Calculating capital requirements for longevity risk in life insurance products. Using an internal model in line with Solvency II

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ABSTRACT

The payments of life insurance products depend on the evolution of future survivor probabilities. The literature has devoted considerable attention to the development of statistical models to forecast future mortality improvements. However, using such statistical models to determine solvency requirements, can be highly time consuming. This is the case in particular when the distribution of discounted cash flows needs to be simulated for a future point in time, and conditional on the information available at that time. The goal of the paper is twofold. First, using an internal model which is in line with the Solvency II proposal we derive a closed form approximation for the capital requirements for different portfolios of life insurance products, in case mortality rates are forecasted by means of the Lee and Carter (1992) model. The approximated distribution reduces computer time. In case of the Cost of Capital approach, where the number of simulations is exponential in the number of years to maturity of the life insurance contract, the approximated distribution allows us to calculate solvency requirements for several life insurance products. Specifically, using a market-to-model model, we calculate the market value of the liabilities and the capital reserve that is needed in order to limit the probability of shortfall within a year to 0.5%. We consider the case where the market value of the liabilities is determined by means of the Cost of Capital approach. Second, using the internal model we quantify the effects of different simplifications made in the Solvency II proposal on the capital requirements.

1. Introduction

Over the last decades, significant improvements in the duration of life have been observed in most countries. For example, over the past three decades, the remaining life expectancy of a male Dutch retiree aged 65 has increased by on average one year per decade. More importantly, there is considerable uncertainty regarding the future development of life expectancy. Whereas the focus of regulators has long been on the risk in financial investments, there is now increasing awareness that accurate quantification and management of the risk in pension and insurance liabilities is equally important. For example, the goal of the Swiss Solvency Test and the Solvency II project (Group Consultatif Actuariel Europeen, 2006) is to redesign financial regulation of insurance companies in Switzerland and the EU, respectively, putting increased emphasis on the valuation and management of pension and insurance liabilities. Specifically, the regulator requires that an insurer holds a reserve in order to limit the probability of underfunding in a one year horizon to 0.5%, where underfunding occurs if the value of the assets is less than the value of the liabilities.

In this paper we develop a methodology in line with the Solvency II proposal to determine reserve requirements for systematic longevity risk in life insurance products. A complicating factor is case of reserve requirements for systematic longevity risk is that there is no liquid market for longevity-linked assets or liabilities, and so no market price is observed. Hence, the value of the liabilities needs to be calculated using a market-to-model approach. Solvency II proposes to define the value of the liabilities as the sum of the *best estimate value of the liability* (BEL), and a *market value margin* (MVM). The latter component can be interpreted as a risk premium, and should be determined following the *Cost of Capital* (CoC) approach. The idea is that the risk premium for a risky liability is determined by the amount of capital the holder of the risk should hold in order to be able to pay the liabilities with a high degree of certainty. Our goal in this paper is twofold. First, it has been argued extensively (see, e.g., Ulm (2009)) that determining capital requirements in line with the Solvency II proposal as described above is technically complex. The main complication is that the value of the liabilities in any given period depends on the required solvency capital, which, in turn, depends on the value of the liabilities as well as the probability distribution of the value of the liabilities in the next period, and so on. This implies that backward induction is needed to determine the current value of the liabilities. In addition, the value of the liabilities in a future period t depends on the probability distribution of future death probabilities, conditional on information available at time t. There is a wide variety of mortality forecast models that can be used to simulate the probability distribution of future survival rates, e.g., Lee and Carter (1992) Renshaw and Haberman (2006), Cairns, Blake, and Dowd (2006), Currie, Durban, and Eilers (2004). However, the number of simulations needed in the backward induction algorithm increases exponentially in the length of the run-off period, which can typically be very long. This makes a simulation approach computationally intractable. Therefore, we develop a closed form approximation for the probability distribution of future mortality rates, conditional on information available at that time, starting from the Lee-Carter (1992)-model.[1] This closed form approximation enables us to determine capital requirements within reasonable time. Second, the regulator allows to use a simplified approach that does not require recursive evaluations, and in which the value of the required solvency capital at future dates is approximated by a current estimate. By comparing our results to those that follow from this simplified approach, we quantify the impact of simplifying assumptions on the capital requirements.

The paper is organized as follows. In Section 2 we discuss the Solvency II proposal for minimum capital requirements. In Section 3 we discuss the proposed simplifications to calculate the capital requirements in the Solvency II proposal, as well as two alternative simplifying approaches. In Section 4 we shortly discuss the Lee-Carter (1992)-model and we present the closed form approximation of the distribution of the underlying quantities of interest in life insurance products, i.e., the number of survivors, number of deaths, one-year mortality probability, and one-year survivor probability. We then use these approximations to obtain the a closed form approximation for the distribution of the discounted present value of the liabilities. In Section 5 we calculate the capital requirements for different portfolios of life insurance products, using both the internal model as well as the simplifications. Section 6 concludes.

2. Direct approach in an internal model of Solvency II

The goal of this paper is to calculate the value of liabilities with longevity risk and the reserve requirement for these liabilities. In addition, we evaluate different approximations to calculate the value and the reserve requirement for these liabilities. In this section we describe the method to calculate the value and reserve requirement for a liability in the Cost of Capital approach. The Cost of Capital approach is based on a funding ratio approach, which is described in Section 2.1. Using the funding ratio approach we determine the value of the liabilities and the capital requirements for a one period liability in Section 2.2. In Section 2.3 we generalize the results to a multi period liability.

[1] We focus on the Lee-Carter model because it is widely used.

2.1. Funding ratio approach

The underlying idea of the Solvency II directive proposal is that insurers should hold an amount of capital that enables them to absorb unexpected losses and meet the obligations towards policy-holders at a high level of equitableness. The calculation of this requirement is to be made on the basis of the Value at Risk (VaR) calculation at the $1-\alpha$ percentile (in Solvency II α is set at 0.5%) for the time period of one year. Specifically, the regulator requires that the capital held by an insurer is such that the probability that the funding ratio falls below 1 within a year is lower than α, where the funding ratio is defined as the ratio of the value of the assets over the value of the liabilities, i.e., the funding ratio in year $t+1$ is given by:

$$FR_{t+1} = \frac{A_{t+1}}{L_{t+1}}.$$

The funding ratio in year $t+1$ is a random variable at time t, because both the market value of the assets at time $t+1$ and the market value of the liabilities at time $t+1$ are random variables at time t. The Solvency II requirements imply that an insurer should hold at least initial assets with value $A_t^\star$, defined as:

$$A_t^\star = \min \left\{ A_t | \mathbb{P}_t \left(\frac{A_{t+1}}{L_{t+1}} < 1 \right) \leq \alpha \right\}, \tag{1}$$

where $\mathbb{P}_t(\cdot)$ denote the time-t probability distribution which represents all possible sources of risk. The required capital in excess of the value of the liabilities is referred to as the *Solvency Capital Requirement*[2] (SCR), i.e.,

$$SCR_t \equiv A_t^\star - L_t. \tag{2}$$

The minimum required capital $A_t^\star$ and the buffer SCR_t depend on the evolution of the assets over time. Let r_t be the (stochastic) return on the assets between time t and $t+1$ and $\widetilde{L}_t$ be the (aggregate) payment of the life insurance products[3] at the end of year t, then the value of the assets

