, if the number Y of survivors is not observed, it may be eliminated from the dynamic programming algorithm, if it is also uncontrollable and independent of the other components. Thus, Y enters the dynamic programming algorithm through the cost function c_T only. Hence, assuming a full state vector $X = (Y, Z^{tr})^{tr}$, where Z represents all state components except for Y, we adopt the modified costs with Y averaged out,

$$\widetilde{c}_T(z) = \int_{\mathcal{Y}} c_T(z, y) P_T(dy).$$

Here, P_T is the probability distribution for Y_T. Finally, we have the dynamic programming algorithm

$$J_t(z) = \min_{h \in \mathcal{H}(z)} \left[\int_{\mathcal{Z}} J_{t+1}(\widetilde{z}) Q(d\widetilde{z}|z, h) \right],$$

where $\mathcal{H}(z) = \{h \in \mathcal{H}(z, y) | y \in \mathcal{Y}\}$, i.e. feasible controls expressed through the observable state component Z_t, $t = 0, 1, \dots, T$, only.

The restriction to information on the number of survivors is also studied in case of stochastic survival probabilities. For this application, a standard Bayesian updating procedure is adopted with the survival probabilities being Beta distributed with a uniform prior.

4 Computational procedures

4.1 Introduction

In this section, a computational procedure for shortfall probability hedging is proposed for handling the general case with an arbitrary number of periods and an arbitrary number of policy-holders. A similar procedure for expected shortfall hedging is commented.

We start out in Section 4.2 with a straightforward discretization with respect to capital and strategy (number of stocks). This approach is extremely computationally intensive and is thus unsuitable for practical applications. However, due to its simplicity, it is useful for verification of more advanced alternatives. We refer to the approach as the *brute force approach*.

We proceed in Section 4.3 by observing that the value function for shortfall probability hedging is piecewise constant considered as a function of capital for given stock value (in each step). Alternatively formulated, the value function as a function of the capital can attain a finite number of different values only in each step. We obtain a dramatic reduction in the computation effort as compared to the above brute force approach by utilizing this property systematically for propagating the point selection in the backward recursion in the dynamic programming algorithm and in the minimization within each time step. We refer to the new procedure as the *discrete properties approach*.

It is important to note that the discrete properties approach does not provide a discretization of a continuous problem but rather utilizes discrete properties of the problem and thus leads to calculations that are exact within the limitations of floating point arithmetics.

Further optimization of the algorithms for large scale problems is considered to be outside the scope of this paper. Naturally, it would then be necessary to give up the property that calculations be exact in the above sense by introducing resolution limits on capital and strategies. Moreover, one could consider the vast literature on sub-optimal methodologies, see e.g. Bertsekas (2005), possibly combined with the discrete properties approach.

A detailed derivation of the discrete properties approach for shortfall probability hedging, see Section 4.3 and Appendix A, leads to a dynamic programming algorithm, see Proposition 4.1. For expected shortfall hedging, see Remark 4.2.

4.2 Brute force approach

The straightforward way for solving the multi-period problem by dynamic programming is by introducing a discretization of capital and strategy. Irrespective of the criterion, e.g. shortfall probability or expected shortfall hedging, the value function as a function of v^* is treated as constant between discretization points for v^*. The basis for the brute force approach is outlined below. The method is mainly interesting for verification of and as a first step to more advanced approaches. Since the value of the stock only takes a finite number of values, discretization is irrelevant for the stock. We introduce:

$$\begin{aligned} v_\ell^* &= v_{min} + \ell \Delta v^*, \quad \ell = 0, 1, \ldots, L_v, \\ h_\ell^* &= h_{min}^* + \ell \Delta h^*, \quad \ell = 0, 1, \ldots, L_h, \end{aligned}$$

where (discounted) capital v^* is limited to the interval $[v_{min}^*, v_{max}^*]$ and is discretized with resolution $\Delta v^* = (v_{max}^* - v_{min}^*)/L_v$. Similarly, the strategy is limited to the interval $[h_{min}^*, h_{max}^*]$ with resolution $\Delta h^* = (h_{max}^* - h_{min}^*)/L_h$. For notational convenience, we introduce

$$s_{t,u}^* := s_0(1+\widetilde{b})^u(1+\widetilde{a})^{t-u}, \quad t = 0, 1, \ldots, T, \quad u = 0, 1, \ldots, t, \tag{4.1}$$

which is the (discounted) stock price at time t, given that the number of jumps upwards is u within time steps $1, 2, \ldots, t$, and s_0 is the value of stock at time 0. Hence, for $m = 0, 1, \ldots, L_v$, we have for $t = 0, 1, \ldots, T-1$, the dynamic programming equation corresponding to (3.4),

$$\begin{aligned} J_t(v_m^*, u, y) = \min_{h \in \{h_0^*, h_1^*, \ldots, h_{L_h}^*\}} \Big[\sum_{k=0}^{y} p_{k|y} \qquad\qquad\qquad\qquad & (4.2) \\ \times \Big(p J_{t+1}(v_m^* + h\widetilde{b}s_{t,u}^*, u+1, k) + (1-p) J_{t+1}(v_m^* + h\widetilde{a}s_{t,u}^*, u, k) \Big) \Big], & \end{aligned}$$

where $s_{t,u}^*$ is represented by u in the value function J_t and $p_{k|y}$ is the probability of k surviving policy-holders at the end of a time step, given that the number is y at the start of the step. It is noted that the J-values entering the sum in (4.2) are found by locating the capital value intervals at the prior time $t+1$ (in the recursion) that contain $v_m^* + h\tilde{b}s_{t,u}^*$ and $v_m^* + h\tilde{a}s_{t,u}^*$ corresponding to the case where the value of the stock jumps up and down, respectively.

4.3 Discrete properties approach for shortfall probability hedging

Single-period case

We calculate the shortfall probability after one period starting with y policy-holders, see also Section 2.4. We have the shortfall probability

$$\begin{aligned} P_{sf}(h) &= \sum_{k=0}^{y}(p1_{\{v^*+h\widetilde{b}s^*<k\widetilde{f}(s^*+\widetilde{b}s^*)\}} + (1-p)1_{\{v^*+h\widetilde{a}s^*<k\widetilde{f}(s^*+\widetilde{a}s^*)\}})p_{k|y} \qquad (4.3) \\ &= \sum_{k=0}^{y}(p1_{\{h<h_{b,k}\}} + (1-p)1_{\{h>h_{a,k}\}})p_{k|y}, \end{aligned}$$

where $p_{k|y}$ is the probability of k survivors after one period starting with y survivors and

$$h_{b,k} = \frac{k\widetilde{f}(s^*+\widetilde{b}s^*)-v^*}{\widetilde{b}s^*}, \quad k=0,1,\dots,y, \qquad (4.4)$$

$$h_{a,k} = \frac{k\widetilde{f}(s^*+\widetilde{a}s^*)-v^*}{\widetilde{a}s^*}, \quad k=0,1,\dots,y. \qquad (4.5)$$

