Actuarial Pricing for Minimum Death Guarantees in Unit-Linked Life Insurance: A Multi-Period Capital Allocation Problem

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Abstract

We analyze an actuarial approach for the pricing and reserving of minimum death guarantees in unit-linked life insurance. After summarizing some results on mono-period risk measurement, we explain two possible strategies to deal with multi-period capital allocation problems. The first one uses no future information whereas the second one does. We explain how a cash-flow model can be used to perform the actuarial pricing and how we model the guarantee we are investigating. Next, we describe a simulation strategy which can be used to derive approximate distribution functions for the future reserves, capitals and total solvency levels in the approach where future information is used. From this strategy, average future capitals, which are needed in the cash-flow model, can be calculated. We test the simulation strategy and we compare both multi-period approaches.

1 Introduction

In [Frantz et al. (2003)], a comparison between the so-called actuarial and financial approaches for pricing and reserving for unit-linked life insurance contracts with minimum death guarantee is made. In the actuarial approach, one does not apply a financial hedging strategy. Instead, capital is allocated to ensure for the necessary security. The authors propose to use a cash-flow model to perform the actuarial pricing. In such a model, the average future capitals are needed. These capitals depend upon the capital allocation strategy which is applied by the (re)insurer.

Since the risk of a minimum death guarantee is typically a risk to which one is exposed for multiple years, it is possible to review reserves and capitals when new information about the underlying asset and the mortality becomes available. We describe two reserving and capital allocation strategies. The first takes no future information into account. Hence, at time 0, the reserves and the capitals which will be kept in the future are fixed. This strategy has some computational advantages but does not seem very rational. By taking the initial capital sufficiently large, a lot of security can be foreseen, but from the results for the second strategy, we will see it is very likely we maintain a capital which is too large in the future when applying the first strategy. Hence, in the cash-flow model, one will end up with a premium which is unnecessarily high. The second strategy takes future information into account on a yearly basis. Such a strategy seems more rational but is quite cumbersome to calculate. Approximations need to be made and we suggest a strategy to do this. Our approach allows us to estimate distribution functions for the future reserves and total solvency levels. This provides a lot of information about the risk which is borne by the (re)insurer reserving in an actuarial way.

Instead of working with the log-normal model to simulate the underlying asset, as in [Frantz et al. (2003)], we use the regime-switching log-normal model with two regimes as described in [Hardy (2001)].

The remainder of this paper is organized as follows. In section 2, we define some well-known risk measures and repeat some of their properties. We then see how these risk measures can be used for mono-period capital allocation purposes in section 3. We suggest two multi-period capital allocation strategies in section 4. Section 5 explains how actuarial pricing can be performed using a cash-flow model. In section 6, we discuss how we modelled the unit-linked contract with minimum death guarantee. After explaining a simulation strategy for using future information in section 7.2, we immediately verify the part of the strategy dealing with the underlying asset in section 8. Finally, we compare the two multi-period strategies for the minimum death guarantee in section 9, where we also analyze the approximated distribution functions of the total solvency levels and the reserves in the approach using future information. We conclude in section 10.

2 Well-known Risk Measures

We first define some well-known risk measures.

Definition 1 (Value-at-Risk) *For any $p \in (0,1)$, the VaR at level p is defined and denoted by*

$$Q_p[X] = \inf\{x \in \mathbb{R} | F_X(x) \geq p\}, \tag{1}$$

where $F_X(x)$ denotes the distribution function of X.

The VaR at level p of a risk X can be interpreted as the value at which there is only $1-p$ % probability that the risk will have an outcome larger than that value.

Definition 2 (Tail Value-at-Risk) *For any $p \in (0,1)$, the TVaR at level p is defined and denoted by*

$$TVaR_p[X] = \frac{1}{1-p}\int_p^1 Q_q[X]dq. \tag{2}$$

Definition 3 (Conditional Tail Expectation) *For any $p \in (0,1)$, the CTE at level p is defined and denoted by*

$$CTE_p[X] = E[X|X > Q_p[X]]. \tag{3}$$

The CTE at level p of a loss X can be interpreted as the average of all possible outcomes of X which are above $Q_p[X]$. As pointed out in [Dhaene et al. (2004-1)], the TVaR and the CTE are the same for all p-levels if the distribution function of X is continuous.

In definition 4, we list some well-known properties that risk measures may or may not satisfy.

Definition 4 (Properties for Risk Measures) *A risk measure ρ is called:*

1. *Monotone if for all random variables X and Y: $X \leq Y$ (X is smaller or equal than Y with probability 1) implies $\rho[X] \leq \rho[Y]$,*
2. *Positive homogeneous if for all random variables X and for all $a > 0$, one has $\rho[aX] = a\rho[X]$,*
3. *Translation invariant if for all random variables X and for all $b \in \mathbb{R}$, one has $\rho[X+b] = \rho[X]+b$,*
4. *Subadditive if for all random variables X and Y, one has $\rho[X+Y] \leq \rho[X]+\rho[Y]$.*

A risk measure which satisfies all the properties in definition 4 is called coherent. In table 1, we summarize which of the above properties are fulfilled for the VaR, CTE and TVaR.

Table 1: Some properties of some well-known risk measures.

Property	VaR	CTE	TVaR
Monotone	+	+	+
Positive Homogeneity	+	+	+
Translation Invariance	+	+	+
Subadditivity	-	-	+

Most of the properties in table 1 are easy to verify. For a proof of the subadditivity of the TVaR and for counterexamples, we refer to [Dhaene et al. (2004-1)].
Both the TVaR and the CTE are popular risk measures since they are coherent or coherent for continuous risks. One of their drawbacks is that they do not take the risk below the p-level into account. Possible solutions for this problem have been suggested (see e.g. [Wang (2001)]).

In literature (see e.g. [Dhaene et al. (2003)], [Dhaene et al. (2004-2)] and [Goovaerts et al. (2004)]), it has been suggested that the properties from definition 4 may not always be desirable.

3 Mono-Period Capital Allocation

Assume we are exposed to a random loss X. We define the pure premium PP as the average $\mathrm{E}[X]$ of X and assume this pure premium is held as a reserve. For the moment we assume X is an insurance loss to which one is exposed for only one period and we do not take into account discounting. For risky business, the pure premium itself will of course not guarantee that at the end of the period, there is enough security to withstand the losses which may occur. Therefore, companies exposed to risks need to hold capital as a safety margin against possible bad outcomes. We denote the capital as K. We define the total solvency level TSL as the sum of the reserve and the capital. Hence, $TSL = PP + K$. The different stakeholders of a company may have another opinion about the amount of capital which should be allocated for a risk X.

1. For the clients, whose interests are in general taken care of by the control authorities, it is important that the company is able to pay all losses with a very high probability. Therefore we may expect they will be in favor of a high capital. At first, we could think they will only agree with a total solvency level of $\max[X]$. One needs however to be aware of the fact that capital also has a cost. Since this cost has to be paid by the clients, it can be understood that clients are willing to accept capitals smaller than $\max[X]$.
2. Shareholders investing in a company want to be rewarded for their investments. Suppose a group of well-diversified investors provides a capital in a company supporting a risk X. If there is a profit at the end of the period, it will be to their advantage that the capital was as small as possible, since the benefit then leads to a higher return on the capital. If they are well-diversified, they do not really care about solvability.
3. For a lot of companies, and certainly for reinsurers, their rating is very important. The rating of a reinsurance company will be heavily influenced by the amount of capital available to withstand possible large losses.