[2]In Solvency II the SCR for life underwriting risk is decomposed into seven different risk factors, including longevity risk. The seven different risk factors are: revision risk, mortality risk, longevity risk, disability risk, lapse risk, expense risk, and catastrophe risk. In this paper we focus on the effect of longevity risk.

[3]In order to calculate the current SCR_t an insurer should incorporate all risk, hence also non-systematic longevity risk. The non-systematic longevity risk arises since the number of deaths conditional on the mortality probabilities is still a random variable. In this paper we focus on the effect of systematic longevity risk, i.e., the effect of uncertainty in survival probabilities and neglige the effect of non-systematic longevity risk, since it is well-known that non-systematic longevity risk becomes negligible in large pools (see, e.g., Olivieri 2001; Olivieri and Pitacco 2003; and Harí et al. 2008), whereas systematic longevity risk does not decrease with portfolio size.

next year is defined by:

$$A_{t+1} = A_t \cdot (1 + r_t) - \widetilde{L}_t,$$

which implies that:

$$\mathbb{P}_t \left(\frac{A_{t+1}}{L_{t+1}} < 1 \right) = \mathbb{P} \left((1 + r_t) \cdot A_t - \widetilde{L}_t < L_{t+1} \right). \tag{3}$$

Now it follows immediately from (2) and (3) that the required solvency capital is given by:

$$SCR_t = \mathbb{Q}_{1-\alpha,t} \left[\frac{\widetilde{L}_t + L_{t+1}}{1 + r_t} \right] - L_t, \tag{4}$$

where $\mathbb{Q}_{1-\alpha,t}[X]$ is the $1-\alpha$ percentile of the random variable X, given the information available at time t.

In order to be able to determine the required solvency capital, it remains to specify how the value of the liabilities in year t, as well as the probability distribution of the value of the liabilities in year $t+1$, is determined. In absence of a liquid market, there is no obvious unique way to determine these market values. Solvency II proposes to determine liability values on the basis of a mark-to-model approach, in which the value consists of the *Best Estimate of the Liabilities* plus a *Market Value Margin* (MVM), where the latter is seen as a risk premium, i.e.,

$$L_t \equiv BEL_t + MVM_t. \tag{5}$$

The best estimate value is defined as the expected present value of all future payments, i.e.,

$$BEL_t \equiv \sum_{s \geq 0} \mathbb{E}_t \left[\widetilde{L}_{t+s} \right] \cdot P_t^{(s)}, \tag{6}$$

where $\mathbb{E}_t[\cdot]$ is the expectation conditional on the information available at time t, and $P_t^{(s)}$ denotes the time t price of a zero coupon bond that matures at time $t+s$.

The market value margin is intended to reflect the cost associated with holding the required solvency capital in the current and any future period.[4] The idea is that because the return on assets that need to be kept as a reserve is generally lower than the return on "free assets", the holder of a risky liability requires a price of taking the risk as a compensation for not

[4]See QIS 4., TS.II.A.29.

being able to invest the reserve as a free asset.[5] For the calculation of the value of the liabilities with a non-tradeable risk, such as longevity risk, in Solvency II the return on the assets are set to the risk-free return. Hence, the stochastic return in equation (3) is replaced by the risk-free return. Because the focus of this paper is on the effect of longevity risk on the price and reserve of life insurance liabilities, in the remainder of the paper we assume that the insurer invest only in a risk-free asset. When the insurer invest (part of) his assets in a risky asset this will not affect the value of the liabilities, but it will affect the reserve requirements. The intention of the regulator is that the market value margin equals the cost $CoC + r_t^{rf}$ charged to the present value of the current and all future values of SCR_t, i.e.,

$$MVM_t = \sum_{\tau \geq 0} \left(CoC + r_{t+\tau}^{rf} \right) \cdot SCR_{t+\tau} \cdot P_t^{(\tau)}. \tag{7}$$

This, however, is impossible since the value of the required solvency capital in a current period is currently unknown.

2.2. One-period liabilities

For the sake of intuition, let us first describe the determination of the required solvency capital under the Cost of Capital approach in a given year t, in a setting where the last payment occurs at the end of year t, so that $L_{t+1} = 0$. Then, the market value margin is defined as a cost of capital of $CoC\%$ in excess of the risk-free rate charged on the required solvency capital as given in equation (7). Given (4), (5), (6), and the fact that $L_{t+1} = 0$ then implies that:

$$\begin{aligned} L_t &= BEL_t + \left(CoC + r_t^{rf} \right) \cdot \left(\mathbb{Q}_{1-\alpha,t} \left[\widetilde{L}_t \cdot P_t^{(1)} \right] - L_t \right) \\ &= \left(\frac{1}{1 + CoC + r_t^{rf}} \right) \cdot BEL_t + \left(\frac{CoC + r_t^{rf}}{1 + CoC + r_t^{rf}} \right) \cdot \mathbb{Q}_{1-\alpha,t} \left[\widetilde{L}_t \cdot P_t^{(1)} \right], \end{aligned} \tag{8}$$

so that the required solvency capital is given by:

$$\begin{aligned} SCR_t =& \mathbb{Q}_{1-\alpha,t} \left[\widetilde{L}_t \cdot P_t^{(1)} \right] - L_t, \\ =& \left(\frac{1}{1 + CoC + r_t^{rf}} \right) \cdot \left(\mathbb{Q}_{1-\alpha,t} \left[\widetilde{L}_t \cdot P_t^{(1)} \right] - BEL_t \right). \end{aligned} \tag{9}$$

Thus, in a one period setting, we obtain closed form expressions for the value of the liabilities, and the required solvency capital.

[5] The non-systematic longevity risk is diversifiable and thus has price zero.

