BERMUDAN SWAPTIONS IN HULL-WHITE ONE-FACTOR MODEL: ANALYTICAL AND NUMERICAL APPROACHES

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Abstract. A popular way to value (Bermudan) swaption in a Hull-White or extended Vasicek model is to use a tree approach. In this note we show that a more direct approach through iterated numerical integration is also possible. A brute force numerical integration would lead to a complexity exponential in the number of exercise dates in the base of the number of points (p^N). By carefully choosing the integration points and their order we can reduce it to a complexity pN^2 versus a quadratic $(pN)^2$ in the tree. We also provide a semi-explicit formula that leads to a faster converging implementation.

1. Introduction

Bermudan swaptions are compounded options. At each exercise date you can or enter into a swap or keep your right up to the next exercise date. In the martingale approach to option pricing, the options prices are obtained through expectation. One way to numerically compute the expectation, if the distribution of the underlying random variable is known, is to perform a numerical integration.

Consequently one way to price Bermudan swaptions is to perform a series of numerical integrations representing embedded integrals. The complexity of this computation is exponential in the number of dates; for p points at each date and N dates, one has approximatively p^N computations to do.

In a (trinomial) tree approach, where for one expiry date to the next one uses p steps, the final point number is around $2pN$ and the computation number around $(Np)^2$. So the comparison for a usual number of points and dates is clearly in favor of the later. Possible numbers would be $p = 1000$ and $N = 10$, giving 10^{30} computations for the multi-integral and 10^8 for the tree.

In the case of swaptions, the number of iterated integrals can be reduced by one by using an explicit formula [3] for the last optionality which is of European type. Even if the explicit formula is usually faster than the numerical computation it involved the solution of a non-linear equation. Fortunately it is possible to solve the equation only once and to use the result for the different points of the numerical integral. A way to achieve this for Bermudan swaptions with only two expiry dates was presented in [4].

But even by reducing the number of expiry dates by one, the brute numerical integration is not very efficient. In this note we describe a way to reduce the number of computations in the last integration to $2p(N-1)$ and the total to $pN(N-1)$.Using the previous example with $p = 1000$ and $N = 10$, the number of computation is less than 10^5. The reduction is possible thanks to a carefull choice of equidistant points in the integration and the separability condition on the volatility.

The use of equidistance points can be viewed as similar to the tree approach. But for a (theoretical) binomial tree with 50% probability on each branche, and 1000 points, the extreme points have a probability of $(1/2)^{1000} \sim 10^{-300}$. While in numerical integration it is possible to *cut* the discretisation at your choosing. In our implementation we chose extreme points such that the

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probability outside them is $1/p$ (10^{-3} for 1000 points). The tree approach consequently spends a lot of time on almost useless (tiny probability) computations.

Similar approach in the literature, even if this one was developed independently, can be found in Gandhi and Hunt [1]. They also propose a numerical integration approach to Bermudan swaption in Hull-White model and recombining properties based on equally spaced points. They work on the short rate while we use a more direct approach on the discount factors.

In some sense our approach can also be linked to what Rebonato [7] call *long jump* technique. Computations are done only at price sensitive dates (no intermediary points) and between those date the diffusion is done analytically.

Also like for the 2-Bermudan swaption [4], it is possible to write explicitly the part of the value corresponding to the exercise into a swap at the first date. In practice, for a lot of options, this first option contains most of the value. By computing in an explicit formula the majority of the value we achieve a better convergence of the results. The speed is not improved by this semi-explicit formulation as to estimate the part on which the explicit method apply one need to compute the numerical value for all the points.

The results presented here are valid for Heath-Jarrow-Morton models satisfying the separability condition (H2). The models used in practice that satisfy this condition are the Hull-White and the Ho-Lee models, with the former being the more frequent, hence the title of the article.

2. Model, hypothesis and preliminary results

The model and main hypothesis used in this paper are the same that in [3].

We use a model for $P(t,u)$, the price at t of the zero-coupon bond paying 1 in u. We will describe this for all $0 \leq t, u \leq T$, where T is some fixed constant.

