YIELD OPTION PRICING IN THE GENERALIZED COX-INGERSOLL-ROSS MODEL

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Abstract : In this paper, we provide pricing formulae for both European and American yield options in the generalized Cox-Ingersoll-Ross (1985) single-factor term structure model with time-dependent parameters. Our results are established by forward-neutral pricing. The law of the generalized CIR short term interest rate process under the forward-neutral probability is obtained by using results on the relation between the generalized square-root process and squared Bessel processes with time-varying dimension.

Keywords : Extended CIR model, European and American yield option pricing, forward-neutral probability, Bessel process with time-varying dimension.

1. Introduction

It is well-known that the Cox-Ingersoll-Ross model (1985) has many appealing advantages over other single factor interest rate models as it is derived in a general equilibrium framework, as it is quite tractable and as the short term interest rate process has empirically relevant properties. Indeed, the Cox-Ingersoll-Ross short term interest rate process remains positive, is mean-reverting and the absolute variance of the interest rate increases with the interest rate itself.
In order to adapt the Cox-Ingersoll-Ross model to be more consistent with the current term structure of interest rates, Hull and White (1990) introduced an extension of the Cox-Ingersoll-Ross model with time-dependent parameters. Working on a probability space $\left(\Omega,\left(F_t\right)_{t\geq 0},P\right)$ which satisfies the usual conditions and where $\left(W_t\right)_{t\geq 0}$ is a Wiener process under P, the instantaneous interest rate process $\left(r_t\right)_{t\geq 0}$ is defined by the stochastic differential equation

$$dr_t = \left(\alpha(t) - \beta(t)r_t\right)dt + \sigma(t)\sqrt{r_t}\,dW_t$$

for some positive bounded functions $\alpha(t), \beta(t)$ and $\sigma(t)$. In order to evaluate discount bonds, Hull and White (1990) derived a partial differential equation and used numerical methods to solve it.
By using the separation of variable approach, Jamshidian (1995) obtained in this extended CIR model the prices of discount bonds and of call options on discount bonds in case of $\alpha(u)/\sigma^2(u) = \delta$ being a constant.
Maghsoodi (1996) studied the relationship between the extended CIR model and the integer-dimensional Ornstein-Uhlenbeck processes and derived the dynamics of the extended CIR term structure under the no-arbitrage condition. Using this results and by forward-neutral pricing, he derived a closed bond option valuation formula in case of $\alpha(u)/\sigma^2(u) = \delta$ being constant.

Delbaen and Shirakawa (1996) studied the general extended Cox-Ingersoll-Ross model ECIR($\delta(t)$) with no constraints on the time-dependent functions $\alpha(t)$, $\beta(t)$ and $\sigma(t)$, except the technical assumptions that $\inf_{t\geq 0}\sigma(t)>0$ and that $\sigma(t)$ is continuously differentiable with respect to t , which we assume to hold from now on. Considering squared Bessel processes with time-varying dimensions, they obtained in the general ECIR($\delta(t)$) model arbitrage-free prices of discount bonds and bond options.

The purpose of this paper is to derive the prices of yield options in the ECIR($\delta(t)$) model by assuming that the market is complete and arbitrage-free. Nowadays, both European and American options on yields are incorporated in different interest-rate derivatives like e.g. interest-rate caps, floors, locks and options on interest-rate swaps. Also a lot of financial institutions propose certificates of deposit that guarantee a minimal renewal yield if the certificate is rolled over at maturity, which means that in fact, they offer a hidden put option on the yield at maturity.
Longstaff (1990) derived a closed-form expression for the European yield option price in the Cox-Ingersoll-Ross model by using the yield as the relevant state variable and by using a separation method. He noticed that these options differ from options on bonds or stocks in that the yield call values can be less than their intrinsic value and can be a decreasing function of the underlying yield.
Chesney, Elliott and Gibson (1993) studied the pricing of American yield options in the Cox-Ingersoll-Ross framework by using properties of Bessel bridge processes.
In this paper, we derive the prices of both European and American yield options in the ECIR($\delta(t)$) model and this by using the forward-neutral probability (see e.g. El Karoui and Geman (1994) ; Geman, El Karoui and Rochet (1995) or Jamshidian (1990, 1993)). First, we show by using results of Delbaen and Shirakawa (1996) and of Maghsoodi (1996) that under the forward-neutral probability, the ECIR($\delta(t)$) short term interest rate is distributed as a rescaled time-changed squared Bessel process with a time-dependent dimension. This fact leads to straightforward pricing of both European and American yield options and in this way, the results of respectively Longstaff (1990) and Chesney et al. (1993) are generalized.