2.3. Multi-period liabilities

In the previous section we determined the value of liabilities and the capital reserve for liabilities in the Cost of Capital approach with a run-off time of one period. However, the run-off time for the liabilities of life insurance products is generally much longer than one year. The intention of the regulator is that the market value margin equals the cost $CoC + r_t^{rf}$ charged to the present value of the current and all future values of SCR_t as given in equation (7). This, however, is impossible since the value of the required solvency capital in a current period is currently unknown. To solve this problem, the liabilities are treated as if the run-off period has length one, and the value of the liabilities in this single period is given by $\widetilde{L}_t + L_{t+1}$. The interpretation is that the insurer can sell the liabilities at the end of the year at price equal to L_{t+1}. This effectively transforms the problem to a one-period problem as described above. Specifically, let $BEL_t^{1 \text{ period}}$ denote the current best estimate value of the liabilities, given that at the end of the year they will be sold at price L_{t+1}, i.e.,

$$BEL_t^{1 \text{ period}} = \mathbb{E}_t\left[\left(\widetilde{L}_t + L_{t+1}\right) \cdot P_t^{(1)}\right]. \tag{10}$$

Then it follows immediately from (4), (8), and $BEL_t^{1 \text{ period}}$ as defined in equation (10), that the current market value of the liabilities is given by:

$$\begin{aligned} L_t =& BEL_t^{1 \text{ period}} + \left(CoC + r_t^{rf}\right) \cdot SCR_t \\ =& \left(\frac{1}{1 + CoC + r_t^{rf}}\right) \cdot \mathbb{E}_t\left[\left(\widetilde{L}_t + L_{t+1}\right) \cdot P_t^{(1)}\right] + \left(\frac{CoC + r_t^{rf}}{1 + CoC + r_t^{rf}}\right) \cdot \mathbb{Q}_{1-\alpha,t}\left[\left(\widetilde{L}_t + L_{t+1}\right) \cdot P_t^{(1)}\right. \end{aligned} \tag{11}$$

Note that even though only the capital charge for the first period appears explicitly in the expression for the value of the liabilities, capital charges for holding the risk in later years are included through their effect on the distribution of the market value of the liabilities next year. Indeed, recursive evaluation of (11) shows that:

$$L_t = BEL_t + MVM_t = BEL_t + \sum_{\tau \geq 0} \left(CoC + r_{t+\tau}^{rf}\right) \cdot \mathbb{E}_t\left[SCR_{t+\tau}\right] \cdot P_t^{(\tau)}, \tag{12}$$

where

$$SCR_{t+\tau} = \mathbb{Q}_{1-\alpha,t+\tau}\left[\left(\widetilde{L}_{t+\tau} + L_{t+\tau+1}\right) \cdot P_{t+\tau}^{(1)}\right] - L_{t+\tau}. \tag{13}$$

Thus, the value of the liabilities equals the current best estimate, plus a market value margin that equals the present value of the *expected* cost of capital, $\mathbb{E}_t\left[SCR_{t+\tau}\right]$, associated with holding the liability in future periods.

Determination of the value of life insurance liabilities and the the required capital reserve using simulations would be very time consuming. Therefore, in Section 3 we describe often used simplifications of the equations for the value of the liabilities, as given in equations (12) and (13). Another approach to determine the value of the liabilities is to make simplifications on the distribution of interest, such that these are closed form distributions. This is described in Section 4. Then, also the quantiles and expectation of the variables have a closed form distribution, which leads to a closed form formula of equations (12) and (13). Hence, there is then no need for the time consuming simulations.

3. Approximations of the calculation of capital requirements in the CoC-approach

To simulate the value of MVM in the CoC approach might require many simulations, since it requires the current expectation of future SCR. Recall from (12) that we have to determine:

$$\mathbb{E}_t\left[SCR_{t+\tau}\right] = \mathbb{E}_t\left[\mathbb{Q}_{1-\alpha,t+\tau}\left[\left(\widetilde{L}_{t+\tau} + L_{t+\tau+1}\right)\cdot P^{(1)}_{t+\tau}\right] - L_{t+\tau}\right],$$

which is computationally intensive. This is because, conditional on information available at time t, one needs to determine the $1-\alpha$ quantile of $\widetilde{L}_{t+\tau} + L_{t+\tau+1}$ for a large number of realizations of death rates at time $t+\tau$. The future SCR itself depends on the conditional expectation of future SCR, which requires many simulations to accurately estimate the value. Solvency II allows to make simplification when not materially different from the result which would result from a more accurate valuation process.[6] In this section we HIER propose some simplifications to calculate the capital requirements in the internal model, based on simplification proposed in Solvency II which might reduce the simulation time. In this paper we also investigate the effect of some simplifications to calculate the MVM. The simplifications of the value of the liabilities in the CoC-approach are due to approximations of the expected value of future values of the SCR:

$$\widetilde{SCR}_{t+\tau} \approx \mathbb{E}_t\left[SCR_{t+\tau}\right],$$

where $\widetilde{SCR}_{t+\tau}$ represents a deterministic approximation of the required solvency capital in period $t+\tau$.

A_1 *Longevity shock approach.* In order to calculate the capital requirements for longevity risk in the Solvency II proposal, QIS 4 (sections TS.XI.C

[6] According to section TS.II.C.16 in QIS 4 the SCR can be calculated using either a direct application of SCR formulae or using the proposed simplifications.(see QIS 4, TS.II.A.35)

and TS.XII.D.28) proposes a simplified approach. In the simplified approach, the required solvency capital in any future period $t+\tau$, $SCR_{t+\tau}$, is defined as the *change* in the net asset value at time $t+\tau+1$ due to a (permanent) 25% decrease in mortality probabilities for each age, compared to their current best estimates.[7][8] The net asset value in year $t+\tau+1$, i.e., the value of assets minus the value of the liabilities, is given by:

$$A_{t+\tau+1} - L_{t+\tau+1} = \left(1 + r_t^{rf}\right) \cdot A_{t+\tau} - (\widetilde{L}_{t+\tau} + L_{t+\tau+1}).$$

it follows that:

$$\widetilde{SCR}_{t+\tau} = \widetilde{L}_{t+\tau}^{BE} + L_{t+\tau+1}^{BE} - \left(\widetilde{L}_{t+\tau}^{SHOCK} + L_{t+\tau+1}^{SHOCK}\right),$$

where $\widetilde{L}_t^{BE}$ and L_{t+1}^{BE} represent the expected value of payments in the current year and the market value of the liabilities in the next year, respectively, in a scenario in which future one-year death probabilities are equal to their current best estimate value, and $\widetilde{L}_t^{SHOCK}$ and L_{t+1}^{SHOCK} represent the expected value of payments in the current year and the market value of the liabilities in the next year, respectively, in a scenario in which future death probabilities are equal to 75% of their current best estimate value. Because in both scenarios, future death probabilities are taken as given (and are either equal to the current best estimate value or 75% of the current best estimate value), the corresponding market values at date $t+1$ are equal to the present value of all future payments, given the death probabilities in the corresponding scenario. Therefore,