We see from (4.3) that $\min_h[P_{sf}(h)]$ is determined by the ordering of the joint sequence consisting of both $h_{a,k}$ and $h_{b,k}$, $k=0,1,\dots,y$. Hence, we may change v^* without affecting $\min_h[P_{sf}(h)]$ as long as the ordering of $h_{a,k}$ and $h_{b,k}$, $k=0,1,\dots,y$ is not changed. The idea is essentially to identify values v^* at (time $T-1$) at which the ordering of the $h_{a,k}$'s and $h_{b,k}$'s is changed. The ordering is changed for v^*-values for which $h_{a,j}=h_{b,k}$, $j=0,1,\dots,y$ and $k=0,1,\dots,y$, i.e. values where

$$\frac{j\widetilde{f}(s^*+\widetilde{a}s^*)-v^*}{\widetilde{a}s^*} = \frac{k\widetilde{f}(s^*+\widetilde{b}s^*)-v^*}{\widetilde{b}s^*},$$

considered separately for different pairs j,k. Thus, we need to focus on values for v^* on the form

$$\frac{j\widetilde{b}\widetilde{f}(s^*+\widetilde{a}s^*)-k\widetilde{a}\widetilde{f}(s^*+\widetilde{b}s^*)}{\widetilde{b}-\widetilde{a}} =: v_{jk}^{*,T-1} \quad ,j=0,1,\dots,y, \quad k=0,1\dots,y. \qquad (4.6)$$

It is noted that the above outlined procedure for selection of v^*-values according to (4.6) can alternatively be obtained as a special case of the procedure for an arbitrary step in the multi-period problem, see Appendix A.1.2.

We introduce the strictly increasing sequence $v_\ell^{*,T-1}$, $\ell=0,1,\dots,L_{T-1}$ where $v_0^{*,T-1} < v_1^{*,T-1} < \cdots < v_{L_{T-1}}^{*,T-1}$ are obtained by rearrangement of $v_{jk}^{*,T-1}$, $j=0,1,\dots,y, k=0,1\dots,y$ and skipping multiples. Hence, $L_{T-1}+1 \le (y+1)^2$ (equality corresponding to no multiples). In order to deal properly with intervals and not only points between intervals, we add a point corresponding to a capital value greater than any of the above values. We choose to add $v_{L_{T-1}+1}^{*,T-1}=\infty$.

Now we are able to limit the number of capital values to be considered in value functions, i.e.,

$$J_{T-1}(v^*,s^*,y) = \sum_{\ell=1}^{L_{T-1}+1} 1_{\{v^*\in[v_{\ell-1}^{*,T-1};v_\ell^{*,T-1})\}} J_{T-1}(\zeta_{\ell,T-1},s^*,y), \qquad (4.7)$$

where $\zeta_{\ell,T-1} \in [v_{\ell-1}^{*,T-1};v_\ell^{*,T-1})$ can be chosen freely. The single step (or first step) determination of capital values is a special case of the general procedure depicted for an arbitrary step in Figure 2. The input points appearing on the vertical axis for the first step are: $v_\ell^{*,t,u+1} = \ell\widetilde{f}(s^* + \widetilde{b}s^*), \ell = 0,\ldots,y$, and $v_\ell^{*,t,u} = \ell\widetilde{f}(s^* + \widetilde{a}s^*), \ell = 0,\ldots,y$.

Multi-period case

We present a proposition for the multi-period case mainly based on the observation that the limitation of capital into a finite number of values in the one-period case propagates to the subsequent time steps in the backward recursion (not necessarily the same capital values and same number of capital values). The results are formulated as Proposition 4.1, and Remarks 4.2–4.4 below.

Proposition 4.1 *The minimum shortfall probability $P_{sf,min} = \min \mathrm{Prob}(\widetilde{f}(S_T^*)Y_T > V_T^*)$, given that the initial capital is $v_m^{*,0,0}$, is $J_0(v_m^{*,0,0}, n)$, which is determined from the subsequent dynamic programming algorithm. The corresponding optimal number of stocks to be purchased at time 0 is the h-value by which $J_0(v_m^{*,0,0}, n)$ is attained. The dynamic programming algorithm:*

$$J_T(v_m^{*,T,u}, y) = 1_{\{y>m\}}, \tag{4.8}$$

where $v_m^{*,T,u} = m\widetilde{f}(s_{T,u}^*)$, $y = 0,1,\ldots,n$, $m = 0,1,\ldots,n$, and $u = 0,1,\ldots,T$. For $t = 0,1,\ldots,T-1$, we have:

$$\begin{aligned} J_t(v_m^{*,u,t}, y) = &\min_{h\in H_a^{(t+1,u)}(v_m^{*,u,t})\cup \widetilde{H}_b^{(t+1,u)}(v_m^{*,u,t})} \Bigg[\sum_{k=0}^{y} p_{k|y}\Big[1_{\{h\in \widetilde{H}_b(v_m^{*,u,t})\}}\times \\ &\sum_{\ell=0}^{L_{t+1,u+1}} 1_{\{h=h_{b,\ell}^{(t+1,u+1)}(v_m^{*,u,t})\}}\Big(pJ_{t+1}(v_\ell^{*,t+1,u+1}, k) + (1-p)J_{t+1}(v^{*,t+1,u+1}(\ell_{a,m}^{(t+1,u)}), k)\Big) \\ + &\sum_{\ell=0}^{L_{t+1,u}} 1_{\{h=h_{a,\ell}^{(t+1,u)}(v_m^{*,u,t})\}}\Big(pJ_{t+1}(v^{*,t+1,u}(\ell_{b,m}^{(t+1,u+1)}), k) + (1-p)J_{t+1}(v_\ell^{*,t+1,u}, k)\Big)\Big]\Bigg], \end{aligned} \tag{4.9}$$

for $m = 0,1,\ldots,L_{t,u}$ with $v_m^{*,t,u}$ the strictly increasing sequence corresponding to

$$v_{\widetilde{m},\hat{m}}^{*,t,u} = \frac{\widetilde{b}v_{\widetilde{m}}^{*,t+1,u} - \widetilde{a}v_{\hat{m}}^{*,t+1,u+1}}{\widetilde{b} - \widetilde{a}}, \quad \widetilde{m} = 0,1,\ldots,L_{t+1,u}, \ \hat{m} = 0,1,\ldots,L_{t+1,u+1},$$

and

$$\begin{aligned} h_{a,\ell}^{(t+1,u)}(v^*) &= \frac{v_\ell^{*,t+1,u} - v^*}{\widetilde{a}s_{t,u}^*}, \quad \ell = 0,1,\ldots,L_{t+1,u}, \\ h_{b,\ell}^{(t+1,u+1)}(v^*) &= \frac{v_\ell^{*,t+1,u+1} - v^*}{\widetilde{b}s_{t,u}^*}, \quad \ell = 0,1,\ldots,L_{t+1,u+1}, \\ \ell_{a,m}^{(t+1,u)} &= \max\{\ell | v_\ell^{*,t+1,u} \le v_m^{*,t,u} + h_{b,\ell}^{(t+1,u+1)}(v_m^{*,t,u})\widetilde{a}s_{t,u}^*\}, \\ \ell_{b,m}^{(t+1,u+1)} &= \max\{\ell | v_\ell^{*,t+1,u+1} \le v_m^{*,t,u} + h_{a,\ell}^{(t+1,u)}(v_m^{*,t,u})\widetilde{b}s_{t,u}^*\}, \\ v^{*,t,u}(\ell) &= v_\ell^{*,t,u}, \\ H_a^{(t+1,u)}(v^*) &= \{h_{a,0}^{(t+1,u)}(v^*), h_{a,1}^{(t+1,u)}(v^*), \ldots, h_{a,L_{t+1,u}}^{(t+1,u)}(v^*)\}, \\ H_b^{(t+1,u)}(v^*) &= \{h_{b,0}^{(t+1,u+1)}(v^*), h_{b,1}^{(t+1,u+1)}(v^*), \ldots, h_{b,L_{t+1,u+1}}^{(t+1,u+1)}(v^*)\}, \\ \widetilde{H}_b^{(t+1,u)}(v^*) &= H_b^{(t+1,u)}(v^*) \setminus H_a^{(t+1,u)}(v^*). \end{aligned}$$

Proof: see Appendix A.