Using a risk measure ρ, the capital K can be determined as $\rho[X - PP]$. If ρ is translation invariant, then $K = \rho[X] - PP$. Take for example the risk measure ρ equal to the VaR at level p. By the meaning of this risk measure, we know that by taking $K = \mathrm{Q}_p[X - PP] = \mathrm{Q}_p[X] - PP$, we will be able to pay up to the p% largest possible losses. If we take $K = \mathrm{CTE}_p[X - PP] = \mathrm{CTE}_p[X] - PP$, then we know that we will be able to pay all losses until the average loss above $\mathrm{Q}_p[X]$, the p % quantile of X.
Both the VaR and CTE are very popular risk measures. One of the reasons for their success is certainly the nice interpretation that can be attached to them. An important question which arises when we want to use these risk measures is the p-level at which we want or need to calculate them. If we suppose the regulators only look at the amount of the total solvency level, then the higher the p-level, the better. It has been suggested in literature (see e.g. [Dhaene et al. (2004-1)], [Dhaene et al. (2004-2)] and [Goovaerts et al. (2004)]) that regulators should also look at the cost of the capital which is held.

4 Multi-Period Capital Allocation

We first introduce some notation. We denote 0 as the time at which the insurance is written and T as the time at which the last liabilities are possible. Suppose we are exposed to a risk

$$X = (X_1, \ldots, X_T), \tag{4}$$

where X_i is the outcome for the risk at the end of year t, for $t \in \{1, \dots, T\}$. Denote the discounting factor for year t as Y_t. We define

$$Z = (Z_0 = X_1 e^{-Y_1}, \dots, Z_{T-1} = X_T e^{-\sum_{t=1}^{T} Y_t}). \tag{5}$$

Z is the vector of all future yearly losses, discounted to the start of the first year. Furthermore, we introduce

$$D = (D_0 = \sum_{t=0}^{T-1} Z_t, D_1 = \sum_{t=1}^{T-1} Z_t e^{Y_1}, \dots, D_{T-1} = Z_{T-1} e^{\sum_{t=1}^{T-1} Y_t}), \tag{6}$$

the vector of the discounted future costs at the start of the different years $t \in \{1, \dots, T\}$.

As time passes, important information may become available. In the context of death guarantees for unit-linked products for instance, this information could be the value of the underlying asset, the number of survivors and the discounting factor. Suppose the information is described by a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$. We propose two possible strategies. The first is explained in section 4.1 and does only use the information available at the start for setting (future) capitals and reserves. The second is explained in section 4.2. In this strategy, future information is taken into account on a regular basis. In both strategies, we use a time period of one year after which reserves and capitals may change. Theoretically, this can easily be generalized to e.g. biannual, monthly or quarterly revisions. Of course, taking more revisions will slow down the algorithms, especially for the second method, as can be understood from section 7.2. In both strategies, we determine for the start of each year a reserve and a capital. We assume that if at the end of the previous year, the reserve or the capital are not available, they are enlarged up to the level which is required by the strategy. For enlarging reserves, capital can be used.

In reality, reserves and, if necessary, capitals will of course be used when costs occur. For the ease of later computations (see section 5), we assume all payments are made at the end of the year in which the costs occur.

4.1 Approach Not Using Future Information

At time 0, we hold the pure premium

$$PP = E[D_0|\mathcal{F}_0] = R_0, \tag{7}$$

as a reserve. In addition, we could decide to hold a capital

$$K_0 = \rho[D_0 - R_0|\mathcal{F}_0], \tag{8}$$

where ρ is a risk measure. When not using future information, we could define the reserves which will be kept in the future as:

$$R_t = \mathrm{E}[D_t|\mathcal{F}_0], t \in \{1, \dots, T-1\}, \tag{9}$$

so the reserve at the start of year t is equal to the average discounted future costs at time $t-1$, given the information at time 0. Using the same risk measure as at time 0, and not incorporating future information, we can define

$$K_t = \rho[D_t - R_t|\mathcal{F}_0], t \in \{1, \dots, T-1\}. \tag{10}$$

The total solvency level at t is then defined and denoted as

$$TSL_t = R_t + K_t, t \in \{1, \dots, T-1\}. \tag{11}$$

It is very unlikely this approach will in reality be applied for reserving and capital allocation purposes. An insurer having future information available will normally take this into account. When the information is such that future costs are becoming more likely, then ruin becomes more likely if this is not taken into account. On the other hand, if future information is positive and this is not taken into account, we keep a level of capital which may be too high and which could be used for other business. The use of this strategy should be seen rather as a fast computational approximation in the cash-flow model of the strategy described in section 4.2 than as a strategy one will really apply for reserving and capital allocation purposes (see section 5 for more details). One of the purposes of this paper is to investigate how close this strategy is to the one using future information.

4.2 Approach Using Future Information

At time 0, we apply the same strategy as in section 4.1. At time $t \in \{1, \ldots, T-1\}$, we define the reserve as

$$R_t = \mathrm{E}[D_t|\mathcal{F}_t], \tag{12}$$

where $\mathcal{F}_t$ denotes the information available at the start of year t. For the capital, we take

$$K_t = \rho[D_t - R_t|\mathcal{F}_t], t \in \{1, \ldots, T-1\}. \tag{13}$$

As seen from time 0, R_t and K_t are random variables, since $\mathcal{F}_t$ is still unknown. At t however, R_t and K_t can be determined using the same methods as those which are used to determine pure premium and the capital at time 0. The total solvency level at t is again defined and denoted as

$$TSL_t = R_t + K_t, t \in \{1, \ldots, T-1\}. \tag{14}$$

The advantage of this approach is that the reserve and the capital are regularly adapted to new information. This also means that when the new information is bad, more reserves and capital than actually being available may need to be allocated. In other words, if the safety margin is not high enough as compared to the new information, it is enlarged with new capital. Of course, this is subject to the assumption that one is able to allocate new capital if necessary. Three questions are very pertinent in this context:

1. Is it probable that, at a given moment in the future, new capital will need to be allocated in order to obtain a total solvency level which is sufficiently high?
2. Given that new capital needs to be allocated, how large can this amount be?
3. Can a company if necessary allocate additional capital to a risk?

In general, we may say that when new capital needs to be allocated in the future, we can face problems of ruin. One may indeed question if one will find shareholders willing to invest in a product which is performing bad and leading to losses on the capital with a substantial probability. Therefore, we may argue that a situation where it is probable that high amounts of capital need to be reinjected in the business is not preferable. For an answer to the first two questions in the context of minimum death guarantees, we refer to section 9.3.

The answer to the third question depends upon a number of factors. If a certain amount of capital is provided to the whole company and the management can choose how this is used, then it may not be impossible to enlarge in the future the original capital which was allocated to a certain line of business.

5 Actuarial Pricing Using a Cash-Flow Model

Using a cash-flow model, we can derive an actuarial pricing strategy taking into account the cost of the capital which is provided by the shareholders.