When the discount curve $P(t,.)$ is absolutely continuous and positive, which is something that is always the case in practice as the curve is constructed from rates and by some kind of interpolation, there exists $f(t,u)$ such that

$$P(t,u) = \exp\left(-\int_t^u f(t,s)ds\right). \tag{1}$$

The idea of Heath-Jarrow-Morton [2] was to exploit this property by modelling f as

$$df(t,u) = \mu(t,u)dt + \sigma(t,u)dW_t$$

for some suitable (possibly stochastic) μ and σ.

Here we use a similar model, but we restrict ourself to non-stochastic coefficients. The exact hypothesis on the volatility term σ is described by (H2). We don't need all the technical refinement used in their paper or similar one, like the one described in [5] in the chapter on *dynamical term structure model*. So instead of describing the conditions that lead to such a model, we suppose that the *conclusions* of such a model are true. By this we mean we have a model, that we call a *HJM one-factor model*, with the following properties.

Let $A = \{(s,u) \in \mathbb{R}^2 : u \in [0,T] \text{ and } s \in [0,u]\}$. We work in a filtered probability space $(\Omega, F, \mathbb{P}^{\mathsf{real}}, (\mathcal{F}_t))$. The filtration $\mathcal{F}_t$ is the (augmented) filtration of a one-dimensional standard Brownian motion $(W^{\mathsf{real}})_{0\leq t\leq T}$.

H1: There exists $\sigma : [0,T]^2 \to \mathbb{R}^+$ measurable and bounded[1] with $\sigma = 0$ on $[0,T]^2 \setminus A$ such that for some process $(r_s)_{0\leq t\leq T}$, $N_t = \exp(\int_0^t r(s)ds)$ forms, with some measure $\mathbb{N}$, a

[1] Bounded is too strong for the proof we use, some L^1 and L^2 conditions are enough, but as all the examples we present are bounded, we use this condition for simplicity.

numeraire pair[2] (with Brownian motion W_t),

$$\begin{aligned} df(t,u) &= \sigma(t,u)\int_t^u \sigma(t,s)ds\; dt - \sigma(t,u)dW_t \\ dP^N(t,u) &= P^N(t,u)\int_t^u \sigma(t,s)ds\; dW_t \end{aligned}$$

and $r(t) = f(t,t)$.

The notation $P^N(t,s)$ designates the numeraire rebased value of P, i.e. $P^N(t,s) = N_t^{-1}P(t,s)$. To simplify the writing in the rest of the paper, we will use the notation

$$\nu(t,u) = \int_t^u \sigma(t,s)ds.$$

Note that ν is increasing in u, measurable and bounded.

To be able to use the explicit formula for the valuation of the European swaptions, we will also use the following hypothesis.

H2: The function σ satisfies $\sigma(t,u) = g(t)h(u)$ for some positive functions g and h.

Note that this condition is essentially equivalent to the condition (H2) of [3] but written on σ instead of on ν. The condition on ν was $\nu(s,t_2) - \nu(s,t_1) = f(t_1,t_2)g(s)$.

We want to price some option in this model. For this we recall the generic pricing theorem [5, Theorem 7.33-7.34].

Theorem 1. *Let V_T be some $\mathcal{F}_T$-measurable random variable. If V_T is attainable, then the time-t value of the derivative is given by $V_t^N = V_0^N + \int_0^t \phi_s dP_s^N$ where ϕ_t is the strategy and*

$$V_t = N_t \,\mathrm{E}^{\mathbb{N}}\left[V_T N_T^{-1} \middle| \mathcal{F}_t \right].$$

We now state two technical lemmas that were presented in [4].

Lemma 1. *Let $0 \le t \le u \le v$. In a HJM one factor model, the price of the zero coupon bond can be written has,*

$$P(u,v) = \frac{P(t,v)}{P(t,u)} \exp\left(-\frac{1}{2}\int_t^u \left(\nu^2(s,v) - \nu^2(s,u)\right) ds + \int_t^u \left(\nu(s,v) - \nu(s,u)\right) dW_s \right).$$

Lemma 2. *In the HJM one factor model, we have*

$$N_u N_v^{-1} = \exp\left(-\int_u^v r_s ds \right) = P(u,v) \exp\left(\int_u^v \nu(s,v) dW_s - \frac{1}{2}\int_u^v \nu^2(s,v) ds \right).$$

We give the pricing formula for swaptions for a future time ([4, Theorem 2].

