Fair Distribution of Assets in Life Insurance

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Abstract

When a life insurance company distributes assets between the equity capital and the portfolio of insured, possible periodic guarantees to the insured must be covered whenever possible. Hence, depending on the development of the financial market and the portfolio of insured, the equity capital may experience periods with low or even negative payoffs. In the worst case scenario, where the guarantee can not be covered, the company is declared bankrupt, and the entire equity capital is lost. To compensate the owners for the risk of low returns on equity capital, the equity capital should be accumulated by a rate, which exceeds the riskfree rate in periods, where the investment return and development of the insurance portfolio allows for such a high return on equity capital. We consider an insurance company with a very simple insurance portfolio: It consists of either capital insurances or pure endowments. The financial market is described by a Black–Scholes model. Given an investment strategy for the company, the principle of no arbitrage gives an equation for the fair additional payoff to the equity capital in periods, when such an additional payoff is possible. The investment strategies considered are: A buy and hold strategy and a strategy with constant relative portfolio weights, both with and without stop-loss in case solvency is threatened. To investigate the magnitude of the fair additional rate of interest and the dependence on parameter values, initial distribution of capital and investment strategy, we supply numerical results.

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1 Introduction

When issuing life insurance contracts with a guarantee, the insurance companies are exposed to a risk, since the guarantee must be covered whenever possible. The two most common types of guarantees are: A maturity guarantee, where the company guarantees a minimal total accumulation for the entire duration of the contract, and guaranteed periodic accumulation factors (guaranteed periodic interest rates), where the company guarantees a minimal accumulation factor for each period. Even though the most common type of guarantee in Denmark is a maturity guarantee, we consider the case of guaranteed periodic accumulation factors, since it allows us to consider each accumulation period independently. When guaranteeing periodic accumulation factors, the equity capital of the company might experience low or even negative payoffs in periods with low returns on investments and/or an adverse development of the insurance portfolio. In the extreme case, where the guarantee can not be covered, the interest of the insured take precedence over the interests of the company, and all assets are paid to the insured.

Guaranteed periodic accumulation factors implicitly introduce a string of European call options on the investment gain in the insurance contract. Historically, the guarantees have in practice been chosen far out of the money, and therefore they have been ignored when pricing the insurance contracts. However, the decreasing interest rates in recent years has caused the guarantees to become an important element of some old contracts. This, in turn, has increased the importance for correct pricing of the options imbedded in the insurance contracts, see e.g. Briys and de Varenne (1997), Aase and Persson (1997), Miltersen and Persson (1999) and Bacinello (2001). In practice, insurance companies use a bonus account for undistributed surplus in order to smooth the accumulation factors over time. When including a bonus account, the price of an insurance contract depends on the bonus mechanism. For some different possible bonus mechanisms and their impact on prices, see Grosen and Jørgensen (2000), Hansen and Miltersen (2002) and Miltersen and Persson (2003). Another feature encountered in practice is the possibility for the insured to surrender, which is included e.g. in Grosen and Jørgensen (2000). The bankruptcy of major life insurance companies in England and Japan have also underlined the importance of including the risk of the company defaulting. This is done in Briys and de Varenne (1997).

The main purpose of the above mentioned papers is essentially to obtain the arbitrage free price of an insurance contract by considering the development of the insurance contract until termination. The aim of the present paper is slightly different from that of pricing individual contracts. Here, the goal is to determine a fair distribution of assets between the owners of the insurance company and the portfolio of insured at the end of each accumulation period. Thus, the model considered is essentially a 1-period model with one accumulation period. In the model the accumulation factor, announced by the company prior to the accumulation period, is viewed as an exogenous parameter. Hence, we avoid the modelling of the announced accumulation factor, which is quite difficult since competition seems to play a major role in the decision process. In contrast to many companies, we do not view the announced accumulation factor as binding. Thus, the actual and announced accumulation factors may differ when experiencing poor investment returns and/or an adverse development of the insurance portfolio. To determine the distribution of the assets between the deposit, the bonus reserve and the equity capital at the end of the accumulation period, we define a distribution scheme. Within this scheme, the only

unknown parameter is the interest rate used, in addition to the riskfree interest rate, to accumulate the equity capital in periods when possible. We assume that the company is allowed to invest in a financial market described by a Black-Scholes model. This market is known to be complete and arbitrage free. A distribution scheme is considered as fair, if it does not introduce arbitrage possibilities for the owners or the insurance portfolio. When considering a portfolio of capital insurances, the distribution scheme depends entirely on the development of the financial market, and since the financial market is complete and arbitrage free, we can derive a simple equation, which has to be fulfilled by a distribution scheme in order not to introduce arbitrage possibilities. Thus, we are able to find an equation for the unique fair additional interest rate. For a portfolio of pure endowments the distribution scheme depends on both the financial market and the development of the insurance portfolio. Hence, we are in an incomplete market. Thus, infinitely many equivalent martingale measures exist, such that the principle of no arbitrage yields infinitely many possible equations from which to derive a fair distribution. However, for a fixed equivalent martingale measure, we again have a unique equation for the fair additional interest rate. Since the equations derived for the fair additional interest rate are implicit equations, we have to use numerical techniques to derive the result. Hence, in contrast to other papers including bonus accounts, no simulation is necessary.

We point out that the results in this paper for the fair additional interest rate are based on a simple financial model with constant interest rate and a deterministic mortality intensity. Hence, we only take the financial risk associated with investments in stocks and the unsystematic mortality risk into account. The fair additional interest rate would be larger if we were to add interest rate risk and/or systematic mortality risk to the model. Note that we distinguish between systematic mortality risk, referring to the future development of the underlying mortality intensity, and unsystematic mortality risk, referring to a possible adverse development of the insured portfolio with known mortality intensity, see Dahl (2004). Furthermore expenses and the associated risk have been disregarded in the study. In addition to the measurable risks mentioned above one could consider operational risk as well. Thus, the fair additional interest rate determined in this paper serves as a lower bound for the fair additional interest rate in practice.

The paper is organized as follows: In Section 2, a simplified balance sheet and a short description of the accounts are given. The financial model and the relevant financial terminology is introduced in Section 3. In Section 4, a company with an insurance portfolio of capital insurances is considered. Given different investment strategies, we decompose the terminal equity capital into payoffs from standard options, such that each investment strategy leads to an equation for the fair additional interest rate. Section 5 studies the case of a portfolio of pure endowments. In this case, the value for the fair additional interest rate depends on the chosen equivalent martingale measure. Since the equations obtained in Sections 4 and 5 for the the fair additional interest rate are implicit equations only, we supply numerical results in Section 6. In Section 7 we discuss some possible changes to the distribution mechanism and their impact on the results. A discussion on the realism and versatility of the model is given in Section 8, and Section 9 concludes the study.

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2 The balance sheet

To describe the assets and liabilities of the insurance company we use the following simplified balance sheet.

Assets	Liabilities
A	V
	U
	E
A	A

The asset side consists of the account A only, while the liability side is comprised of three accounts: V, U and E. The bottom line of the balance sheet just states that the assets and liabilities must balance, i.e. $V + U + E = A$. We now give a detailed description of the individual accounts.

Account V (the deposit) is the total deposit of the insurance portfolio. The deposit is allocated to the insured on an individual basis. In case of a capital insurance or a pure endowment, the individual deposit at time of termination is the sum paid to the insured. Whenever an insurance contract states a guaranteed periodic accumulation factor, the guarantee applies to the deposit. Capital allocated to the deposit belongs to the individual owning the actual account, and cannot be transferred to the deposit of another insured or other accounts on the liability side.

Account U (the bonus reserve) is the undistributed surplus allocated to the insurance portfolio as a whole. It is used by the company to smooth deposit accumulation factors over time. Capital allocated to the bonus reserve cannot freely be transferred to the equity capital. Such a transfer may only take place as a payment to the equity capital for the risk associated with the insurance contracts.

Account E (the equity capital) is the capital belonging to the owners of the company.

Account A (the assets) describes the value of the assets of the insurance company. We assume that the insurance company invests in the financial market described in Section 3. In order to consider the risk associated with the insurance contracts only, we assume that the company invests the amount E_0 in the savings account, and the amount $V_0 + U_0$ in an admissible strategy $\varphi = (\vartheta, \eta)$ with value process $\mathcal{V}(\varphi)$. Thus, at time t, $t \in [0, T]$, we have

$$A_t = e^{rt} E_0 + \frac{\mathcal{V}_t(\varphi)}{\mathcal{V}_0(\varphi)}(V_0 + U_0).$$

It now follows from the following argument that we without loss of generality may assume that

$$V_0 + U_0 = \mathcal{V}_0(\varphi) \qquad (2.1)$$

such that $A_t = e^{rt} E_0 + \mathcal{V}_t(\varphi)$. Assume that (2.1) does not hold. Then the self-financing strategy given by

$$\widetilde{\varphi} = \frac{V_0 + U_0}{\mathcal{V}_0(\varphi)} \varphi = \left(\frac{V_0 + U_0}{\mathcal{V}_0(\varphi)} \vartheta, \frac{V_0 + U_0}{\mathcal{V}_0(\varphi)} \eta \right)$$

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fulfills

$$e^{rt}E_0 + \mathcal{V}_t(\widetilde{\varphi}) = e^{rt}E_0 + \frac{V_0 + U_0}{\mathcal{V}_0(\varphi)}\mathcal{V}_t(\varphi) = A_t, \quad t \in [0,T].$$

A similar simplified balance sheet is used in Grosen and Jørgensen (2000), Hansen and Miltersen (2002) and Miltersen and Persson (2003). However, the number of accounts on the liability side of the balance sheet, and their interpretation varies.

3   The financial model

We consider a financial market described by the standard Black–Scholes model. Here, the market consists of two traded assets: A risky asset with price process S and a riskfree asset with price process B. The risky asset is usually referred to as a stock and the riskfree asset as a savings account. The price processes are defined on a probability space $(\Omega, \mathcal{F}, P)$, and the P-dynamics of the price processes are given by

$$dS_t = \alpha S_t dt + \sigma S_t d\widetilde{W}_t, \quad S_0 > 0, \tag{3.1}$$
$$dB_t = rB_t dt, \quad B_0 = 1,$$

where $(\widetilde{W}_t)_{0 \leq t \leq T}$ is a standard Wiener process on the interval $[0,T]$ under P, with T being a fixed finite time horizon. The coefficient σ is a strictly positive constant, while α and r are non-negative constants. The filtration $\mathbb{G} = (\mathcal{G}_t)_{0 \leq t \leq T}$ is the P-augmentation of the natural filtration generated by (B, S), i.e. $\mathcal{G}_t = \mathcal{G}_t^+ \vee \mathcal{N}$, where $\mathcal{N}$ is the σ-algebra generated by all P-null sets and

$$\mathcal{G}_t^+ = \sigma\{(B_u, S_u), u \leq t\} = \sigma\{S_u, u \leq t\} = \sigma\{\widetilde{W}_u, u \leq t\}.$$

Here, we have used the strict positivity of σ in the last equality. We interpret α as the mean rate of return of the stock, σ as the standard deviation of the rate of return and r as the short rate of interest. The constant ν defined by $\nu = \frac{\alpha - r}{\sigma}$ is known as the market price of risk associated with S. It is well-known, see e.g. Musiela and Rutkowski (1997), that in the Black–Scholes model, the probability measure Q^0 defined by

$$\frac{dQ^0}{dP} \equiv O_T = e^{-\nu \widetilde{W}_T - \frac{1}{2}\nu^2 T}$$

is the unique equivalent martingale measure. Hence, Q^0 is a probability measure equivalent to P under which all discounted price processes on the financial market are (local) martingales. The Q^0-dynamics of the price processes are

$$dS_t = rS_t dt + \sigma S_t dW_t, \quad S_0 > 0, \tag{3.2}$$
$$dB_t = rB_t dt, \quad B_0 = 1,$$

where $(W_t)_{0 \leq t \leq T}$ is a standard Wiener process on the interval $[0,T]$ under Q^0.

A trading strategy is an adapted process $\varphi = (\vartheta, \eta)$ satisfying certain integrability conditions. The pair $\varphi_t = (\vartheta_t, \eta_t)$ is interpreted as the portfolio held at time t. Here, ϑ_t and η_t, respectively, denote the number of stocks and the discounted deposit in the savings account in the portfolio at time t. The value process $\mathcal{V}(\varphi)$ associated with φ is given by

$$\mathcal{V}_t(\varphi) = \vartheta_t S_t + \eta_t B_t.$$

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A strategy φ is called self-financing if

$$\mathcal{V}_t(\varphi) = \mathcal{V}_0(\varphi) + \int_0^t \vartheta_u dS_u + \int_0^t \eta_u dB_u.$$

Thus, the value at any time t of a self-financing strategy is the initial value added trading gains from investing in stocks and interest earned on the deposit in the savings account; withdrawals and additional deposits are not allowed during $(0, T)$. A self-financing strategy $\varphi = (\vartheta, \eta)$ is called admissible if $(\vartheta, \eta) \geq 0$, which guarantees that $\mathcal{V}_t(\varphi) \geq 0$ P-a.s. for all $t \in [0, T]$. We restrict the investment strategies of the insurance company to admissible strategies. A self-financing strategy is a so-called arbitrage if $\mathcal{V}_0(\varphi) = 0$ and $\mathcal{V}_T(\varphi) \geq 0$ P-a.s. with $P(\mathcal{V}_T(\varphi) > 0) > 0$. A contingent claim (or a derivative) in the model $(B, S, \mathbb{G})$ with maturity T is a $\mathcal{G}_T$-measurable, Q^0-square integrable random variable X. A contingent claim is called attainable if there exists a self-financing strategy such that $\mathcal{V}_T(\varphi) = X$, P-a.s. An attainable claim can thus be replicated perfectly by investing $\mathcal{V}_0(\varphi)$ at time 0 and investing during the interval $[0, T]$ according to the self-financing strategy φ. Hence, at any time t, there is no difference between holding the claim X and the portfolio φ_t. In this sense, the claim X is redundant in the market, and from the assumption of no arbitrage it follows that the price of X at time t must be $\mathcal{V}_t(\varphi)$. Thus, the initial investment $\mathcal{V}_0(\varphi)$ is the unique arbitrage free price of X. If all contingent claims are attainable, the model is called complete and otherwise it is called incomplete. It is well-known from the financial literature, see e.g. Björk (1998), that the Black–Scholes model is complete and arbitrage free, and that the discounted value process associated with any self-financing trading strategy is a Q^0-martingale. Throughout the paper, we denote by S^* the discounted stock price and by $\mathcal{V}^*(\varphi)$ the discounted value process. Furthermore we use the asterisk to denote that a constant or function has been multiplied by e^{-rT}, i.e. discounted from time T to time 0.

4 Capital insurances

Consider a life insurance company whose insurance portfolio constitutes capital insurances exclusively. A capital insurance pays out a sum insured at a specified time, whether the insured is alive or not. For simplicity we assume that no payments between the company and the insured take place during $(0, T)$. In this case, we can disregard the individual contracts and focus on the total insurance portfolio.

The aim of this section is, while respecting the general terms of the contracts, to determine an arbitrage free distribution of the assets at time T among the accounts on the liability side. We shall refer to such a distribution as fair, see Section 4.2 for more details. We assume that all insured are promised the same accumulation factor G_T on the deposit in the period $(0, T)$. In practice, we often have $G_T \geq 1$. The consequence of the guarantee is that the total deposit should be at least $G_T V_0$ at time T whenever possible. If the company is unable to cover the guarantee, all assets are allocated to the deposit and paid to the insured in cash, while the company is declared bankrupt.

Remark 4.1 Two possible choices for the guaranteed accumulation factor are $1 + gT$ and e^{gT}, depending on whether g is expressed in terms of a periodical or a continuously compounding rate. If $T = 1$ and time is measured in years, then $G_1 = 1 + g$ corresponds

to a guaranteed annual interest rate of g.