Remark 4.2 An analogous dynamic programming algorithm can be derived for the expected shortfall criterion. The capital value point selection is similar, whereas the use of constant J-functions between selected points is replaced by linearly varying J-functions.

Remark 4.3 According to the proof, see Appendix A.1.2, we have $L_{t,u} + 1 \leq (L_{t+1,u} + 1)(L_{t+1,u+1} + 1)$ with equality in case of no multiples. Note that the number of points increases exponentially. Hence, it is is highly advantageous with regard to performance to carry out a screening after calculation of $J(v_m^{*,t,u})$, $m = 0, 1, \ldots, L_{t,u}$, only keeping $v_m^{*,t,u}$ where the J-function as a function of v^* is discontinuous. After the screening (and renumbering), a reduced vector is kept for the backward recursion for the following dynamic programming step.

Remark 4.4 If the non-negativity constraint on capital, see Section 2.1, is imposed, this is reflected in corresponding constraints on the strategy h. For each h-value, it is checked that the implied capital in the previous step is non-negative, both for jumps up and down for the stock value. If this condition is not satisfied, the h-value is discarded.

Accounting for the number of survivors Y, too, in the capital value points selection procedure might seem relevant since fewer survivors imply fewer points to be included. However, we do not know Y (looking ahead from time 0) and hence must account for the case that all policy-holders survive. Moreover, introducing Y into the point selection procedure complicates matters severely and has not been used. The procedure according to Proposition 4.1 is described in the following. For details, see Appendix A.

Points selection
The procedure for deriving a set of capital values to be considered at time t based on the corresponding values for time $t + 1$ is illustrated in Figure 2. Here, one sees the capital $v_0^{*,t+1,u}, \ldots, v_2^{*,t+1,u}$ and $v_0^{*,t+1,u+1}, \ldots, v_3^{*,t+1,u}$ for u and $u + 1$ jumps upwards realized at time $t + 1$. In this algorithm, $v_0^{*,t+1,u} = v_0^{*,t+1,u+1} = 0$. The values are determined by the points selection procedure in the dynamic programming algorithm. The three lines with slope $-1/(\widetilde{a}s^*_{t,u})$ through one of the points with $v^* = 0$ and $h = v_0^{*,t+1,u}, \ldots, v_2^{*,t+1,u}$ represent $h_{a,\ell}^{*t+1,u}(v^*)$, $\ell = 0, \ldots, 2$, in Proposition 4.1. Similarly, the four lines with slope $1/(\widetilde{b}s^*_{t,u})$ through one of the points with $v^* = 0$ and $h = v_0^{*,t+1,u+1}, \ldots, v_3^{*,t+1,u+1}$, represent $h_{b,\ell}^{(*,t+1,u+1)}(v^*)$, $\ell = 0, \ldots, 3$. The functions $h_{a,\ell}^{(*,t+1,u)}$ and $h_{b,\ell}^{(*,t+1,u+1)}$ determine the number of stocks h to be purchased at time t such that, given that the stock price jumps down and up, respectively, we obtain the ℓ'th capital value according to the points selection procedure for time $t + 1$ in the recursion. Next, we find the intersections between the lines corresponding to $h_{a,\ell}^{(*,t+1,u)}(v^*)$ and $h_{b,\ell}^{(*,t+1,u+1)}(v^*)$. The v^*-values in the intersection points, i.e. $(v_0^{*,t,u}, \ldots, v_8^{*,t,u}) = (v_0, \ldots, v_8)$ in the current example, constitute the set of v^*-values to be considered in the dynamic programming algorithm for time t assuming u upward jumps of the stock up to this time. It is noted that since the v^*-value from the intersection generated by $v_2^{*,t+1,u+1}$ and $v_2^{*,t+1,u}$ coincides with the one from the intersection generated by $v_1^{*,t+1,u+1}$ and $v_1^{*,t+1,u}$, it is discarded (both intersections result in $v_2^{*,t,u} = v_2$ only).

Minimization within each time step
We exploit that we only need to consider the strategies $h_{b,\ell}^{(*,t+1,u+1)}(v^*)$, in case the stock

value jumps up, see Appendix A.1.3. The reason is that we then obtain capital values in the previous step that are selected according to the points selection algorithm. Similarly, we need only consider $h_{a,\ell}^{(*,t+1,u)}(v^*)$, if the stock jumps downs. Hence, the strategies $h_{b,\ell}^{(*,t+1,u+1)}(v^*)$ and $h_{a,\ell}^{(*,t+1,u)}(v^*)$ cover all relevant strategies, and we may pick separately from each group (inner sums in (4.9)). However, since it is unknown in advance whether the stock jumps up or down, capital values have to be located if we are considering $h_{b,\ell}^{(*,t+1,u+1)}(v^*)$, in case the stock jumps down (analogously for $h_{b,\ell}^{(*,t+1,u+1)}(v^*)$). Located capital values appear e.g. in the second J-function in the first inner sum in (4.9).

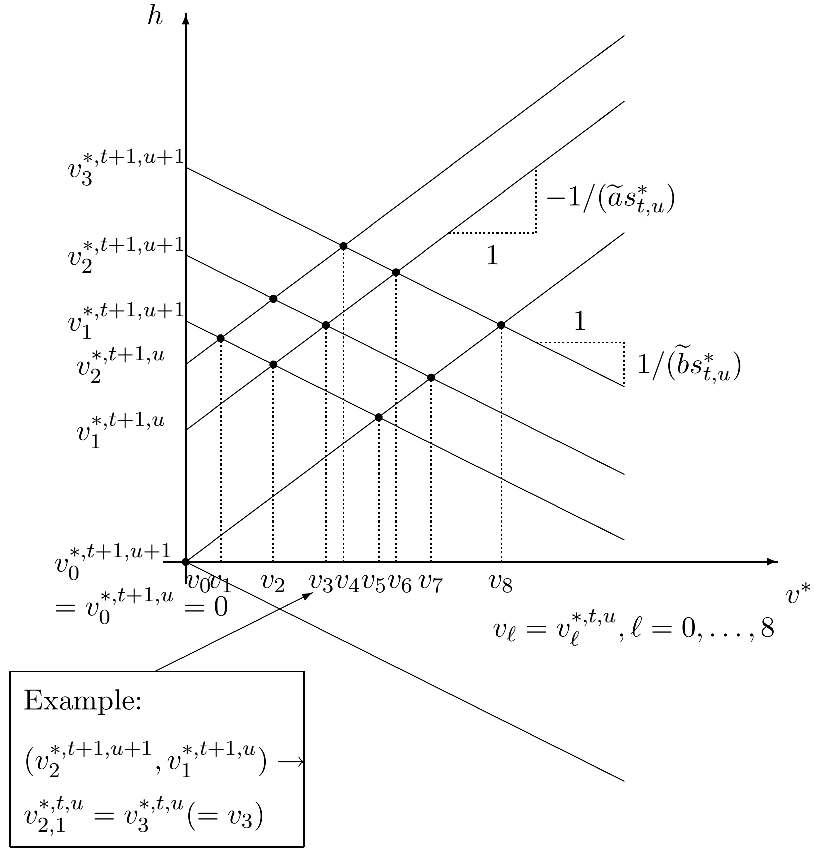

Figure 2: *Illustration of backward recursion for selection of capital values to be considered in the dynamic programming algorithm, Proposition 4.1.*