The paper is organized as follows. In section 2, we briefly recall from Delbaen and Shirakawa (1996) the arbitrage-free discount bond price and the risk-neutral probability in the ECIR($\delta(t)$) model. Using their results, we further study the dynamics and the law of the ECIR($\delta(t)$) process under the forward-neutral probability, as Maghsoodi (1996) did in case of $\alpha(u)/\sigma^2(u)=\delta$ being constant. These results lead in section 3 to straightforward pricing of European yield options in the ECIR($\delta(t)$) model and to the recovery of the formula of Longstaff (1990) in the CIR model. In section 4, we turn in the ECIR($\delta(t)$) model to the American yield call decomposition as studied by Chesney, Elliott and Gibson (1993). Using the forward-neutral probability and the results of section 2, the early exercise premium can be reformulated. Section 5 concludes the paper.

2. The law of the ECIR($\delta(t)$) process under the forward-neutral measure

In this section, we briefly recall some results from Delbaen & Shirakawa (1996) and Maghsoodi (1996) in order to obtain the law of the ECIR($\delta(t)$) spot rate under the forward-neutral probability. These results will lead in the following sections to straightforward pricing of yield options.

Let $P(t,T)$ denote the price at time t of the T- maturity discount bond. In the following, we use the subscripts 1 and 2 to denote partial derivatives with respect to the first and second variable respectively. For notational use, time variables appear not always between parentheses but also as subscript.

Delbaen and Shirakawa (1996) showed that the arbitrage-free discount bond price in the ECIR($\delta(t)$) model equals

$$P(t,T) = \exp\left\{\int_t^T \frac{2\alpha_u \eta_1(u,T)du}{\sigma_u^2 \eta(u,T)}\right\} \exp\left\{\frac{2\eta_1(t,T)}{\sigma_t^2 \eta(t,T)} r_t\right\}$$

where $\eta(u,T)$ is a solution for the differential equation :

$$\begin{cases} \eta_{11}(u,T) - \left(\beta_u + \lambda_u + 2\dfrac{\sigma_u^{'}}{\sigma_u}\right)\eta_1(u,T) - \dfrac{1}{2}\sigma_u^2 \eta(u,T) = 0 \\ \eta_1(T,T) = 0, \quad \eta(t,T) = k > 0 \end{cases}$$

for some $k > 0$ and where λ_u is such that $-\lambda_u \sqrt{r_u} / \sigma_u$ is the market price of risk, defining the risk-neutral martingale measure (see e.g. Harrison and Pliska (1981)). In fact, the proof that $(\rho_t)_{t \geq 0}$ with

$$\rho_t = \exp\left\{-\int_0^t \frac{\lambda_u \sqrt{r_u}}{\sigma_u} dW_u - \frac{1}{2}\int_0^t \frac{\lambda_u^2 r_u}{\sigma_u^2} du\right\}$$

is a martingale is not trivial and we refer the interested reader to Delbaen and Shirakawa (1996) for the details that there exists indeed a risk-neutral measure $\overline{P}$ defined by

$$\overline{P}(A) = E[1_A \rho_t] \text{ for all } A \in F_t .$$

For notational use and following Cox, Ingersoll and Ross (1985), we denote

$$P(t,T) = A(t,T)\exp\{-B(t,T)r_t\} \qquad 0 \leq t \leq T$$

with $A(t,T)$=1 and $B(T,T) = 0$.