$$\begin{aligned}
\widetilde{SCR}_{t+\tau} &= \widetilde{L}_{t+\tau}^{shock} - \widetilde{L}_{t+\tau}^{BE} + \sum_{s\geq 0} \frac{\widetilde{L}_{t+\tau+1+s}^{SHOCK}}{(1+r)^{s+1}} - \sum_{s\geq 0} \frac{\widetilde{L}_{t+\tau+1+s}^{BE}}{(1+r)^{s+1}} \\
&= \sum_{s\geq 0} \frac{\widetilde{L}_{t+\tau+s}^{SHOCK} - \widetilde{L}_{t+\tau+s}^{BE}}{(1+r)^{s}}. \qquad (14)
\end{aligned}$$

where $\widetilde{L}_{t+\tau}^{BE}$ represents the expected value of payment in period $t+\tau$ in the best estimate scenario, and $\widetilde{L}_{t+\tau}^{SHOCK}$ represents the expected value of payment in period $t+\tau$ in the shock scenario.

A_2 *Best estimate scenario.* Equation (12) shows that the MVM in the internal model is equal to the expected sum of the current and future

[7]See QIS 4, TS.II.A.10 and TS.II.B

[8]The permanent decrease of 25% is based on ICAS submission in the UK. The average stress test for longevity risk an insurer in the UK used was 18% with a range of between 5% and 35% in 2004. More recent ICAS submissions in the UK are believed to have shown an assumed decrease of around 25% in mortality probabilities.

SCR multiplied with a constant, whereas in the MVM calculated with the Solvency II simplifications, as given in approximation A_1, only depends on the best estimate scenario and a shock to this best estimate scenario. Therefore, in Section 5 we also calculate MVM using the internal model with the simplification that the current MVM equals the sum of the discounted SCR in the best estimate scenario, i.e., using equation (12) for MVM. As an alternative, we consider the case where the current expectation of the future SCR's is replaced by the value of the future $SCR_{t+\tau}$ in the best estimate scenario for death rates at time $t+\tau$, i.e.,

$$\widetilde{SCR}_{t+\tau} = \mathbb{Q}^{BE}_{1-\alpha,t+\tau}\left[\left(\widetilde{L}_{t+\tau} + L_{t+\tau+1}\right) \cdot P_t^{(\tau+1)}\right] - L^{BE}_{t+\tau},$$

where $\mathbb{Q}^{BE}_{1-\alpha,t+\tau}[X]$ is the $1-\alpha$ quantile of X in the best estimate scenario at time $t+\tau$. In Section 5 we calculate capital requirements with the CoC approach for the MVL.

A_3 *Fraction of current SCR.* Another approximation which is often used in order to calculate the capital requirements with the CoC approach, and also proposed in Solvency II, is to take

$$\frac{\widetilde{SCR}_{t+\tau}}{SCR_t} = \frac{\mathbb{E}_t\left[BEL_{t+\tau}\right]}{BEL_t}, \tag{15}$$

i.e., the SCR_{t+s} relative to BEL_{t+s} equal to SCR_t relative to BEL_t for all s, as given in equation (16). The idea behind this simplification is that the future SCR as fraction of the best estimate of the liabilities is equal to the current SCR as a fraction of the current best estimate of the liabilities. Then, the market value margin is given by: [CHECK]

$$\begin{aligned} MVM_t &\approx \left(CoC + r_t^{rf}\right) \cdot \sum_{\tau \geq 0} \frac{\mathbb{E}_t\left[\sum_{s\geq 0} \widetilde{L}_{t+\tau} \cdot P_t^{(\tau)}\right]}{BEL_t} \cdot \frac{SCR_t}{\left(1 + r_{t+\tau}^{rf}\right)^{\tau}} \\ &= \left(CoC + r_t^{rf}\right) \cdot \frac{\sum_{\tau\geq 0}(\tau+1)\left(\mathbb{E}_t\left[\widetilde{L}_{t+\tau}\right] \cdot P_t^{(\tau)}\right)}{\sum_{\tau\geq 0}\mathbb{E}_t\left[\widetilde{L}_{t+\tau}\right] \cdot P_t^{(\tau)}} \cdot SCR_t \\ &= \left(CoC + r_t^{rf}\right) \cdot Dur \cdot SCR_t, \end{aligned} \tag{16}$$

where Dur is the duration of the best estimate of the liabilities. This simplification is often used see, for example, the Swiss Solvency Test (SST) to calculate the MVM in the Cost of Capital approach.[9]

[9]QIS 4, in TS.II.C.26, also proposed a simplification which is close to equation (16). However, in QIS 4 the modified duration, i.e. the duration divided by one plus the yield to maturity, is used instead of the duration. Hence, the proposed MVM in QIS 4 is a factor one plus the yield to maturity lower than the MVM using equation (16). Since the interpretation of the MVM calculated with the duration has a clear interpretation we use this one in the calculation.

In this paper we investigate the effect of using this second approximation to calculate the MVM using equation (16) in the internal model.

4. Model for longevity risk in life insurance products

In this section we briefly describe the method to calculate the capital requirements and the life insurance products. To obtain the capital requirements for longevity risk, we need to obtain the (joint) distribution of the future value of the assets and the future market value of the liabilities. The capital requirements using simulations might require lots of simulations in case of the Cost of Capital approach in order to obtain the market value of the liabilities. Therefore, in this section we consider a closed form expression to approximate longevity risk in life insurance products. In Subsection 4.1 we describe the uncertainty in future forces of mortality, which influences the reserve requirements for life insurance products. The uncertainty in future survivor probabilities is quantified using the Lee-Carter (1992) model, see Lee and Carter (1992). In Subsection 4.2 we describe the life insurance products and the approximations in order to obtain a closed form expression of the sum of the discounted cash flows. In Subsection 4.3 we obtain a closed form approximation for the value of the liabilities in the Cost of Capital approach using the approximations made in Subsection 4.2.