5 Numerical examples

In this section, a number of numerical examples are presented. Throughout, we impose the non-negativity constraint on capital. In Section 5.1, an example is presented considering all times and time steps with one policy-holder and four periods. It is demonstrated that the strategies for the first time step are equal for the shortfall probability and the expected shortfall hedging criteria and that the strategies corresponding to the same outcomes of the financial market (number of jumps upwards of stock) in subsequent time steps are quite similar. Moreover, comparisons with the quadratic hedging criterion indicate that the above criteria imply that it is optimal to buy more stocks than according to quadratic hedging.

In Section 5.2, a number of examples are presented focusing on the first time (time 0) and the first time step (time step 1). Firstly, expected shortfall hedging with 8 or 9 policy-holders and one period is treated. It seems that adding a single policy-holder may affect the optimal strategy significantly. Secondly, the focus is on observing the number of survivors or not. An example calculation (not reported herein) for shortfall probability hedging with two policy-holders and two periods indicates that the effect from observations is moderate. However, taking the survival probability to be stochastic with a uniform prior, see Section 3.1, in an example with three policy-holders and four periods shows that the importance of observing the number of policy-holders *can* be high. Finally, an example with three policy-holders and three periods, covering both shortfall probability and expected shortfall hedging, is reported. It is demonstrated that the value for p is of great significance. Moreover, the results from shortfall probability and from expected shortfall hedging are qualitatively not very different but there are differences with regard to complexity of strategy versus start capital and the effect from optimal investment relative to no action.

Unless otherwise stated, the following parameter values are applied: relative (signed) stock jump down $a = -0.10$, relative stock jump up $b = 0.15$, probability of stock jumping up $p = 0.7$, risk-free interest rate $r = 0$, purely financial claim at time T, $f(S_T) = \max(S_T, K)$ with $K = 100$, initial value of stock $s_0 = 100$, mortality-intensity $\mu = 0.25$, and initial capital $v_0 = 100$.

5.1 Example considering all times and steps

We study the case with one policy-holder and four periods. The aim is to illustrate optimal strategies and the corresponding shortfall probabilities. Depending on how the financial market develops, the strategies and minimization results for all time steps are shown. The strategy also reflects the development of the number of survivors. However, since we impose the non-negativity constraint on capital, zero survivors trivially lead to zero shortfall probability. Hence, we only show results conditional on survival of the (one) policy-holder.

Even though strategies are non-unique, we may set up rules for selection of a single strategy in each time step, thus obtaining a unique strategy. Without much consideration on the choice of such a rule, we adopt the following principle: Out of all optimal strategies, we select the one which is numerically smallest. Value functions are found for a single capital value in each time point, where this capital value follows from a *forward* calculation as opposed to the backward calculation in the dynamic programming algorithm.

In Figure 3, optimal strategies and minimum shortfall probabilities from shortfall probability hedging are shown. Optimal strategies and corresponding shortfall probabilities are shown above and below the tree nodes. It is noted that whereas the value of the stock can be represented by a recombining tree, the present trees are not recombining.

At time points 0, 1, and 2, super-hedging does not occur. At time point 3, it occurs only if the stock value has jumped up in all three of the preceding time steps (jump sequence 111). At time point 4, 7 jump sequences imply super-hedging. Out of 6 sequences with 2 jumps up, only the 0011 and 0101 sequences give super-hedging, i.e. the two sequences with two jumps up that are located as late as possible. Moreover, all of the 4 sequences with 3 jumps up and the one sequence with 4 jumps up lead to super-hedging.

Calculations as in Figure 3 based on expected shortfall have been carried out but are not included here. However, it is mentioned that the two hedging criteria give the same

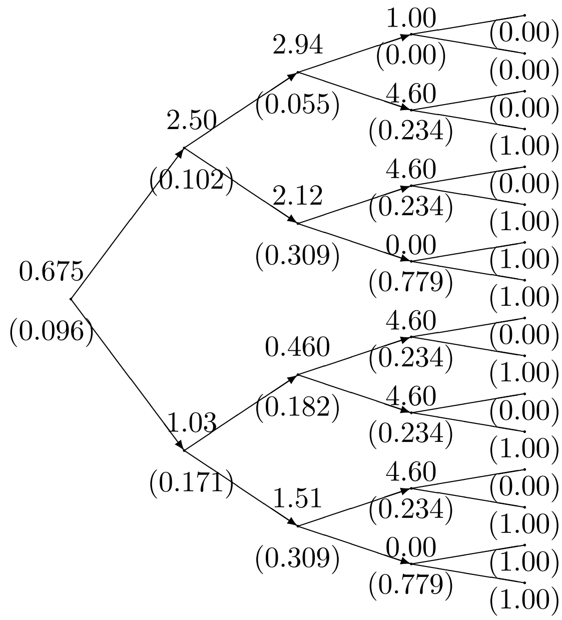

Figure 3: *Tree showing the optimal strategy (the upper numbers) and the minimum shortfall probability (lower numbers) in case of shortfall probability hedging in example with one policy-holder and four periods. The results are conditional on survival of the policy-holder (up to the time in question).*

strategy at time 0. Furthermore, the strategies according to the two hedging criteria are not very different at later time points, and the outcomes of the financial market that lead to super-hedging are common for the two criteria.

In Figure 4, strategies corresponding to shortfall probability hedging and quadratic hedging, as described in Møller (2001a) (interest rate set to 0), are compared. It appears that the initial number of stocks is more than three times bigger for shortfall probability hedging than quadratic hedging. At later time points, this factor is in most cases even bigger. However, later time points are not directly comparable, since the quadratic hedging strategy depicted is not self-financing, i.e. it is based on different capital at later time points. Intuitively, the observation that shortfall probability hedging results in a larger strategy than quadratic hedging may be explained by the fact that with quadratic hedging, a large strategy may lead to a large amount of redundant capital which is punished just as much as lack of capital. With the shortfall probability strategy, such considerations on redundant capital are irrelevant, since redundant capital is not punished according to this strategy.

5.2 Illustrations for the first time point and time step

Eight or nine policy-holders and one period
In this section, we study two cases with a single period, one with 8 and one other with 9 policy-holders. The purpose is to demonstrate that significant differences in optimal strategies can occur in cases that appear to be very similar.