It is well-known that under the risk-neutral measure, the discounted bond price is a martingale. Indeed, the ECIR($\delta(t)$) bond price dynamics follow

$$dP(t,T) = r(t)P(t,T)dt - P(t,T)B(t,T)\sigma(t)\sqrt{r(t)}d\overline{W}_t$$

with $(\overline{W}_t)_{t \geq 0}$ a Brownian motion under the risk-neutral measure $\overline{P}$.

Following e.g. El Karoui and Geman (1994) ; Geman, El Karoui and Rochet (1995) or Jamshidian (1990, 1993), the forward-neutral probability of maturity T is defined by the following formula where X_s is an arbitrary F_s measurable random variable :

$$E^T\left[X_s \middle| F_t\right] = \overline{E}\left[X_s \frac{\exp\left\{-\int_t^s r_u du\right\} P(s,T)}{P(t,T)} \middle| F_t \right] \qquad \forall\, t \le s < \infty.$$

Under this measure, forward rates as well as forward prices become martingales.

Maghsoodi (1996) derived that under P^T, the short-term interest rate process still follows an extended square root process but with a new reversion rate, namely

$$dr_t = \left(\alpha(t) - B_{2,1}(t,T)B_2(t,T)^{-1} r_t\right)dt + \sigma(t)\sqrt{r_t}\,dW_t^T \qquad (1)$$

with $\left(W_t^T\right)_{t\ge 0}$ a Brownian motion under the forward-neutral measure P^T.

The following lemma states that under the forward-neutral probability, the law of the short rate process can be expressed in terms of a squared Bessel process with a time-varying dimension. We denote the squared Bessel process with time-varying dimension function $\delta: \Re^+ \to \Re^+$ by $X_t^{(\delta)}$ which follows the stochastic differential equation :

$$dX_t^{(\delta)} = 2\sqrt{X_t^{(\delta)}}\,dW_t + \delta_t dt.$$

Lemma 1

Under the forward-neutral probability P^T, the ECIR($\delta(t)$) spot rate is distributed as

$$r(u) \overset{law}{=} \kappa^T(t,u)^{-2}\,\overline{r}\left(\frac{1}{4}\int_t^u \sigma^2(w)\kappa^T(t,w)^2 dw\right)$$

where

$$\kappa^T(t,u)^2 = \frac{B_2(u,T)}{B_2(t,T)}$$

and where $\overline{r}$ is a squared Bessel process with time-varying dimension δ^t with

$$\delta^t(u) = \frac{4\left(\alpha \circ \nu_{t,.}^{-1}\right)_u}{\left(\sigma^2 \circ \nu_{t,.}^{-1}\right)_u}$$

where

$$\nu_{t,u} = \frac{1}{4}\int_t^u \sigma^2(w)\kappa^T(t,w)^2 dw$$

Proof

Using the relation between an extended square-root process and squared Bessel processes with time-varying dimension (see corollary 3.1 of Delbaen-Shirakawa (1996)) in case of the dynamics of the ECIR($\delta(t)$) process under the forward-neutral probability P^T (see (1)) implies that

$$\left\{r_u; t \le u, r_t = r\right\} \overset{law}{=} \left\{\vartheta_u X_{\nu_u}^{(\delta)}; t \le u, X_{\nu_t}^{(\delta)} = \frac{r}{\vartheta_t}\right\}$$

where

$$\vartheta_u = \exp\left\{-\int_0^u \frac{B_{2,1}(w,T)}{B_2(w,T)}dw\right\} = \frac{B_2(0,T)}{B_2(u,T)}$$

$$\nu_u = \frac{1}{4}\int_0^u \frac{\sigma^2(w)}{\vartheta_w}dw$$

$$\delta_u = \frac{4\left(\alpha \circ \nu^{-1}\right)_u}{\left(\sigma^2 \circ \nu^{-1}\right)_u}.$$