5.1 Cash-Flows

In a cash-flow model, one models the average in-and outflows, taking the point of view of the shareholders. Assume the risk-free rate r is constant and equal to the rate of return on the bonds. Assume the corporate tax rate is equal to γ and the return on the stocks, after taxation, is equal to δ.

We then have the following average outflows:

1. Net mean claim payments: $c_s(1-\gamma)$, where $s \in \{1/12, 2/12, \ldots, T\}$. Hence, costs can arise at the end of every month. For the ease of later computations, we assume claims which occur in a certain year are only paid at the end of that year. Hence, at the end of year t, the reinsurer has to pay $c_t(1-\gamma)$, where

$$c_t = \sum_{s=1}^{12} e^{\frac{(12-s)r}{12}} c_{t-1+s/12}, t \in \{1, \ldots, T\}. \tag{15}$$

2. Net mean change in the technical provisions: $\Delta p_t(1-\gamma)$, where $t \in \{0, \ldots, T\}$
3. Mean change in the allocated capital: Δk_t, where $t \in \{0, \ldots, T\}$

We have the following inflows:

1. Net mean return on the provisions: $R_t(p)(1-\gamma)$, where

$$R_t(p) = p_{t-1}(e^r - 1), t \in \{1, \ldots, T\}. \tag{16}$$

2. Net mean return on the capital: $R_t(k)$, where

$$R_t(k) = k_{t-1}(e^\delta - 1) \text{ and } t \in \{1, \ldots, T\}. \tag{17}$$

3. Net premium income: $TFP(1-\gamma)$, where TFP denotes the technico-financial premium which has to be determined.

It is at this point that we can understand the possible usefulness of the strategy described in section 4.1. To calculate the average in-and outflows, we need to determine the average reserves p_t and the average capitals k_t for all $t \in \{0, \ldots, T\}$. For the reserves, due to the iterativity property of the expectation, we have

$$p_t = \mathrm{E}[R_t] = \mathrm{E}[\mathrm{E}[D_t|\mathcal{F}_t]] = \mathrm{E}[D_t|\mathcal{F}_0], \text{ for all } t \in \{0, \ldots, T\}. \tag{18}$$

Hence, with respect to the reserves, there is no difference in the cash-flow model between the approach described in section 4.1 and section 4.2. Assume the capitals are calculated using the conditional tail expectation. For all $t \in \{1, \ldots, T\}$, we then have

$$k_t = \mathrm{E}[K_t] = \mathrm{E}[\mathrm{CTE}_p[D_t - R_t|\mathcal{F}_t]] \neq \mathrm{CTE}_p[D_t - R_t|\mathcal{F}_0]. \tag{19}$$

We could use $\mathrm{CTE}_p[D_t|\mathcal{F}_0]$ as an approximation of k_t but we certainly need to verify whether this approximation is good.

5.2 Cost of Capital

We assume the cost of capital is constant and given. To determine this, one could e.g. use the classical Capital Asset Pricing Model.

5.3 Technico-Financial Premium

When we discount all future cash-flows with the cost of capital, the technico-financial premium is the value which makes the sum of the discounted inflows equal to the sum of the discounted outflows. Hence, the technico-financial premium is the value which solves equation (20)

$$\begin{aligned} &\sum_{t=0}^{T} e^{-tCOC} \left[\Delta p_t - R_t(p) + \sum_{s=1}^{12} e^{\frac{(12-s)r}{12}} c_{t-1+s/12} \right] (1-\gamma) \\ = &\sum_{t=0}^{T} e^{-tCOC} \left[R_t(k) - \Delta k_t \right] + TFP(1-\gamma), \end{aligned} \tag{20}$$

with the convention that $R_0(p) = R_0(k) = 0$ and $c_s = 0$ for all $s \leq 0$. Now suppose the average reserves are equal to the future mean claim payments, discounted at the risk free rate, i.e.

$$p_t = \sum_{s=1}^{12(T-t)} e^{-rs/12} c_{t+s/12}. \tag{21}$$

Using (21) it can be verified that the term $\Delta p_t - R_t(p) + \sum_{s=1}^{12} e^{\frac{(12-s)r}{12}} c_{t-1+s/12}$ in (20) is equal to

$$P_0 \quad \text{if} \quad t = 0, \tag{22}$$
$$0 \quad \text{if} \quad t \in \{1, \ldots, T\}. \tag{23}$$

The assumption that costs are only paid at the end of the year was taken in order to obtain (23). Using (22) and (23), we can write the technico-financial premium as

$$TFP = P_0 + \frac{\sum_{t=0}^{T} e^{-tCOC}[\Delta k_t - R_t(k)]}{1-\gamma}. \tag{24}$$

This means the technico-financial premium consists of two parts:

1. The reserve P_0 taken at time 0.
2. A loading for the average amounts of capital k_t which are allocated at the start of each year $t \in \{1, \ldots, T\}$.

To calculate expression (24), we need the average capitals at the start of the different years. In the approach not using future information the future capitals are fixed at time 0. When using future information, the future capitals are unknown at time 0. In section 7, we will explain how the average capitals can be approximated when pricing a minimum death guarantee. We first describe how we can model the unit-linked product with minimum death guarantee.

6 Modelling a Unit-Linked Life Insurance Contract With Minimum Death Guarantee

We use the parameters as summarized in appendix A, unless explicitly mentioned otherwise.

6.1 Product Description

We suppose there is a group of $N_I = 1000$ insured aged $x = 50$ which all invest $C = 1$ into a risky asset $(S_t)_{t\geq 0}$. The insurer gives a guarantee of $K = 1$ in case the insured dies before retirement. We suppose this guarantee is reinsured but the same estimations could be made by an insurer pricing the guarantee in an actuarial way himself. Hence, in case an insured dies at time t, the reinsurer will pay $(K - S_t)_+$, to the insurer, who will pass this through to the beneficiaries of the insured, along with the value of the units at time t. Note that $(K - S_t)_+$ is equal to the payoff function of a European put option with strike price K and maturity date t on the underlying asset $(S_i)_{i\in[0,t]}$. We suppose there is no surrender and there is an age of retirement $x_R = 65$ after which no payments are made. At retirement, the insurer reimburses the value of the investments to the insured who are still alive, without guarantee.

6.2 Financial Model

We use the regime-switching log-normal (RSLN) model with 2 regimes as described in [Hardy (2001)]. This model provides us with monthly log-returns. We denote the regime applying to the interval $[s, s+1)$ as κ_s. Hence $\kappa_s \in \{1, 2\}$. In a certain regime κ_s we assume the return Y_s satisfies:

$$Y_s = \log(\frac{S_{s+1}}{S_s}|\kappa_s) \sim N(\mu_{\kappa_s}, \sigma_{\kappa_s}). \tag{25}$$

Furthermore, the transitions between the regimes are assumed to follow a Markov Process characterized by the matrix P of transition probabilities

$$p_{ij} = Pr[\kappa_{s+1} = j|\kappa_s = i], \text{for } i, j \in \{1, 2\}. \tag{26}$$

As parameters, we use the parameters estimated using the maximum log-likelihood techniques explained in [Hardy (2001)]. We use the S&P 500 data (total returns) from January 1960 to December 2003. The maximum log-likelihood (MLE) estimates are given in table 2 along with the mean and the standard deviation of the parameters estimated using the Markov Chain Monte Carlo (MCMC) methodology explained in [Hardy (2002)]. We generated 10000 parameters using the Markov Chain Monte Carlo method to get an idea of the uncertainty of the parameters. The log-likelihood is equal to 931.31 for the maximum log-likelihood estimates and 931.05 for the Markov Chain Monte Carlo means.