$\square$

Remark 4.2 For the company to survive in the long run, we should have $G_T \leq e^{rT}$. However, since we are interested in short term conditions only, also the reverse situation is relevant.

$\square$

To be consistent with common practice, the company at time 0 announces a deposit accumulation factor K_T, $K_T \geq G_T$, by which they intend to accumulate the deposit at time T. In contrast to G_T, we do not consider K_T as legally binding. Hence, at time T the company is allowed to use an accumulation factor different from K_T for the actual accumulation. However, using an accumulation factor different from K_T affects the credibility of the company, and thus, it is not done frequently in practice. In order to model this reluctancy in a simple way without removing the possibility of using an accumulation factor different from K_T, we assume that the company uses K_T unless the value of the risky investments at time T, $\mathcal{V}_T(\varphi)$, is less than $K_T V_0 (1+\gamma)$. Here, the factor γ, $\gamma \geq 0$, is the proportion of the deposit which is the target for the minimal bonus reserve, as decided by the management of the insurance company.

To compensate the equity capital for the exposure to the financial risk inherent in capital insurances, we introduce the parameter ρ, which represents the interest rate credited to the equity capital in addition to the riskfree interest rate, whenever such an additional return is possible.

4.1 Distribution scheme

The distribution scheme used by the company to distribute assets at time T between the three accounts on the liability side depends on the development of the assets of the company and hence on the financial market. We distinguish between the following three situations for the development of the assets:

1. $A_T < G_T V_0$: In this case, the company does not have sufficient assets to cover the guarantee. Since the interest of the insured takes priority over the interest of the owners of the company, all capital is allocated to the deposit, and the equity is set to 0, that is

$$
\begin{aligned}
V_T &= A_T, \\
E_T &= 0, \\
U_T &= 0.
\end{aligned}
$$

2. $G_T V_0 \leq A_T < K_T V_0 (1+\gamma) + e^{rT} E_0$: Here, the assets are sufficient to meet the guaranteed accumulation of the deposit. However, using the announced deposit accumulation factor would leave the company with a bonus reserve less than the target for the minimal bonus reserve, γV_T. Hence, the company chooses to accumulate the deposit by the guaranteed accumulation factor G_T. This way, the company obtains

the maximal possible bonus reserve, which in some cases exceeds γV_T. The equity capital at time T is given by the equity capital at time 0 accumulated with the interest rate $r + \rho$ or the total assets deducted the deposit at time T, whichever is smallest. The bonus account is calculated residually as the assets subtracted the deposit and the equity capital. This leads to the following distribution:

$$V_T = G_T V_0,$$
$$E_T = \min\left(e^{(r+\rho)T} E_0, A_T - V_T\right),$$
$$U_T = A_T - V_T - E_T.$$

3. $e^{rT} E_0 + K_T V_0 (1 + \gamma) \leq A_T$: This outcome leaves the company with a bonus reserve larger than γV_T after accumulating the deposit with K_T. The distribution is given by an expression similar to the one in case 2 with G_T substituted by K_T. Thus

$$V_T = K_T V_0,$$
$$E_T = \min\left(e^{(r+\rho)T} E_0, A_T - V_T\right),$$
$$U_T = A_T - V_T - E_T.$$

Note that in the distribution scheme we first use the bonus reserve to cover the accumulation of the deposit, and if this is insufficient, we then use the equity capital.

In the distribution scheme, the only unknown parameter is ρ. Hence, determining the fair distribution scheme reduces to determining the fair value of ρ. Since $E_T \leq e^{(r+\rho)T} E_0$, a necessary requirement for a distribution scheme to be arbitrage free is $\rho \geq 0$. Hence, the referral to ρ as the *additional* rate of return. Furthermore, we immediately observe from the distribution scheme that E_T is non-decreasing in ρ for all ω. If further A_T is stochastic, i.e. if the company invests some capital in the risky asset, then $P(A_T - V_T \geq e^{(r+\rho)T} E_0) > 0$ for all finite ρ. Hence the set of ω's for which E_T is strictly increasing in ρ has a positive probability. We thus have that the fair value of ρ, if it exists, is unique.

4.2 Fair distribution

A distribution scheme is said to be fair if it does not introduce arbitrage opportunities for the insurance company or the insurance portfolio. Since the size of the accounts on the liability side of the balance sheet depends on the development of the financial market only, we can view E_T and $V_T + U_T$ as contingent claims in the complete and arbitrage free market $(B, S, \mathbb{G})$. Hence, the claims E_T and $V_T + U_T$ have unique prices. Thus, the distribution scheme is fair, if

$$E_0 = e^{-rT} \mathrm{E}^{Q^0}\left[E_T\right], \tag{4.1}$$

and

$$V_0 + U_0 = e^{-rT} \mathrm{E}^{Q^0}\left[V_T + U_T\right]. \tag{4.2}$$

Note that since we are interested in the distribution of assets between the insurance portfolio as a whole and the company, and not between the insured individuals, we do not

distinguish between the deposit and the bonus reserve in (4.2). Depending on the bonus strategy of the company, the individual contracts may or may not be fair, but for the insured portfolio as a whole the contracts are fair if (4.2) is fulfilled. Since the assets are invested in a self-financing portfolio we have

$$\mathrm{E}^{Q^0}[e^{-rT}A_T] = A_0,$$

such that (4.1) is satisfied if and only if (4.2) is satisfied. Hence, determining the fair value of ρ, if it exists, amounts to solve (4.1) with respect to ρ.

4.3 Buy and hold strategy

Consider a buy and hold strategy, which is the simplest example of a self-financing strategy. In the buy and hold strategy the company invests ϑS_0 and η, respectively, in the risky and the riskfree asset at time 0 and no trading takes place until time T. Hence, the value at time T of the risky portfolio is

$$\mathcal{V}_T(\varphi) = \vartheta S_T + \eta e^{rT}.$$

Assume the company follows a buy and hold strategy with $\vartheta > 0$, i.e. with some investments in the risky asset. We now derive an implicit expression for the fair value of ρ by decomposing the value of the equity capital at time T into payoffs from standard European options on the stock.

Define the quantities s_1 and s_2 as the values of S_T which solve the two equations

$$G_T V_0 = e^{rT} E_0 + \vartheta S_T + \eta e^{rT}, \tag{4.3}$$

and

$$K_T V_0 (1 + \gamma) = \vartheta S_T + \eta e^{rT}, \tag{4.4}$$

respectively. Hence, s_1 is the lowest stock price at time T, which does not lead to bankruptcy of the insurance company, while s_2 is the lowest stock price for which, the company uses K_T as accumulation factor. Solving (4.3) and (4.4) for S_T we get

$$s_1 = \frac{G_T V_0 - \eta e^{rT} - e^{rT} E_0}{\vartheta}, \tag{4.5}$$

and

$$s_2 = \frac{K_T V_0 (1 + \gamma) - \eta e^{rT}}{\vartheta}. \tag{4.6}$$

Note that even though the stock price is positive, s_1 and s_2 might be negative. If s_1 is negative, the capital invested in the savings account is sufficient to ensure that the company is not bankrupted, whereas a negative value for s_2 corresponds to the case, where the capital invested in the savings account is sufficient to ensure that the company always uses K_T to accumulate the deposit. Using s_1 and s_2, we can rewrite the value of the equity capital at time T as

$$E_T^B = 1_{(S_T < s_1)} E_T^B + 1_{(s_1 \le S_T < s_2)} E_T^B + 1_{(s_2 \le S_T)} E_T^B \equiv E_T^{B1} + E_T^{B2} + E_T^{B3}.$$

Here, the superscript B indicates that we are working with a buy and hold strategy. The expressions for the equity capital in the different situations can be found in Section 4.1. Since E_T^{B1} is the equity capital in case of bankruptcy, it is equal to 0.

In order to decompose E_T^{B2}, we first recall that

$$E_T^{B2} = 1_{(s_1 \leq S_T < s_2)} \min\left(e^{(r+\rho)T}E_0, \mathcal{V}_T(\varphi) + e^{rT}E_0 - G_T V_0\right). \tag{4.7}$$

In order to calculate (4.7), we determine s_3 which is the maximum value of S_T for which

$$\mathcal{V}_T(\varphi) + e^{rT}E_0 - G_T V_0 \leq e^{(r+\rho)T}E_0. \tag{4.8}$$

Hence s_3 is the largest value for the stock price at time T for which the assets are insufficient to accumulate the equity capital with interest rate $r + \rho$, if we accumulate the deposit with G_T. Solving (4.8) we get

$$s_3 = \frac{(e^{\rho T} - 1)e^{rT}E_0 + G_T V_0 - \eta e^{rT}}{\vartheta}. \tag{4.9}$$

Rewriting s_3 as

$$s_3 = s_1 + \frac{e^{(r+\rho)T}E_0}{\vartheta},$$

and using that $\min(r, \rho) > -\infty$ and $\vartheta > 0$ we observe that $s_3 > s_1$, such that inserting in (4.7) gives

$$\begin{aligned}
E_T^{B2} &= 1_{(s_1 \leq S_T < \min(s_2, s_3))}\left(e^{rT}E_0 + \mathcal{V}_T(\varphi) - G_T V_0\right) + 1_{(\min(s_2, s_3) \leq S_T < s_2)}e^{(r+\rho)T}E_0 \\
&= 1_{(s_1 \leq S_T < \min(s_2, s_3))}\vartheta\left(S_T - s_1\right) + 1_{(\min(s_2, s_3) \leq S_T < s_2)}e^{(r+\rho)T}E_0 \\
&= \vartheta\Bigg((S_T - s_1)^+ - (S_T - \min(s_2, s_3))^+ - (\min(s_2, s_3) - s_1)\,1_{(\min(s_2, s_3) < S_T)}\Bigg) \\
&\quad + e^{(r+\rho)T}E_0\left(1_{(\min(s_2, s_3) \leq S_T)} - 1_{(s_2 \leq S_T)}\right). \tag{4.10}
\end{aligned}$$

Thus, E_T^{B2} can be decomposed into two terms. The first term is the number of stocks multiplied by the difference between the payoff from two European call options with strikes s_1 and $\min(s_2, s_3)$ subtracted the payoff from a so-called binary cash call option with strike $\min(s_2, s_3)$ and cash $\min(s_2, s_3) - s_1$. The second term is the equity capital accumulated with interest rate $r + \rho$ multiplied by the difference between the payoff from two binary cash call options with strikes $\min(s_2, s_3)$ and s_2. For a description of the these and other options mentioned in this paper see Musiela and Rutkowski (1997).

In order to decompose E_T^{B3} we first determine s_4, which is the largest value of S_T for which

$$e^{rT}E_0 + \mathcal{V}_T(\varphi) - K_T V_0 \leq e^{(r+\rho)T}E_0.$$

Solving for S_T we get

$$s_4 = \frac{(e^{\rho T} - 1)e^{rT}E_0 + K_T V_0 - \eta e^{rT}}{\vartheta}. \tag{4.11}$$

The interpretation of s_4 is similar to that of s_3, however here the deposit is accumulated with K_T. Calculations similar to those for E_T^{B2} give

$$
\begin{aligned}
E_T^{B3} &= 1_{(s_2 \leq S_T)} \min \left(e^{(r+\rho)T} E_0, e^{rT} E_0 + \mathcal{V}_T(\varphi) - K_T V_0 \right) \\
&= 1_{(s_2 \leq S_T < \max(s_2,s_4))} \vartheta \left(S_T - s_5 \right) + 1_{(\max(s_2,s_4) \leq S_T)} e^{(r+\rho)T} E_0 \\
&= \vartheta \left((S_T - s_2)^+ - (S_T - \max(s_2,s_4))^+ + 1_{(s_2 \leq S_T)}(s_2 - s_5) \right. \\
&\quad \left. - 1_{(\max(s_2,s_4) \leq S_T)} \left(\max(s_2,s_4) - s_5 \right) \right) + 1_{(\max(s_2,s_4) \leq S_T)} e^{(r+\rho)T} E_0, \qquad (4.12)
\end{aligned}
$$

where we have used the notation

$$
s_5 = \frac{K_T V_0 - \eta e^{rT} - e^{rT} E_0}{\vartheta}. \qquad (4.13)
$$

Hence, E_T^{B3} can be decomposed into two terms as well. The first term is the number of stocks multiplied by the payoff from known European options, and the second term is the equity capital accumulated with interest rate $r + \rho$ multiplied by the payoff from a binary cash call option. Denote by BCC and C, respectively, the price of a *binary cash call* and a *call option*. It is well-known that BCC and C are given by

$$
BCC(x, S_0, \sigma) = E^{Q^0} \left[e^{-rT} 1_{(x \leq S_T)} \right] = \begin{cases} e^{-rT} \Phi \left(\frac{\log\left(\frac{S_0}{x}\right) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right), & x > 0, \\ e^{-rT}, & x \leq 0, \end{cases}
$$

and

$$
\begin{aligned}
C(x, S_0, \sigma) &= E^{Q^0} \left[e^{-rT} (S_T - x)^+ \right] \\
&= \begin{cases} S_0 \Phi \left(\frac{\log\left(\frac{S_0}{x}\right) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right) - e^{-rT} x \Phi \left(\frac{\log\left(\frac{S_0}{x}\right) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right), & x > 0, \\ S_0 - e^{-rT} x, & x \leq 0, \end{cases}
\end{aligned}
$$

where Φ denotes the distribution function for the standard normal distribution. To simplify notation, we use the short hand notation $BCC(x)$ and $C(x)$ in expressions involving many option prices with the same initial value and volatility. Applying criterion (4.1) we obtain the following proposition

Proposition 4.3
If a company invests according to a buy and hold strategy the fair value of ρ satisfies

$$
\begin{aligned}
E_0 &= e^{(r+\rho)T} E_0 \bigg(BCC(\min(s_2,s_3)) - BCC(s_2) + BCC(\max(s_2,s_4)) \bigg) \\
&\quad + \vartheta \bigg(C(s_1) - C(\min(s_2,s_3)) + C(s_2) - C(\max(s_2,s_4)) \\
&\quad - (\min(s_2,s_3) - s_1) BCC(\min(s_2,s_3)) \\
&\quad + (s_2 - s_5) BCC(s_2) - (\max(s_2,s_4) - s_5) BCC(\max(s_2,s_4)) \bigg),
\end{aligned}
$$

where $s_1 - s_5$ are given by (4.5), (4.6), (4.9), (4.11) and (4.13) and all option prices are calculated using initial value S_0 and volatility σ.

If $\vartheta = 0$, all assets are invested in the savings account. Hence, the value at time T of the assets is deterministic and equal to $A_T = e^{rT}A_0$. In this case we obtain the following result for the fair value of ρ.

Proposition 4.4
If a company invests in the savings account only, a fair value of ρ must satisfy

1. *If $e^{rT}A_0 < G_T V_0$ then no values of ρ exist for which the distribution scheme fair.*

2. *If $G_T V_0 \le e^{rT}A_0 < K_T V_0(1 + \gamma) + e^{rT}E_0$, then the distribution scheme is fair, if either of the following apply*

   (a) $e^{rT}E_0 < e^{rT}A_0 - G_T V_0$ and $\rho = 0$.
   (b) $G_T = e^{rT}\frac{V_0 + U_0}{V_0}$ and $\rho \ge 0$.

3. *If $K_T V_0(1+\gamma) + e^{rT}E_0 < e^{rT}A_0$, then the distribution scheme is fair, if either of the following apply*

   (a) $e^{rT}E_0 < e^{rT}A_0 - K_T V_0$ and $\rho = 0$.
   (b) $K_T = e^{rT}\frac{V_0 + U_0}{V_0}$ and $\rho \ge 0$.

Proof of Proposition 4.4: See Appendix A.