In Figure 5, the optimal strategy h versus start capital v is shown for 9 policy-holders. In the following, we focus on large initial values for the initial capital such that the non-negativity constraint on capital is not active. Hence, we consider initial capital values of

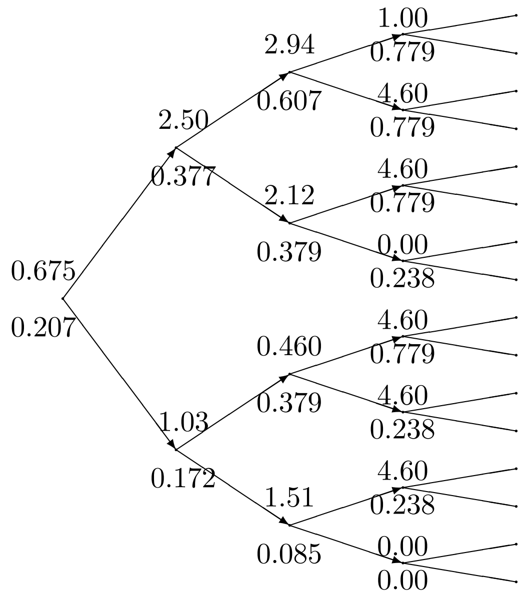

Figure 4: *Tree showing optimal strategies according to shortfall probability hedging (the upper numbers) and quadratic hedging (lower numbers) in example with one policy-holder and four periods. The results are conditional on survival of the policy-holder (up to the time in question).*

400 or larger. It appears that for $n = 9$ policy-holders, it is always optimal to choose a strategy which hedges survival of all n policy-holders when the stock value jumps up, except for two intervals of capital, where the triangles in the figure are located below the line defining the bottom sides the other triangles. Below, we refer to the two v-intervals by 1 and 2 (1 for the interval with the larges v-values). In these intervals, the survival of only $n-1 = 8$ policy-holders, is hedged in case the stock value jumps up. In case of $n = 8$ policy-holders (not shown), no such exceptions occur, i.e. in the considered capital levels, it is always optimal to hedge all n policy-holders.

The above mentioned differences can be justified by looking at the probabilities more closely. Below we show that for $n = 9$, referring to the above v-intervals, 1 and 2:

1 It is more optimal to 1-i) hedge $n-1$ survivors, irrespective of the outcome for the stock than to 1-ii) hedge n and $n-2$ survivors provided that the stock value jumps up and down, respectively.

2 It is more optimal to 2-i) hedge $n-1$ and $n-2$ survivors provided that the stock value jumps up and down, respectively, than to 2-ii) hedge n and $n-3$ survivors provided that the stock value jumps up and down, respectively.

With $p_{y|n}$, the probability of y survivors, given that the number of policy-holders is n, we find the shortfall probability using the strategies 1-i) and 1-ii) introduced above:

$$\begin{aligned} P_{sf,1i} &= pp_{n|n} + (1-p)p_{n|n} = p_{n|n}, \\ P_{sf,1ii} &= p \times 0 + (1-p)(p_{n-1|n} + p_{n|n}) = (1-p)(p_{n-1|n} + p_{n|n}). \end{aligned}$$

Strategy i) is more optimal than ii) if

$$\Delta P_{sf,1} =: P_{sf,1ii} - P_{sf,1i} = (1-p)p_{n-1|n} - pp_{n|n} > 0.$$

In our case with $p_{n|y}$ being the Binomial probability distribution function with parameters (n, q) (and argument y), we find $\Delta P_{sf,1} = q^{n-1}[(1-p)(1-q)n - pq]$, i.e. $P_{sf,1} > 0$, i.e. strategy 1-i) is more optimal than 1-ii), for $n \geq n_1$ where $n_1 = \min\{n \in I\!\!N | n \geq \frac{pq}{(1-p)(1-q)}\}$. With $q = \exp(-0.25)$ and $p = 0.7$, we find $n_1 = 9$. Hence, for $n = 8$ policy-holders, strategy 1-ii) is optimal, whereas strategy 1-i) is optimal for $n = 9$. Similarly, one finds that strategy 2-i) is more optimal than 2-ii) for $n \geq n_2$ where $n_2 = \min\{n \in I\!\!N | n \geq \frac{2pq^2}{(1-q)^2}\}$. Again, with $q = \exp(-0.25)$ and $p = 0.7$, $n_2 = 9$.

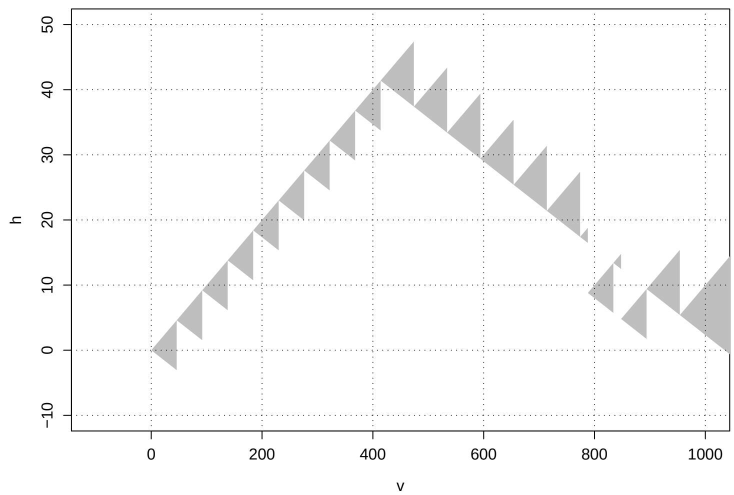

Figure 5: *Optimal strategy h versus start capital v in example with 9 policy-holders and 1 period using shortfall probability hedging.*

Three policy-holders and four periods - importance of restrictions to observations
We adopt a stochastic survival probability as discussed in Section 3.1. We consider 4 periods and 3 policy-holders. Shortfall probabilities in case of observations and no observations are shown in Figure 6. A considerable reduction of the shortfall probability obtained through observations is seen. For start capital 200 and above, the probability is roughly reduced by 50 %. Naturally, choosing a uniform prior as in this example would be too conservative in most applications. The interpretation is then that for the case considered, we have roughly an upper bound for the effect from observations.

Three policy-holders and three periods
Here we study an example with minimum shortfall probabilities and expected shortfall and corresponding strategies for the case with three policy-holders and three periods.

Optimal strategies for shortfall probability hedging are given in Figure 7 for $p = 0.7$. The corresponding minimum shortfall probabilities are shown in Figure 8 for $p = 0.7$ and $p = 0.5$, including shortfall probabilities with no action for comparison. It appears that the relationship between non-unique strategies and start capital level is rather complicated.

Even though the shortfall probability without investment is higher for $p = 0.7$ than for $p = 0.5$, the minimum shortfall probability from optimal investments is much lower with $p = 0.7$ than $p = 0.5$, see Figure 8, as expected due to the more efficient control possibility.