4.1. Lee-Carter model

In this section we present a model for the uncertainty in the mortality rates. To forecast the one-year mortality probabilities we will use the Lee-Carter model with some additional assumptions. Let $x \in \{0, 1, ..., MA\}$ be the age, where MA is the maximum attainable age (set at 110), let $g \in \{M, F\}$ be the gender, where M is males and F is females, let t be the base year, and let $T \in \{1, 2, 3, ...\}$ be the number of years since the base year. The Lee-Carter model assumes that the age and time dependent one-year force of mortality with age dependent parameters a_x^g and b_x^g, and time dependent parameter k_T^g, is given by:

$$\mu_{x,t+T}^g = \exp\left(a_x^g + b_x^g k_{t+T}^g + \epsilon_{x,t+T}^g\right), \tag{17}$$

where the $\epsilon_{x,t+T}^g$ represent the measurement errors. Introduce $\epsilon_{t+T}^g = \left(\epsilon_{1,t+T}^g, ..., , \epsilon_{MA,t+T}^g\right)'$, $g \in \{M, F\}$, $T \in \{1, 2, 3, ...\}$. Then the Lee-Carter model assumes

$$\epsilon_{t+T} = \begin{pmatrix} \epsilon_{t+T}^M \\ \epsilon_{t+T}^F \end{pmatrix} \overset{i.i.d}{\sim} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma_\epsilon^{MM} & \Sigma_\epsilon^{MF} \\ \Sigma_\epsilon^{FM} & \Sigma_\epsilon^{FF} \end{pmatrix}\right). \tag{18}$$

Following typical findings in the empirical literature, see, e.g. Lee and Carter (1992), Renshaw and Haberman (2006), and Booth, Hyndman, Tickle, and de Jong (2006) we postulate that the evolution over time of k_{t+T}^g, $g \in \{M, F\}$, $T \in \{1, 2, 3, ...\}$, can be described by a random walk with drift

$$k_{t+T}^g = k_{t+T-1}^g + c^g + e_{t+T}^g, \tag{19}$$

with k_t^g given, and with

$$e_{t+T} = \begin{pmatrix} e_{t+T}^M \\ e_{t+T}^F \end{pmatrix} \overset{i.i.d}{\sim} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \left(\sigma_e^M\right)^2 & \rho_e \sigma_e^M \sigma_e^F \\ \rho_e \sigma_e^M \sigma_e^F & \left(\sigma_e^F\right)^2 \end{pmatrix}\right), \tag{20}$$

independent of ϵ_{t+T}. To avoid identifiability problems, we set $\sum_x b_x^g = 0$, and $k_t^g = 0$. Combining this with equations (17) and (19) we have, for $T \geq 1$,

$$\log\left(\mu_{x,t+T}^g\right) = a_x^g + T b_x^g c^g + b_x^g \sum_{s=1}^{T} e_{t+s}^g + \epsilon_{x,t+T}^g. \tag{21}$$

We set[10]

$$\log\left(\mu_{x,t}^g\right) = a_x^g + \epsilon_{x,t}^g. \tag{22}$$

To complete the model description, we assume uncertainty in the parameters c^F and c^M, i.e., we assume

$$c = \begin{pmatrix} c^M \\ c^F \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_c^M \\ \mu_c^F \end{pmatrix}, \begin{pmatrix} \left(\sigma_c^M\right)^2 & \rho_c \sigma_c^M \sigma_c^F \\ \rho_c \sigma_c^M \sigma_c^F & \left(\sigma_c^M\right)^2 \end{pmatrix}\right), \tag{23}$$

independent of ϵ_{t+T} and e_{t+T}. We do not include parameter uncertainty in a_x since this represents uncertainty in the current mortality probabilities and not the uncertainty in the changes in the future mortality probabilities. The uncertainty in the parameter b_x is small. Lee and Carter (1992) show that for the forecast of the mortality rates of 75 years the fraction of variance which is due to uncertainty in the parameter b_x is less then 2%. Moreover, we found that the fraction of the standard deviation of the survival probabilities which is due to uncertainty in b_x for an individual of 65 is less than 1%. So we also do not include the uncertainty in b_x.

[10]We set the last observation of the mortality force equal to its observation in order to prevent jump-off bias, see Booth, Maindonald, and Smith (2002).

4.2. Approximation of present value of liabilities

In this section we describe the life insurance products and the approximation in order to obtain the value of the payments of liabilities. Given the uncertainty in the mortality rates, also the sum of the discounted future payments of life insurance products is currently stochastic.

We will use the closed form expression of longevity risk in life insurance products for the following two types of liabilities:

1. An *old age pension* consisting of a nominal yearly payment of 1 at the beginning of a year in which the participant reaches the age of 65, with a last payment in the year (s)he dies;

2. A *partner pension* consisting of a nominal yearly payment of 1 at the beginning of a year in which the participant dies, with a last payment in the year his (her) partner dies.

Let x, y represents the age class of respectively the insured and partner of the insured if present, $g, g\prime$ the gender of respectively the insured and partner of the insured if present, and p be the partner indicator, with $p = 1$ in case a partner is present, and $p = 0$ otherwise, $r_i = \max\{65 - x, 1\}$, the number of years until the first old-age retirement payment. Let ${}_{\tau}p^{g}_{x,t}$ denote the probability that an individual aged x at time t with gender g will survive at least τ years and let $P_t^{(\tau)}$ denote the current market value of one unit to be paid at time $t + \tau$, i.e., the market value of a zero coupon bond maturing at time $t + \tau \geq t$. Define the vector $\widetilde{L}_{pp,\tau}$ with the payments of each pension product in year $t + \tau$ and the vector $\overline{L}$ with the sum of the discounted payments of each pension product, i.e., the expectation of $\overline{L}$ is the best estimate of the liabilities. For our two products we have:

1. *Old-age pension:*

$$\widetilde{L}_{oa,\tau}\left(x,g,t\right) = {}_{\tau}p^{g}_{x,t}, \qquad \overline{L}_{oa}\left(x,g,t\right) = \sum_{\tau=r_i}^{MA-x} \left[{}_{\tau}p^{g}_{x,t} \times P_t^{(\tau)}\right].$$

2. *Partner pension:*

$$\widetilde{L}_{pp,\tau}\left(x,g,1,y,g\prime,t\right) = \left(1 - {}_{\tau}p^{g}_{x,t}\right) \times {}_{\tau}p^{g\prime}_{y,t}, \qquad \overline{L}_{pp}\left(x,g,1,y,g\prime,t\right) = \sum_{\tau=1}^{MA-y} \left[\left(1 - {}_{\tau}p^{g}_{x,t}\right) \times {}_{\tau}p^{g\prime}_{y,t} \times P_t^{(\tau)}\right].$$

In order to obtain a closed form approximation of longevity risk in the discounted cash flows for a portfolio of life insurance products we make three approximation steps. The first one is that we approximate the lognormal distribution of the forces of mortality by a normal one, where the parameters are set such that the first two moments of the two distributions match. As a consequence of this approximation the distribution of the one-year survivor probabilities and the probabilities of surviving τ years are approximately lognormal. The second approximation step considers the distribution of the one-year probability of dying (which is equal to one minus the one-year probability of surviving) and for the distribution of dying within τ years (which is equal to one minus the τ years probability of surviving). We assume that these variables, which each are equal to one minus a lognormal distributed variable, are approximately lognormally distributed. The parameters for this approximation are also set such that the first two moments of these distributions match. The third approximation step is that the sum of discounted payments of the different life insurance products of the different years (which are each by approximation lognormally distributed) is approximated using the lognormal distribution. Asymptotically, when the variance goes to zero, one can show that the sum of lognormal distributed variables is indeed lognormal distributed, see e.g., Dufresne (2004). For a more detailed description of the approximations and the closed form expression of longevity risk in life insurance products we refer to Appendix 7.1.