Proposition 4.4 has the following interpretation: If the assets and hence the accounts on the liability side are deterministic at time T the distribution scheme is fair if only if $E_T = e^{rT}E_0$. Since this is intuitively clear, the proposition is not particularly interesting and stated for completeness, only.

We end this section with a result for the probability of ruin of the company at time T.

Proposition 4.5
The probability, $p_{ruin}(\varphi)$, that a company, using the buy and hold strategy φ, is ruined at time T is given by

$$p_{ruin}(\varphi) = \Phi \left(\frac{\log\left(\frac{s_1}{S_0}\right) - \left(\alpha - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \right).$$

Proof of Proposition 4.5: The company is ruined at time T if $A_T < G_T V_T$. Hence,

$$p_{ruin}(\varphi) = P\left[A_T < G_T V_0\right] = P\left[S_T < \frac{G_T V_T - \eta e^{rT} - e^{rT}E_0}{\vartheta}\right] = P\left[S_T < s_1\right].$$

Here, we have used the definition of s_1 from (4.5). The result now follows by inserting the solution, $S_T = S_0 e^{(\alpha - \sigma^2/2)T + \sigma\widetilde{W}_T}$, to the stochastic differential equation for the dynamics of S under P given in (3.1).

$\square$

4.4 Constant relative portfolio weights

Now consider the case where the company continuously adjusts the investment portfolio, such that at all times, $t \in [0, T]$, the proportion $\delta \in [0, 1]$ of the portfolio value is invested in stocks. Hence,

$$\vartheta_t S_t = \delta \mathcal{V}_t(\varphi) \text{ and } \eta_t B_t = (1 - \delta)\mathcal{V}_t(\varphi).$$

In this case the dynamics under Q^0 of the value process of the self-financing strategy are

$$\begin{aligned}
d\mathcal{V}_t(\varphi) &= \vartheta_t dS_t + \eta_t dB_t \\
&= \vartheta_t(rS_t dt + \sigma S_t dW_t) + \eta_t r B_t dt \\
&= r\mathcal{V}_t(\varphi)dt + \delta\sigma\mathcal{V}_t(\varphi)dW_t.
\end{aligned}$$

We note that the dynamics of the value process are of the same form as the dynamics of the stock price. For $\delta > 0$ calculations similar to those for a buy and hold strategy give

Proposition 4.6
When investing in a portfolio with constant relative portfolio weights the fair value of ρ solves the following equation

$$\begin{aligned}
E_0 = e^{(r+\rho)T} E_0 &\left(BCC(\min(v_2, v_3)) - BCC(v_2) + BCC(\max(v_2, v_4)) \right） \\
&+ C(v_1) - C(\min(v_2, v_3)) + C(v_2) - C(\max(v_2, v_4)) \\
&- (\min(v_2, v_3) - v_1)BCC(\min(v_2, v_3)) \\
&+ (v_2 - v_5)BCC(v_2) - (\max(v_2, v_4) - v_5)BCC(\max(v_2, v_4)),
\end{aligned}$$

where

$$\begin{aligned}
v_1 &= G_T V_0 - e^{rT} E_0, \\
v_2 &= K_T V_0(1 + \gamma), \\
v_3 &= (e^{\rho T} - 1)e^{rT} E_0 + G_T V_0, \\
v_4 &= (e^{\rho T} - 1)e^{rT} E_0 + K_T V_0, \\
v_5 &= K_T V_0 - e^{rT} E_0,
\end{aligned}$$

and all option prices are calculated with initial value $V_0 + U_0$ and volatility $\delta\sigma$.

If $\delta = 0$ we are in exactly the same situation as in the buy and hold strategy with $\vartheta = 0$, so Proposition 4.4 applies.

Note that under P the dynamics of the value process for a self-financing strategy with constant relative portfolio weights are

$$\begin{aligned}
d\mathcal{V}_t(\varphi) &= \vartheta_t dS_t + \eta_t dB_t \\
&= \vartheta_t(\alpha S_t dt + \sigma S_t d\widetilde{W}_t) + \eta_t r B_t dt \\
&= (r + \delta(\alpha - r))\mathcal{V}_t(\varphi)dt + \delta\sigma\mathcal{V}_t(\varphi)d\widetilde{W}_t.
\end{aligned}$$

This leads to the following proposition for the probability of ruin at time T.

Proposition 4.7
The probability of ruin, $p_{ruin}(\varphi)$, is given by

$$p_{ruin}(\varphi) = \Phi\left(\frac{\log\left(\frac{v_1}{V_0+U_0}\right) - \left(r + \delta(\alpha - r) - \frac{1}{2}(\delta\sigma)^2\right)T}{\delta\sigma\sqrt{T}}\right).$$

4.5 Buy and hold with stop-loss if solvency is threatened

Consider the case where the regulatory institutions set a solvency requirement for the insurance company. As in practice, the requirement considered here is a requirement on the equity capital. After accumulating the deposit at time T, the equity capital should be at least a proportion β of the deposit, i.e. $E_T \geq \beta V_T$. Since the solvency requirement must be satisfied at the end of each accumulation period we know that $E_0 \geq \beta V_0$. Otherwise the company would have been declared insolvent already. If further $e^{rT}E_0 \geq \beta K_T V_0$ and $A_0 \geq e^{-rT}G_T V_0(1 + \beta)$ the company may avoid insolvency by rebalancing the risky portfolio to include investments in the savings account only, if the value of the assets reaches the lower boundary

$$A_t = e^{-r(T-t)}G_T V_0(1 + \beta). \tag{4.14}$$

Now assume the company invests in a buy and hold strategy as introduced in Section 4.3, until a possible intervention. In this case we can write (4.14) in terms of the discounted stock price

$$S_t^* = \frac{e^{-rT}G_T V_0(1 + \beta) - \eta - E_0}{\vartheta} \equiv H.$$

Remark 4.8 The stop-loss criterion in (4.14) is just one of many possible criterions. If $V_0 + U_0 \geq e^{-rT}K_T V_0(1 + \gamma)$ the alternative criterion $\mathcal{V}_t(\varphi) = e^{-r(T-t)}K_T V_0(1 + \gamma)$ in addition to solvency also guarantees that K_T is used as accumulation factor.

$\square$

Decomposing the equity capital we first distinguish between whether the company has intervened or not

$$E_T^{BS} = 1_{(\inf_{0\leq t\leq T} S_t^* \leq H)}E_T^{BS} + 1_{(\inf_{0\leq t\leq T} S_t^* > H)}E_T^{BS} \equiv E_T^{BS1} + E_T^{BS2}.$$

Here, the superscript BS indicates that the company uses a buy and hold strategy with stop-loss. When $\inf_{0\leq t\leq T} S_t^* \leq H$ the asset value is deterministic and equal to $G_T V_0(1+\beta)$, such that

$$E_T^{BS1} = 1_{(\inf_{0\leq t\leq T} S_t^* \leq H)}\min\left(e^{(r+\rho)T}E_0, \beta G_T V_0\right) = 1_{(\inf_{0\leq t\leq T} S_t^* \leq H)}\beta G_T V_0.$$

Here, we have used that $e^{rT}E_0 \geq \beta K_T V_0 \geq \beta G_T V_0$ in both equalities and $\rho \geq 0$ in the last equality. We recognize this as the payoff from a down-and-in barrier option on the discounted stock price with the deterministic payoff $\beta G_T V_0$ when knocked in. When $\inf_{0\leq t\leq T} S_t^* > H$ it holds in particular that

$$S_T > \frac{G_T V_0(1 + \beta) - \eta e^{rT} - e^{rT}E_0}{\vartheta} \equiv s_1^\beta.$$

The assumptions on the equity capital and the fact that $\rho \geq 0$ gives that $s_3 \geq s_1^\beta$. Hence, calculations similar to those leading to (4.10) and (4.12) gives

$$E_T^{BS2} = 1_{(\inf_{0 \leq t \leq T} S_t^* > H)} \left(e^{(r+\rho)T} E_0 \left(1_{(\min(s_2^*, s_3^*) \leq S_T^*)} - 1_{(s_2^* \leq S_T^*)} + 1_{(\max(s_2^*, s_4^*) \leq S_T^*)} \right) \right.$$

$$+ \vartheta \left(e^{rT} \left(\left(S_T^* - s_1^{\beta,*} \right)^+ - (S_T^* - \min(s_2^*, s_3^*))^+ + (S_T^* - s_2^*)^+ - (S_T^* - \max(s_2^*, s_4^*))^+ \right) \right.$$

$$+ \left(s_1^\beta - s_1 \right) 1_{(s_1^{\beta,*} < S_T^*)} - (\min(s_2, s_3) - s_1) 1_{(\min(s_2^*, s_3^*) < S_T^*)}$$

$$\left. \left. + (s_2 - s_5) 1_{(s_2^* \leq S_T^*)} - 1_{(\max(s_2^*, s_4^*) \leq S_T^*)} (\max(s_2, s_4) - s_5) \right) \right).$$

Thus, the equity capital can be written in terms of payoffs from barrier options on the discounted stock price. To indicate that an option is written on the discounted stock price, we equip the option price by an asterisk ($*$). When working with barrier options we equip the notation for the corresponding European option, or 1 in case of a deterministic value, with two letters as a sub- or superscript depending on whether we are dealing with a *down* or an *up* barrier option. The first letter is the barrier and the second describe whether we are dealing with an *out*, denoted O, or an *in*, denoted I, option. Using Björk (1998, Theorem 13.8) we are able to write prices of the relevant barrier options on the discounted stock price in terms of prices of European options. For $S_0 > H$ we obtain the following option prices: A down-and-out option with payoff 1

$$1_{HO}^*(S_0, \sigma) = E^{Q^0} \left[e^{-rT} 1_{(\inf_{0 \leq t \leq T} S_t^* > H)} \right]$$

$$= \begin{cases} e^{-rT} \Phi \left(\frac{\log\left(\frac{S_0}{H}\right) - \frac{1}{2}\sigma^2 T}{\sigma \sqrt{T}} \right) - \left(\frac{S_0}{H}\right) e^{-rT} \Phi \left(\frac{\log\left(\frac{H}{S_0}\right) - \frac{1}{2}\sigma^2 T}{\sigma \sqrt{T}} \right), & H > 0, \\ e^{-rT}, & H \leq 0, \end{cases}$$

a down-and-out binary cash call option

$$BCC_{HO}^*(x, S_0, \sigma) = E^{Q^0} \left[e^{-rT} 1_{(\inf_{0 \leq t \leq T} S_t^* > H)} 1_{(x \leq S_T^*)} \right]$$

$$= \begin{cases} BCC^*(x, S_0, \sigma) - \frac{S_0}{H} BCC^* \left(x, \frac{H^2}{S_0}, \sigma \right), & 0 < H \leq x, \\ 1_{H0}^*(S_0, \sigma), & \max(0, x) \leq H, \\ BCC^*(x, S_0, \sigma), & H \leq 0 < x, \\ e^{-rT}, & \max(x, H) \leq 0, \end{cases}$$

and a down-and-out call option

$$C_{HO}^*(x, S_0, \sigma) = E^{Q^0} \left[e^{-rT} 1_{(\inf_{0 \leq t \leq T} S_t^* > H)} (S_T^* - x)^+ \right]$$

$$= \begin{cases} C^*(x, S_0, \sigma) - \frac{S_0}{H} C^* \left(x, \frac{H^2}{S_0}, \sigma \right), & 0 < H \leq x, \\ C^*(x, S_0, \sigma), & H \leq 0 < x, \\ e^{-rT}(S_0 - x), & \max(H, x) \leq 0, \\ C^*(H, S_0, \sigma) - \frac{S_0}{H} C^* \left(H, \frac{H^2}{S_0}, \sigma \right) + (H - x) 1_{HO}^*(S_0, \sigma), & \max(0, x) \leq H. \end{cases}$$

Here the prices BCC^* and C^* can be calculated from the formulas for BBC and C using

$$BCC^*(x, S_0, \sigma) = BCC(e^{rT} x, S_0, \sigma),$$

$$C^*(x, S_0, \sigma) = e^{-rT} C(e^{rT} x, S_0, \sigma).$$

For $S_0 \leq H$ all down-and-out options have a price equal to 0. For a down-and-in option with payoff 1 we have for all S_0

$$1_{HI}^*(S_0, \sigma) = e^{-rT} - 1_{HO}^*(S_0, \sigma).$$

The following proposition now follows from applying criterion (4.1).

Proposition 4.9
If a company follows a buy and hold strategy with stop-loss in case solvency is threatened the fair value of ρ must satisfy

$$E_0 = 1_{HI}^* \beta G_T V_0$$

$$+ e^{(r+\rho)T} E_0 \bigg(BCC_{HO}^* \left(\min\left(s_2^*, s_3^*\right)\right) - BCC_{HO}^* \left(s_2^*\right) + BCC_{HO}^* \left(\max(s_2^*, s_4^*)\right) \bigg)$$

$$+ \vartheta \bigg(e^{rT} \Big(C_{HO}^* \left(s_1^{\beta,*}\right) - C_{HO}^* \left(\min\left(s_2^*, s_3^*\right)\right) + C_{HO}^* \left(s_2^*\right) - C_{HO}^* \left(\max(s_2^*, s_4^*)\right) \Big)$$

$$+ \left(s_1^\beta - s_1\right) BCC_{HO}^* \left(s_1^{\beta,*}\right) - \left(\min\left(s_2, s_3\right) - s_1\right) BCC_{HO}^* \left(\min\left(s_2^*, s_3^*\right)\right)$$

$$+ (s_2 - s_5) BCC_{HO}^* \left(s_2^*\right) - \left(\max(s_2, s_4) - s_5\right) BCC_{HO}^* \left(\max(s_2^*, s_4^*)\right) \bigg),$$

where all option prices are calculated with initial value S_0 and volatility σ.

4.6 Constant relative amount δ in stocks until solvency is threatened

Now assume that a company, whose assets at time 0 fulfill $A_0 \geq e^{-rT} G_T V_0 (1 + \beta)$, initially invests in a portfolio with constant relative portfolio weights as described in Section 4.4. As in Section 4.5 the company rebalances the investment portfolio to include the riskfree asset only, the first time (4.14) holds. Written in terms of the discounted value process of the investment portfolio the company rebalances the portfolio if

$$\mathcal{V}_t^*(\varphi) = e^{-rT} G_T V_0 (1 + \beta) - E_0 \equiv \widetilde{H}.$$

As in Section 4.5 we know that $E_0 \geq \beta V_0$ and further assume that $e^{rT} E_0 \geq \beta K_T V_0$. The following proposition now follows from Proposition 4.9 in the same way as Proposition 4.6 followed from Proposition 4.3

Proposition 4.10
For a company investing in a portfolio with constant relative portfolio weights until solvency is threatened the fair value of ρ must satisfy

$$E_0 = 1_{\widetilde{H}I}^* \beta G_T V_0$$

$$+ e^{(r+\rho)T} E_0 \left(BCC_{\widetilde{H}O}^* \left(\min\left(v_2^*, v_3^*\right)\right) - BCC_{\widetilde{H}O}^* \left(v_2^*\right) + BCC_{\widetilde{H}O}^* \left(\max(v_2^*, v_4^*)\right) \right)$$

$$+ e^{rT} \left(C_{\widetilde{H}O}^* \left(v_1^{\beta,*}\right) - C_{\widetilde{H}O}^* \left(\min\left(v_2^*, v_3^*\right)\right) + C_{\widetilde{H}O}^* (v_2^*) - C_{\widetilde{H}O}^* (\max(v_2^*, v_4^*)) \right)$$

$$+ \left(v_1^\beta - v_1\right) BCC_{\widetilde{H}O}^* \left(v_1^{\beta,*}\right) - \left(\min\left(v_2, v_3\right) - v_1\right) BCC_{\widetilde{H}O}^* \left(\min\left(v_2^*, v_3^*\right)\right)$$

$$+ (v_2 - v_5) BCC_{\widetilde{H}O}^* (v_2^*) - \left(\max(v_2, v_4) - v_5\right) BCC_{\widetilde{H}O}^* \left(\max(v_2^*, v_4^*)\right),$$

where all option prices are calculated with initial value $V_0 + U_0$ and volatility $\delta\sigma$ and

$$v_1^\beta = G_T V_0 (1 + \beta) - e^{rT} E_0.$$

5 Pure endowments

We now consider a company whose insurance portfolio consists of pure endowments. To carry out the study we first extend the probabilistic model to include the development of a portfolio of insured lives. This is done following the approach in Møller (1998).

