Systematic risk modelisation in credit risk insurance

Frédéric Planchet[*] **Jean-François Decroocq**[Ψ] **Fabrice Magnin**[α]

ISFA - Laboratoire SAF [β]
Université de Lyon - Université Claude Bernard Lyon 1
Groupe EULER HERMES [*]
WINTER & Associés [γ]

Abstract

The purpose of this paper is to propose a realistic and operational model to quantify the systematic risk in credit risk insurance. The model presented is built on the basis of classical credit risk model in which the joint laws of the risk factors become non Gaussian. We discuss also the way to take into account the ability for the insurer to mitigate the risk.

KEYWORDS: Credit insurance, Mitigation correlation factors, Insurance underwriting, Credit risk, NORTA method, default probability,

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[*] Frédéric Planchet is a Doctor in Management Science, a professor at ISFA and an actuarial associate at WINTER & Associés. Contact : fplanchet@winter-associes.fr.

[Ψ] Jean-François Decroocq is Chief Risk Officer at EULER HERMES. Contact : jean-francois.decroocq@eulerhermes.com.

[α] Fabrice Magnin is an actuary at WINTER & Associés. Contact : fmagnin@winter-associes.fr.

[β] Institut de Science Financière et d'Assurances (ISFA) - 50 avenue Tony Garnier - 69366 Lyon Cedex 07 - France.

[*] Euler Hermes - 1 rue Euler 75008 Paris - France.

[γ] WINTER & Associés 18 avenue Félix Faure - 69007 Lyon - France.

1. Introduction

To improve the classical model used to quantify the systematic risk in credit risk insurance (*cf.* BONTI, DECROOCQ [2008]) it is also possible on one hand to take into account non Gaussian joint laws of the risk factors and on the other hand to take into account the ability for the insurer to mitigate the risk.

Note that the business of credit risk is short-term and that an important part of it is risk mitigation. The total loss over a certain time period is considered as a random variable of the model and the most basic credit information used in the model is an exposure. An exposure amount or limit is an estimate for the claim at the time of default and is mapped to a single buyer group and a single policyholder. Additional attributes of an exposure are:

- A severity rate
- A default probability
- A business area or organizational unit

A tree-like structure can be defined for business areas so that risk measures are aggregated and/or allocated at different levels. As default events for a set of buyer groups are not independent, a factor model is introduced relating the buyer group's ability to pay to so-called systematic and specific risk factors. It is a linear relationship. Each buyer group is characterized by the following attributes:

- A map to one or more systematic risk factors given by weights (numbers between 0 and 1).
- A number called r^2 giving the portion of a buyer group's ability to pay variance due to the variance resulting from systematic factors in combination with their weights.

One of the main issues for credit insurance modeling is also the application of a specific and appropriate correlations structure which represents properly the systematic risk and the activity of credit insurance. Default correlations determine the likelihood of joint default of two or more companies over a given period.

As it is done in MKMV (*cf.* ZHANG [2008] and al. for a presentation), the most current assumption about dependency is that if two firms operate similarly facing some factors then their assets are correlated. It means the approach is based on the correlation of assets.

KMV uses 120 basic factors: Global Economy, Industry, Country, *etc*. The algorithm makes a regression of those indices in an orthogonal matrix built from asset returns. For each firm, it builds a composite factor based on Country and Industry defining this way a systematic risk induced by common factors. The part not explained by the regression is the specific risk of the company.

Note that KMV processes only simple counterparts what limits its use to estimations on aggregated portfolios for the credit insurance.

2. The classical model

2.1. Presentation

A natural way to model the risk in a credit insurance business (for example to estimate the risk capital required) is to consider the default probability of the companies. The credit insurer claim is also mapped to a single buyer (in fact a single buyer group) and a single policyholder (*cf.* DECROOCQ, PLANCHET, MAGNIN [2008]).

The parties and roles in credit insurance business are also:

Parties	Explanation
Credit insurer	The insurance company (protection seller).
Policyholder	The insured (protection buyer) paying for protection against non-payment by its buyer.
Buyer	The origin of credit risk (the reference name). The policyholder receives indemnification from credit insurer in case the company (the buyer) defaults.

2.1.1. Indicator of default and ability to pay

Each buyer is characterized by a default probability and given default probabilities the classical model uses the fact that a company is defaulting if its ability to pay is lower than a certain threshold.