Analogous results based on expected shortfall hedging are reported in Figure 9 and 10. The

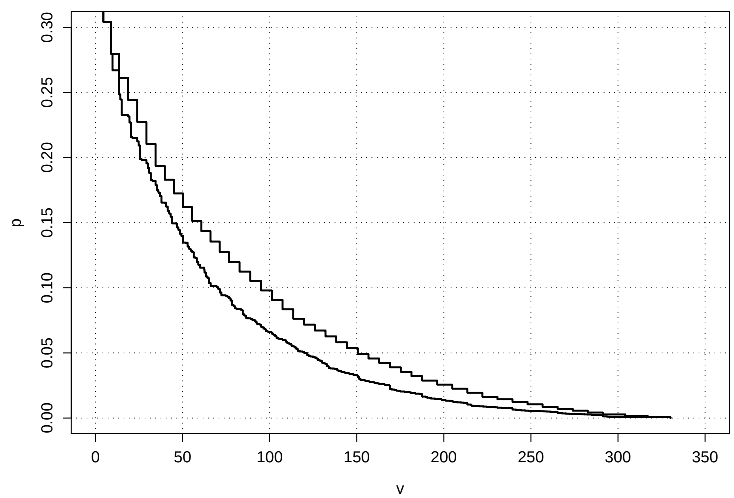

Figure 6: *Minimum shortfall probability p versus start capital v in example with three policy-holders and four periods. The one-period survival probability is stochastic with a prior uniformly distributed on [0;1]. From top: 1) optimal investment without observation of survivors, and 2) optimal investment with observation of survivors.*

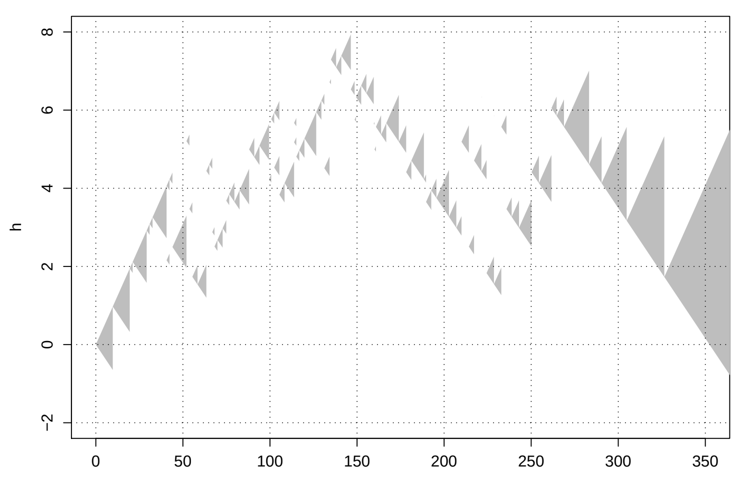

Figure 7: *Optimal strategy h versus start capital v in example with 3 policy-holders and 3 periods using shortfall probability hedging. Probability of the stock value jumping up* $p = 0.7$.

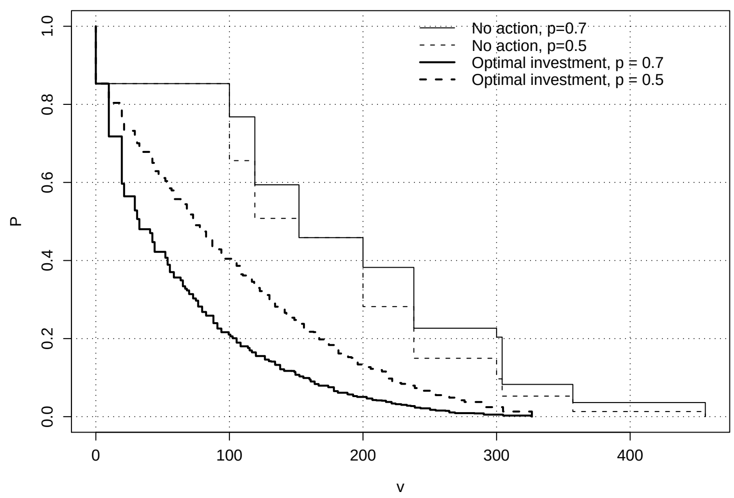

Figure 8: *Minimum shortfall probability P versus start capital v in example with 3 policy-holders and 3 periods using shortfall probability hedging. The probability of stock jumping up is $p = 0.5$ or $p = 0.7$. For comparison, shortfall probabilities without investment are included.*

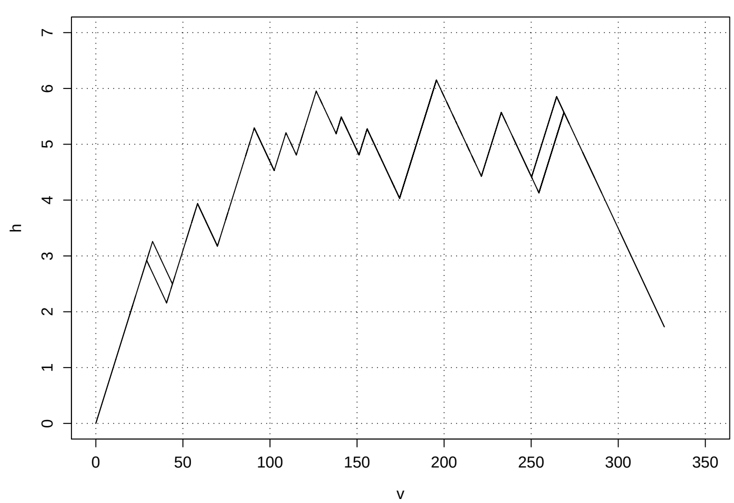

Figure 9: *Optimal strategy h versus start capital v in example with 3 policy-holders and 3 periods using expected shortfall hedging. Probability of the stock value jumping up $p = 0.7$. Super-hedging strategies are not shown (same as for shortfall probability hedging, see Figure 7). Areas that are enclosed by lines should be interpreted as zones with optimal h versus v.*

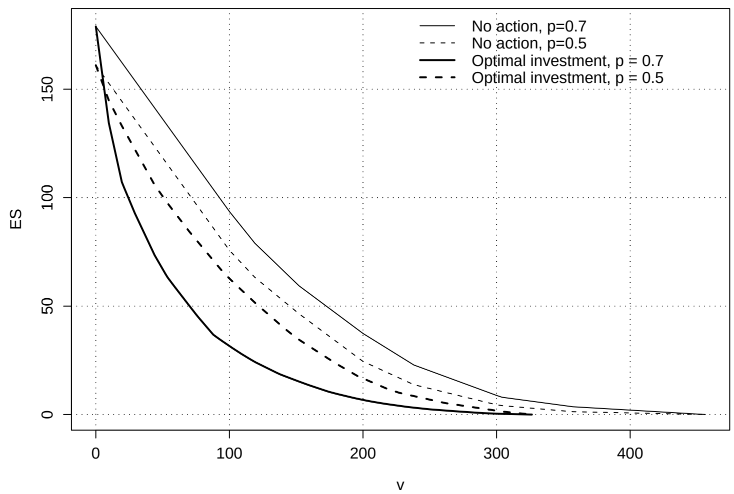

Figure 10: *Minimum expected shortfall ES versus start capital v in example with 3 policyholders and 3 periods using expected shortfall hedging. The probability of stock jumping up is $p = 0.5$ or $p = 0.7$. For comparison, expected shortfalls ES without investment are included.*

qualitative findings are similar to those above from shortfall probability hedging. However, the optimal strategies from expected shortfall hedging as functions of start capital are more unique and less complicated. Moreover, the gain with respect to the expected shortfall from optimal investment as compared to no action is very moderate for $p = 0.5$.

In conclusion, the results indicate that the choice of p-value is crucial for optimal strategies, shortfall probabilities, and expected shortfalls. Different values for sizes of jumps in stock values (derived from parameters a and b) have not been considered here. However, it is obvious that parameter triples (a, b, p) that imply the same expected value for the value of the stock after a jump (for given start value), will give significantly different results for varying p values.