5.1 The model for the insurance portfolio

Consider an insurance portfolio consisting at time 0 of Y_0 insured lives of the same age, say x. We assume that the individual remaining lifetimes at time 0 of the insured are described by a sequence $T_1, \ldots, T_{Y_0}$ of i.i.d. non-negative random variables defined on $(\Omega, \mathcal{F}, P)$. We further make the natural assumption that the distribution of T_i is absolutely continuous and $P(T_i > T) > 0$, such that the mortality intensity μ_{x+t} is well-defined on $[0, T]$. The survival probability from time 0 to t, $t \in [0, T]$ for one individual in the insurance portfolio is given by

$$_t p_x \equiv P(T_i > t) = e^{-\int_0^t \mu_{x+u} du}.$$

Denote by $_t q_x$ the probability of death from time 0 to t, i.e. $_t q_x = 1 - {_t p_x}$. Now define the processes $Y = (Y_t)_{0 \le t \le T}$ and $N = (N_t)_{0 \le t \le T}$ by

$$Y_t = \sum_{i=1}^{Y_0} 1_{(T_i > t)} \text{ and } N_t = \sum_{i=1}^{Y_0} 1_{(T_i \le t)}.$$

Then Y_t and N_t, respectively, denote the number of survivors and the number of deaths in the insurance portfolio at time t. The filtration $\mathbb{H} = (\mathcal{H}_t)_{0 \le t \le T}$ is the P-augmentation of the natural filtration generated by N, i.e. $\mathcal{H}_t = \mathcal{H}_t^+ \vee \mathcal{N}$, where

$$\mathcal{H}_t^+ = \sigma\{N_u, u \le t\}.$$

Since the probability of two individuals dying at the same time is 0, then N is a 1-dimensional counting process. The i.i.d. assumption on the remaining lifetimes further gives that N is an $\mathbb{H}$-Markov process. The stochastic intensity process $\lambda = (\lambda_t)_{0 \le t \le T}$ of N under P can now be informally defined by

$$\lambda_t dt \equiv E^P[dN_t | \mathcal{H}_{t-}] = (Y_0 - N_{t-})\mu_{x+t} dt.$$

Thus, the probability of experiencing a death in the portfolio in the next short interval is the number of survivors multiplied by the probability of one person dying. It is well-known that the process M defined by

$$M_t = N_t - \int_0^t \lambda_u du = N_t - \int_0^t (Y_0 - N_{u-})\mu_{x+u} du, \quad 0 \le t \le T,$$

is an $\mathbb{H}$-martingale under P.

5.2 The combined model

Now introduce the filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ for the combined model of the economy and the insurance portfolio by

$$\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t.$$

Assume that the economy is stochastically independent of the development of the insurance portfolio, i.e. $\mathcal{G}_t$ and $\mathcal{H}_t$ are independent. This ensures that the properties of M and W are inherited in the larger filtration $\mathbb{F}$.

We now address the choice of equivalent martingale measure in the combined model. For any $\mathbb{F}$-predictable function h, $h > -1$, we can define a likelihood process $L = (L_t)_{0 \leq t \leq T}$ by

$$dL_t = L_{t-} h_t dM_t,$$
$$L_0 = 1,$$

and construct a new measure equivalent to P by

$$\frac{dQ^h}{dP} = O_T L_T. \tag{5.1}$$

We note that $h = 0$ corresponds to Q^0 defined in Section 3. The measure Q^h defined by (5.1) is a probability measure if $\mathrm{E}^{Q^0}[L_T] = 1$, or equivalently, if $\mathrm{E}^P[O_T L_T] = 1$. To preserve the independence between $\mathcal{G}_t$ and $\mathcal{H}_t$ under Q^h we restrict ourselves to functions h which are $\mathbb{H}$-predictable. Under this additional assumption, all measures Q^h defined by (5.1) are equivalent martingale measures if $\mathrm{E}^P[L_T] = 1$, see Møller (1998) for the necessary calculations. Girsanov's theorem for point processes, see e.g. Andersen et al. (1993), gives that the stochastic intensity process $\lambda^h = (\lambda_t^h)_{0 \leq t \leq T}$ for N under Q^h is given by

$$\lambda_t^h = (1 + h_t)\lambda_t = (Y_0 - N_{t-})(1 + h_t)\mu_{x+t}.$$

Hence, changing measure from Q^0 to Q^h can be interpreted as changing the mortality intensity from μ_{x+t} to $(1 + h_t)\mu_{x+t}$. With this interpretation the survival probability under Q^h is given by

$$_t p_x^h = Q^h(T_i > t) = e^{-\int_0^t (1+h_u)\mu_{x+u}du}.$$

The probability of death under Q^h is given by $_t q_x^h = 1 - {_t p_x^h}$. We note that if h is on the form $h(t, N_{t-})$ then N is a Markov process under Q^h as well as under P. Since no unique equivalent martingale measure exists for the combined model, not all contingent claims in $(B, S, \mathbb{F})$ have unique prices. However, since $(B, S, \mathbb{G})$ is complete, all contingent claims depending only on the financial market still have unique prices. To find unique prices for contingent claims depending on the development of the insurance portfolio, we henceforth consider a fixed, but arbitrary, equivalent martingale measure Q^h. Motivated by the strong law of large numbers, the measure Q^0, corresponding to risk neutrality with respect to unsystematic mortality risk, is frequently used in the literature to price insurance contracts with financial risk, see e.g. Aase and Persson (1994) and Møller (1998). Møller (1998) also recognizes Q^0 as the minimal martingale measure for the considered model.

5.3 The development of the deposit in a 1-period model

Now assume all insured in the portfolio introduced in Section 5.1 have purchased identical pure endowments with termination at time T or later. If premiums are paid before or at time 0 and the portfolio of insured lives develop exactly as expected, the portfolio-wide deposit at time T is given by

$$V_T^{det} = \mathcal{E}_T V_0^{det}.$$

Here, $\mathcal{E}_T \in \{G_T, K_T\}$ is the deposit accumulation factor for the interval $(0, T]$, and the superscript *det* refers to a deterministic development of the insured portfolio. Dividing by the number of expected survivors we obtain an expression for the development of the deposit of one insured surviving to time T

$$V_T^{ind} = \mathcal{E}_T V_0^{ind} \frac{1}{_T p_x}.$$

Thus, the portfolio-wide deposit at time T is given by

$$V_T = Y_T V_T^{ind} = Y_T \mathcal{E}_T V_0^{ind} \frac{1}{_T p_x}. \tag{5.2}$$

5.4 Distribution scheme

Using (5.2) we define a distribution scheme, similar to the distribution scheme from Section 4.1, which is used by the company in case of a portfolio of pure endowments:

1. $A_T < Y_T G_T V_0^{ind} \frac{1}{_T p_x}$: Here, the assets are insufficient to meet the guaranteed deposit at time T for all the survivors in the insured portfolio. Hence, the company is declared bankrupt and all capital is allocated to the deposit.

$$V_T = A_T,$$
$$E_T = 0,$$
$$U_T = 0.$$

2. $Y_T G_T V_0^{ind} \frac{1}{_T p_x} \leq A_T < Y_T K_T V_0^{ind} \frac{1}{_T p_x}(1 + \gamma) + e^{rT} E_0$: In this case the assets are sufficient to meet the guarantee. However, accumulating with the announced accumulation factor leaves the company with a bonus reserve less than the minimal target, γV_T. Thus, as in the case of capital insurances the company uses G_T to accumulate. Similarly to Section 4.1 the capital is distributed as follows

$$V_T = Y_T G_T V_0^{ind} \frac{1}{_T p_x},$$
$$E_T = \min\left(e^{(r+\rho)T} E_0, A_T - V_T\right),$$
$$U_T = A_T - V_T - E_T.$$

3. $e^{rT} E_0 + Y_T K_T V_0^{ind} \frac{1}{_T p_x}(1 + \gamma) \leq A_T$: Here, the investments and the development of the insurance portfolio allow the company to accumulate using the announced

deposit rate and still have a bonus reserve above the minimal target. The distribution is similar to the one above with G_T replaced by K_T

$$V_T = Y_T K_T V_0^{ind} \frac{1}{{}_T p_x},$$

$$E_T = \min \left(e^{(r+\rho)T} E_0, A_T - V_T \right),$$

$$U_T = A_T - V_T - E_T.$$

Note that we by the above distribution scheme implicitly consider the mortality intensity as guaranteed, since it is used even if the portfolio of insured behaves worse than anticipated. Thus, in the present situation the additional interest rate ρ is a compensation for both financial and unsystematic mortality risk. As in the case of capital insurances, the equity capital is only used to cover the accumulation of the deposit if the payoff generated by the deposit and bonus reserve is insufficient.

5.5 Fair distribution

From Section 5.4 we note that E_T and $V_T + U_T$ can be viewed as contingent claims in the combined model $(B, S, \mathbb{F})$. As in the case of capital insurances, we define the distribution scheme as fair if it does not include an arbitrage possibility for either the company or the portfolio of insured, i.e. if

$$E_0 = e^{-rT} \mathrm{E}^{Q^h} \left[E_T \right], \tag{5.3}$$

and

$$V_0 + U_0 = e^{-rT} \mathrm{E}^{Q^h} \left[V_T + U_T \right]. \tag{5.4}$$

The relation

$$E^{Q^h} [e^{-rT} A_T] = A_0,$$

now ensures that (5.3) holds if and only if (5.4) holds, such that we may consider (5.3) only.

Using the law of iterated expectations we can write (5.3) as

$$
\begin{aligned}
E^{Q^h} \left[E_T \right] &= E^{Q^h} E^{Q^h} \left[E_T | \mathcal{H}_T \right] \\
&= \sum_{n=0}^{Y_0} \binom{Y_0}{n} ({}_T p_x^h)^n ({}_T q_x^h)^{Y_0 - n} E^{Q^h} \left[\mathbb{1}_{\left(A_T < n G_T V_0^{ind} \frac{1}{{}_T p_x} \right)} 0 \right. \\
&\quad + \mathbb{1}_{\left(n G_T V_0^{ind} \frac{1}{{}_T p_x} \le A_T < n K_T V_0^{ind} \frac{1}{{}_T p_x} (1+\gamma) + e^{rT} E_0 \right)} \min \left(e^{(r+\rho)T} E_0, A_T - n G_T V_0^{ind} \frac{1}{{}_T p_x} \right) \\
&\quad + \left. \mathbb{1}_{\left(n K_T V_0 \frac{1}{{}_T p_x} (1+\gamma) \le \mathcal{V}_T(\varphi) \right)} \min \left(e^{(r+\rho)T} E_0, A_T - n K_T V_0^{ind} \frac{1}{{}_T p_x} \right) \right]. \tag{5.5}
\end{aligned}
$$

Recall that with respect to the financial market all measures Q^h are identical. Thus, the expectation can be viewed as a weighted average of $Y_0 + 1$ portfolios of capital insurances with initial deposit $n V_0^{ind} \frac{1}{{}_T p_x}$, $n = 0, 1, \ldots, Y_0$, respectively. Hence, most calculations necessary to derive an implicit equation for ρ are identical to those already carried out in Section 4.

Remark 5.1 Note that since all insured have identical contracts, the individual contracts are fair if the bonus reserve at the time of purchase was 0 and a possible bonus reserve at time of termination is distributed among the survivors in the insurance portfolio.

□

5.6 Buy and hold

When the company follows a buy and hold strategy the fair value of ρ is given by the following proposition

Proposition 5.2
If an insurance company, whose portfolio consists of Y_0 pure endowments, follows a buy and hold strategy, then the fair value of ρ satisfies

$$
\begin{aligned}
E_0 = \sum_{n=0}^{Y_0} \binom{Y_0}{n} (_T p_x^h)^n (_T q_x^h)^{Y_0-n} \Bigg(& e^{(r+\rho)T} E_0 \Big(BCC(\max(s_2^n, s_4^n)) \\
& + BCC(\min(s_2^n, s_3^n)) - BCC(s_2^n) \Big) \\
& + \vartheta \Big(C(s_1^n) - C(\min(s_2^n, s_3^n)) + C(s_2^n) - C(\max(s_2^n, s_4^n)) \\
& - (\min(s_2^n, s_3^n) - s_1^n) BCC(\min(s_2^n, s_3^n)) \\
& + (s_2^n - s_5^n) BCC(s_2^n) - (\max(s_2^n, s_4^n) - s_5^n) BCC(\max(s_2^n, s_4^n)) \Big) \Bigg),
\end{aligned}
$$

where

$$
s_1^n = \frac{nG_T V_0^{ind} \frac{1}{_T p_x} - \eta e^{rT} - e^{rT} E_0}{\vartheta},
$$

$$
s_2^n = \frac{nK_T V_0^{ind} \frac{1}{_T p_x}(1+\gamma) - \eta e^{rT}}{\vartheta},
$$

$$
s_3^n = \frac{(e^{\rho T} - 1) e^{rT} E_0 + nG_T V_0^{ind} \frac{1}{_T p_x} - \eta e^{rT}}{\vartheta},
$$

$$
s_4^n = \frac{(e^{\rho T} - 1) e^{rT} E_0 + nK_T V_0^{ind} \frac{1}{_T p_x} - \eta e^{rT}}{\vartheta},
$$

$$
s_5^n = \frac{nK_T V_0^{ind} \frac{1}{_T p_x} - \eta e^{rT} - e^{rT} E_0}{\vartheta}.
$$

Here, all option prices are calculated using initial value S_0 and volatility σ.

Again we are interested in the probability that the company is ruined at time T.

Proposition 5.3
The probability of ruin, $p_{ruin}(\varphi)$, at time T for a company following a buy and hold strategy is

$$
p_{ruin}(\varphi) = \sum_{n=0}^{Y_0} \binom{Y_0}{n} (_T p_x)^n (_T q_x)^{Y_0-n} \Phi \left(\frac{\log\left(\frac{s_1^n}{S_0}\right) - \left(\alpha - \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}} \right).
$$

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Proof of Proposition 5.3: Using iterated expectations we get

$$p_{ruin}(\varphi) = P\left[A_T < Y_T G_T V_0^{ind} \frac{1}{T p_x}\right]$$

$$= E^P\left[P\left[A_T < Y_T G_T V_0^{ind} \frac{1}{T p_x}\,\middle|\, \mathcal{H}_T\right]\right]$$

$$= \sum_{n=0}^{Y_0} \binom{Y_0}{n} (_T p_x)^n (_T q_x)^{Y_0-n} P\left[A_T < n G_T V_0^{ind} \frac{1}{T p_x}\right].$$

The result now follows immediately from Proposition 4.5 and the definition of s_1^n.

$\square$

5.7 Constant relative portfolio

In the case of investments in a portfolio with constant relative portfolio weights we obtain the following proposition from (5.5).