Label	Explanation
J	Set of abilities to pay (buyer groups), $j \in J = \{1,...,n\}$.
L	Set of buyers, $l \in L = \{1,...,m\}$.
I_l^D	Indicator of default for the buyer $l \in L$.
π_l	Default probability for the buyer $l \in L$.
Z_j	Ability to pay for the buyer group $j \in J$.
d_l	Default threshold (or value at risk for ability to pay) for the buyer $l \in L$.

The table shows sets and variables that are used to describe the model. Capital letters are used for sets and random variables.

The buyer's default indicator is defined for each buyer by:

$$\forall l \in L \,,\; I_l^D = 1_{]-\infty, d_l[}\left(Z_j\right)$$

Given default probability π, a simulated default event is given by:

$$\forall l \in L, \quad Z_j < d_l$$

Where d_l solves the equation:

$$P\left(I_l^D = 1\right) = P\left(Z_j < d_l\right) = \pi_l$$

Z_j models the ability to pay of the buyer group $j \in J$ over a fixed time period.

2.1.2. Systematic and specific risks

As explained in the introduction, the KMV algorithm makes a regression of factors in an orthogonal matrix and, for each firm, it builds a composite factor based on Country and Industry. It defines a systematic risk induced by common factors and a specific risk induced by the part not explained by the regression.

Two strong assumptions are underlying this approach:

- The specific risk is individual and completely independent from the systematic risk.
- There is no correlation between two specific risks.

Label	Explanation
N	Set of systematic risk factors (countries, industries, etc.), $\nu \in \mathrm{N} = \{1,...,k\}$.
R	Systematic risk factor common to all abilities to pay. $R = \left(R_\nu\right)_{\nu \in \mathrm{N}}$ is a random variable vector of size k which is distributed $N\left(0,\delta\right)$.
δ	Covariance matrix of R. $\delta = \left(\delta_{\nu_1,\nu_2}\right)_{\nu_1 \in \mathrm{N}, \nu_2 \in \mathrm{N}}$ is a constant matrix of size $k \times k$ where $\delta_{\nu_1,\nu_2} = 0$ if $\nu_2 > \nu_1$.
w	Weights between abilities to pay and systematic risk factors. $w = \left(w_{j,\nu}\right)_{j \in J, \nu \in \mathrm{N}}$ is a constant matrix of size $n \times k$. $\forall j \in J$, $w_j = \left(w_{j,\nu}\right)_{\nu \in \mathrm{N}}$ is a constant vector of size k.
ε	Specific risk factor defined for each ability to pay. $\varepsilon = \left(\varepsilon_j\right)_{j \in J}$ is a random variable vector (independent) of size n which is distributed $N\left(0,1\right)$.
α	Adjustment factors (portions) for the combination of systematic and specific risk factors in a linear relation. $\alpha = \left(\alpha_j\right)_{j \in J}$ is a constant vector of size n.

r^2	Portions of buyer groups' abilities to pay variance due to the variance resulting from systematic risk factors in combination with their weights. $r^2 = \left(r_j^2\right)_{j\in J}$ is a constant vector of size n.

The table shows sets and variables that are used to describe the model. Capital letters are used for sets and random variables.

The vector Z is defined as:

$$Z = \alpha \cdot w \cdot R + (1-\alpha)\cdot \varepsilon$$

We have:

$$\forall j \in J\ ,\ Z_j = \alpha_j \cdot \sum_{\nu=1}^{k} w_{j,\nu} \cdot R_\nu + \left(1-\alpha_j\right)\cdot \varepsilon_j$$

Where $R \sim N(\mu,\delta)$ and $\varepsilon_j \sim N(0,1)$ which is corresponding with the original model used by the MKMV software.

The defined number called r^2 giving the portion of a buyer group's ability to pay variance due to the variance resulting from systematic factors in combination with their weights is introduced with the relation:

$\forall j \in J$,

$$r_j^2 = \frac{Var\left(\alpha_j \cdot \left(w_j\right)^T \cdot R\right)}{Var\left(Z_j\right)}$$

By correcting the covariance matrix of R and the weights between abilities to pay and systematic risk factors we can transform the relation into the final form:

$$Z = w \cdot R + \varepsilon$$

2.1.3. Simulation of ability to pay

As inputs of the model, we have the features r^2, w, δ and π. The thresholds d can be calculated and the variables R and ε can be estimated to simulate the ability to pay, which is very simple : to simulate $R \sim N(\mu,\delta)$ (a Gaussian vector of size k), it is possible to come down to the simulation of $\varepsilon = \left(\varepsilon_1,\ldots,\varepsilon_k\right)^T$ with $\forall \nu = 1,\ldots,k$, ε_ν i.i.d. $N(0,1)$ random variables thanks to the relation :

$$R = \mu + M \cdot \varepsilon$$

where M is triangular with positive coefficients on the diagonal such as $\delta = (M)^T \cdot M$ (Cholesky decomposition).