Theorem 2. *Suppose we work in the HJM one-factor model with a volatility term of the form (H2). Let $\theta \le t_0 < \cdots < t_n$, $c_0 < 0$ and $c_i \ge 0$ ($1 \le i \le n$). The price of an European receiver swaption, with expiry θ on a swap with cash-flows c_i and cash-flow dates t_i is given at time t by the $\mathcal{F}_t$-measurable random variable*

$$\sum_{i=0}^n c_i P(t,t_i) N(\kappa + \alpha_i)$$

where κ is the $\mathcal{F}_t$-measurable random variable defined as the (unique) solution of

$$\sum_{i=0}^n c_i P(t,t_i) \exp\left(-\frac{1}{2}{\alpha_i}^2 - \alpha_i \kappa \right) = 0 \qquad (2)$$

and

$${\alpha_i}^2 = \int_t^\theta \left(\nu(s,t_i) - \nu(s,\theta)\right)^2 ds.$$

[2]See [5] for the definition of a numeraire pair. Note that here we require that the bonds of *all* maturities are martingales for the numeraire pair $(N, \mathbb{N})$.

The price of the payer swaption is

$$-\sum_{i=0}^{n} c_i P(t,t_i) N(-\kappa-\alpha_i)$$

The following result describes the change of probability for conditional expectation (find a reference!).

Theorem 3. *Let X be a random variable, $\mathcal{G}$ be a sub-σ-algebra and ξ be the Radon-Nikodym derivative $\frac{dQ}{dP}$, then*

$$\mathrm{E}^P\left[\xi|\,\mathcal{G}\right]\mathrm{E}^Q\left[X|\,\mathcal{G}\right]=\mathrm{E}^P\left[X\xi|\,\mathcal{G}\right].$$

3. Main result

The notations we use to describe the swaption are the following. The N expiry dates are $0<\theta_1<\theta_2<\cdots\theta_N$ and to simplify some notations we set $\theta_0=0$. For each expiry i $(1\leq i\leq N)$ the swap that can be enter into by exercising the option in θ_i has n_i fixed coupons. The swap is represented by its cash-flow equivalent $(t_{i,j},c_{i,j})_{j=0,\ldots,n_i}$. The date $t_{i,0}$ is the swap start date and $t_{i,j}$ $(j=1,\ldots,n_i)$ are the fix coupon dates. The amounts $c_{i,0}$ are -1[3], $c_{i,j}$ $(j=1,\ldots,n_i-1)$ are the coupons and c_{i,n_i} is the final coupon plus 1 for the notional.

For the study of the swaptions we will use a change of probability. We define $\nu^{\#}(s)=\nu(s,\theta_{i+1})$ for $s\in[\theta_i,\theta_{i+1})$ and the Dolean exponential of its stochastic integral

$$L_t=\mathcal{E}\left(\int_0^t \nu^{\#}(s)dW_s\right)=\exp\left(\int_0^t \nu^{\#}(s)dW_s-\frac{1}{2}\int_0^t {\nu^{\#}}^2(s)ds\right).$$

Theorem 4. *Suppose we work in a HJM one-factor model with a volatility structure of the form (H2). Consider a N-Bermudan swaption with expiry dates $\theta_1<\theta_2<\cdots<\theta_N$ on swaps represented by $(t_{i,j},c_{i,j})_{i=1,\ldots,N;j=0,\ldots,n_i}$. Let $\alpha_{i,j,k}$ $(i=1,\ldots,N; j=0,\ldots,n_i, k=1,\ldots,i)$ be the positive number defined by*

$$ {\alpha_{i,j,k}}^2=\int_{\theta_{k-1}}^{\theta_k}\left(\nu(s,t_{i,j})-\nu(s,\theta_k)\right)^2 ds.$$