Writing down the conditional Laplace transform of $\vartheta_u X^{(\delta)}_{\nu_u}$ conditional on $X^{(\delta)}_{\nu_t} = \frac{r}{\vartheta_t}$ (see theorem 2.2 of Delbaen & Shirakawa (1996)), one finds that (like in the proof of theorem 4.1 of Delbaen & Shirakawa (1996)) :

$$\left\{r_u; t \le u, r_t = r\right\} \overset{law}{=} \left\{\vartheta_u X^{(\delta)}_{\nu_u}; t \le u, X^{(\delta)}_{\nu_t} = \frac{r}{\vartheta_t}\right\}$$

$$\overset{law}{=} \left\{\vartheta_{t,u} X^{(\delta^t)}_{\nu_{t,u}}; t \le u, X^{(\delta^t)}_0 = r\right\}$$

where

$$\vartheta_{t,u} = \frac{B_2(t,T)}{B_2(u,T)}$$

$$\nu_{t,u} = \frac{1}{4}\int_t^u \sigma^2(w)\kappa^T(t,w)^2 dw$$

$$\delta^t(u) = \frac{4\left(\alpha \circ \nu_{t,\cdot}^{-1}\right)_u}{\left(\sigma^2 \circ \nu_{t,\cdot}^{-1}\right)_u}$$

which proves the lemma. Q.e.d.

3. European yield options

In this section, we study the arbitrage-free price of a European call option of maturity T and strike price or exercise yield K on a τ - maturity yield

$$Y(T,T+\tau) = \frac{-1}{\tau}\ln P(T,T+\tau).$$

The payoff function of this call is $\left(Y(T,T+\tau)-K\right)_+$ and its arbitrage-free price at time t equals

$$C_E\left(Y(T,T+\tau),K,t,T\right) = \overline{E}_{t,r}\left[\exp\left(-\int_t^T r_u du\right)\left(Y(T,T+\tau)-K\right)_+\right].$$

From this expression, it is clear that an increase of the underlying yield influences both the discount factor and the payoff of the option. It is possible that the decrease of the discount factor, implied by an increase of the underlying yield, dominates the corresponding increase of the payoff such that as a whole, the yield option decreases. The hedging implications of these features have been studied by Longstaff (1990).

In the Cox-Ingersoll-Ross model, Longstaff (1990) derived an explicit formula of the price of a European yield call which matures at T and can be exercised at K by using the τ - maturity yield as a state factor in stead of the short-term interest rate. This procedure can be justified since for a constant maturity τ, the τ- maturity yield is a linear function of the spot rate. Further, Longstaff uses the separation method.

Using the forward-neutral probability, we do not only recover the same formula as Longstaff (1990) in case of the CIR model, but we also obtain expressions in the ECIR($\delta(t)$) model. The advantage of the forward-neutral probability in comparison with the risk-neutral probability, is that under the risk-neutral probability, the expectation has to be taken of the product of two dependent terms, namely of the discount factor and the payoff, whereas the definition of the forward-neutral probability solves this problem.

Theorem 1

Consider the ECIR($\delta(t)$) term structure model. For $0 \le t \le T \le T+\tau$, the price at time t of a European option on a τ- maturity yield, expiring at T with exercise price K is given by

$$C_E\left(Y(T,T+\tau),K,t,T\right) = P(t,T)\frac{B(T,T+\tau)}{\tau}E_{t,r}^{T}\left[r_T 1_{\left(r_T \ge r^*\right)}\right] - \left(\frac{\ln A(T,T+\tau)}{\tau}+K\right)P(t,T)P_{t,r}^{T}\left(r_T \ge r^*\right)$$

where

$$r^* = \frac{\tau K + \ln A(T,T+\tau)}{B(T,T+\tau)}$$

and where the conditional Laplace transform (for $\lambda > 0$) of r_T under P^T is given by

$$E_{t,r}^{T}\left[\exp\left(-\lambda r_T\right)\right] = \exp\left\{\frac{-\lambda \vartheta_{t,T} r}{1+2\lambda\vartheta_{t,T}\nu_{t,T}} - \int_0^{\nu_{t,T}} \frac{\lambda\vartheta_{t,T}\delta_u^t}{1+2\lambda\vartheta_{t,T}\left(\nu_{t,T}-u\right)}du\right\}$$

with $\vartheta_{t,T}$, $\nu_{t,T}$ and δ_u^t as in lemma 1.