To simulate $R \sim N(\mu,\delta)$ we also have to respect the following steps:

- Simulation of each realization ε_ν of $\varepsilon_\nu \sim N(0,1)$, $\forall \nu = 1,...,k$.
- Calculation of M with $\forall \nu = 1,\dots,k$:

$$m_{\nu,\nu} = \sqrt{\delta_{\nu,\nu} - \sum_{\iota=1}^{\nu-1} m_{\nu,\iota}^2}$$

And for $\nu_2 = 1,...,k$ and $\nu_1 = 2,...,k$:

$$m_{\nu_1,\nu_2} = \frac{\delta_{\nu_1,\nu_2} - \sum_{\iota=1}^{\nu_2-1} m_{\nu_1,\iota} m_{\nu_2,\iota}}{m_{\nu_2,\nu_2}}$$

- Calculation of each realization r_ν of R_ν, $\forall \nu = 1,\dots,k$:

$$r_\nu = \mu_\nu + \sum_{\iota=1}^{\nu} m_{\nu,1} \cdot \varepsilon_\iota$$

- **Calculation of the value at risk for a normal random variable**

To solve as a function of d the equation $P(Z \le d) = \pi$ where Z is a Gaussian random variable $N(E(Z),Var(Z))$, we use:

$$d = E(Z) + \sqrt{Var(Z)} \cdot \Phi^{-1}(\pi)$$

With Φ^{-1} is the inverse of the normal distribution function and we approximate $\Phi^{-1}(\pi)$ with the Moro algorithm. The main drawback of the Gaussian hypothesis is that it leads *a priori* to an underestimation of high level quantiles. As we show below, this hypothesis can be relaxed.

2.2. Discussion

An internal model is based in one way or another on the description of a certain number of risk factors, exogenous or endogenous, and of their structure of dependence. Since work founders of the modern portfolio theory in 1952, the assumption that the risk factors are jointly distributed according to a Gaussian distribution was largely used. This assumption brings back the analysis of the dependence between the factors worthy of correlations existing between them.

Recent interest for the nonlinear dependence led to many criticisms of the Gaussian assumption, and this in an all the more relevant way when it is a question of estimating a quantile of a high nature (99,5 % for example) within the framework of the calculation of a SCR. Indeed, this assumption results in underestimating the extreme events, and it is easy to build examples showing the strong sensitivity of the level of the SCR to the shape of the tail of distribution. The specification of the tail is largely dependent on the choice of law but also of a proper correlation model adapted to the business for large portfolios attached to the assessment.

But is it necessary for as much giving up the Gaussian assumption? An attentive examination led to a more moderate appreciation on the illustrative examples could give rise to think. One can already observe that it makes it possible to integrate in modeling a high number of factors: for example, in the present paper, the model developed by KMV for credit insurance integrates thus approximately 120 factors in calculation on the part of systematic risk in the default probability of a buyer. To integrate as many factors with models using of the non Gaussian copulas seems for the moment out of reach. The correlation, with all the limits of this concept, remains a means accessible and simple to quantify the intensity and the direction of the link between the factors.

To circumvent the undervaluation induced by the choice of the Gaussian assumption, two approaches seem possible to us:

- To gauge the matrix of correlations to compensate for this effect by worsening the intensity of certain critical connections;
- To preserve the assumption of Gaussian dependence, but to use non Gaussian marginal laws for the factors.

This last approach, known under the name of method NORTA ("normal to anything") seems to us to have to be privileged in the reflections around the modeling of a structure of at the same time rich and operational dependence. This method applied certainly at small portfolios where correlation matrix can be easily adjusted. In the case of credit insurance, the method is a complementary approach to benchmark and adjust tail distribution compared to parameters assumptions done with current Gaussian model and specific correlation estimates.