Table 2: Parameter estimates for S&P 500.

Parameter	Mean (MLE)	Mean (MCMC)	StDev (MCMC)
μ_1	0.0135	0.0129	0.0020
μ_2	-0.0109	-0.0116	0.0105
σ_1	0.0344	0.0351	0.0021
σ_2	0.0645	0.0697	0.0118
p_{12}	0.0483	0.0438	0.0192
p_{21}	0.1985	0.2329	0.0970

In table 2, we detect an important standard deviation for the μ_2, the p_{12} and the p_{21} parameter. In what follows, we will use the maximum log-likelihood estimates.

6.3 Mortality Model

To model mortalities, we use the Gompertz-Makeham approach. The survival probability of a person aged x is then described as

$$ {}_tp_x = \exp\left(-\alpha t - \frac{\beta e^{\gamma x}(e^{\gamma t}-1)}{\gamma}\right), \tag{27}$$

for some $\alpha > 0$, $\beta > 0$ and γ. We use the first set of Gompertz-Makeham parameters in table 5 up to age 65 and the second set for ages higher than 65. As for the underlying asset, we model mortalities on a monthly basis.

7 Simulation Strategies

The main aim of the simulation strategies is to obtain the average reserves and capitals which will be kept in the two approaches. For the approach using future information, some other useful information can be retained.

7.1 Approach not Using Future Information

Modelling the approach not using future information is not so difficult. At time 0, we make N_S simulations for the underlying asset and for the mortality. From these simulations, $p_t = \mathrm{E}[D_t|\mathcal{F}_0]$ and $k_t = \rho[D_t|\mathcal{F}_0]$ can immediately be calculated, for all $t \in \{0, \ldots, T\}$.

7.2 Approach Using Future Information

7.2.1 A First Strategy

First we describe a naive simulation strategy which could be used to incorporate future information (see figure 1). At time 0 we make N_S simulations going from 0 to T and we use these to determine the distribution function of D_0, the discounted future costs at time 0. At time 1, we then obtain N_S values for the underlying asset and N_S values for the number of people who are still alive. This leads to $N = N_S^2$ combinations. From each of these combinations, we should then make new simulations from 1 to T to determine N distribution functions from which reserves and capitals can be calculated. p_1 and k_1 are determined as the averages of these N reserves and capitals respectively. For $t \in \{2, \ldots, T\}$, p_t and k_t could be determined using a similar strategy.

It can easily be understood that the approach described above will require a huge amount of computation time. Assume we start with 10^4 simulations for the underlying asset and for the mortality. To obtain the reserves and capitals at time 1, we then have to make e.g. 10^4 simulations for each of the 10^8 possible combinations for the underlying asset and the mortality. Hence from the start of year 2, we need to make 10^{12} simulations up to year T. If this is a period of e.g. 10 years, then this means a total of 1.2 10^{14} simulations. At the start of the following years, the same has to be done.

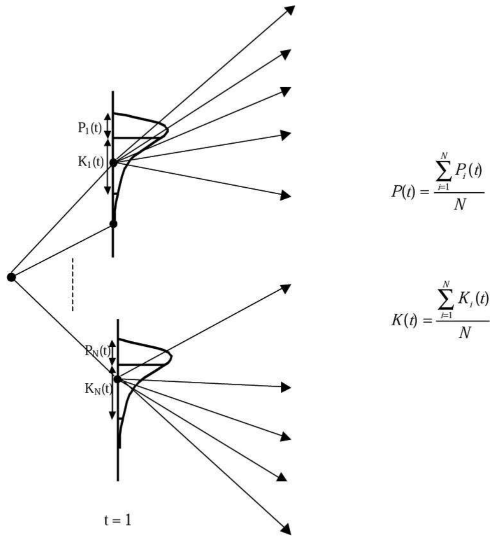

Figure 1: Tree simulations to obtain p_t and k_t.

7.2.2 Two-Period Approximation Strategy

For the reasons explained in section 7.2.1, we will turn to an approximation strategy. This strategy will be explained in general in section 7.2.3 but we first explain the ideas in a concrete two-period setting. Eventually, we wish to calculate the average reserves and capitals which will be kept in the strategy using future information at time 0 and time 1. Suppose we make 10^4 simulations for the underlying asset and the mortality from time 0 to time 2. Since p_0 and k_0 can simply be calculated from this set of simulations and p_1 could be determined using the iterativity property of the expectation (see 19), we will focus on how k_1 can be approximated.

We have 10^4 simulations for the underlying asset and for the number of survivors at time 1. Denote these as $S_1^{(k)}$ and $N_1^{(l)}$ respectively, where k and l in $\{1,\ldots,10^4\}$. We could calculate k_1 as:

$$k_1 = \mathrm{E}[\rho[D_1 - R_1|\mathcal{F}_1]] \quad (28)$$

$$= \frac{1}{10^8}\sum_{k=1}^{10^4}\sum_{l=1}^{10^4}\rho[D_1 - R_1|S_1 = S_1^{(k)}, N_1 = N_1^{(l)}], \quad (29)$$

where N_1 denotes random variable describing the number of people who are still alive at time 1. To calculate $\rho[D_1 - R_1|S_1 = S_1^{(k)}, N_1 = N_1^{(l)}]$, a new set of simulations could be made. As it is explained above, this method will be very cumbersome to calculate when dealing with a larger number of periods.

The solution we propose consists of the following two simplifications.

1. On the one hand, we will retain a only limited number of values for the underlying asset and a limited number of classes for the mortality to take future information into account at time 1.

   (a) Suppose we retain 500 simulations for the underlying asset. To select these, first order the 10^4 simulations from the original set. Then calculate the average of the 20 smallest simulations and denote this as $S_1^{(a,1)}$. From the remaining set of simulations, again calculate the average of the 20 smallest and denote this as $S_1^{(a,2)}$. Continue until a set of 500 values for the underlying asset at time 1 is obtained.

   (b) Suppose we use 5 classes for the mortality. To obtain these classes, simply order the 10^4 simulations from the original set. Let $N_1^{(o,1)}$ denote the set of the 2000 smallest simulations. Let $N_1^{(o,2)}$ denote the set of the 2000 smallest simulations after the set $N_1^{(o,1)}$ has been taken from the original set of simulations. Continue until five classes are obtained.

2. On the other hand, we will avoid resimulating by using the original simulations.
   (a) For each retained value for the underlying asset $S_1^{(a,k)}$, $k \in \{1, \ldots, 500\}$, we will use the returns from the original simulations to build a set of 10^4 simulations starting at $S_1^{(a,k)}$.
   (b) For the mortality, we will simply copy 5 times the simulations from time 1 to time 2, starting at the different values within $N_1^{(o,l)}$, to build a set of 10^4 simulations starting at the values within $N_1^{(o,l)}$.