Proof

Using the forward-neutral measure P^T, the call option becomes

$$C_E\left(Y(T,T+\tau),K,t,T\right) = P(t,T)E_{t,r}^{T}\left[\left(Y(T,T+\tau)-K\right)_+\right].$$

By definition and by the notation above, the yield of maturity τ at time T equals

$$Y(T,T+\tau) = -\frac{\ln A(T,T+\tau)}{\tau} + \frac{B(T,T+\tau)}{\tau}r_T$$

and a simple calculation leads to the smallest value of r_T such that the call is not worthless :

$$r^* = \frac{\tau K + \ln A(T,T+\tau)}{B(T,T+\tau)}$$

and to the expression

$$C_E\left(Y(T,T+\tau),K,t,T\right) = P(t,T)\frac{B(T,T+\tau)}{\tau}E_{t,r}^{T}\left[r_T 1_{\left(r_T \ge r^*\right)}\right] - \left(\frac{\ln A(T,T+\tau)}{\tau}+K\right)P(t,T)P_{t,r}^{T}\left(r_T \ge r^*\right).$$

As noticed in lemma 1, under the forward-neutral probability

$$\left\{r_u; t \le u, r_t = r\right\} \overset{law}{=} \left\{\vartheta_{t,u} X_{\nu_{t,u}}^{(\delta^t)}; t \le u, X_0^{(\delta^t)} = r\right\}$$

with the notations of above. Therefore, the conditional Laplace transform of r_T follows immediately from theorem 2.2 of Delbaen & Shirakawa (1996).

Q.e.d.

In the following corollary, we recover in a straightforward way the formula of Longstaff in the Cox-Ingersoll-Ross model and this by using the forward-neutral probability as in the theorem above. Therefore, this corollary is a nice example of the advantage of using the forward-neutral probability.

Corollary

Consider the CIR term structure model. For $0 \le t \le T \le T+\tau$, the price at time t of a European option on a τ - maturity yield, expiring at T with exercise price K is given by

$$C_E\left(Y(T,T+\tau),K,t,T\right) = P(t,T)\left[Y(T,T+\tau)\,\varepsilon\, Q(g,\delta+4,\eta) - KQ(g,\delta,\eta) + \psi\right]$$

where

$$\psi = A'(\tau)Q(g,\delta,\eta) + \alpha B(t,T)B'(\tau)Q(g,\delta+2,\eta) - \varepsilon A'(\tau)Q(g,\delta+4,\eta)$$

and where $Q(g,\delta,\eta)$ denotes the complementary noncentral χ^2 distribution with δ degrees of freedom and non-centrality parameter η, with

$$\delta = \frac{4\alpha}{\sigma^2}$$

$$\eta = \frac{4\gamma^2 \exp\left(\gamma(T-t)\right)B(t,T)}{\sigma^2\left(\exp\left(\gamma(T-t)\right)-1\right)^2} r_T$$

$$g = \frac{4\left(K - A'(\tau)\right)}{\sigma^2 B(t,T)B'(\tau)}$$

and where the following notation has been used :

$$\varepsilon = \left(\frac{\gamma \exp\left(\gamma(T-t)/2\right)B(t,T)}{\exp\left(\gamma(T-t)\right)-1}\right)^2$$

$$A(t,T) = \left[\frac{2\gamma \exp\left\{(\beta+\lambda+\gamma)(T-t)/2\right\}}{(\beta+\lambda+\gamma)\left(\exp\gamma(T-t)-1\right)+2\gamma}\right]^{2\alpha/\sigma^2}$$