Our approach to obtain the distribution of the discounted future cash flows of life insurance products is different from the comonotonic quantile-additivity approach developed in Denuit and Dhaene (2007) and Denuit (2008), since it does not require the assumption of comonotonic random variables (i.e., the mortality rates) and monotonic function of these variables. In order to investigate hedging effects between two (or more) types of insured (for example, males and females, high and low income insureds, or insureds from different countries) we want to allow for some (positive) correlation, but not perfectly. In order to investigate the hedging effects of different products, such as partner pension liabilities or death benefits liabilities, we also want to allow that the variable of interest may be a non-monotonic function of the death rates. Moreover, in our model we do not need the restriction that all the age specific parameters β_x should have the same sign, which may not be the case see e.g. Brouhns, Denuit, and Vermunt (2002) where the sign of the β_x switch sign at the age of 95 for males and 97 for females using Belgian mortality data.

For the two different life insurance products we compared the distribution characteristics of the closed form expression with the distribution characteristics of 100,000 simulations of the discounted cash flows without the approximations. In the simulations the one-year survival probabilities are given by:

$$
\begin{aligned}
p^g_{x,t+T} &= \exp\left(-\mu^g_{x,t+T}\right), \\
{}_\tau p^g_{x,t+T} &= \prod_{s=0}^{\tau-1} p^g_{x,t+T+s},
\end{aligned}
$$

where the parameters as given in equations (17)–(23). In Appendix 7.1 the 95% confidence intervals for the simulation error for the 75%, 90%, and 99.5% percentile of the discounted cash flows are given. We observe that the difference in characteristics of the distribution of discounted cash flows for the different life insurance products are close to the results of the simulations. The main difference in characteristics of the distribution using the simulations and the closed form approximation is the skewness. This leads to an increase in the difference between the simulations and the closed form expression when approximating the tail of the distribution.

4.3. Approximation for the CoC approach

In this section we describe the approximation for the market value of the liabilities in the Cost of Capital approach. In order to simulate the CoC capital requirements one needs for each year until the last payment the MVL (and the SCR) conditional on the information until that year. In addition, the MVL in a given year depends on the distribution of MVL in the succeeding years conditional on the available information in that year. It requires not too much computer time is in the year before the last payments are made to simulate the capital requirements in the Cost of Capital approach. This can be done by simulating N times the forces of mortality at the time the last payment is made. To simulate the CoC capital requirements one year earlier, one should make N simulations of the forces of mortality at the time the second last payment is made. In addition, to calculate the MVL in the year before the last payment is made, for each of these N simulations we need the MVL. In order to calculate the MVL, for each of the simulated paths of the forces of mortality next year one needs the SCR next year, for which one should do also N simulations for each path of the forces of mortality next year. For pension liabilities, which are typically contracts with a payments of possibly a long duration, this backward induction algorithm becomes too time consuming to simulate. The capital requirements in the Cost of Capital approach depend on the joint distribution of the payments in a year and the market value of the liabilities after the payment conditional on the information in the previous year. Given this joint distribution one can calculate the capital requirements using a backward induction algorithm. For the joint distribution we additionally need to have the distribution of the market value of the liabilities. Whereas the market price of the liabilities are known at a given point in time, given the information at that time, the market price of the liabilities in the next year in stochastic. In the year before the last payment will be made, given the information upon that year, the market value of the liabilities is deterministic and depends on the distribution of the payments in the last year, which is approximately lognormally distributed. In the year before, i.e., two years before the last payment is made, the market value of the liabilities in the succeeding year, i.e., the year before the last payment is made, is stochastic. Since the payments in the last year are lognormally distributed, the distribution of the market price of the liabilities in the succeeding year is also lognormally distributed. Using that the sum of two lognormal distributed random variables is again lognormally distributed (see, e.g., Dufresne, 2004), the market value of the liabilities next year plus the liability payments within a year is again lognormally distributed. Using this backward induction algorithm we obtain a closed form approximation for the capital requirements in the Cost of Capital approach. For a more detailed description we refer to Appendix 7.2, where we derive the closed form approximated distribution of the market value of a portfolio of life insurance products over time. In addition, in Appendix 7.2 we compare the aggregated discounted differences in the best estimate scenarios between the SCR using simulations of the mortality probabilities and using the approximations. We find for different portfolios of life insurance products that the aggregated discounted difference is small, approximately 1%. The discounted sum of SCR multiplied with the cost of capital rate gives the MVM hence, the results indicate that the error using the model for calculating the MVL is small.

5. Solvency capital requirements

In this section we will use the closed form approximation to calculate capital requirements for a portfolio of life insurance products and compare the results with the simplified approach in the Solvency II proposal. Let $\delta(j)$ be the vector with portfolio weights for portfolio j. To calculate the capital requirements and to illustrate the effect of the assumptions made in the simplified approach, we will use the following four portfolios:

$\delta(1)$: 100% male, aged 65, with old age pension;

$\delta(2)$: 100% female, aged 65, with old age pension;

$\delta(3)$: 50% male, aged 65, with old age pension and 50% female, aged 65, with old age pension liabilities;

$\delta(4)$: 50% male ,aged 65, with old age pension and partner pension, and 50% female, aged 65, with old age pension and partner pension. The partner is of the opposite gender with age 65, the partner pension payments are 70% of the old-age pension payments.

In order to focus on the effect of longevity risk, we assume that the return on assets within a year is deterministic and equal to 4%, and we let the term structure of interest rates be constant and deterministic at 4%, resulting in $P_t^{(\tau)} = \frac{1}{1.04^\tau}$ for all t, and τ. To estimate the parameters of the distribution of the future mortality rates we use US, UK, and Dutch age and gender specific mortality data from 1970 to 2006. A detailed description of the method to estimate the parameters in the Lee-Carter model and the parameter estimates are given in Appendix **??**. In Subsection 5.1 we first calculate the capital requirements using the Solvency II simplifications. Then we calculate the capital requirements using the internal approach for the Cost of Capital approach. In order to obtain the capital requirement we need to set the parameters T, α, FR_{min}, and CoC. These parameters are set following the Solvency II proposal, i.e., T is set at one year, α is set at 0.005, FR_{min} is set equal to one, and CoC is set equal to 6%. In Section 5.2 we use the simplifications, as described in Section 3, in order to give insight to why these simplifications lead to different required capital reserves.