Let $V_{\theta_k}^{N-k}$ be the value of the $(N-k)$-Bermudan swaption at time θ_k $(k=0,\ldots,N-1)$. Let $\tilde{V}_{\theta_k}^{N-k}=N_{\theta_0}N_{\theta_k}^{-1}V_{\theta_k}^{N-k}L_{\theta_k}^{-1}$. The $\tilde{V}_k$ are $\mathcal{F}_{\theta_k}$-measurable random variables given recursively by

$$\tilde{V}_{\theta_{k-1}}^{N-(k-1)}=\mathrm{E}^{\#}\left[\max\left(\sum_{j=0}^{n_k} c_{k,j}P(0,t_{k,j})\exp\left(-\frac{1}{2}\sum_{l=1}^{k}\alpha_{k,j,l}^2-\sum_{l=1}^{k}\alpha_{k,j,l}X_l\right),\tilde{V}_{\theta_k}^{N-k}\right)\middle|\,\mathcal{F}_{\theta_{k-1}}\right]$$

and $\tilde{V}_{\theta_{N-1}}^1$ is the European swaption given by

$$\tilde{V}_{\theta_{N-1}}^1=\sum_{j=0}^{n_N} c_{N,j}P(0,t_{N,j})\exp\left(-\frac{1}{2}\sum_{l=1}^{N-1}\alpha_{N,j,l}^2-\sum_{l=1}^{N-1}\alpha_{N,j,l}X_l\right)N(\kappa+\alpha_{N,j,N})$$

with κ given by

$$\sum_{j=0}^{n_N} c_{N,j}P(0,t_{N,j})\exp\left(-\frac{1}{2}\sum_{l=1}^{N}\alpha_{N,j,l}^2-\sum_{l=1}^{N-1}\alpha_{N,j,l}X_l-\alpha_{N,j,N}\kappa\right)=0 \tag{3}$$

and the X_l are independent $\mathcal{F}_l$-measurable $\mathbb{P}^{\#}$-standard normal random variable.

[3] It is $-K$ for a bond option of strike K.

Proof. Using the generic pricing formula we have for $k = 0, \dots, N-2$,

$$V_{\theta_{k-1}}^{N-(k-1)} = N_{\theta_{k-1}} \,\mathrm{E}\left[N_{\theta_k}^{-1} \max\left(\sum_{j=0}^{n_k} c_{k,j} P(\theta_k, t_{k,j}), V_{\theta_k}^{N-k} \right) \middle| \mathcal{F}_{\theta_{k-1}} \right].$$

Using the explicit formula for European swaptions [3],

$$V_{\theta_{N-1}}^{1} = \sum_{j=0}^{n_N} c_{N,j} P(\theta_{N-1}, t_{N,j}) N(\kappa + \alpha_{N,j,N})$$

where κ is the solution of

$$\sum_{j=0}^{n_N} c_{N,j} P(\theta_{N-1}, t_{N,j}) \exp\left(-\frac{1}{2}\alpha_{N,j,N}^2 - \alpha_{N,j,N}\kappa \right) = 0.$$

Using the lemmas 1 and 2 of [4] recursively k times together with the definition of L_t, we have that

$$N_{\theta_0} N_{\theta_k}^{-1} P(\theta_k, t_{i,j}) = L_{\theta_k} P(0, t_{i,j}) \exp\left(-\frac{1}{2} \sum_{l=1}^{k} \alpha_{i,j,l}^2 - \sum_{l=1}^{k} \alpha_{i,j,l} X_l \right).$$

where the X_l are independent $\mathcal{F}_l$-measurable standard normal random variables with respect to the probability $\mathbb{P}^{\#}$. To obtain the result we used Girsanov Theorem [6, Section 4.2.2, p. 72] with $\nu^{\#}$. The random variables X_l are the same for all i and j thanks to the property (H2) of the volatility function.

With this result and using the result on conditional expectation Theorem 3, we can rewrite the value of the options

$$\begin{aligned}
\tilde{V}_{\theta_{k-1}}^{N-(k-1)} &= N_{\theta_0} N_{\theta_{k-1}}^{-1} V_{\theta_{k-1}}^{N-(k-1)} L_{\theta_{k-1}}^{-1} \\
&= L_{\theta_{k-1}}^{-1} N_{\theta_0} \,\mathrm{E}\left[N_{\theta_k}^{-1} \max\left(\sum_{j=0}^{n_k} c_{k,j} P(\theta_k, t_{k,j}), V_{\theta_k}^{N-k} \right) \middle| \mathcal{F}_{\theta_{k-1}} \right] \\
&= \mathrm{E}^{\#}\left[\max\left(\sum_{j=0}^{n_k} c_{k,j} P(0, t_{k,j}) \exp\left(-\frac{1}{2} \sum_{l=1}^{k} \alpha_{k,j,l}^2 - \sum_{l=1}^{k} \alpha_{k,j,l} X_l \right), \tilde{V}_{\theta_k}^{N-k} \right) \middle| \mathcal{F}_{\theta_{k-1}} \right]
\end{aligned}$$