$$B(t,T) = \frac{2\left(\exp\gamma(T-t)-1\right)}{(\beta+\lambda+\gamma)\left(\exp\gamma(T-t)-1\right)+2\gamma}$$

$$\gamma = \left(\left(\beta+\lambda\right)^2 + 2\sigma^2\right)^{1/2}$$

$$A'(\tau) = -\frac{\ln A(T,T+\tau)}{\tau}$$

$$B'(\tau) = \frac{B(T,T+\tau)}{\tau}.$$

Proof

In the CIR model, it is well-known (see Cox, Ingersoll and Ross (1985)) that the arbitrage-free discount bond price equals

$$P(t,T) = A(t,T)\exp\{-B(t,T)r_t\} \qquad\qquad 0 \le t \le T$$

with $A(t,T)$ and $B(t,T)$ as above.

Using the fact that

$$Y(T,T+\tau) = A'(\tau) + B'(\tau)r_T$$

and the forward-neutral probability, the European yield call can be rewritten as

$$C_E\left(Y(T,T+\tau),K,t,T\right) = P(t,T)B'(\tau)E_{t,r}^T\left[r_T 1_{(r_T \ge r^*)}\right] + \left(A'(\tau)-K\right)P(t,T)P_{t,r}^T\left(r_T \ge r^*\right).$$

From the results of Maghsoodi (1996) or by taking constant parameters in lemma 1, it is easy to derive that the short-term interest rate follows under the forward-neutral probability a non-central χ^2 distribution :

$$\frac{\delta\, r_T}{\alpha B(t,T)} \overset{law}{=} \chi^2(\delta,\eta)$$

with δ and η as above. Straightforward calculations now lead to the result.

Q.e.d.

4. American yield options

Chesney, Elliott and Gibson (1993) studied in the Cox-Ingersoll-Ross framework, besides the pricing of American bond options, also the pricing of an American yield option with exercise yield K which expires at date T and whose underlying instrument is the τ-maturity yield $Y(\tau)$. They obtained quasi-analytical formulae by using properties of Bessel bridge processes.

Following Bensoussan (1984) and Karatzas (1988, 1989), Chesney, Elliott and Gibson (1993) showed that the American yield call equals the smallest supermartingale majorant of the discounted payoff, which is the Snell envelope

$$C\left(Y(\tau),K,t,T\right) = ess\sup_{t\le\tau_1\le T} \overline{E}_{t,r}\left[\exp\left(-\int_t^{\tau_1} r_u du\right)\left(Y(\tau_1,\tau_1+\tau)-K\right)_+ \mid F_t\right].$$

This optimal stopping representation can be expressed as a function of two components : the price of the European yield and the early exercise premium. Indeed, Chesney, Elliott and Gibson (1993) showed that the approach of Bensoussan and Lions (1982), Jacka (1991) and Myneni (1992) in case of American stock options can be applied to derive the yield call's decomposition.

In case of the ECIR($\delta(t)$) model, an application of Itô's lemma for piecewise convex functions implies :

$$C\left(Y(\tau),K,t,T\right) =$$

$$C_E\left(Y(T,T+\tau),K,t,T\right) - \int_t^T \overline{E}_{t,r}\left[\exp\left(-\int_t^s r_u du\right) L_s\left(Y(s,s+\tau)-K\right)1_{(r(s)>r^*(s))}\right]ds$$

where

$$L_s(.) = \frac{1}{2}\sigma_s^2 r_s \frac{\partial^2(.)}{\partial r^2} + \left(\alpha_s - r_s(\beta_s+\lambda_s)\right)\frac{\partial(.)}{\partial r} + \frac{\partial(.)}{\partial t} - r_s(.)$$

and where $r^*(.)$ denotes the critical interest rate, the smallest value at which the exercise of the yield call becomes optimal. The determination of this critical interest rate is still an open question and is beyond the scope of this paper.