5.1. The capital requirements

In this section we will calculate the capital requirements using the Cost of Capital approach to calculate the market value and the reserve of life insurance products with longevity risk. In the recent versions of QIS it is proposed to calculate the market value of liabilities for which there is no liquid market using the Cost of Capital (CoC) approach. The MVM in the CoC approach depends on the current SCR and all future SCR. The MVM is given by a CoC-percentage times the sum of the current SCR and discounted future SCR. To simulate the MVM and the SCR in the CoC approach requires too many simulations. Therefore, in this section we quantify the effect of the simplification as described in Section 3 and we determine the MVM and SCR in an internal model using the approximated distribution. In QIS 4[11] the CoC-rate is set at 6% in excess of the riskfree interest rate. The reaction of the industry to the Solvency proposal[12], that the CoC-percentage was set too high. Therefore, in our calculations we also use the CoC-percentage of 4% in addition of the proposed 6%. First, we determine the capital requirements for the four portfolios using the simplifications proposed in Solvency II, i.e., using simplification A_1. Table 1 displays the capital reserves in the simplified approach for the four portfolios.

[11]See QIS 4, TS.II.C.14.

[12]See the FSA UK country report and CEA (the European insurance and reinsurance federation)

Table 1
Table with capital requirements using the Solvency II proposal.

	$\frac{A_t^\star \text{ - BEL}_t}{\text{BEL}_t}$	$\frac{\text{MVM}_t}{\text{BEL}_t}$	$\frac{\text{SCR}_t}{\text{BEL}_t}$	$\frac{A_t^\star \text{ - BEL}_t}{\text{BEL}_t}$	$\frac{\text{MVM}_t}{\text{BEL}_t}$	$\frac{\text{SCR}_t}{\text{BEL}_t}$
	CoC = 6%			CoC = 4%		
	US					
$\delta(1)$	20.59%	10.66%	9.93%	17.04%	7.10%	9.93%
$\delta(2)$	17.12%	9.28%	7.84%	14.03%	6.19%	7.84%
$\delta(3)$	18.75%	9.93%	8.83%	15.44%	6.62%	8.83%
$\delta(4)$	16.47%	9.33%	7.13%	13.36%	6.22%	7.13%
	UK					
$\delta(1)$	20.83%	10.71%	10.13%	17.26%	7.14%	10.13%
$\delta(2)$	17.13%	9.27%	7.86%	14.04%	6.18%	7.86%
$\delta(3)$	18.86%	9.94%	8.92%	15.55%	6.63%	8.92%
$\delta(4)$	16.84%	9.50%	7.33%	13.67%	6.34%	7.33%
	NL					
$\delta(1)$	20.33%	10.25%	10.07%	16.91%	6.83%	10.07%
$\delta(2)$	15.72%	8.57%	7.15%	12.86%	5.71%	7.15%
$\delta(3)$	17.81%	9.33%	8.47%	14.70%	6.22%	8.47%
$\delta(4)$	15.89%	8.94%	6.95%	12.91%	5.96%	6.95%

This table displays the capital reserve, as percentage of the best estimate of the liabilities, for four different portfolios of life insurance products for the US, UK, and the Netherlands. The market value of the liabilities is set according the simplifications proposed in Solvency II, using a Cost of Capital rate of 6% and 4%.

We observe the following:

i) *The capital requirements are significant.* For an insured aged 65 years, depending on the portfolio composition and the CoC percentage, an insurer should hold between 13% and 21% of the best estimate of the liabilities in order to fulfill the capital requirements in the Solvency II proposal. This is due to the large decrease in mortality probabilities of 25% for calculating the current and future SCR's.

ii) *The SCR is independent of the CoC-rate.* This is due to the definition of the SCR in Solvency II. The current SCR is only affected by a change in the best estimate of the liabilities and not by the market value of the liabilities.

iii) *For a CoC percentage of 4% and 6%, the MVM is approximately of the same order as the SCR.* The MVM include a compensation for the reserves hold at any time of the contract. Hence, this is due to the combination of a CoC percentage of 4% or 6% in combination with a long runn-off period of the liabilities.

Next, we determine the capital requirements for the different portfolios using the internal model and using simplifications made to the internal model. The capital requirements for the four portfolio using the three different CoC-percentages and the MVM using the two simplifications to the internal model are given in Table 2. The second till the fourth column represents the capital requirements using the internal model. The fifth column represents the MVM relative to the BEL determined using A_2 as given in equation (15) and the sixth column represents the MVM relative to the BEL determined using A_3 as given in equation (16). Table 8 displays the capital requirements using a CoC-percentage of 4%.

Table 2
Table with capital requirements using the CoC-approach.

	Internal model			A_2	A_3
	$\frac{A_t^* - \text{BEL}_t}{\text{BEL}_t}$	$\frac{\text{MVM}_t}{\text{BEL}_t}$	$\frac{\text{SCR}_t}{\text{BEL}_t}$	$\frac{\text{MVM}_t}{\text{BEL}_t}$	$\frac{\text{MVM}_t}{\text{BEL}_t}$
	US				
$\delta(1)$	2.77%	1.09%	1.69%	0.99%	1.08%
$\delta(2)$	1.31%	0.83%	0.49%	0.76%	0.34%
$\delta(3)$	1.85%	0.86%	1.00%	0.79%	0.66%
$\delta(4)$	1.54%	0.77%	0.78%	0.71%	0.55%
	UK				
$\delta(1)$	4.31%	1.65%	2.69%	1.50%	1.68%
$\delta(2)$	2.97%	1.54%	1.46%	1.42%	0.99%
$\delta(3)$	3.48%	1.52%	1.99%	1.40%	1.30%
$\delta(4)$	2.96%	1.36%	1.62%	1.26%	1.13%
	NL				
$\delta(1)$	2.06%	1.06%	1.02%	0.97%	0.61%
$\delta(2)$	4.52%	1.97%	2.58%	1.84%	1.76%
$\delta(3)$	2.69%	1.17%	1.53%	1.10%	0.99%
$\delta(4)$	2.62%	1.16%	1.47%	1.11%	1.02%

This table displays the capital reserve in the internal model, as percentage of the best estimate of the liabilities, for four different portfolios of life insurance products for the US, UK, and the Netherlands. The market value of the liabilities is set according the internal model with a Cost of Capital rate of 6%. The last two columns display the MVM_t using the simplifications. The column A_2 refers to the second simplification as given in equation (15), and the column A_3 refers to the third simplification as given in equation (16).