Similarly by replacing P in the equation defining κ, we obtain an implicit definition of κ which depend on X_l:

$$\sum_{j=0}^{n_N} c_{N,j} P(0, t_{N,j}) \exp\left(-\frac{1}{2} \sum_{l=1}^{N} \alpha_{N,j,l}^2 - \sum_{l=1}^{N-1} \alpha_{N,j,l} X_l - \alpha_{N,j,N}\kappa \right) = 0.$$

And for the European swaption we obtain

$$N_{\theta_0} N_{\theta_{N-1}} V_{\theta_{N-1}}^{1} L_{\theta_{N-1}}^{-1} = \sum_{j=0}^{n_N} c_{N,j} P(0, t_{N,j}) \exp\left(-\frac{1}{2} \sum_{l=1}^{N-1} \alpha_{N,j,l}^2 - \sum_{l=1}^{N-1} \alpha_{N,j,l} X_l \right) N(\kappa + \alpha_{N,j,N}).$$

□

Like for 2-Bermudan swaption we can write explicitly the expected value for the exercise at θ_1. We obtain the following semi-explicit valuation theorem.

Theorem 5 (Semi-explicit formula)**.** *Let μ be defined by*

$$\mu = \min_{y \in \mathbb{R}} \left\{ \sum_{j=0}^{n_1} c_{1,j} P(0, t_{1,j}) \exp\left(-\frac{1}{2}\alpha_{1,j,1}^2 - \alpha_{1,j,1} y \right) \le \tilde{V}_{\theta_1}^{N-1}(y) \right\}$$

with the convention that if the set is empty, $\mu = +\infty$ and if the set a no minimum, $\mu = -\infty$.

The value of the Bermudan swaption of the previous theorem can then be written as

$$\begin{aligned} V_0^N &= \sum_{j=0}^{n_1} c_{1,j} P(0,t_{1,j}) N(\mu+\alpha_{1,j,1}) \\ &\quad + \mathrm{E}\left[\mathbb{1}(X_1 \geq \mu) \max\left(\sum_{j=0}^{n_1} c_{1,j} P(0,t_{1,j}) \exp\left(-\frac{1}{2}\alpha_{1,j,1}^2 - \alpha_{1,j,1} X_1 \right), \tilde{V}_{\theta_1}^{N-1} \right) \right] \end{aligned}$$

with $\tilde{V}_{\theta_1}^{N-1}$ defined in the previous theorem.

Knowing if μ is non trivial ($-\infty < \mu < \infty$), even in a particular case, is not obvious. Also the uniqueness (or non uniqueness) of the number for which we have an equality is a non-trivial question.

4. Numerical implementation

Suppose we have computed $\alpha_{i,j,k}$.

4.1. **Kappa.** We first review the computation of κ. From the definition it seems that equation (3) needs to be solved for each draw of $\{X_l\}$. In the next lemma we show that this is not the case.

Lemma 3. *Under the hypothesis of Theorem 4, the solution κ of (3) is given by*

$$\kappa = \frac{1}{\beta_N}\left(\Lambda - \sum_{l=1}^{N-1} \beta_l X_l \right)$$

where Λ is the (unique) solution of

$$\sum_{j=0}^{n_N} c_{N,j} P(0,t_{N,j}) \exp\left(-\frac{1}{2}\sum_{l=1}^{N} \alpha_{N,j,l}^2 - H(t_{N,j})\Lambda \right) = 0$$

where $H(t) = \int_0^t h(s)ds$ and $\beta_l = \sqrt{G(\theta_l) - G(\theta_{l-1})}$ for $G(t) = \int_0^t g^2(s)ds$ (h and g are defined in (H2)).