Let's come back to the assumptions underlying the standard modeling and the specific situation of credit insurance:

- **Asset correlations estimates**

This approach is clearly introducing additional correlations compared to default/non default approach of credit insurance. The measurements of return of assets are amplifying the volatility. If some macro economics are well embedded in the market prices there are also other non welcomes adds-on as market liquidity or dilution of capital etc.

When situation becomes really bad, in general credit insurers have already written down any exposure for some times.

Asset correlations have been mainly derived from public firms when private companies demonstrate different behaviors that are not reflected in these estimates. If this approach can be sufficient for an investment portfolio it can be different for other portfolio.

- **Fixed asset correlations**

Asset correlations are supposed fixed for a given period and for all the distribution of the factors. This can be clearly challenged.

- **Assumptions**

The factors are assumed to be orthogonal with the specific risks. Depending of groups set up, trade and industrial operations this assumptions can be strong especially for credit insurance.

No evidence has been identified so far on this point.

- **Completeness of factors**

Most modeling include now more factors but it still appears that more dimensions would be needed as size of companies, legal structure, dependencies, financial structure etc.

3. The proposed model

At first we propose to relax the Gaussian hypothesis for the joint distribution of the factors and to use the NORTA approach (CARIO and NELSON [1997] for the initial presentation and STANHOPE [2005] for improvements and comments).

3.1. Non Gaussian distribution of the risk factors

In the standard model we assume that $R \sim N(\mu, \delta)$. For simplification we are rather interested in the vector R' defines by $\forall \nu \in N$, $R'_\nu = \frac{R_\nu - \mu_\nu}{\delta_{\nu,\nu}}$ ($R' \sim N(\mu', \delta')$ with $\mu'_\nu = 0$ and $\delta_{\nu,\nu} = 1$).

We assume now that if the dependence structure of $(R'_1, .., R'_k)$ remains Gaussian, that is:

$$(R'_1, .., R'_k) = \left(F_1^{-1}\left(\phi(Y_1)\right), .., F_k^{-1}\left(\phi(Y_k)\right)\right)$$

with:

- $(Y_1, .., Y_k)$ a Gaussian random variable with covariance Σ_Y. Σ_Y is in fact a correlation matrix and is completely defined by $\frac{k(k-1)}{2}$ coefficients ;

- The F_v may be non Gaussian, for example log-normal ;
- ϕ is the cumulative distribution function of a normalized Gaussian variable.

We thus have to compute Σ_Y in a way that leads to δ' when we compute the correlations of $\left(R_1',..,R_k'\right)$.

We observe that if $r'_{v_1,v_2} = E\left[Y_{v_1}Y_{v_2}\right]$ we have $\delta_{v_1,v_2} = \omega_{v_1,v_2}\left(r'_{v_1,v_2}\right)$ where:

$$\omega_{v_1,v_2}\left(r'\right) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F_{v_1}^{-1}\left(\phi\left(y_{v_1}\right)\right) F_{v_2}^{-1}\left(\phi\left(y_{v_2}\right)\right) h\left(r', y_{v_1}, y_{v_2}\right) dy_{v_1}\, dy_{v_2} \,,$$

and:

$$h\left(r', x, y\right) = \frac{1}{2\pi\left(1-r'\right)} \mathbf{exp}\left[\frac{x^2+y^2-2r'xy}{2\left(1-r'\right)}\right].$$

This leads to $\frac{k\left(k-1\right)}{2}$ equations in r'_{v_1,v_2}. In practice it would be more efficient to work directly with the rank correlation, that is to use $\left(F_1\left(R_1'\right),..,F_k\left(R_k'\right)\right)$ because we obtain in this case an explicit form for r'_{v_1,v_2} :

$$r'_{v_1,v_2} = 2\,\mathbf{sin}\left(\frac{\pi\delta'_{v_1,v_2}}{6}\right).$$

Let us now focus on the application of some specific parameters of credit insurance and their influence over the one year horizon risk capital computation:

1. Credit insurance is short term and credit risk mitigation
2. Policies include insurance features limiting the risk (franchise, maximum liability)
3. Portfolio consideration

3.2. Credit risk mitigation

Credit insurance has a unique capability to smooth the risks through the cycle and therefore to reduce its exposure on the worst situation. The following illustration is built with simulated data:

Fig. 1 : *The credit insurer mitigation smooth the Loss ratio over the cycle*

It can easily be demonstrated that the correlation with macro factors at loss ratio level is modified by the credit insurer capability to mitigate the risk. Because credit insurance is short term and exposure can be cancelled on short term, the modelling approach must be set up appropriately to avoid including double counting. The question would be how to introduce the credit insurer capability to mitigate the risk.