The average capital at time 1 can then be approximated as:

$$k_1 = \mathrm{E}[\rho[E_1 - R_1|S_1, N_1]] \approx \frac{1}{2500} \sum_{k=1}^{500} \sum_{l=1}^{5} \rho[D_1 - R_1|S_1 = S_1^{(a,k)}, N_1 = N_1^{(o,l)}], \tag{30}$$

where $\rho[D_1 - R_1|S_1 = S_1^{(a,k)}, N_1 = N_1^{(o,l)}]$ represents the risk measure ρ which is calculated using the set of simulations for the underlying asset starting at $S_1^{(a,k)}$ and the set of simulations for the mortality starting at the different values within $N_1^{(o,l)}$ which are obtained as described above.

7.2.3 General Approximation Strategy

Start by making a set of N_S simulations from 0 to T. For each $t \in \{1, \ldots, T-1\}$, we then have N_S values for S_t, the underlying risky asset at time t and for N_t, the number of people who are still alive at time t. Instead of now using all the values, we could decide to retain a number of values representing the simulations for S_t. For the mortality, we will arrange the simulations in a number of classes as will be described below. Since the influence of S_t on the estimated costs is a lot higher than that of N_t and since more variation in S_t than in N_t is possible, it is very reasonable to take more values for S_t than classes for N_t. We assume we take the same number of values and classes for every $t \in \{1, \ldots, T-1\}$. Hence, we can write the number of values we retain for S_t as N_A and the number of classes we retain for N_t as N_O. We suppose both N_A and N_O divide N_S. Taking N_A and N_O smaller than N_S will result in a faster algorithm than the one which is described in section 7.2.1.

Along with the number of values and classes we want to retain, we also have to decide upon how they ought to be selected among all obtained values. Furthermore, we also need a strategy to make the estimations for the future, given the information about the underlying asset and about the mortality at a certain point in time. Both for the underlying asset and for the mortality, we will use a method where no resimulation is required.

First we describe the strategy applied for the underlying asset.

1. Make N_S simulations for the underlying asset S_s from 0 to T. Denote these simulations as $\{S_s^{(i)}|i \in \{1, \ldots, N_S\}\}$, where $s \in \{0, 1/12, 2/12, \ldots, T\}$. We call this the original set of simulations. Since we only review the capitals and the reserves on a yearly basis, we are only interested in the situation at the start of each year. Denote the value of the underlying asset at the start of year t in the ith simulation as $S_t^{(i)}$, where $i \in \{1, \ldots, N_S\}$ and $t \in \{1, \ldots, T-1\}$.

2. For all $t \in \{1, \ldots, T\}$, order the set $\{S_t^{(i)}|i \in \{1, \ldots, N_S\}\}$. Assume now rank numbers are the same as the numbers in the superscript. This means $S_t^{(1)}$ is the smallest simulation at t, $S_t^{(2)}$ the second smallest and so on.

3. For all $t \in \{1, \ldots, T\}$, divide $\{S_t^{(i)}|i \in \{1, \ldots, N_S\}\}$ into N_A classes as follows
$$\{\{S_t^{(1)}, \ldots, S_t^{(N_S/N_A)}\}, \ldots, \{S_t^{((N_A-1)N_S/N_A+1)}, \ldots, S_t^{N_S}\}\}.$$

4. Calculate the average value of every class and retain this value as a representing value for this class. Denote these values as $\{S_t^{(a,1)}, \ldots, S_t^{(a,N_A)}\}$, for all $t \in \{1, \ldots, T-1\}$. The a in the superscript indicates that $S_t^{(a,i)}$ is a retained value for the underlying asset.

5. For all $i \in \{1, \ldots, N_A\}$, all simulations starting at $S_t^{(i')}$ ($i' \in \{1, \ldots, N_S\}$) are rescaled such that $S_t^{(i')} = S_t^{(a,i)}$. Hence for all $i \in \{1, \ldots, N_A\}$ and $t \in \{1, \ldots, T\}$, we obtain a set of N_S simulations starting at $S_t^{(a,i)}$ without resimulating.

For the underlying asset, we are able to perform a rescaling strategy as described in the fifth item above. Hence, we can use the same returns for the future as the returns from the original simulations. It is not possible to proceed like this for the mortality. To simulate mortalities, we need to know at which number of insured we start because this has an immediate impact on the simulations. Furthermore, survival probabilities change as the age of the insured changes. Since the simulation of the number of people in the portfolio takes a computation time which is considerably longer than for the simulation of the underlying asset, it will take too many time to resimulate from the selected classes. Therefore, for the mortality, we use the following simulation strategy.

1. Generate N_S simulations for the number of people in the portfolio from 0 to T. Denote these simulations as $\{N_s^{(i)}|i \in \{1,\dots,N_S\}\}$, where $s \in \{0, 1/12, 2/12, \dots, T\}$. We call this set the original set of simulations. Again, we are only interested in the situation at the start of each year. Denote the number of people in the portfolio at the start of year t in the ith simulation as $N_t^{(i)}$, where $i \in \{1,\dots,N_S\}$ and $t \in \{1,\dots,T-1\}$.
2. For all $t \in \{1,\dots,T-1\}$, order $\{N_t^{(i)}|i \in \{1,\dots,N_S\}\}$. Assume now rank numbers are the same as the numbers in the superscript and retain the location of each simulation in the ordered sets in the original set of simulations.
3. For all $t \in \{1,\dots,T-1\}$, divide $\{N_t^{(i)}|i \in \{1,\dots,N_S\}\}$ into N_O classes as follows
$$\{\{N_t^{(1)},\dots,N_t^{(N_S/N_O)}\},\dots,\{N_t^{((N_A-1)N_S/N_O+1)},\dots,N_t^{N_S}\}\}.$$
Denote these classes as $\{N_t^{(o,1)},\dots,N_t^{(o,N_O)}\}$, for all $t \in \{1,\dots,T-1\}$. Note that a class $N_t^{(o,k)}$, with $k \in \{1,\dots,N_O\}$ consists of N_S/N_O values.
4. Because we have retained the location of the ordered values in the original set of simulations, starting at each value in a class, we are able to extract a simulation for the mortality from the original set of simulations. For each class, we then have N_S/N_O simulations. We then just copy these simulations N_O times and use these as an approximation of a set of N_S simulations from t to T for a given class.

Once we have completed these steps, with some abuse of notation ($N_t^{(o,l)}$ is not one value but a class of values close to each other), we are able to calculate an approximation for the average reserves and capitals in the strategy using future information using expression (31) and (32)

$$p_t = \mathrm{E}[\mathrm{E}[D_t|S_t,N_t]] \approx \frac{1}{N_A N_O}\sum_{k=1}^{N_A}\sum_{l=1}^{N_O}\mathrm{E}[D_t|S_t = S_t^{(a,k)}, N_t = N_t^{(o,l)}] \tag{31}$$

$$k_t = \mathrm{E}[\rho[D_t - R_t|S_t,N_t]] \approx \frac{1}{N_A N_O}\sum_{k=1}^{N_A}\sum_{l=1}^{N_O}\rho[D_t - R_t|S_t = S_t^{(a,k)}, N_t = N_t^{(o,l)}]. \tag{32}$$

Note that due to the iterativity property of the expectation, we have

$$p_t = \mathrm{E}[\mathrm{E}[D_t|S_t,N_t]] = \mathrm{E}[D_t|S_0,N_0]. \tag{33}$$

Hence, we can test the simulation strategy by verifying if the approximations obtained by (31) are close to $\mathrm{E}[D_t|S_0,N_0]$, for all $t \in \{1,\dots,T-1\}$. We will do this in section 9.1.