Proof. By definition

$$\nu(t,u) = \int_t^u \sigma(t,s)ds = \int_t^u h(s)ds\, g(t) = (H(u) - H(t))\, g(t).$$

From there we have that

$$\alpha_{N,j,l}^2 = \int_{\theta_{l-1}}^{\theta_l} (\nu(s,t_{N,j}) - \nu(s,\theta_l))^2 ds = \beta_l^2 (H(t_{N,j}) - H(\theta_l))^2.$$

The last two terms in the equation (3) are

$$-\sum_{l=1}^{N-1} \alpha_{N,j,l} X_l - \alpha_{N,j,N}\kappa = -\left(\sum_{l=1}^{N-1} \beta_l X_l + \beta_N \kappa \right) H(t_{N,j}) + \left(\sum_{l=1}^{N-1} \beta_l X_l H(\theta_l) + \beta_N \kappa H(\theta_N) \right).$$

The second last term in this last expression being independent of j, it can be simplified in the equation (3) and we obtain the result. $\square$

4.2. **Some notation.** To shorten the writing we use the following notations ($1 \le k \le N-1$):

$$Y_k = \sum_{l=1}^{k} \beta_l X_l, \qquad Z_l = \exp\left(\beta_l X_l H(\theta_l)\right),$$
$$W_{N-1} = \sum_{j=0}^{n_N} c_{N,j} P(0, t_{N,j}) \exp\left(-\frac{1}{2}\sum_{l=1}^{N-1} \alpha_{N,j,l}^2 - H(t_{N,j}) Y_{N-1}\right) N(\kappa(Y_{N-1}) + \alpha_{N,j,N}),$$
$$T_k = \sum_{j=1}^{n_k} c_{k,j} P(0, t_{k,j}) \exp\left(-\frac{1}{2}\sum_{l=1}^{k} \alpha_{k,j,l}^2 - H(t_{k,j}) Y_k\right).$$

With those notations, the swap that compose the term of the max in the description of $\tilde{V}_{\theta_{k-1}}$ is

$$S_{\theta_k}^k = T_k Z_k \prod_{l=1}^{k-1} Z_l$$

and

$$\tilde{V}_{\theta_{N-1}} = W_{N-1} \prod_{l=1}^{N-1} Z_l.$$

Now let

$$W_{k-1} = \mathrm{E}^{\#}\left[Z_k \max(T_k, W_k) \right| \mathcal{F}_{k-1}]$$

then

$$\tilde{V}_{\theta_{k-1}}^{N-(k-1)} = W_{k-1} \prod_{l=1}^{k-1} Z_l. \tag{4}$$

4.3. **How to compute the integral.** To compute the nested integrals we sample $\beta_l X_l$ using $2p+1$ equally spaced points $[-p\epsilon, \dots, p\epsilon]$. Then Y_k is sampled with $2pk+1$ equally spaced points $[-pk\epsilon, \dots, pk\epsilon]$. The Z_l are sampled directy from X_l.

Note that $W_{N-1} = W_{N-1}(Y_{N-1})$ and we can easily compute the $2p(N-1)+1$ values for W_{N-1}.

We are interested by $V_{\theta_0}^N = \tilde{V}_{\theta_0}^N = W_0$. We only need to compute (recursively) the W_k.

The T_k depend only on Y_k and are sampled with $2pk+1$ points. We suppose that they have been computed from the Y_k.

The W_{k-1} are expected values depending on Z_k, T_k and W_k. To compute the m-th point of the sample of W_{k-1} we compute the expected value over the $2p+1$ points of Z_k multiplied by the $2p+1$ points that symmetrically surround the m-th point of $\max(T_k, W_k)$.

To do so we need the weight of the points, i.e. the weight of the points of X_l. This weight is the probability under $\beta_l X_l$ of the interval centered on $k\epsilon$ and length ϵ (except the initial and final intervals that have one extremity at infinity).

Remark: The implementation uses explicitely the fact that the points of the different $\beta_l X_l$ are equidistant with the same distance. The number of points could potentially be different but this would introduce some complication in the algorithm.