Proposition 5.4
For a company investing in a portfolio with constant relative portfolio weights the fair value of ρ is the solution to the following equation

$$E_0 = \sum_{n=0}^{Y_0} \binom{Y_0}{n} (_T p_x^h)^n (_T q_x^h)^{Y_0-n} \Bigg(e^{(r+\rho)T} E_0 \Big(BCC(\max(v_2^n, v_4^n))$$

$$+ BCC(\min(v_2^n, v_3^n)) - BCC(v_2^n) \Big)$$

$$+ C(v_1^n) - C(\min(v_2^n, v_3^n)) + C(v_2^n) - C(\max(v_2^n, v_4^n))$$

$$- (\min(v_2^n, v_3^n) - v_1^n) BCC(\min(v_2^n, v_3^n))$$

$$+ (v_2^n - v_5^n) BCC(v_2^n) - (\max(v_2^n, v_4^n) - v_5^n) BCC(\max(v_2^n, v_4^n)) \Bigg),$$

where

$$v_1^n = n G_T V_0^{ind} \frac{1}{T p_x} - e^{rT} E_0, \tag{5.6}$$

$$v_2^n = n K_T V_0^{ind} \frac{1}{T p_x} (1 + \gamma), \tag{5.7}$$

$$v_3^n = \left(e^{\rho T} - 1\right) e^{rT} E_0 + n G_T V_0^{ind} \frac{1}{T p_x},$$

$$v_4^n = \left(e^{\rho T} - 1\right) e^{rT} E_0 + n K_T V_0^{ind} \frac{1}{T p_x},$$

$$v_5^n = n K_T V_0^{ind} \frac{1}{T p_x} - e^{rT} E_0.$$

All option prices above are calculated using initial value $V_0 + U_0$ and volatility $\delta\sigma$.

Calculations similar to the case of investments in a buy and hold strategy gives the following result for the ruin probability.

Proposition 5.5
If a company, whose insurance portfolio consists of pure endowments, invests in a portfolio with constant relative portfolio weights, then the probability of ruin at time T is given by

$$p_{ruin}(\varphi) = \sum_{n=0}^{Y_0} \binom{Y_0}{n} (_T p_x)^n (_T q_x)^{Y_0-n} \Phi \left(\frac{\log\left(\frac{v_1^n}{V_0+U_0}\right) - \left(r + \delta(\alpha-r) - \frac{1}{2}(\delta\sigma)^2\right)T}{\delta\sigma\sqrt{T}} \right).$$

5.8 Buy and hold with stop-loss if solvency is threatened

Assume the solvency requirement determined by the regulatory institutions is given by $E_T \geq \beta Y_T V_T^{ind}$. Hence, $E_0 \geq \beta Y_0 V_0^{ind}$, since the company otherwise would be insolvent already at time 0. Here, we further assume that the initial assets of the company fulfills

$$A_0 \geq e^{-rT} Y_0 G_T V_0^{ind} \frac{_T p_x^h}{_T p_x}(1+\beta).$$

To avoid accumulating with K_T in situations where this leads to insolvency, we require that $e^{rT} E_0 \geq \beta K_T Y_0 V_0^{ind} \frac{1}{_T p_x}$. Thus, the factor $\frac{1}{_T p_x}$ makes the assumption on the initial equity capital stronger than in the case of capital insurances. At time 0 the company invests in a buy and hold strategy. However, to decrease the probability of insolvency the company rebalances the investment portfolio to include investments in the savings account only, if the assets hit the lower boundary

$$A_t = \mathrm{E}^{Q^h}\left[e^{-r(T-t)} Y_T G_T V_0^{ind} \frac{1}{_T p_x}(1+\beta) \right] = e^{-r(T-t)} Y_0 G_T V_0^{ind} \frac{_T p_x^h}{_T p_x}(1+\beta). \quad (5.8)$$

Thus, disregarding the information at time t about the development of the insurance portfolio the company rebalances the portfolio if the value of the solvency requirement is equal to the assets. The advantage of (5.8) is that it can be written as

$$S_t^* = \frac{e^{-rT} Y_0 G_T V_0^{ind} \frac{_T p_x^h}{_T p_x}(1+\beta) - \eta - E_0}{\vartheta} \equiv Z.$$

Hence, as in Section 4 the requirement on the assets can be transformed into a barrier problem for the discounted stock price with a constant barrier.

Remark 5.6 A natural extension of (5.8) is to take the development of the insurance portfolio into account. This gives the criterion

$$A_t = \mathrm{E}^{Q^h}\left[e^{-r(T-t)} Y_T G_T V_0^{ind} \frac{1}{_T p_x}(1+\beta) \,\middle|\, \mathcal{F}_t \right] = e^{-r(T-t)} Y_t G_T V_0^{ind} \frac{_{T-t} p_{x+t}^h}{_T p_x}(1+\beta). \quad (5.9)$$

However, this criterion does not allow us to write the problem as a constant barrier problem. Both criterion (5.8) and (5.9) leave the company with a positive probability of insolvency. To avoid insolvency almost surely, we could assume that

$$A_0 \geq e^{-rT} Y_0 G_T V_0^{ind} \frac{1}{_T p_x}(1+\beta),$$

and use the intervention criterion

$$A_t = e^{-r(T-t)} Y_t G_T V_0^{ind} \frac{1}{_T p_x}(1+\beta),$$

which corresponds to assuming that all insured persons, which are alive at time t survive

to time T.

$\square$

In order to use (5.5) we consider a fixed number of survivors, say n. Given the number of survivors the equity capital can be decomposed into a term $E_T^{n,BS1}$, which is different from 0 if the company has intervened and a term $E_T^{n,BS2}$, which is non-zero if the company has not intervened. For $E_T^{n,BS1}$ we obtain

$$
\begin{aligned}
E_T = 1_{(\inf_{0 \le t \le T} S_t^* \le Z)} \Bigg(&1_{\left(Y_0 G_T V_0^{ind} \frac{T p_x^h}{T p_x}(1+\beta) < n G_T V_0^{ind} \frac{1}{T p_x} \right)} 0 \\
&+ 1_{\left(n G_T V_0^{ind} \frac{1}{T p_x} \le Y_0 G_T V_0^{ind} \frac{T p_x^h}{T p_x}(1+\beta) < n K_T V_0^{ind} \frac{1}{T p_x}(1+\gamma)+e^{rT} E_0 \right)} \\
&\times \min\left(e^{(r+\rho)T} E_0, Y_0 G_T V_0^{ind} \frac{T p_x^h}{T p_x}(1+\beta) - n G_T V_0^{ind} \frac{1}{T p_x} \right) \\
&+ 1_{\left(n K_T V_0^{ind} \frac{1}{T p_x}(1+\gamma) \le Y_0 G_T V_0^{ind} \frac{T p_x^h}{T p_x}(1+\beta) - e^{rT} E_0 \right)} \\
&\times \min\left(e^{(r+\rho)T} E_0, Y_0 G_T V_0^{ind} \frac{T p_x^h}{T p_x}(1+\beta) - n K_T V_0^{ind} \frac{1}{T p_x} \right) \Bigg) \\
= 1_{(\inf_{0 \le t \le T} S_t^* \le Z)} \Bigg(&1_{(v_6 < v_1^n)} 0 + 1_{(v_1^n \le v_6 < v_2^n)} \min\left(e^{(r+\rho)T} E_0, v_6 - v_1^n \right) \\
&+ 1_{(v_2^n \le v_6)} \min\left(e^{(r+\rho)T} E_0, v_6 - v_5^n \right) \Bigg),
\end{aligned}
$$

where v_1^n and v_2^n are given by (5.6) and (5.7), respectively, and

$$
v_6 = Y_0 G_T V_0^{ind} \frac{T p_x^h}{T p_x}(1+\beta) - e^{rT} E_0.
$$

For $E_T^{n,BS2}$ the calculations in Section 4.5 applies. Thus, we get

Proposition 5.7
In the situation with stop-loss the fair value of ρ must satisfy

$$
\begin{aligned}
E_0 = \sum_{n=0}^{Y_0} \binom{Y_0}{n} (_T p_x^h)^n (_T q_x^h)^{Y_0-n} \Bigg(&1_{ZI}^* \Big(1_{(v_1^n \le v_6 < v_2^n)} \min\left(e^{(r+\rho)T} E_0, v_6 - v_1^n \right) \\
&+ 1_{(v_2^n \le v_6)} \min\left(e^{(r+\rho)T} E_0, v_6 - v_5^n \right) \Big) + e^{(r+\rho)T} E_0 \Big(BCC_{ZO}^*\left(\max(s_2^{n,*}, s_4^{n,*}) \right) \\
&+ BCC_{ZO}^*\left(\min\left(s_2^{n,*}, s_3^{n,*} \right) \right) - BCC_{ZO}^*\left(s_2^{n,*} \right) \Big) \\
&+ \vartheta \Big(e^{rT} \left(C_{ZO}^*(s_1^{\beta,n,*}) - C_{ZO}^*\left(\min\left(s_2^{n,*}, s_3^{n,*} \right) \right) + C_{ZO}^*(s_2^{n,*}) - C_{ZO}^*(\max(s_2^{n,*}, s_4^{n,*})) \right) \\
&+ (s_1^{\beta,n} - s_1^n) BCC_{ZO}^*(s_1^{\beta,n,*}) - (\min\left(s_2^n, s_3^n \right) - s_1^n) BCC_{ZO}^*\left(\min\left(s_2^{n,*}, s_3^{n,*} \right) \right) \\
&+ (s_2^n - s_5^n) BCC_{ZO}^*(s_2^{n,*}) - (\max(s_2^n, s_4^n) - s_5^n) BCC_{ZO}^*\left(\max(s_2^{n,*}, s_4^{n,*}) \right) \Big) \Bigg),
\end{aligned}
$$

where

$$s_1^{\beta,n} = \frac{nG_T V_0^{ind}\frac{1}{Tp_x}(1+\beta) - \eta e^{rT} - e^{rT}E_0}{\vartheta}.$$

Here, all option prices are calculated with initial value S_0 and volatility σ.

The probability of insolvency for a company following the investment strategy described above is given in the following proposition

Proposition 5.8
For a company following a buy and hold strategy with stop-loss the probability of insolvency is given by

$$p_{ins}(\varphi) = \sum_{n=0}^{Y_0} \binom{Y_0}{n} (_Tp_x)^n (_Tq_x)^{Y_0-n} 1_{(Y_0 _Tp_x^h < n)} \left(\Phi\left(\frac{\log\left(\frac{s_1^{\beta,n,*}S_0}{Z^2}\right) - \left(\alpha - r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \right) \right.$$

$$\left. + \left(\frac{Z}{S_0}\right)^{\frac{2(\alpha-r)}{\sigma^2}-1} \left(1 - \Phi\left(\frac{\log\left(\frac{s_1^{\beta,n,*}S_0}{Z^2}\right) - \left(\alpha - r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \right) \right) \right).$$

Proof of Proposition 5.8: See Appendix D.

5.9 Constant relative amount δ in stocks until solvency is threatened

Consider the same set-up as in Section 5.8. The only difference is that the company invests in a strategy with constant relative portfolio weights until a possible intervention. Written in terms of the discounted value of the portfolio including risky investments the rebalancing takes place the first time

$$\mathcal{V}_t^*(\varphi) = e^{-rT}Y_0 G_T V_0^{ind}\frac{Tp_x^h}{Tp_x}(1+\beta) - E_0 \equiv \widetilde{Z}.$$

The result now follows from calculations similar to those in Section 5.8.

Proposition 5.9
In the situation with stop-loss the fair value of ρ must satisfy

$$E_0 = \sum_{n=0}^{Y_0} \binom{Y_0}{n} (_Tp_x^h)^n (_Tq_x^h)^{Y_0-n} \left(1_{\widetilde{ZI}}^* \left(1_{(v_1^n \le v_6 < v_2^n)} \min\left(e^{(r+\rho)T}E_0, v_6 - v_1^n\right) \right.\right.$$

$$+ 1_{(v_2^n \le v_6)} \min\left(e^{(r+\rho)T}E_0, v_6 - v_5^n\right)\right) + e^{(r+\rho)T}E_0\left(BCC_{\widetilde{ZO}}^*\left(\max(v_2^{n,*}, v_4^{n,*})\right)\right.$$

$$+ BCC_{\widetilde{ZO}}^*\left(\min\left(v_2^{n,*}, v_3^{n,*}\right)\right) - BCC_{\widetilde{ZO}}^*\left(v_2^{n,*}\right)\right)$$

$$+ e^{rT}\left(C_{\widetilde{ZO}}^*(v_1^{\beta,n,*}) - C_{\widetilde{ZO}}^*\left(\min\left(v_2^{n,*}, v_3^{n,*}\right)\right) + C_{\widetilde{ZO}}^*(v_2^{n,*}) - C_{\widetilde{ZO}}^*(\max(v_2^{n,*}, v_4^{n,*}))\right)$$

$$+ (v_1^{\beta,n} - v_1^n)BCC_{\widetilde{ZO}}^*(v_1^{\beta,n,*}) - \left(\min\left(v_2^n, v_3^n\right) - v_1^n\right)BCC_{\widetilde{ZO}}^*\left(\min\left(v_2^{n,*}, v_3^{n,*}\right)\right)$$

$$+ (v_2^n - v_5^n)BCC_{\widetilde{ZO}}^*(v_2^{n,*}) - \left(\max(v_2^n, v_4^n) - v_5^n\right)BCC_{\widetilde{ZO}}^*\left(\max(v_2^{n,*}, v_4^{n,*})\right)\right),$$

where

$$v_1^{\beta,n} = nG_T V_0^{ind} \frac{1}{T p_x}(1+\beta) - e^{rT}E_0.$$

Here, all option prices are calculated with initial value $V_0 + U_0$ and volatility $\delta\sigma$.

Calculations similar to those leading to Proposition 5.8 now gives

Proposition 5.10
For a company following a strategy with constant relative portfolio weights with stop-loss the probability of insolvency is given by

$p_{ins}(\varphi)$

$$= \sum_{n=0}^{Y_0} \binom{Y_0}{n} (_T p_x)^n (_T q_x)^{Y_0-n} 1_{(Y_0 \, _T p_x^h < n)} \left(\Phi \left(\frac{\log\left(\frac{v_1^{\beta,n,*}(V_0+U_0)}{Z^2}\right) - \left(\delta(\alpha-r) - \frac{1}{2}(\delta\sigma)^2\right)T}{\delta\sigma\sqrt{T}} \right) \right.$$

$$\left. + \left(\frac{\widetilde{Z}}{V_0+U_0} \right)^{\frac{2\delta(\alpha-r)}{(\delta\sigma)^2}-1} \left(1 - \Phi \left(\frac{\log\left(\frac{v_1^{\beta,n,*}(V_0+U_0)}{\widetilde{Z}^2}\right) - \left(\delta(\alpha-r) - \frac{1}{2}(\delta\sigma)^2\right)T}{\delta\sigma\sqrt{T}} \right) \right) \right).$$

6 Numerical results

Since we obtain implicit equations for ρ only, we now resort to numerical techniques to obtain fair values of ρ. In order to do so we rewrite the expressions for the fair value of ρ on the form $\rho = f(\rho)$ for some function f and use iterations to find fix points for f. For all numerical calculations we assume that time is measured in years and let $T = 1$. Before turning to the results we recall the following notation:

Symbol	Interpretation
V	Portfolio-wide deposit
U	Bonus reserve
E	Equity capital
S	Stock price
$\mathcal{V}(\varphi)$	Value of investment portfolio φ
ρ	Fair additional rate of return to equity capital
r	Riskfree interest rate
σ	Volatility of stock
G_T	Guaranteed accumulation factor
K_T	Announced accumulation factor
γ	Target for minimal bonus reserve per deposit
T	Length of accumulation period
β	Solvency requirement on equity capital per deposit
ϑ	Number of stocks held in a buy and hold strategy
δ	Constant proportion invested in stocks
Y	Number of survivors in insurance portfolio
h	Market attitude towards unsystematic mortality risk

6.1 Dependence on investment strategy

In this section we fix the parameters $r = 0.06$, $\sigma = 0.2$, $G_T = 1.045$, $K_T = 1.06$ and $\gamma = 0.1$ and consider the dependence of ρ on the investment strategy.
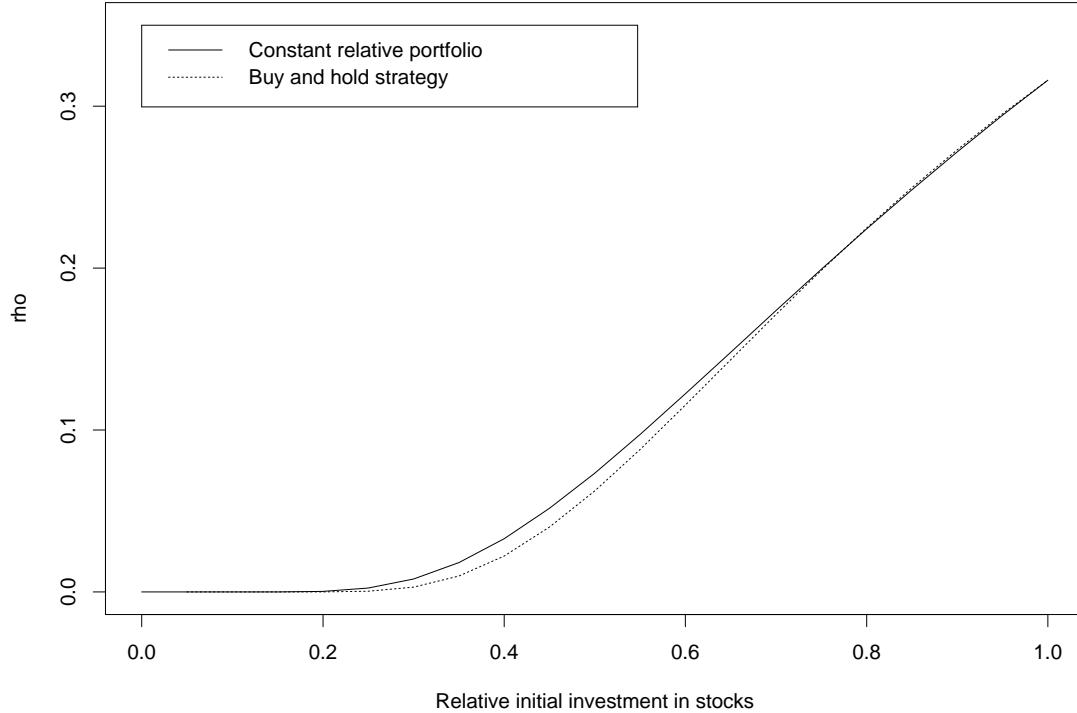

Figure 1: ρ as a function of the relative initial investment in stocks.