8 Verification of the Simulation Strategy for the Underlying Asset for a European Put Option

In [Hardy (2001)], theoretical formulas for the VaR and CTE of a European put option with payoff function $X = (G - S_n)_+$ on a risky asset $(S_s)_{s\in[0,n]}$ are derived under the log-normal model. As pointed out by Hardy, when $p < \Pr[S_n > G]$, the definition of the CTE given by (3) does not give suitable results. Therefore, she redefines the CTE as

$$\mathrm{CTE}_p[X] = \frac{(1-\beta')\mathrm{E}[X|X > \mathrm{Q}_p[X]] + (\beta' - p)\mathrm{Q}_p[X]}{1-p}, \tag{34}$$

where $\beta' = \max\{\beta | \mathrm{Q}_p[X] = \mathrm{Q}_\beta[X]\}$. In what follows, we will use definition (34) for the CTE. When calculating the CTE from simulations, we take $\mathrm{CTE}_p[X]$ equal to the average of the $(1-p)\%$ worst outcomes. If $S_n \sim LN(n\mu, \sqrt{n}\sigma)$, then for $p \geq \Pr[S_n > G]$

$$\mathrm{CTE}_p[X] \quad = \quad G - S_0 \frac{e^{n\mu + n\sigma^2/2}}{1-p} \Phi(-\Phi^{-1}(p) - \sqrt{n}\sigma). \tag{35}$$

On the other hand, if $p < \Pr[S_n > G]$, one can verify the CTE is given by

$$\mathrm{CTE}_p[X] \quad = \quad \frac{1 - \Pr[S_n > G]}{1-p} \mathrm{CTE}_{\Pr[S_n > G]}[X]. \tag{36}$$

To test the simulation strategy explained in section 7.2 for the underlying asset under the log-normal model, we wish to calculate

$$\mathrm{E}[\mathrm{CTE}_p[(G - S_n)_+ | S_y]], \text{ where } y \in \{1, \ldots, T\}, \tag{37}$$

the expectation of the CTE of the European put option, given the value of the underlying asset at $y \in \{1, \ldots, T\}$, both theoretically and using the strategy explained in section 7.2.

First we calculate the theoretical value of (37), as if $p \geq \Pr[S_n > G | S_y]$ were valid for all possible values S_y as a rough approximation of $\mathrm{E}[\mathrm{CTE}_p[(G - S_n)_+ | S_y]]$. Hence we can write

$$\begin{aligned}
\mathrm{E}[\mathrm{CTE}_p[(G - S_n)_+ | S_y]] \quad &\approx \quad \int_0^\infty \left(G - x \frac{e^{(\mu + \sigma^2/2)(n-y)}}{1-p} \Phi(-\Phi^{-1}(p) - \sqrt{n-y}) \right) f_{S_y}(x) dx && (38) \\
&= \quad G - \frac{e^{(\mu + \sigma^2/2)(n-y)}}{1-p} \Phi(-\Phi^{-1}(p) - \sqrt{n-y}) \int_0^\infty x f_{S_y}(x) dx && (39) \\
&= \quad G - \frac{e^{(\mu + \sigma^2/2)n}}{1-p} \Phi(-\Phi^{-1}(p) - \sqrt{n-y}). && (40)
\end{aligned}$$

The approximation made in (38) will improve for high values of p but even for very high p-values it will still need to be corrected to get a good approximation, if we go into the future. To perform this correction, we first divide the possible values of S_y into N_A classes each of $1/N_A$ probability mass and we calculate for every class the values $S_y^{(a,i')}$, where $i' \in \{1, \ldots, N_A\}$, as explained in the strategy for the underlying asset in section 7.2. For all classes where $p < \Pr[S_n > G | S_y = S_y^{(a,i')}]$, we subtract

$$\frac{1}{N_A} \left(G - S_y^{(a,i')} \frac{e^{(\mu + \sigma^2/2)(n-y)}}{1-p} \Phi(-\Phi^{-1}(p) - \sqrt{n-y}) \right) \tag{41}$$

from the approximation (38) to correct for the mistake by ignoring the condition $p \geq \Pr[S_n > G | S_y]$. We then add

$$\frac{1}{N_A} \frac{1 - \Pr[S_n > G | S_y = S_y^{(a,i')}]}{1-p} \mathrm{CTE}_{\Pr[S_n > G | S_y = S_y^{(a,i')}]}[X]. \tag{42}$$

This is a correction using the adapted theoretical definition of the CTE from (34).

In table 3, we summarize the results of this test for the CTE at level $p = 0.99$ for a maturity guarantee of $G = 1 = S_0$ after 10 years. Given the information we have at 0, we wish to compare the average simulated and theoretical CTE at level 0.99 of the maturity guarantee at the start of every year, taking into account the information about the underlying asset we have at that moment. For the simulated average (A_S), we used the method described in section 7.2 with $N_A = 500$. We made 15000 simulations of the underlying asset for 10 years where the returns are log-normally distributed with parameters $\mu = 0.085$ and $\sigma = 0.20$. In column 1, we specify the year at the start of which the average CTE's at level 0.99 are compared. In the fourth column, we calculate the difference between A_S and A_T, relative to A_T. In column 5, we specify the number of $S_y^{(a,i')}$'s for which we had to use the corrections described by (41) and (42), when calculating the theoretical average (A_T).

Table 3: Test average CTE at level 0.99 of a maturity guarantee.

Year	A_T	A_S	$\frac{\lvert A_S - A_T\rvert}{A_T}$	Nb. Corr.
1	0.557	0.568	2.0%	0
2	0.508	0.512	0.8%	2
3	0.453	0.450	0.7%	25
4	0.397	0.399	0.5%	68
5	0.339	0.345	1.8%	118
6	0.286	0.297	3.8%	171
7	0.234	0.246	5.1%	217
8	0.183	0.183	0.0%	267
9	0.132	0.138	4.5%	315
10	0.082	0.088	7.3%	366

We see that for all compared years, the theoretical and average CTE are fairly close. We also see that the number of corrections increases with increasing year number, which is logical since $\mu > 0$. At year 10, the number of corrections is almost 75%. In these circumstances we may wonder if our test is as reliable as for the cases where no corrections are made. For each $y \in \{1, \ldots, 9\}$, the correction process is performed by using (41) and (42) to discretize the part of the integral corresponding with $\mathrm{E}[\mathrm{CTE}_p[(G - S_n)_+|S_y]]$ where $p < \Pr[S_n > G|S_y]$. In the discretizing process, we use the simulations of the underlying asset, but for the rest, we use theoretical formulas.

9 Comparison of the two Multi-Period Strategies for a Minimum Death Guarantee

We compare the two strategies explained in section 4 in the context of a unit-linked minimum death guarantee upon three criteria. First we analyze the average reserves and capitals. Then we compare the Technico-financial premiums which are obtained using the cash-flow model explained in section 5. Finally, we also investigate the estimated distribution functions for the reserves and the capitals in the strategy using future information. This provides an answer to the first two questions raised at the end of section 4.2. All comparisons are made for the portfolio and parameters as mentioned in table 5.