The factor model remains the same but the ability of credit insurance to mitigate the risk must be introduced as an additional dimension to the Risks factors as shown in the figure below:

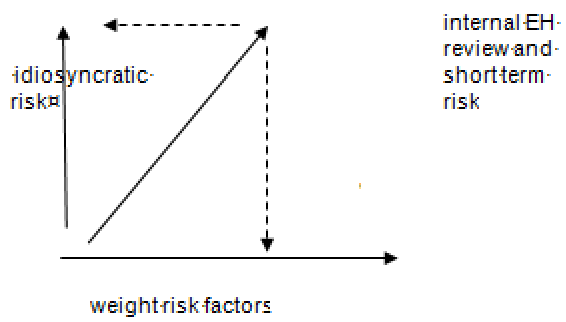


Fig. 2 : *Illustration of MKMV Global Correlation Factor Model*

This additional dimension reduces the risk and therefore the weight of the initial risk factors. Outside the fact that the credit insurer demonstrates its ability to go through the cycle, he is also organized to review individually the buyers monitored by geographical area and industry. It means exposure is already restricted (or will be at short notice in case of downturn) on the buyer presenting and explaining the most sensitivity to sectors/industry.

Without credit insurance the probability of default is higher and explained by Risk factors:

$$Z = w \cdot R + \varepsilon$$

Then the modelling, assumed in a simplified way, that the risk explained by the factors and by the specific risk will be weighted according to the r^2 to simulate the global risk of each buyer. r^2 is defined as the share of the variance of the global risk by the factors model and it means that:

$\forall j \in J$,

$$r_j^2 = \frac{Var\left(\alpha_j \cdot \left(w_j\right)^T \cdot R\right)}{Var\left(Z_j\right)}$$

For the credit insurer, this factor is calibrated to take into account the specificities of risk mitigation of credit insurance. Let assume CR the new dimension and includes it as a factor and then define the global risk of a buyer as follow:

$$\tilde{Z} = \tilde{w} \cdot \tilde{R} + \varepsilon$$

where:

- $\tilde{R}$ made of the previous matrix of factors and CR factors;
- $\tilde{w}$ is a new set of weight according for $\tilde{R}$;
- the correlations previously defined are completed with the correlation coefficients attached to the new dimension CR .

It can easily be shown that r_j^2 decreases because:

$$\mathrm{Cov}\left(\tilde{w}_{j,\nu} \cdot \tilde{R}_\nu, \tilde{w}_{j,k+1} \cdot CR\right) \leq 0$$

what was already illustrated below. In addition the specific risk is only going to be reduced partly and to simplify CR can be considered orthogonal to ε .

The consequence is that:

$$\mathrm{Cov}\left(\tilde{w}_{j,\nu}\cdot\tilde{R}_{\nu},\tilde{w}_{j,k+1}\cdot CR\right)=-\lambda\cdot Var\left(\tilde{w}_{j,\nu}\cdot\tilde{R}_{\nu}\right)$$

which is the reduction of correlation impact due to credit insurance. The parameter λ define the global ability for the insurer to mitigate the risk.

The variance of the factor is lower with credit insurance and the r^2 is corrected accordingly.

Let consider another aspect of the credit insurance ability to mitigate the risk through some considerations of probability of default and underwriting capabilities which will impact the stress situation.

In the case of a *k*-factors model the composite factors of a company is defined as a sum of industry and country indices:

$$\forall j\in J\ ,\ \rho_j=\sum_{\nu=1}^{k}w_{j,\nu}\times R_\nu\ \text{ (and } Z_j=\rho_j+\varepsilon_j\text{)}$$