For now we assume the initial capital is distributed as follows: $V_0 = 100$, $U_0 = 10$ and $E_0 = 10$. Figure 1 then shows the dependence of ρ on the relative initial investment in stocks for a buy and hold strategy and a constant relative portfolio. The relative initial investment in stocks is given by $\kappa = \vartheta S_0 / \mathcal{V}_0(\varphi)$ for the buy and hold strategy and by δ for the constant relative portfolio. The observations to be made from Figure 1 are twofold. Firstly, ρ is an increasing function of the relative initial investment in stocks for both investment strategies. This is not surprising, since ρ is a measure for the risk of the insurance company and investing in stocks increases the risk. Secondly, comparing the two investment strategies, we observe that for a relative initial investment in stocks between 0.2 and 0.7 the fair value of ρ is slightly higher when investing in a constant relative portfolio rather than following a buy and hold strategy. This may be explained by the fact that when investing in a portfolio with constant relative portfolio weights a decrease in the stock price leads to additional investments in stocks and hence an increase in the capital at risk. Comparing the strategies we also note that the values of ρ coincide in the extremes where none or all capital is invested in stocks. This relies on the fact that the strategies coincide in these two situations.

In order to investigate the dependence of β we consider a buy and hold strategy with stop-
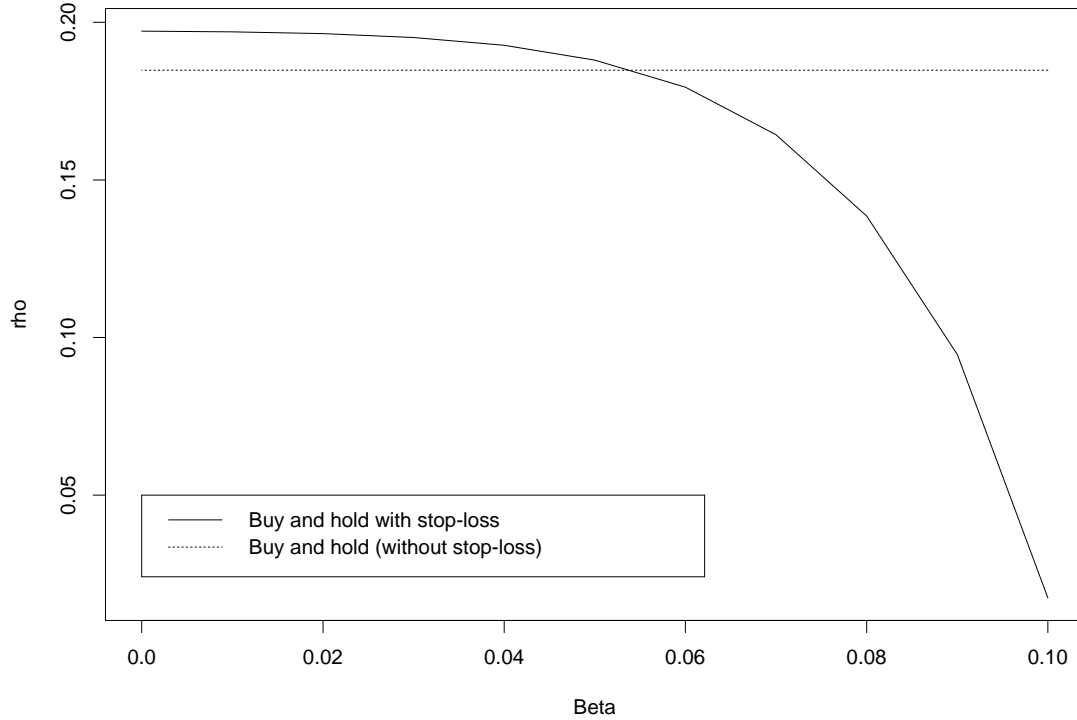
Figure 2: ρ as a function of β.

loss. The initial distribution of capital is changed such that $U_0 = 5$, since the dependence is more obvious in this case. The dependence of ρ on the required solvency margin β is now shown in Figure 2 for $\kappa = 0.5$. We observe that ρ is a decreasing function of β. This is also intuitively clear since increasing β, within the restrictions given in Section 4.5, increases the minimum payoff to the equity capital and hence decreases the risk of the company. For comparison Figure 2 also includes a horizontal line showing the fair value for an ordinary buy and hold strategy. Comparing the two strategies we observe that for low values of β the stop-loss strategy leads to higher values of ρ than the strategy without stop-loss. The reason for this is, that for low values of β the equity capital receives a payoff in case of intervention which is so low that at the time of a possible intervention the expected increase in the payoff from continuing the buy and hold strategy outweighes the risk of an even smaller payoff.

6.2   Dependence on parameters

For a company following a buy and hold strategy we now consider the dependence of ρ on the parameters r, σ, G_T, K_T and γ for a fixed initial distribution of capital. To study the dependence on r we let $\sigma = 0.2$, $G_T = 1.045$, $K_T = 1.06$, $\gamma = 0.1$, $V_0 = 100$, $U_0 = 10$ and $E_0 = 10$. Figure 3 then shows the dependence on r for $\kappa \in \{0.10, 0.25, 0.50\}$. The values of κ are chosen to illustrate a company with a conservative, a moderate and an aggressive investment strategy, respectively. We observe that ρ is a decreasing function of r for all
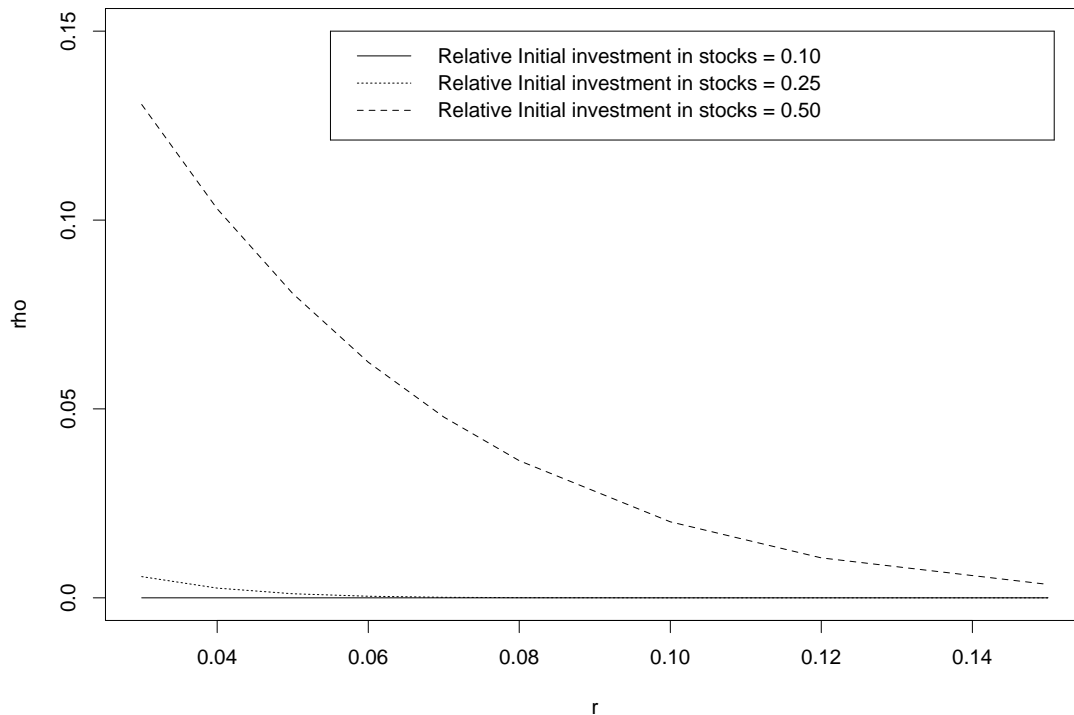
Figure 3: ρ as a function of r.

values of κ. This is also expected since increasing the riskfree interest rate lowers the probability of investment returns below the guaranteed/announced accumulation factor, hence decreasing the risk of the insurance company. Fixing $r = 0.06$ and letting $U_0 = 5$ and $E_0 = 5$, we now turn to the dependence on the guaranteed accumulation factor, G_T. The low values of E_0 and U_0 are chosen in order to observe a dependence on G_T for low values of κ. Figure 4 now shows the dependence on G_T for the same values of κ as above, i.e. $\kappa \in \{0.10, 0.25, 0.50\}$. We observe that ρ is an increasing function of G_T for all three values of κ. The positive dependence of ρ on G_T is intuitively clear, since the larger the guarantee to the insured, the more risky the contract is for the company.

For a company investing in a constant relative portfolio the constants δ and σ only enter the implicit equations for ρ as $\delta\sigma$, hence varying σ is identical to varying δ. Thus, we observe from Figure 1 that ρ is an increasing function of σ. This seems intuitively clear since increasing the volatility of the stocks increases the risk of the company. Investigating the dependence of ρ on γ, we find that ρ essentially is independent of γ. However, a slight negative dependence has been observed for high levels of volatility, low values of γ and an equity capital which is large compared to the bonus reserve. That ρ is a decreasing function of γ may be explained by the fact that increasing γ increases the probability of accumulating using G_T. Hence for some outcomes of the stock price there is a small increase in the payoff to the equity capital, whereas all other outcomes give the same payoff. Since the dependence is very small and in most cases non-existent, we have left out a figure illustrating this. Regarding the relationship between ρ and K_T we find that ρ only depends on K_T if V_0 and E_0 are large compared to U_0 and the investment strategy

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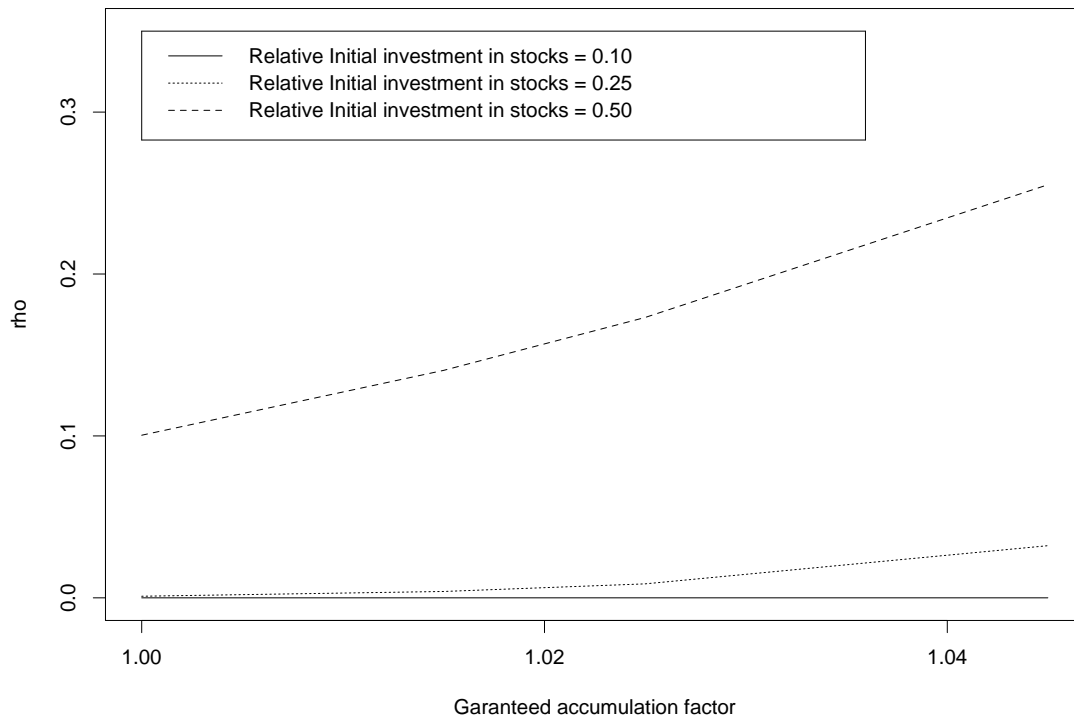Figure 4: ρ as a function of G_T.

is quite risky. In this case plotting ρ as a function K_T shows a shape similar to a 2. order polynomial with branches pointing downwards. The dependence may be explained by the fact that, when increasing K_T the payoff to the insurance portfolio increases if K_T is used as accumulation factor, but at the same time the probability of accumulation with K_T decreases. Thus, the risk of the company is a tradeoff between two factors working in opposite directions, such that the value of K_T for which the maximum value of ρ is obtained depends on V_0 and U_0. Since the equity capital in practice is much smaller than the deposit, we conclude that ρ for practical purposes is independent of K_T, and doing so, we leave out a graph showing the uninteresting case where a dependence is found.