5. Convergence and stability

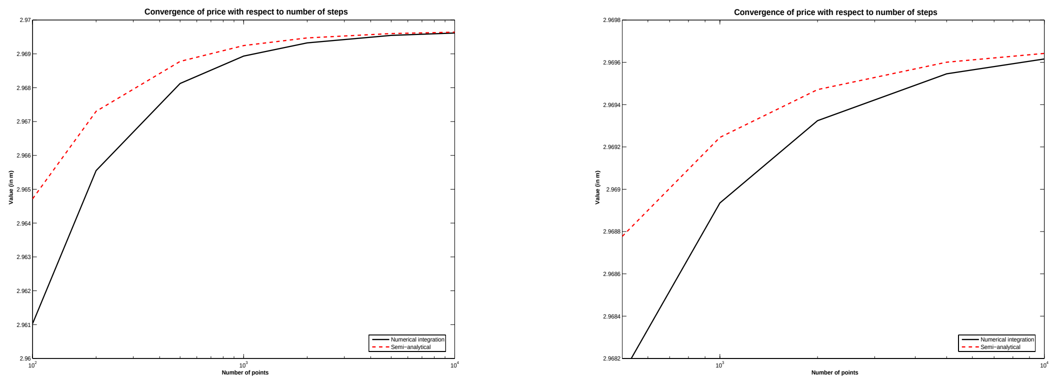
It was shown in a previous article [4] that numerical integration and semi-analytical approaches are faster and more precise than the classical tree approach for 2-Bermudan swaptions. In particular it was shown that in the tree case the delta figures are unstable and gamma figures meaningless.

For the general case that the tree approach will be more efficient and we concentrate only on the numerical integration.

All the computations are done with a 1y x 5y Bermudan swaption with annual expiry dates. There are five expiries between one and five years from now and the swaps to be entered into have tenors between five and one year with a common final maturity six year from now. The coupon is set at 3% and the swaption notional is 100m. The curve is flat at 3%.

The semi-analytical approach is marginally slower. The reason is that one has to first compute all the points of the numerical integration to estimate the μ. Once this is done the points on one side of μ are used in the numerical integration and the other side are disregarded.

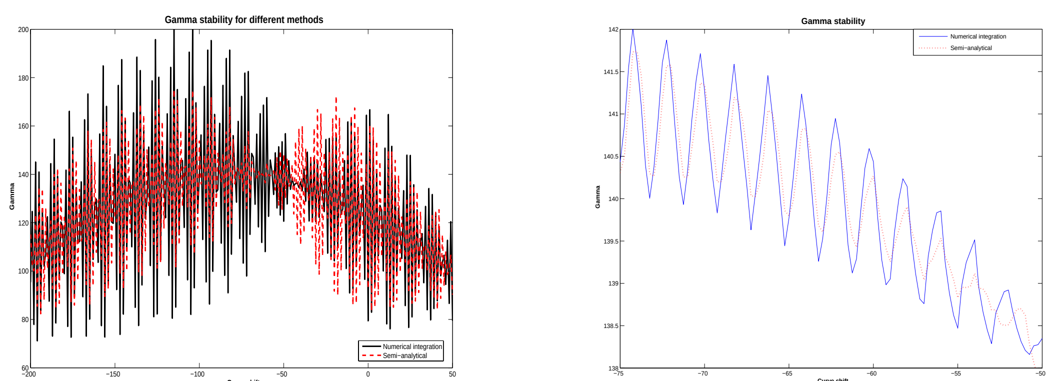
But in term of precision the improvement is significant. The semi-analytical precision is equivalent to the purely numerical one with 50% more points.



(a) Full picture (b) Enlarged picture

FIGURE 1. Price convergence for numerical integration and semi-analytical methods.

We now analyse the stability of the computations. From previous results we know that the most demanding figure is the gamma, this is why we concentrate only on that one. The *gamma profile* is obtain for the two approaches. The profile consists on computing the gamma for curves obtain from the base curve by parallel shifts. The gamma profile is a good graphical estimate of the stability of the computation to market changes.



(a) Gamma profile with 200 step-points (b) Gamma profile with 2000 step-points

FIGURE 2. Gamma profile for the numerical and semi-analytical methods with 200 and 2000 step-points.

To graphically compare the results to the previous article we give the profile 200 step-equivalent points (5×401 points in total). By opposition to what we have for the tree approach, the shape of the gamma profile is visible. But the imprecission is still important.

The second graph shows a portion of the same profile but with the number of steps increased to 2000 and a more precise scale. The precision is then acceptable with the gamma (second order derivative) varying within a 1% range.

The improvement of the semi-analytical approach is not as important than for the valuation. This is due to the fact that the ossilations are coming from the numerical discritisation of the integration. This discritisation is still present. The error is of the second order changes is of similar magnitude but from a more precise starting point.

Disclaimer: The views expressed here are those of the author and not necessarily those of the Bank for International Settlements.

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