9.1 Average Reserves and Capitals

In figure 2, we compare the average reserves for the parameters as specified in table 5. For all $t \in \{0, T\}$, the dashed line represents $\mathrm{E}[\mathrm{E}[D_t|S_t, N_t]]$, which is approximated by expression 31, with $N_A = 500$ and $N_O = 5$, whereas the full line represents $\mathrm{E}[D_t|S_0, N_I]$. In figure 3, we compare the average capitals. For all $t \in \{0, T\}$, the dashed line represents $\mathrm{E}[\mathrm{CTE}_{0.99}[D_t|S_t, N_t]]$, which is approximated by expression (32), again with $N_A = 500$ and $N_O = 5$. The full line represents $\mathrm{CTE}_{0.99}[D_t|S_0, N_I]$.

In figure 2, we see that the two lines fit very well onto each other. This means that under these conditions, our approximation strategy passes the test of the iterativity property of the expectation. By taking N_A smaller under the same conditions, we would see the iterativity property is less well satisfied. The investigation of some other conditions learned that when costs are more likely (e.g. when $K > S_0$), less values can be taken for the underlying asset to satisfy the iterativity property in the same way. On the other hand, when costs are less likely (e.g. when $K < S_0$), N_A needs to be larger to obtain results of the same quality of those in figure 2.

In figure 3, we see that, just as for the reserves, the capitals of course start at the same value in the two multi-period strategies. From the start of the second year however, the differences between the average capitals in the two approaches start to increase. In the strategy not using future information, the capitals first increase, whereas in the other strategy, they start to decrease immediately.

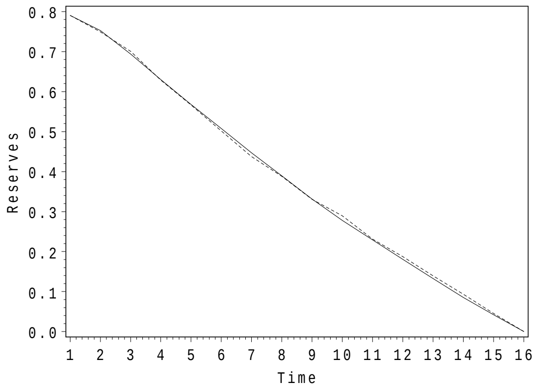

Figure 2: Comparison of the average reserves.

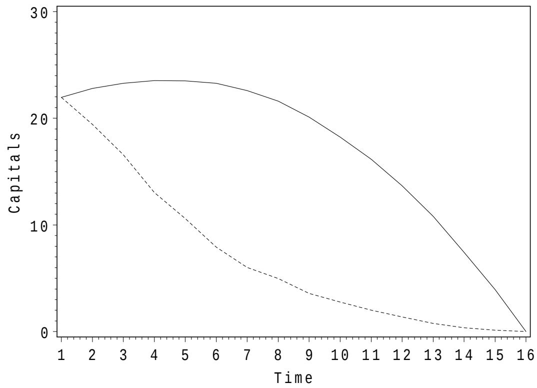

Figure 3: Comparison of the average capitals.

9.2 Technico-Financial Premium

To calculate the technico-financial premium, we can use expression (24), provided (21) is valid. Let C_s denote the random variable describing the costs at the end of each month s, for s in $\{1/12, 2/12, \ldots, T\}$. Since due to the iterativity property of the expectation

$$\begin{aligned} p_t &= \mathrm{E}[D_t|\mathcal{F}_0] && (43) \\ &= \mathrm{E}[\mathrm{E}[D_t|\mathcal{F}_t]] && (44) \\ &= \mathrm{E}[\mathrm{E}[\sum_{s=1}^{12(T-t)} e^{-rs/12} C_{t+s/12}|\mathcal{F}_t]] && (45) \\ &= \sum_{s=1}^{12(T-t)} e^{-rs/12} c_{t+s/12}, && (46) \end{aligned}$$

both in the approach not using future information and the one using future information, (21) is verified.

The differences observed between the average capitals in figure 3 have an important influence on the technico-financial premium. This can be seen in table 4, where we summarize the pure premium (PP), the initial capital (K_0) and the technico-financial premium (TFP) for the two multi-period strategies.

Table 4: Comparison technico-financial premium.

	Approach section 4.1	Approach section 4.2
PP	0.79	0.79
K_0	21.97	21.97
TFP	14.05	6.50

The technico-financial premium in the approach not using future information is more than twice as large as in the approach in which future information is taken into account.

9.3 Approximated Distribution Functions of Future Total Solvency Levels and Reserves

In figure 4, we show the approximated distribution functions of the future total solvency levels, when they are adapted to the information at the start of year $t \in \{1, \ldots, T\}$. The vertical line is the total solvency level at time 0. The higher the line is at the left, the further in time the situation represented.

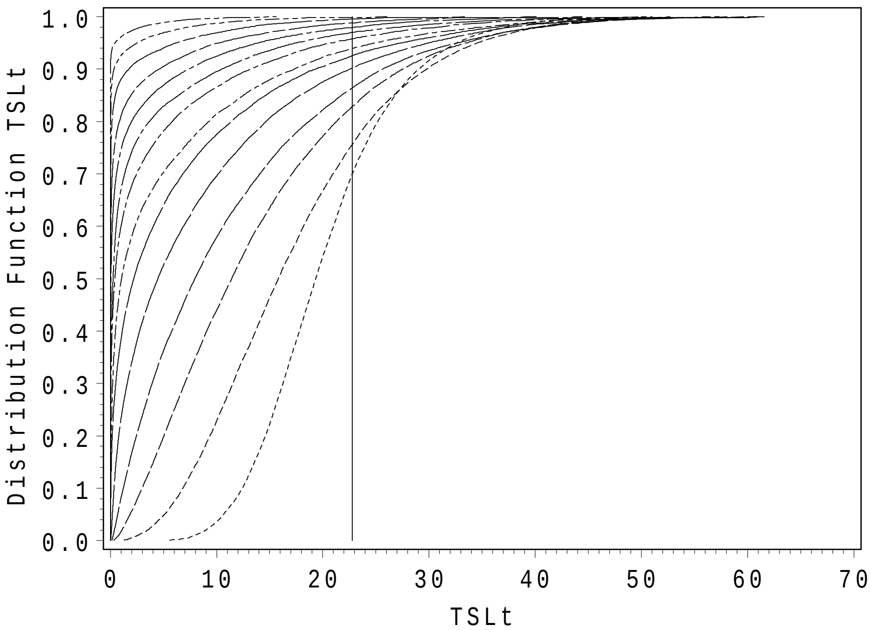

Figure 4: Approximate distribution functions for the total solvency levels at the start of each year when using future information.

We see that in about 30% of the cases, the total solvency level at the start of the second year should be larger than the initial total solvency level. The largest estimations for the required total solvency level at the start of the second year are about twice as large as the initial total solvency level. On the other hand, the lowest estimations are about one third of the initial total solvency level. Hence, already after one year, the differences in mainly the underlying asset can be such that we obtain a very wide range in possible required total solvency levels. At the start of the third year, the largest estimations become even larger but the probability that the total solvency level should be larger than the initial total solvency level decreases to about 25%. Then for a few years, the largest estimations remain at a comparable level. For these years, the lines all lay in the same region for the largest estimations, which makes it difficult to trace the lines separately. We also see that the probabilities that the total solvency levels should be larger than the initial total solvency level continue decreasing. At about half the lifetime of the risk, the largest estimations start to decrease too. From the start of year five on, we already find estimations for the required total solvency level of 0.