6.3 Dependence on initial distribution of capital

To study the dependence of ρ on the initial distribution of capital we fix the parameters $r = 0.06$, $\sigma = 0.20$, $G_T = 1.045$, $K_T = 1.06$ and $\gamma = 0.10$ and consider an insurance company investing according to a buy and hold strategy with $\kappa = 0.25$. Since the value of ρ is indifferent to scaling of the initial distribution of capital, we further fix $V_0 = 100$ and allow E_0 and U_0 to vary. Figure 5 now shows the dependence of ρ on U_0 for different values of E_0. Comparing the graphs for the different values of E_0, we observe that ρ is a decreasing function of the equity capital. However, since ρ is an additional interest rate to the entire equity capital, we still observe an increase in the nominal payment for the increased risk even though ρ is decreasing. A decrease in ρ should thus be interpreted as
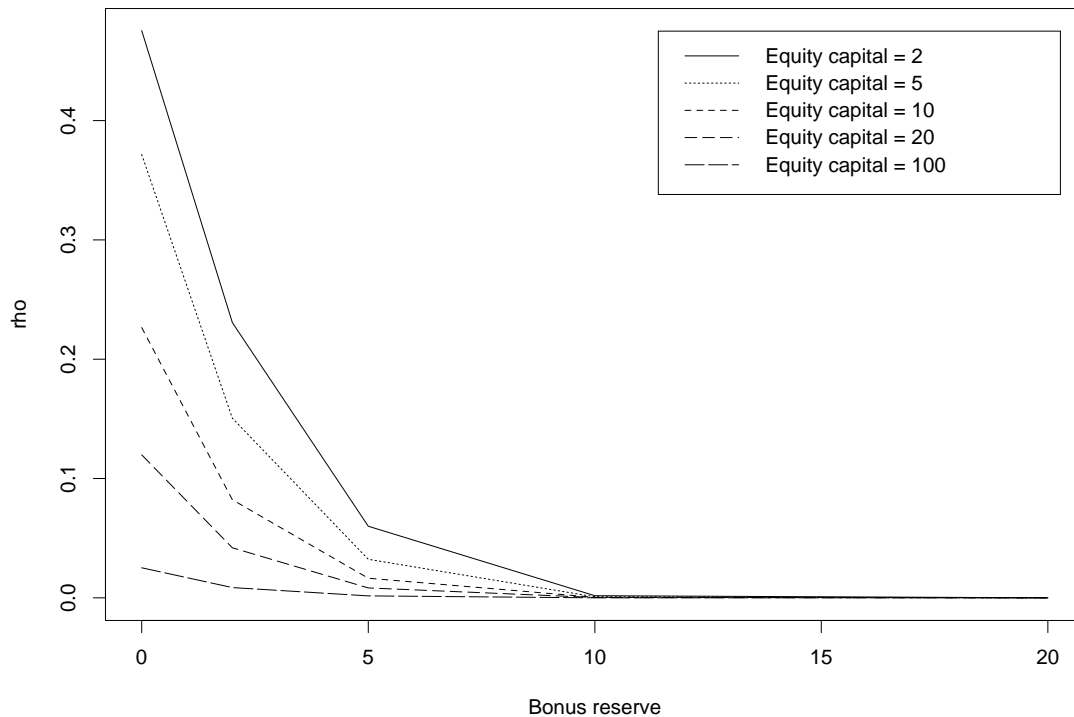
Figure 5: ρ as a function of U_0 for different values of E_0.

6.4 Effect from unsystematic mortality risk

Now consider an insurance company whose insurance portfolio consists of identical pure endowments for a group of persons of age 50. To model the possible deaths of the insured individuals we use a so-called Gompertz–Makeham form for the mortality intensity. Here, the mortality intensity can be written as

$$\mu_{x+t} = a + bc^{x+t}.$$

Here, the parameters, as in the Danish G82 mortality table for males, are given by $a = 0.0005$, $b = 0.000075858$ and $c = 1.09144$. To investigate the dependence on the number of insured and the choice of equivalent martingale measure we assume the company follows a buy and hold strategy with $\kappa = 0.25$ and keep the parameters and initial capital fixed as $r = 0.06$, $G_T = 1.045$, $K_T = 1.06$, $\sigma = 0.20$, $\gamma = 0.10$, $V_0 = 100$, $U_0 = 5$ and $E_0 = 5$. Recall that $V_0 = Y_0 V_0^{ind}$, so the total deposit is held constant while the number of insured
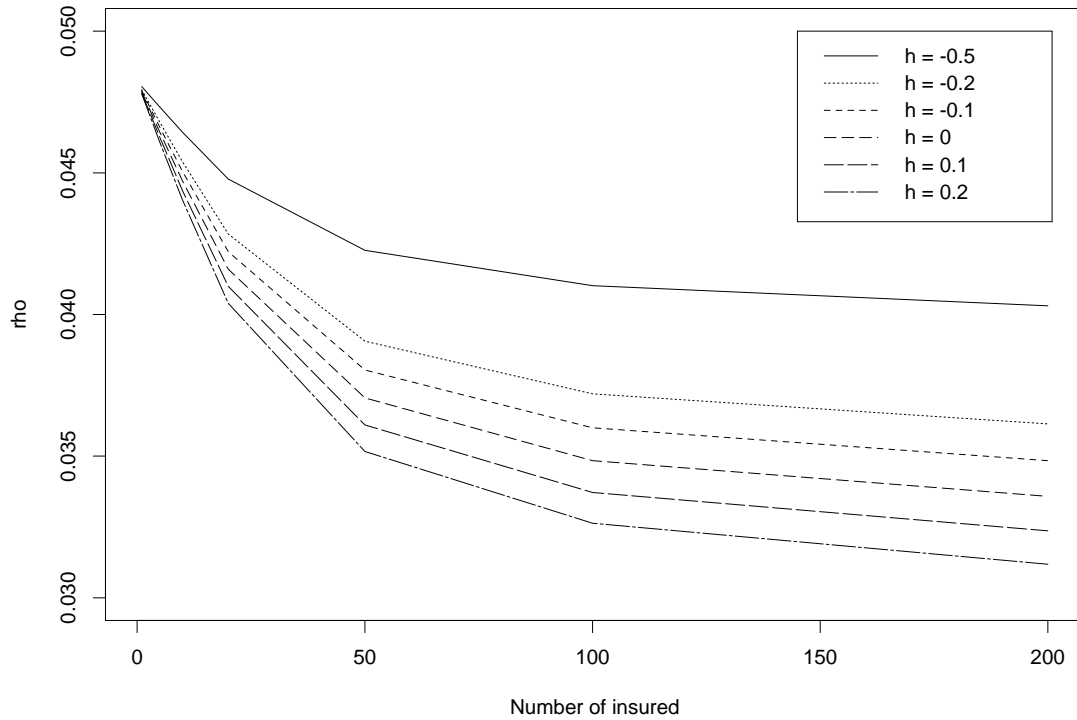
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Figure 6: ρ as a function of Y_0 for different values of h.

individuals increases by decreasing the individual deposits accordingly. From Figure 6 we see that ρ is a decreasing function of the number of insured. This is in correspondence with our intuition, since increasing the size of the insurance portfolio decreases the unsystematic mortality risk. Furthermore, we observe that ρ is a decreasing function of h, and that the dependence on h is an increasing function of the number of insured. That ρ is a decreasing function of h is intuitively clear since decreasing h corresponds to decreasing the market mortality intensity, and hence increase the survival probability in the derivation of ρ. The increasing dependence on h can be explained by the strong law of large numbers, which says that as the number of insured increases the number of survivors concentrate increasingly around the mortality intensity. Hence the mortality intensity used to determine ρ becomes increasingly important as the size of the insured portfolio is increased. It can be shown, see Appendix C, that if the number of insured tends to infinity then ρ converges downwards to the solution in case of capital insurances with G_T and K_T replaced by $G_T \frac{_T p_x^h}{_T p_x}$ and $K_T \frac{_T p_x^h}{_T p_x}$, respectively. Hence considering the case $h = 0$, we see that adding unsystematic mortality risk to a finite insurance portfolio leads to a fair value of ρ, which is higher than the fair value of ρ, 0.0322, obtained for capital insurances.

7   Impact of alternative distribution schemes

In this section we discuss how possible changes in the distribution scheme impact the results for the fair value of ρ.

A major possible change in the distribution scheme would be not to allow *any* transfer of capital from the bonus reserve to the equity capital. In the case where $G_T V_0 \leq A_T < K_T V_0 (1 + \gamma) + e^{rT} E_0$ this would lead to the following expression for the equity capital

$$E_T = \max \left(0, \min \left(e^{(r+\rho)T} E_0, A_T - V_T, A_T - (V_0 + U_0) \right) \right),$$

A similar change of course applies to the situation where $e^{rT} E_0 + K_T V_0 (1 + \gamma) \leq A_T$. Here the last term, which ensures that capital is not transferred from the bonus reserve to the equity capital, might be negative and hence the maximum operator is necessary to ensure that the equity capital is non-negative. Using this model increases the fair values of ρ, since the exposure of the equity capital to risk is larger. The increase is easily seen from the fact, that for a fixed ρ the new model would give an equity capital at time T which always is less or equal to the equity capital in the original model. Hence, a fair value of ρ must be higher. This model has been investigated in detail in the case where the solvency requirement applies to the sum of the equity capital and bonus reserve. Two important differences between this model and the model in the paper are: Firstly, as U_0 tends to infinity ρ converges to a strictly positive number, and secondly the dependence on the solvency parameter is more complex as the equity capital might receive the same negative payoff in case of intervention for different values of β.

Another possibility is to change the distribution scheme, such that the company use K_T to accumulate the deposit if $A_T \geq K_T V_0 (1 + \gamma)$. Thus, the company uses the accumulation factor K_T, providing that this leaves it with a minimum of $\gamma K_T V_0$ in the sum of the bonus reserve and equity capital. This criterion is closely related to a solvency requirement of βV_T on the sum of the equity capital and bonus reserve. Here, however the requirement on the sum of the bonus reserve and equity capital is set by the board of the company and not by legislation. Using this criterion in association with the model above we obtain a strange hump around $E_0 = 20$ for low values of U_0, when investigating the dependence on the size of the equity capital. This may be explained by the fact that with the proposed criterion the accumulation factor for the deposit depends on the initial equity capital. Hence, for some outcomes of the investments different values for the initial equity capital leads to different accumulation factors. Since applying a higher accumulation factor for the same investment return obviously increases the risk of the company, this leads to a positive dependence on the equity capital. The hump around $E_0 = 20$ for low values of U_0 shows that here this effect is more dominant than the otherwise predominant effect that increasing the equity capital decreases ρ. For $\gamma = 0$ the criterion corresponds to the case where the company views the announced accumulation factor as binding unless using K_T instead of G_T would bankrupt the company. In this case we would obviously expect an increase in the fair value of ρ.

If the company views the announced accumulation factor as legally binding the company is bankrupt if $A_T < K_T V_0$, and for $A_T \geq K_T V_0$ the deposit is accumulated using K_T. Applying the proper changes to the distribution scheme all necessary calculations are similar to those already presented. Since viewing K_T as binding obviously increases the risk for the equity capital, this change should lead to higher fair values of ρ.

8 A discussion of the realism and versatility of the model

In this section we comment on the chosen model. First we comment on the chosen probabilistic model and the requirement on the investment strategy. Then we discuss the advantages and versatility of the 1-period model. To end the section we discuss possible extensions.

The assumption that the financial market can be described by a Black–Scholes model is not very realistic, since both the interest rate and the volatility changes stochastically over time. However, if the accumulation period is relatively small the model is likely to be an acceptable approximation to reality. Hence, working with a more advanced financial model would make the results unnecessarily complicated. In the model we assume that the mortality intensity is deterministic, such that only the unsystematic mortality risk is considered. By unsystematic mortality risk we refer to the risk associated with the random development of an insured portfolio with known mortality intensity. Thus, the unsystematic mortality risk is the diversifiable part of the mortality risk. For a more realistic model we could introduce a stochastic mortality intensity as in Dahl (2004). This would allow us to consider the systematic mortality risk, referring to the risk associated with changes in the underlying mortality intensity, as well. Since changes in the underlying mortality intensity affect all insured, the systematic mortality risk is non-diversifiable. On the contrary it increases as the number of similar contracts in the portfolio of insured increases. Hence, if we were to add systematic mortality risk to the model the impact on the fair value of ρ would increase as a function of the length of the accumulation period, T, and the number of insured, Y_0. Since we consider one accumulation period only, the assumption of deterministic mortality intensity is very close to reality and sufficient for our purpose.

In the paper we assume that the company distinguishes between the investments belonging to the equity capital and the investments belonging to the insurance portfolio. Furthermore we assume that the assets belonging to the equity capital are invested in the savings account to keep possible risky investments on behalf of the owners aside from the risk associated with the insurance contracts. If the company does not make this distinction when investing, we may obtain the desired distinction by assuming that the equity capital is invested in the savings account and define the value of the risky portfolio residually as

$$\mathcal{V}_t(\varphi) = A_t - e^{rt} E_0.$$

Now the results in the paper apply immediately for buy and hold strategies for A, whereas an investment strategy for A with constant relative portfolio weights would lead to minor modifications of the results.

Using a model with only one accumulation period has several advantages. Firstly, we, as seen above, can justify working with a relatively simple probabilistic model. Secondly, we are able to define a distribution scheme with only one endogenously given parameter, since we do not have to specify a formula used to anticipate how the company chooses K_T. This is of importance, since in practice the choice of K_T is widely influenced by the competition, and thus, it is difficult to model. As for the versatility of the model we are particularly interested in answers to the following two questions: Does repeated use of the 1-period model yield fairness in a multi-period setting? And if so, what insight does the company gain by repeated use of the model? To answer the first question we consider an

arbitrary sequence of accumulation times $0 = T_0 < T_1 < \ldots < T_n$. For the distribution of the assets to be fair in the multi period model it must hold that

$$E^{Q^h} \left[e^{-rT_n} E_{T_n} \right] = E_0,$$

for an arbitrary, but fixed, equivalent martingale measure, Q^h. If we at each accumulation time, T_i, condition on the information $\mathcal{F}_{T_i}$ we obtain a string of 1-period models. Thus, if we determine the fair value of ρ in each 1-period model we obtain:

$$E^Q \left[e^{-rT_n} E_{T_n} \right] = E^Q \left[e^{-r(T_n - T_{n-1})} E^Q \left[e^{-rT_{n-1}} E_{T_n} \,\middle|\, \mathcal{F}_{T_{n-1}} \right] \right] = E^Q [e^{-rT_{n-1}} E_{T_{n-1}}]$$
$$= \ldots = E_0.$$

Here, the only restriction is, that the initial distribution of capital in one period is the terminal distribution in the preceding period. Hence, it even holds if the model parameters r, σ, γ and β and the investment strategy varies for different accumulation periods. Thus, repeated use of the 1-period criterion for fairness yields fairness in a multi-period setting. Using the model in a multi-period setting the company can obtain confidence bands for the development of the balance sheet and long term ruin probabilities by simulating the development of the financial market and the insurance portfolio. However, using the model for simulation purposes we need to specify a formula, from which the company determines the announced accumulation factor in each period. Furthermore the assumptions about constant parameters in the financial market and a deterministic mortality intensity are less realistic on a long term basis. This however, could be remedied by applying stochastic models to determine the constant interest rate and volatility and deterministic mortality intensity for the next accumulation period.

Some possible extensions of the model are to include different types of insurance contracts, insured of different ages and payments during the accumulation period. However, extending the model to include different types of contracts and different age groups increases the possibility of a systematic redistribution of capital from one group of insured to another.

9   Conclusion

For a company issuing insurance contracts with guaranteed periodic accumulation factors we consider the problem of distributing the assets fairly between the accounts of the insured and the equity capital. To derive a fair distribution we consider a 1-period model representing one accumulation period. In the model the only free parameter in the distribution scheme is the interest rate ρ, paid to the equity capital in addition to the riskfree interest rate, when such an additional rate is possible. Using the principle of no arbitrage, we are able to derive an implicit equation for the fair value of ρ given one of four different investment strategies. Investigating the dependence of ρ on the investment strategy, we observe that a constant relative portfolio is slightly more risky than a buy and hold strategy, and that ρ is an increasing function of the relative initial investment in stocks. In the case of a solvency requirement and stop-loss strategies we find that ρ is a decreasing function of β. Considering the dependence of ρ on the parameters, we observe a positive dependence on the volatility and the guaranteed accumulation factor and a negative dependence on the riskfree interest rate. As for the announced deposit rate and the parameter γ we found that the dependence for practical purposes is non-existent.