Applying the differential generator L_s and using the notation

$$A'(s,\tau) = -\frac{\ln A(s,s+\tau)}{\tau} \text{ and } B'(s,\tau) = \frac{B(s,s+\tau)}{\tau},$$

a simple calculation yields

$$\begin{aligned} &L_s\big(Y(s,s+\tau)-K\big) \\ &= L_s\big(A'(s,\tau)+B'(s,\tau)r_s-K\big) \\ &= -B'(s,\tau)r_s^2 + \left(K + \frac{\partial B'(s,\tau)}{\partial t} - A'(s,\tau) - B'(s,\tau)\big(\beta_s+\lambda_s\big)\right)r_s + \alpha_s B'(s,\tau) + \frac{\partial A'(s,\tau)}{\partial t}. \end{aligned}$$

Introducing some more abbreviating notation such that

$$L_s\big(Y(s,s+\tau)-K\big) = H_1(s,\tau)r_s^2 + H_2(s,\tau)r_s + H_3(s,\tau),$$

one obtains that

$$\begin{aligned} C\big(Y(\tau),K,t,T\big) &= C_E\big(Y(T,T+\tau),K,t,T\big) \\ &\quad - \int_t^T \overline{E}_{t,r}\left[\exp\left(-\int_t^s r_u du\right)\big(H_1(s,\tau)r_s^2 + H_2(s,\tau)r_s + H_3(s,\tau)\big)1_{(r(s)>r^*(s))}\right]ds. \end{aligned}$$

It is at this stage that we propose to use the forward-neutral probability as an alternative of the method of Chesney, Elliott and Gibson (1993) :

$$\begin{aligned} C\big(Y(\tau),K,t,T\big) &= C_E\big(Y(T,T+\tau),K,t,T\big) \\ &\quad - \int_t^T P(t,s)\, E_{t,r}^s\Big[\big(H_1(s,\tau)r_s^2 + H_2(s,\tau)r_s + H_3(s,\tau)\big)1_{(r(s)>r^*(s))}\Big]ds. \end{aligned}$$

As in theorem 1, the law and thus the density function $f_{r(s)}(x)$ of the short term interest rate r_s under the forward-neutral probability is determined by its conditional Laplace transform

$$E_{t,r}^s\big[\exp(-\lambda r_s)\big] = \exp\left\{\frac{-\lambda\vartheta_{t,s}r}{1+2\lambda\vartheta_{t,s}\nu_{t,s}} - \int_0^{\nu_{t,s}} \frac{\lambda\vartheta_{t,s}\delta_u^t}{1+2\lambda\vartheta_{t,s}\big(\nu_{t,s}-u\big)}du\right\}$$

with $\vartheta_{t,s}$, $\nu_{t,s}$ and δ_u^t as in lemma 1, and can therefore be derived numerically. We conclude that the price of an American yield option with exercise yield K which expires at date T and whose underlying instrument is the τ-maturity yield $Y(\tau)$, can be formulated as

$$\begin{aligned} C\big(Y(\tau),K,t,T\big) &= C_E\big(Y(T,T+\tau),K,t,T\big) \\ &\quad - \int_t^T P(t,s) \int_{r^*(s)}^{+\infty}\big(H_1(s,\tau)x^2 + H_2(s,\tau)x + H_3(s,\tau)\big)f_{r(s)}(x)\,ds. \end{aligned}$$

5. Conclusion

Using the forward-neutral probability and results of the papers of Delbaen and Shirakawa (1996) and Maghsoodi (1996), we have shown that it is straightforward to obtain yield option prices in the extended Cox-Ingersoll-Ross model with time-dependent parameters. In this way we have generalized the results of Longstaff (1990) and of Chesney, Elliott and Gibson (1993) who evaluated in the Cox-Ingersoll-Ross model respectively European yield options and American yield options by using other methods.

Analogously, other interest rate derivatives could be evaluated like European compound bond options, futures, forward contracts, floating rate notes, interest rate swaps etcetera. For example, it appears that Jamshididian (1990) has written a (working) paper in which he derives the prices of American default-free bond options in the Cox-Ingersoll-Ross model by using the forward-neutral probability. Such results can be extended in a straightforward way to the ECIR($\delta(t)$) model.