A Appendix

A.1 Proof for the dynamic programming algorithm, Proposition 4.1

The proof considers mainly three issues: i) Single period case generalized, ii) Propagation of points selection procedure, and iii) Minimization within each time step.

A.1.1 Single period case generalized

It is necessary to keep track of the particular s^*-values used for finding v^*_{kj}. As a generalization of (4.6) using $s^*_{t,u}$ defined by (4.1), we introduce

$$\begin{aligned} v^{*,T-1,u}_{jk} &= \frac{j\widetilde{b}\widetilde{f}(s^*_{T-1,u} + \widetilde{a}s^*_{T-1,u}) - k\widetilde{a}\widetilde{f}(s^*_{T-1,u} + \widetilde{b}s^*_{T-1,u})}{\widetilde{b} - \widetilde{a}} \qquad \text{(A.1)} \\ &= \frac{j\widetilde{b}\widetilde{f}(s^*_{T,u}) - k\widetilde{a}\widetilde{f}(s^*_{T,u+1})}{\widetilde{b} - \widetilde{a}}, \quad j = 0, 1, \ldots, y, k = 0, 1 \ldots, y, \end{aligned}$$

where $u = 0, 1, \ldots, T-1$. The $v_{jk}^{*,T-1,u}$-values are rearranged into $v_\ell^{*,T-1,u}$, $\ell = 0, 1, \ldots, L_{T-1,u}$, where $v_1^{*,T-1,u} < v_2^{*,T-1,u} < \cdots < v_{L_{T-1,u}}^{*,T-1,u}$. We set $v_{L_{T-1,u}+1}^{*,T-1,u} = \infty$.

A.1.2 Propagation of points selection procedure

In the following, the state component corresponding to the discounted stock value is replaced by the number u of jumps upwards for the stock (up to the time considered). Hence,

$$J_t(v^*, u, y) = \sum_{\ell=1}^{L_{t,u}+1} 1_{\{v^* \in [v_{\ell-1}^{*,t,u}; v_\ell^{*,t,u})\}} J_t(\zeta_{\ell,t,u}, u, y), \tag{A.2}$$

where $t = T-1$, $\zeta_{\ell,t,u} \in [v_{\ell-1}^{*,t,u}, v_\ell^{*,t,u})$. We may choose $\zeta_{\ell,t,u} = v_{\ell-1}^{*,t,u}$, $\ell = 1, 2, \ldots, L_{t,u}$, $t = 0, 1, \ldots, T$, and $u = 0, 1, \ldots, t$.

It appears that different capital values are to be considered depending on time and the number of jumps upwards of the stock up to the time. In each step of the dynamic programming algorithm, we need to determine

$$\begin{aligned} J_t(v^*, u, y) = \min_h \Bigg[& \sum_{k=0}^{y} p_{y|k} \\ & \times \Bigg\{ p \sum_{\ell=1}^{L_{t+1,u+1}+1} 1_{\{v^* + h\widetilde{b}s_{t,u}^* \in [v_{\ell-1}^{*,t+1,u+1}; v_\ell^{*,t+1,u+1})\}} J_{t+1}(v_{\ell-1}^{*,t+1,u+1}, u+1, k) \\ & +(1-p) \sum_{\ell=1}^{L_{t+1,u}+1} 1_{\{v^* + h\widetilde{a}s_{t,u}^* \in [v_{\ell-1}^{*,t+1,u}; v_\ell^{*,t+1,u})\}} J_{t+1}(v_{\ell-1}^{*,t+1,u}, u, k) \Bigg\} \Bigg]. \end{aligned} \tag{A.3}$$

We introduce:

$$\begin{aligned} h_{a,\ell}^{(t+1,u)}(v^*) &= \frac{v_\ell^{*,t+1,u} - v^*}{\widetilde{a}s_{t,u}^*}, \quad \ell = 0, 1, \ldots, L_{t+1,u}, \\ h_{b,\ell}^{(t+1,u+1)}(v^*) &= \frac{v_\ell^{*,t+1,u+1} - v^*}{\widetilde{b}s_{t,u}^*}, \quad \ell = 0, 1, \ldots, L_{t+1,u+1}, \end{aligned} \tag{A.4}$$

where $t = 0, 1, \ldots, T-1$ and $u = 0, 1, \ldots, t$. Thus:

$$\begin{aligned} J_t(v^*, u, y) = \min_h \Bigg[& \sum_{k=0}^{y} p_{k|y} \\ & \times \Bigg\{ p \sum_{\ell=1}^{L_{t+1,u+1}+1} 1_{\{h \in [h_{b,\ell-1}^{(t+1,u+1)}(v^*); h_{b,\ell}^{(t+1,u+1)}(v^*))\}} J_{t+1}(v_{\ell-1}^{*,t+1,u+1}, u+1, k) \\ + \quad & (1-p) \sum_{\ell=1}^{L_{t+1,u}+1} 1_{\{h \in (h_{a,\ell}^{(t+1,u)}(v^*); h_{a,\ell-1}^{(t+1,u)}(v^*)]\}} J_{t+1}(v_{\ell-1}^{*,t+1,u}, u, k) \Bigg\} \Bigg]. \end{aligned} \tag{A.5}$$

It is seen that as a function of h, each of the two inner sums in (A.5) is piecewise constant. We may rewrite (A.3) as

$$J_t(v^*, u, y) = \min_h \left[\sum_{k=0}^{y} p_{y|k} \Big(p w_{t+1}(v^* + h\widetilde{b}s_{t,u}^*, u+1, k) + (1-p) w_{t+1}(v^* + h\widetilde{a}s_{t,u}^*, u, k) \Big) \right], \tag{A.6}$$

where we have introduced:

$$w_t(v^*, u, y) = \sum_{\ell=1}^{L_{t,u}+1} 1_{\{v^* \in [v_{\ell-1}^{*,t,u}; v_\ell^{*,t,u})\}} J_t(v_{\ell-1}^{*,t,u}, u, y). \tag{A.7}$$

Utilizing that the inner sums in (A.5) are piecewise constant a functions of h, $w_{t+1}(v^* + h\widetilde{a}s_{t,u}^*, u, k)$ and $w_{t+1}(v^* + h\widetilde{b}s_{t,u}^*, u+1, k)$ in (A.6) are illustrated as functions of h in Figure 11. For increasing v^*, $w_{t+1}(v^* + h\widetilde{a}s_{t,u}^*, u, k)$ and $w_{t+1}(v^* + h\widetilde{b}s_{t,u}^*, u+1, k)$ are shifted to the right and left, respectively (exemplified through $v^* = \alpha$ and $v^* = \beta$, where $\beta > \alpha$). The v^*-values for which the J-function in the l.h.s. of (A.5) is discontinuous as a function of v^* (constant between such values) are within the list of v^*-values for which both $w_{t+1}(v^* + h\widetilde{a}s_{t,u}^*, u, k)$ and $w_{t+1}(v^* + h\widetilde{b}s_{t,u}^*, u+1, k)$ as functions of h are discontinuous. The solutions with respect to v^* of $h_{a,\ell}^{(t+1,u)}(v^*) = h_{b,m}^{(t+1,u+1)}(v^*)$, $\ell = 0, 1, \ldots, L_{t+1,u}$ and $m = 0, 1, \ldots, L_{t+1,u+1}$, provide the potential discontinuity v^*-values, i.e.,