To explain why the largest estimations do not continue increasing, we need to be aware of two intuitive boundaries to the costs: the number of insured dying may be expected to be below a certain level and the underlying asset cannot move below 0. The largest estimations correspond to the situations where the underlying asset is low and the number of insured still in the portfolio is high. Hence, as we are getting closer to the boundaries, it becomes unlikely that future developments will result in increasing estimations of the largest future capitals. On the other hand, as time passes, the number of periods in which costs may occur decreases. Together with the previous argument, this explains why the largest required solvency levels start to decrease after some years.

In figure 5, we show the approximated distribution functions of the reserves in the future, when they are adapted to the information at the start of year $t \in \{1, \ldots, T\}$. The vertical line is the reserve which is held at time 0. The higher the line is at the left, the further in time the situation represented. The behavior we detect in figure 5 shows some similarities with what we detected for the total solvency levels but is clearly more extreme. Again, it is difficult to read the situation at the right because the lines coincide. We therefore plot the approximate distribution function in more detail in figure 6.

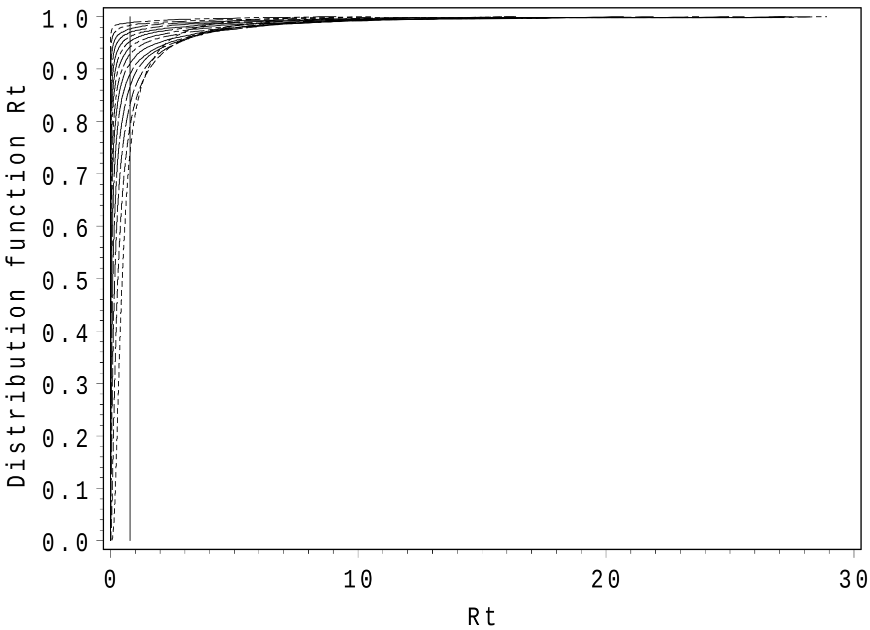

Figure 5: Approximate distribution functions for the reserves at the start of each year when using future information.

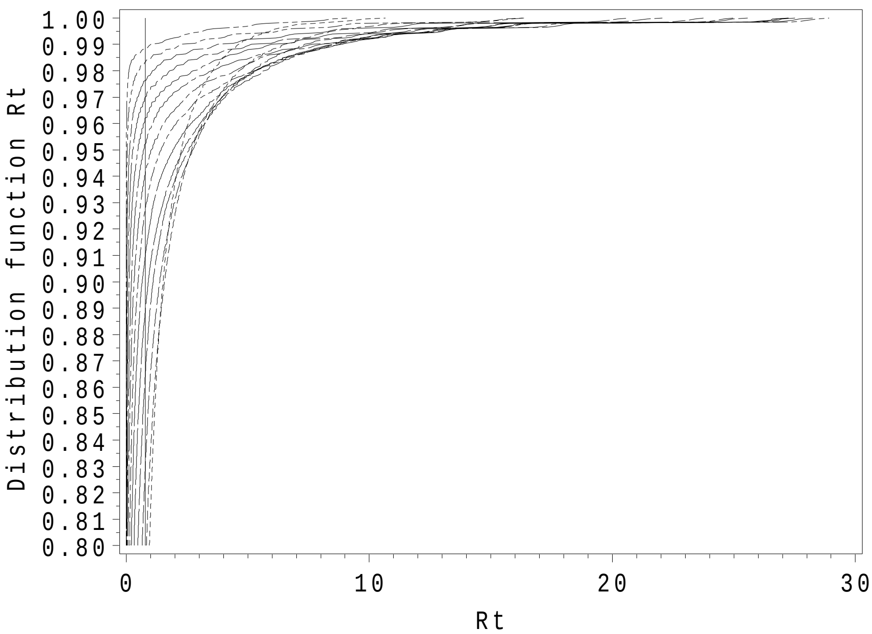

Figure 6: Approximate distribution functions for the reserves at the start of each year when using future information (detailed).

By the start of year four, the largest required reserves can reach values which are in the region of the initial total solvency level, even though this is very improbable. On the other hand, already at the start of the second year, the lowest reserves are estimated equal to 0.

10 Conclusion

In this paper, we suggested a multi-period capital allocation strategy which incorporates future information. We explained how this strategy can be used to develop a cash-flow model to price a multi-period risk. An interesting example of such a risk is a minimum death guarantee in unit-linked life insurance, especially when pricing and reserving is done in an actuarial way. After explaining and verifying a simulation strategy for the approach using future information, we compare it with an approach not using future information. For the average reserves, we see both strategies lead to the same results, due to the iterativity property of the expectation. For the average capitals however, we observe important differences between the two methods. As a consequence, the technico-financial premium for the method not using future information is more than twice as large as in the approach using future information. Our simulation strategy also provides approximations for the distribution functions for the future reserves and capitals. From these, interesting information about possible variations of reserves and capitals can be withdrawn.

A Used Parameters and Notations: Summary

Table 5: Used parameters and notations: summary.

Simulations		
Number of simulations	N_S	15000
Number of values for underlying asset	N_A	500
Number of classes for mortality	N_O	5
Contractual Parameters		
Portfolio composition	$\{N_I, x, C\}$	$\{1000, 50, 1\}$
Initial value underlying asset	S_0	1
Guarantee at death	K	1
Age of retirement	x_R	65
Mortality Parameters		
Gompertz-Makeham parameters	α	0.000591068646661458
	β_{0-65}	0.00000737593571037331
	γ_{0-65}	0.11807173977857
	β_{65-99}	0.000619125291109306
	γ_{65-99}	0.0532009916754107
Parameters Cash Flow model		
Risk free rate	r	0.0425
Tax rate	γ	0.4
Average return on investments in shares	δ	0.0505
Cost of capital	COC	0.085
Parameters RSLN model		
Average log-return in regime 1	μ_1	0.0135
Average log-return in regime 2	μ_2	-0.0109
Volatility in regime 1	σ_1	0.0344
Volatility in regime 2	σ_2	0.0645
Probability to move from regime 1 to 1	p_{11}	0.0483
Probability to move from regime 2 to 1	p_{21}	0.1985

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