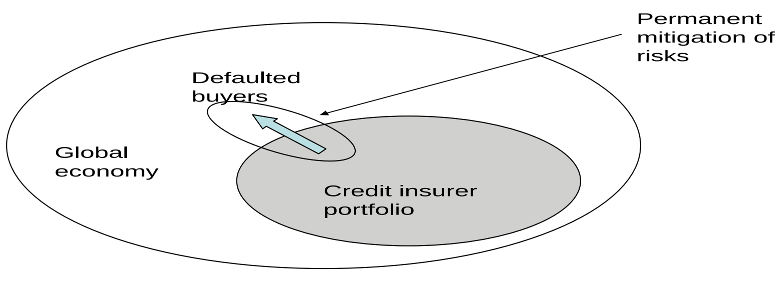
Then $\forall j\in J$, the conditional probability of default π_j^C knowing ρ_j can be written as:

$$\begin{aligned}\pi_j^C\left(\rho_j\right)&=P\left(Z_j\le d_j\left|\rho_j\right.\right)\\&=\Phi\left(\frac{\Phi^{-1}\left(\pi\right)-r_j\rho_j^c}{\sqrt{1-r_j^2}}\right)\end{aligned}$$

where $\rho_j^c=\dfrac{\rho_j-E\left(\rho_j\right)}{\sigma\left(\rho_j\right)}$ is the renormalized version of ρ_j.

Due to its underwriting capability and short term maturity, an adverse change in the economy will be managed before the end of a one year horizon conditional to this event, and the exposure will be reduced.

The probability of default of the portfolio can be illustrated as follow:



It is clear that $\pi_j^C\left(\rho_j\right)$ is conditional to the state of the economy and to the underwriting decisions of the credit insurers.

So this decision capability can be introduced in three different ways:

- By reducing the r^2 ;
- By introducing new factors representing the capability of risk mitigation.
- By using a multi-period modelling

Modified the systematic factors can be easily computed on historical data but it still reduced the real potential impact of a non-manage crisis.

The introduction of new factors representing the underwriting is the preferred solution with the following advantages:

- Maintain the initial factors model and parameters ;
- Introduce new dimensions to represent the initial risk profile of the company and its ability to manage through the cycle. We call the underwriting factors JFD (for Justified Factors Design).

The factors can be introduced in an existing factors model but calibration requires long historical benchmark (STEIN [2002]).

Modelling could be improved by introducing a multi-period structure within the one year horizon with still the necessity to introduce the conditional behaviour of the underwriter to mitigate the risk knowing the state of the economy during the previous period. As a consequence, another way to explain these factors is linked to the overall parameters assessments. A single buyer is correlated to the rest of the companies. The parameters of default and correlation attached to this company are defined according to a sample or a set of other companies. Therefore, when the underwriter modifies the portfolio of buyers by selected the risks, the original sample is modified in such a way that the parameters conditional to the state of the economy should be estimated again on the sample this way modified. The additional correlation factors introduced are an estimate of the conditional effect of underwriting over the time horizon.

It has to be noticed that this situation doesn't happen in a portfolio stable over the time horizon of the computation.

3.3. Policy features impact

The current KMV modelling includes a calibration of maximum liability. The effect is computed for the main policies and disregards for the other. It means the current set up is very conservative. It amplified the impact of any deviation in the set up of correlation instead of reducing it as losses are not properly capped.

This difference can reduce the value of a loss distribution quantile up to more than 20 %. Therefore, it appears that the impact of this factor is much more relevant as it is currently set in the KMV modelling and it could be proposed to reduce the r^2 even further.

3.4. Portfolio consideration

If part of the credit insurer exposure is on large companies, the main part is on small and medium size private companies. A number of analysis shows that default correlation on SME is very small, which is now becoming more obvious according to latest analysis (MKMV has analyzed the integration of this segment in GCor 10/07). Risk mitigation is also more efficient on small companies especially in case of downturn where limits can be more easily operated on this type of companies leading to reduce significantly the losses.

Therefore the global risk of that type of buyer is more linked to its specific risk than explain by the risk factors which are reflected in the r^2 used by the credit insurer.

4. Conclusion

Credit insurance is an insurance activity with unique mitigation capability when managed with a clear credit underwriting organisation. The credit insurer has recognized this capability in its modelling in a proper and conservative way.

The current modelling using KMV system is conservative and r^2 could even be reduced according to some calibration aspect. Modelling must catch the main aspect of the activity. If KMV shows that it can be adjusted to represent some aspects of the business, its computation can only be done with single counterparties limiting the possibility to represent properly the policy features and reinsurance. These two aspects cap the losses and one way to represent them better could be to reduce even further the r^2.

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