$$\frac{v_\ell^{*,t+1,u} - v^*}{\widetilde{a}s_{t,u}^*} = \frac{v_m^{*,t+1,u+1} - v^*}{\widetilde{b}s_{t,u}^*} \Leftrightarrow \tag{A.8}$$

$$v^* = \frac{\widetilde{b}v_\ell^{*,t+1,u} - \widetilde{a}v_m^{*,t+1,u+1}}{\widetilde{b} - \widetilde{a}} := v_{\ell m}^{*,t,u},$$

where $\ell = 0, 1, \ldots, L_{t+1,u}$, $m = 0, 1, \ldots, L_{t+1,u+1}$, $t = 0, 1, \ldots, T-2$, and $u = 0, 1, \ldots, t$. As earlier, we rearrange $v_{\ell m}^{*,t,u}$, $\ell = 0, 1, \ldots, L_{t+1,u}, m = 0, 1, \ldots, L_{t+1,u+1}$ into a strictly increasing sequence $v_\ell^{*,t,u}$, $\ell = 0, 1, \ldots, L_{t,u}$ where $v_1^{*,t,u} < v_2^{*,t,u} < \cdots < v_{L_{t,u}}^{*,t,u}$ skipping multiples, implying that $L_{t,u} + 1 \leq (L_{t+1,u} + 1)(L_{t+1,u+1} + 1)$. We add $v_{L_{t,u}+1}^{*,t,u} = \infty$. Hence, we are evaluating J_t only at the given v^* values.

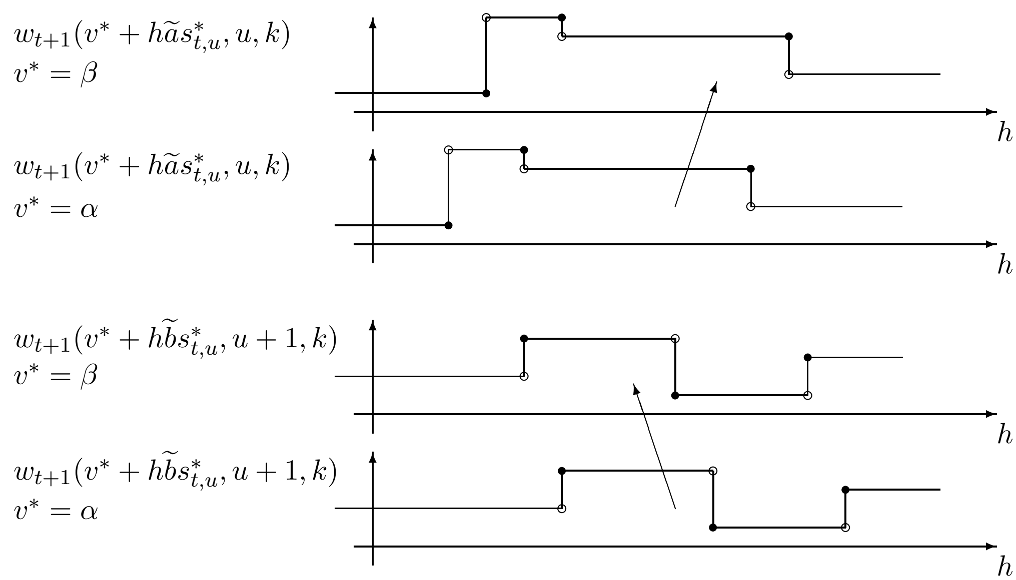

Figure 11: *Illustration of the piecewise constant value function.*

A.1.3 Minimization within each time step

Now we exploit that the two inner sums in (A.5) are piecewise constant with respect to capital in the minimization procedure. Only $h_{b,\ell}^{(t+1,u+1)}(v^*)$, $\ell = 0, 1, \ldots, L_{t+1,u+1}$ need to be considered if the value of the stocks jumps up. Similarly, only $h_{a,\ell}^{(t+1,u)}(v^*)$, $\ell = 0, 1, \ldots, L_{t+1,u}$ need to be considered if the stock value jumps down. Hence, taking e.g. $h = h_{b,\ell}^{(t+1,u+1)}(v^*)$ implies that we immediately have the J-value, $J(v_{t+1,\ell}^{*,t+1,u+1}, u+1, k)$ available in the dynamic programming step to be used if the stock jumps up. However, taking $h = h_{b,\ell}^{(t+1,u+1)}(v^*)$ implies that if the stock jumps down, the J-value is not directly available. Thus, we have to locate the interval defined by two consecutive points amongst $v_\ell^{*,t,u}$, $\ell = 0, 1, \ldots, L_{t+1,u}$, which contains $v_m^{*,t,u} + h_{b,\ell}^{(t+1,u+1)}(v^*)\widetilde{a}s_{t,u}^*$ where we start out with a (discounted) capital value $v_m^{*,t,u}$, $m = 0, 1, \ldots, L_{t,u}$, according to the above backward recursion scheme. The interval in question is

$$[v^{*,t+1,u}(\ell_{a,m}^{(t+1,u)}); v^{*,t+1,u}(1 + \ell_{a,m}^{(t+1,u)})),$$

where we for convenience have introduced the alternative notation $v^{*,t,u}(\ell) = v_\ell^{*,t,u}$ and

$$\ell_{a,m}^{(t+1,u)} = \max\{\ell | v_\ell^{*,t+1,u} \leq v_m^{*,t,u} + h_{b,\ell}^{(t+1,u+1)}(v_m^{*,t,u})\widetilde{a}s_{t,u}^*\}.$$

By choosing $h = h_{a,\ell}^{(t+1,u)}$, $\ell = 0, 1, \ldots, L_{t+1,u}$, we make completely analogous considerations, i.e. we find the interval defined by two consecutive points amongst $v_\ell^{*,t+1,u+1}$, $\ell = 0, 1, \ldots, L_{t+1,u+1}$, containing $v_m^{*,t,u} + h_{a,\ell}^{(t+1,u)}\widetilde{b}s_{t,u}^*$. The interval is

$$[v^{*,t+1,u+1}(\ell_{b,m}^{(t+1,u+1)}); v^{*,t+1,u+1}(1 + \ell_{b,m}^{(t+1,u+1)})),$$

where

$$\ell_{b,m}^{(t+1,u+1)} = \max\{\ell | v_\ell^{*,t+1,u+1} \leq v_m^{*,t,u} + h_{a,\ell}^{(t+1,u)}(v_m^{*,t,u})\widetilde{b}s_{t,u}^*\}.$$

A.1.4 Concluding remarks

Combining Sections A.1.1, A.1.2, and A.1.3 concludes the proof. However, two comments are in place. Is is easily verified that the J-functions in the l.h.s. of (4.8) and (4.9) are only indirectly dependent on u (through capital values, selected according to the algorithm, that are depending on u). Moreover, it is straightforward to verify Assumption 3.2 for an arbitrary step in the dynamic programming algorithm, since the J-functions, at any given time t, are piecewise constant functions with a finite number of values and $g : \mathcal{X} \to \mathcal{H}$ in (3.8) may be chosen as e.g. a piecewise continuous function $\widetilde{g} : \mathcal{X} \to I\!R$, i.e. a measurable function.

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