When considering the initial distribution of capital we find that ρ is an increasing function of the initial deposit and a decreasing function of the equity capital and the bonus reserve. Extending the model to include mortality obviously increases the fair value of ρ, since it adds more uncertainty to the model. As expected we observe that in the case of risk neutrality with respect to unsystematic mortality risk the fair value of ρ is a decreasing function of the number of insured tending to the fair value in the case without mortality. Furthermore we observe that the influence of the market attitude towards mortality risk on the fair value of ρ increases as the number of insured increases.

Acknowledgements

Appendix

A   Proof of Proposition 4.4

If the company invests in the savings account only, the value of the assets at time T is $A_T = e^{rT} A_0$. Since the value of the assets is deterministic, the distribution scheme is fair if and only if $E_T = e^{rT} E_0$. Considering the different intervals in the distribution scheme for the possible outcomes of A_T, we get

1. If $e^{rT} A_0 < G_T V_0$ then $E_T = 0$, so we cannot have $E_0 = e^{-rT} E_T$ if $E_0 > 0$ since $r < \infty$. Thus, no value of ρ gives a fair distribution scheme.

2. If $G_T V_0 \leq e^{rT} A_0 < K_T V_0 (1+\gamma) + e^{rT} E_0$ then each of the two terms in the minimum operator may be the smallest, and we have to consider each of the possibilities.

    (a) If $e^{(r+\rho)T} E_0 \leq e^{rT} A_0 - G_T V_0$ then a fair value of ρ satisfies $E_0 = e^{-rT} e^{(r+\rho)T} E_0$, i.e. $\rho = 0$.

    (b) If $e^{rT} A_0 - G_T V_0 \leq e^{(r+\rho)T} E_0$ then we must have $E_0 = e^{-rT}(e^{rT} A_0 - G_T V_0)$, i.e. $G_T = e^{rT} \frac{V_0 + U_0}{V_0}$. Thus, fair values of ρ must satisfy $e^{rT} E_0 \leq e^{(r+\rho)T} E_0$, i.e. $\rho \geq 0$.

3. If $K_T V_0 (1+\gamma) + e^{rT} E_0 \leq e^{rT} A_0$ then each of the two terms in the minimum operator may be the smallest, and we have to consider each of the possibilities.

    (a) If $e^{(r+\rho)T} E_0 \leq e^{rT} A_0 - K_T V_0$ then a fair value of ρ satisfies $E_0 = e^{-rT} e^{(r+\rho)T} E_0$, i.e. $\rho = 0$.

    (b) If $e^{rT} A_0 - K_T V_0 \leq e^{(r+\rho)T} E_0$ then we must have $E_0 = e^{-rT}(e^{rT} A_0 - G_T V_0)$, i.e. $K_T = e^{rT} \frac{V_0 + U_0}{V_0}$. Thus, fair values of ρ must satisfy $e^{rT} E_0 \leq e^{(r+\rho)T} E_0$, i.e. $\rho \geq 0$.

B Determining the limit as $U_0 \to \infty$

In this appendix we derive the fair value of ρ as the bonus reserve tends to ∞. For simplicity we consider the case of capital insurances. Taking the limit as $U_0 \to \infty$ in criterion (4.1) gives

$$
E_0 = e^{-rT} \lim_{U_0 \to \infty} E^{Q^0} \left[1_{(G_T V_0 \leq A_T < K_T V_0 (1+\gamma) + e^{rT} E_0)} \min \left(e^{(r+\rho)T} E_0, A_T - G_T V_0 \right) \right.
$$
$$
\left. + 1_{(K_T V_0 (1+\gamma) \leq \mathcal{V}_T(\varphi))} \min \left(e^{(r+\rho)T} E_0, A_T - K_T V_0 \right) \right] \tag{B.1}
$$

Assuming $\rho < \infty$ dominated convergence allows us to interchange limit and expectation. Since we consider admissible investment strategies only, we have that $\lim_{U_0 \to \infty} \mathcal{V}_T = \infty$. Hence it holds that

$$
\lim_{U_0 \to \infty} 1_{(G_T V_0 \leq A_T < K_T V_0 (1+\gamma) + e^{rT} E_0)} = 0
$$

and

$$
\lim_{U_0 \to \infty} 1_{(K_T V_0 (1+\gamma) < \mathcal{V}_T(\varphi))} = 1.
$$

Furthermore we have for $\mathcal{E}_T \in \{G_T, K_T\}$ that

$$
\lim_{U_0 \to \infty} \min \left(e^{(r+\rho)T} E_0, A_T - \mathcal{E}_T V_0 \right) = e^{(r+\rho)T} E_0 < \infty.
$$

Hence, in the limit we obtain the following equation

$$
E_0 = e^{-rT} e^{(r+\rho)T} E_0,
$$

such that in the limit $\rho = 0$. This is also intuitively clear, since increasing the bonus reserve decreases the probability of the equity capital suffering a loss, and in the limit where the bonus reserve is infinitely large the equity capital bears no risk and obviously it should not receive an additional payment compared to the riskfree interest rate.

We end this appendix by noting that the assumption $\rho < \infty$ does not impose a restriction, since $\rho = \infty$ cannot be a solution to (B.1). In order to do so we assume $\rho = \infty$ solves (B.1). This in turn would lead to

$$
E_0 = e^{-rT} \lim_{U_0 \to \infty} E^{Q^0} \left[1_{(G_T V_0 \leq A_T < K_T V_0 (1+\gamma) + e^{rT} E_0)} (A_T - G_T V_0) \right.
$$
$$
\left. + 1_{(K_T V_0 (1+\gamma) \leq \mathcal{V}_T(\varphi))} (A_T - K_T V_0) \right]
$$
$$
\geq e^{-rT} \lim_{U_0 \to \infty} E^{Q^0} \left[1_{(K_T V_0 (1+\gamma) \leq \mathcal{V}_T(\varphi))} (A_T - K_T V_0) \right]
$$
$$
= \infty,
$$

where we have used monotone convergence to interchange limit and integration in the last equality and considerations similar to those above to determine the limit. However, since $E_0 < \infty$ we have a contradiction, such that $\rho = \infty$ can not be the solution.

C Determining the limit as $Y_0 \to \infty$

We now determine the convergence of ρ as Y_0 tends to ∞, while keeping $V_0 = Y_0 V_0^{ind}$ fixed. Taking the limit in (5.3) we get

$$
\begin{aligned}
E_0 = e^{-rT} \lim_{Y_0 \to \infty} E^{Q^h} \Bigg[& 1_{(Y_T G_T V_0^{ind} \frac{1}{T p_x} \leq A_T < Y_T K_T V_0^{ind} \frac{1}{T p_x}(1+\gamma) + e^{rT} E_0)} \\
& \times \min \left(e^{(r+\rho)T} E_0, A_T - Y_T G_T V_0^{ind} \frac{1}{T p_x} \right) \\
& + 1_{(Y_T K_T V_0^{ind} \frac{1}{T p_x}(1+\gamma) \leq \mathcal{V}_T(\varphi))} \min \left(e^{(r+\rho)T} E_0, A_T - Y_T K_T V_0^{ind} \frac{1}{T p_x} \right) \Bigg]
\end{aligned}
$$

Assuming that $\rho < \infty$ we can use dominated convergence to interchange limit and integral. Using the strong law of large numbers we have for an arbitrary accumulation factor $\mathcal{E}_T$:

$$
\lim_{Y_0 \to \infty} \left(Y_T \mathcal{E}_T V_0^{ind} \frac{1}{T p_x} \right) = \lim_{Y_0 \to \infty} \left(\frac{Y_T}{Y_0} \mathcal{E}_T V_0 \frac{1}{T p_x} \right) = \mathcal{E}_T V_0 \frac{T p_x^h}{T p_x}, \quad Q^h - a.s.
$$

Since Q^h is identical to Q^0 with respect to the financial market for all h, we obtain the following equation in the limit

$$
\begin{aligned}
E_0 = e^{-rT} E^{Q^0} \Bigg[& 1_{(G_T V_0 \frac{T p_x^h}{T p_x} \leq A_T < K_T V_0 \frac{T p_x^h}{T p_x}(1+\gamma) + e^{rT} E_0)} \min \left(e^{(r+\rho)T} E_0, A_T - G_T V_0 \frac{T p_x^h}{T p_x} \right) \\
& + 1_{(K_T V_0 \frac{T p_x^h}{T p_x}(1+\gamma) \leq \mathcal{V}_T(\varphi))} \min \left(e^{(r+\rho)T} E_0, A_T - K_T V_0 \frac{T p_x^h}{T p_x} \right) \Bigg].
\end{aligned}
$$

This is exactly the equation in the case of capital insurances with G_T replaced by $G_T \frac{T p_x^h}{T p_x}$ and K_T replaced by $K_T \frac{T p_x^h}{T p_x}$. Note in particular, that assuming risk neutrality with respect to unsystematic mortality risk, i.e. $h = 0$ gives the same results in the limit as in the case without mortality.

The calculations above are carried out for an arbitrary h. However in the limit the measures Q^h and P are singular rather than equivalent if $h \neq 0$. Thus, using a Q^h with $h \neq 0$ in an attempt to derive a fair value of ρ for an infinitely large insurance portfolio would thus result in introducing an arbitrage possibility in the model. However, even though the limit result for $h \neq 0$ has no economic interpretation, it still provides useful insight for the dependence of ρ on h for a large portfolio. Furthermore solving the limit equation gives an approximation to the fair value in the case of a large portfolio of pure endowments.

D   Proof of Proposition 5.8

In order to prove Proposition 5.8, we first note that the probability of insolvency can be written as:

$$
p_{ins}(\varphi) = P\left[E_T < Y_T \beta V_T^{ind}\right]
$$

$$
= \sum_{n=0}^{Y_0} \binom{Y_0}{n} (_T p_x)^n (_T q_x)^{Y_0-n} P\left[E_T < n\beta V_T^{ind}\right]
$$

$$
= \sum_{n=0}^{Y_0} \binom{Y_0}{n} (_T p_x)^n (_T q_x)^{Y_0-n} \left(P\left[E_T < n\beta V_T^{ind}, \inf_{0\leq t\leq T} S_t^* > Z\right]\right.
$$

$$
\left. + P\left[E_T < n\beta V_T^{ind}, \inf_{0\leq t\leq T} S_t^* \leq Z\right]\right)
$$

$$
= \sum_{n=0}^{Y_0} \binom{Y_0}{n} (_T p_x)^n (_T q_x)^{Y_0-n} \left(P\left[A_T < n(1+\beta)G_T V_0^{ind}\frac{1}{_T p_x}, \inf_{0\leq t\leq T} S_t^* > Z\right]\right.
$$

$$
\left. + P\left[A_T < n(1+\beta)G_T V_0^{ind}\frac{1}{_T p_x}, \inf_{0\leq t\leq T} S_t^* \leq Z\right]\right)
$$

$$
= \sum_{n=0}^{Y_0} \binom{Y_0}{n} (_T p_x)^n (_T q_x)^{Y_0-n} \left(P\left[S_T^* < s_1^{\beta,n,*}, \inf_{0\leq t\leq T} S_t^* > Z\right]\right.
$$

$$
\left. + P\left[Y_0 G_T V_0^{ind}\frac{_T p_x^h}{_T p_x}(1+\beta) < nG_T V_0^{ind}\frac{1}{_T p_x}(1+\beta), \inf_{0\leq t\leq T} S_t^* \leq Z\right]\right)
$$

$$
= \sum_{n=0}^{Y_0} \binom{Y_0}{n} (_T p_x)^n (_T q_x)^{Y_0-n} \left(P\left[S_T^* < s_1^{\beta,n,*}, \inf_{0\leq t\leq T} S_t^* > Z\right]\right.
$$

$$
\left. + 1_{(Y_0 _T p_x^h < n)} P\left[\inf_{0\leq t\leq T} S_t^* \leq Z\right]\right). \tag{D.1}
$$

Here we first split up according to whether the company intervenes in the third equality. Then we use that $e^{rT}E_0 \geq \beta K_T Y_0 V_0^{ind}\frac{1}{_T p_x}$ in the fourth equality, since this ensures that the company never is insolvent if they accumulate with K_T or if the equity capital at time T is given by $E_T = e^{(r+\rho)T}E_0$. The fifth equality follows from inserting $s_1^{\beta,n,*}$ and the deterministic value of A_T in case of intervention. From (D.1) we observe that if the number of survivors if greater that the Q^h expectation then the company is insolvent in case of intervention, whereas this is not necessarily the case in the situation without intervention. Calculations similar to those in the proof of Björk (1998, Theorem 13.8) give

$$
P\left[S_T^* < s_1^{\beta,n,*}, \inf_{0\leq t\leq T} S_t^* > Z\right] = E^P\left[1_{(S_T^* < s_1^{\beta,n,*})} 1_{(\inf_{0\leq t\leq T} S_t^* > Z)}\right]
$$

$$
= E^P\left[1_{(Z < S_T^* < s_1^{\beta,n,*})}\right] - \left(\frac{Z}{S_0}\right)^{\frac{2(\alpha-r)}{\sigma^2}-1} E^P\left[1_{(Z < \tilde{S}_T^* < s_1^{\beta,n,*})}\right],
$$

where $\widetilde{S}^*$ is a process with the same dynamics as S^*, but with initial value $\widetilde{S}_0^* = \frac{Z^2}{S_0}$. Investigating each term separately we get

$$E^P\left[1_{(Z<S_T^*<s_1^{\beta,n,*})}\right]$$

$$= 1_{(Z<s_1^{\beta,n,*})}\left(P\left[S_T^* < s_1^{\beta,n,*}\right] - P[S_T^* \leq Z]\right)$$

$$= 1_{(Y_0 \, _T p_x^h < n)}\left(\Phi\left(\frac{\log\left(\frac{s_1^{\beta,n,*}}{S_0}\right) - \left(\alpha - r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) - \Phi\left(\frac{\log\left(\frac{Z}{S_0}\right) - \left(\alpha - r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\right),$$

and

$$E^P\left[1_{(Z<\widetilde{S}_T^*<s_1^{\beta,n,*})}\right]$$

$$= 1_{(Y_0 \, _T p_x^h < n)}\left(\Phi\left(\frac{\log\left(\frac{s_1^{\beta,n,*}S_0}{Z^2}\right) - \left(\alpha - r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) - \Phi\left(\frac{\log\left(\frac{S_0}{Z}\right) - \left(\alpha - r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\right).$$

Similarly

$$P\left[\inf_{0\leq t\leq T} S_t^* \leq Z\right] = 1 - E^P\left[1_{(\inf_{0\leq t\leq T} S_t^* > Z)}\right]$$

$$= 1 - E^P\left[1_{(Z<S_T^*)}\right] + \left(\frac{Z}{S_0}\right)^{\frac{2(\alpha-r-\frac{1}{2}\sigma^2)}{\sigma^2}} E^P\left[1_{(Z<\mathcal{S}_T^*)}\right]$$

$$= \Phi\left(\frac{\log\left(\frac{Z}{S_0}\right) - \left(\alpha - r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)$$

$$+ \left(\frac{Z}{S_0}\right)^{\frac{2(\alpha-r)}{\sigma^2}-1}\left(1 - \Phi\left(\frac{\log\left(\frac{S_0}{Z}\right) - \left(\alpha - r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\right).$$

Combining the results we get

$$p_{ins}(\varphi) = \sum_{n=0}^{Y_0}\binom{Y_0}{n}(_T p_x)^n(_T q_x)^{Y_0-n}1_{(Y_0 \, _T p_x^h < n)}\left(\Phi\left(\frac{\log\left(\frac{s_1^{\beta,n,*}S_0}{Z^2}\right) - \left(\alpha - r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\right.$$

$$\left. + \left(\frac{Z}{S_0}\right)^{\frac{2(\alpha-r)}{\sigma^2}-1}\left(1 - \Phi\left(\frac{\log\left(\frac{s_1^{\beta,n,*}S_0}{Z^2}\right) - \left(\alpha - r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\right)\right).$$

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