REFERENCES

Bensoussan A., 1984, « On the Theory of Option Pricing », Acta Appl. Math., **2**, 139-158.

Bensoussan A. and J. L. Lions, 1982, *Applications of Variational Inequalities in Stochastic Control*, New-York : North-Holland.

Carr P., R. Jarrow and R. Myneni, 1992, « Alternative Characterizations of American Put Options », *Mathematical Finance,* **2**, 87-105.

Chesney M., R. J. Elliott and R. Gibson, 1993, « Analytical Solutions for the Pricing of American Bond and Yield Options », *Mathematical Finance*, **3 (3)**, 277-294.

Cox J. C., J. Ingersoll and S. Ross, 1985, « A theory of the Term Structure of Interest Rates », *Econometrica,* **53**, 385-405.

Delbaen F. and H. Shirakawa, 1996, « Squared Bessel Processes and Their Applications for the Square Root Interest Rate Model », *working paper ETH Zentrum.*

El Karoui N. and H. Geman, 1994, « A Probabilistic Approach to the Valuation of General Floating-Rate Notes with an Application to Interest Rate Swaps », *Advances in Futures and Options Research*, **7**, 47-64.

Geman H., N. El Karoui and J.-C. Rochet, 1995, « Changes of Numéraire, Changes of Probability Measure and Option Pricing », *Journal of Applied Probability*, **32**, 443-458.

Harrison J. M. and D. Kreps, 1979, « Martingale and Arbitrage in Multiperiods Securities Markets », *Journal of Economic Theory*, **20**, 381-408.

Harrison J. M. and S. Pliska, 1981, « Martingale and Stochastic Integrals in the Theory of Continuous Trading », *Stoch. Proc. Appl.*, **11**, 215-260.

Heath D., R. Jarrow and A. Morton, 1992, «Bond Pricing and the Term Structure of Interest Rates : A New Methodology for Contingent Claim Valuation », *Econometrica*, **60**, 77-105.

Hull J. and A. White, 1990, « Pricing Interest-Rate Derivative Securities », *Review of Financial Studies,* **3**, 573-592.

Jacka S. D., 1991, « Optimal Stopping and the American Put », *Math. Fin.,* **1**, 1-14.

Jamshidian F., 1990, « An Analysis of American Options», Working paper, Merrill Lynch Capital Markets, New York.

Jamshidian F., 1993, « Bond, Futures and Option Evaluation in the Quadratic Interest Rate Model », Fuji International Finance working paper.

Jamsidian F., 1995, « A Simple Class of Square Root Interest Rate Model », *Sakura Global Capital*, to appear in *Applied Mathematical Finance*.

Johnson N. L. and S. Kotz, 1970, *Distributions in Statistics ; Continuous Univariate Distributions 2*, Boston : Houghton Mifflin.

Karatzas I., 1988, « On the Pricing of American Options », *Appl. Math. Optim.,* **17**, 37-60.

Karatzas I., 1989, « Optimization Problems in the Theory of Continuous Trading », *SIAM J. Control Optim.,* **27**, 1221-1259.

Leblanc B. and O. Scaillet, 1995, « Path Dependent Options on Yields in the Affine Term Structure Model », CREST working paper.

Maghsoodi Y., 1996, « Solutions of the Extended CIR Term Structure and Bond Option Valuation », *Mathematical Finance*, **6 (1)**, 89-109.

Myneni R., 1992, « The Pricing of the American Option », *Ann. Appl. Probab.,* **2**, 1-23.

Revuz D. and M. Yor, 1991, *Continuous Martingales and Brownian Motion,* Berlin : Springer Verlag.

Szatzschneider W. and M. A. Flores-Lopez, « Pricing Interest Rate Derivatives in Extended CIR Model. Martingale Approach », *Anahuac University working paper.*