We observe the following:

i) *The capital requirements are much smaller than in the Solvency II proposal.* This hold for all investigated portfolios and for all three countries. This may indicates that the capital requirements set by the simplified proposal in Solvency II are conservative.

ii) *The SCR marginally decreases when the CoC-rate increases.* The SCR decreases when the CoC-rate increases because a higher CoC-rates lead to a larger part of the one year risks which is covered in the market value of the liabilities. Hence, a higher Cost of Capital-rate leads to a higher expected return of the liabilities within a year, resulting in a lower SCR. However, the MVL increases more than the corresponding decrease of SCR, resulting in a higher total capital requirement for the life insurance products. Although the SCR decreases when the COC-rate increases, the SCR is not strongly affected by the different CoC-rates. This implies that the SCR is relatively robust to the use of method for the calculation of the SCR.

iii) *The MVM calculated using the simplifications A_2 and A_3 generally underestimates the capital requirements using the internal model.* Therefore, these simplifications may not be a preferable way to determine the capital requirements.

In the next section we investigate the sources which results in the difference between the capital requirements using the different approaches.

5.2. The effect of alternative definitions to calculate the capital requirements

In the previous section we observed that the approximations as described in Section 3 leads to different capital requirements than in the internal model. Therefore, in this section we investigate the sources which causes this difference and show their impact. The remainder of this section is organized as follows. First, we will argue why a decrease of 25% in the mortality probabilities may be too conservative. Second, we argue why the second approximation A_2 i.e., calculating the MVM using the discounted SCR in the best estimate scenario, may preform well, but generally leads to an underestimation of the capital requirement. Third, we will show why the third approximation A_3 i.e., calculating the MVM using a constant fraction of SCR relative to BEL over the run-off period, may for some portfolios of life insurance products preform well, but for others not.

Comparing (13) and (14) shows that the difference in the MVM of the internal model and the Solvency II simplification is due to the method to calculate the current and future SCR. In the internal model future mortality probabilities until time $t + \tau$ are stochastic at time t, which leads to stochastic future SCR, whereas the future SCR in the Solvency II is deterministic. A second difference between the MVM of the internal model and the Solvency simplification is that in the internal model the current and future SCR depend on the 99.5% of the discounted value of the payment and the market value next year, whereas in the Solvency II simplification it only depends on a scenario with a decrease in mortality probabilities. Note that, although SCR_t is equivalent to the sum of the present value of the change in the asset value in all future years due to an immediate and permanent shock in the survival probabilities, it cannot be interpreted as the capital requirement in a ruin probability approach for the whole run-off period. The reason is that SCR_t only reflects capital requirements for the uncertainty in the survival probabilities due to changes within a year, but not for uncertainty in the survival probabilities after one year. This can also be seen from the MVM given in equation (14), which is the discounted sum of the current and future SCR multiplied with a cost of capital charge. The capital requirements using the simplified approach depend on the size of the the *longevityshock*. In the document *UNESPA longevity risk investigation* (2009) of the CEA[13], the standard deviations of the annual mortality factor for European mortality probabilities since 1956 are calculated. Using age bands of 10 years and by 5 years ranges. The standard deviation was 1.32%, 1.18%, and 1.01% for the age bands [60–70], [70–80], and [80–90], respectively. Table 3 displays the standard deviation of the annual improvement factor for the time period 1970 till 2006 in US, UK, and the Netherlands. The difference with the calculation of the standard deviation calculated in the *UNESPA longevity risk investigation* (2009) is that we use age bands of 5 years and by 1 year ranges.

Table 3
Table with standard deviation of the annual mortality improvement factor.

	65–69	70–74	75–79	80–84	85–89	90–94	95–99
US Males	1.13%	1.38%	1.50%	1.54%	2.07%	2.74%	2.96%
US Females	1.70%	1.39%	1.78%	1.82%	2.22%	2.77%	2.91%
UK Males	2.08%	2.24%	2.69%	2.37%	3.33%	3.37%	4.50%
UK Females	2.29%	2.19%	2.71%	2.56%	3.21%	3.61%	3.42%
NL Males	2.38%	2.56%	2.80%	3.29%	3.53%	4.25%	5.88%
NL Females	2.38%	2.41%	2.68%	2.94%	3.22%	3.16%	3.74%

This table displays the standard deviation of the age and gender specific mortality improvement factor from 1970 till 2006 for US, UK, and the Netherlands. The standard deviation of the 5 years age band is the average of the five annual age and gender specific mortality improvement factors.

Assuming a normal distribution for the annual improvement factor, the 0.5% quantile of the distribution of the unexpected change in annual improvement factor is minus 2.58 times the standard deviation. From Table 3 we observe that a reduction of 25% in mortality probabilities seems unreasonable high when using the uncertainty in the annual mortality improvement factor from 1970 in all countries. According to historical data and assuming a normal distribution, the shock scenario would have a decrease

[13] CEA is the European insurance and reinsurance federation.

in the annual improvement of between 5% and 10%. The much lower capital reserve calculated in the internal model than using the simplification in Solvency II can be explained by a much lower uncertainty in the future mortality probabilities in the internal model than in the Solvency II shock scenario.

The second approximation A_2 is calculating the MVM using the discounted SCR in the best estimate scenario instead of the currently expected discounted SCR. This leads to a small, i.e., less than 10%, underestimation of the MVM. This occurs because the SCR at time $t + \tau$ does not only depends on the evolution in the survivor probabilities until time t, but also on the evolution in the survivor probabilities between time t and $t + \tau$. In the best estimate scenario, the trend in the survivor probabilities between time t and $t + \tau$ is the same as before time t, whereas in the whole distribution of the possible occurred evolutions of the survivor probabilities the trend in the survivor probabilities between time t and $t + \tau$ generally differs from the trend before time t. A different trend between time t and $t + \tau$ and before time t leads to more uncertainty in the forecasts after time $t + \tau$ conditional on information until time $t + \tau$ than when the trend between time t and $t + \tau$ equals the trend before time t. This implies that using the best estimate scenario instead of the currently expected value values of the SCR typically leads to an underestimation of the uncertainty in the future survivor probabilities which leads to an underestimation of the MVM.

The third approximation A_3 is calculating the MVM using a constant fraction of SCR relative to BEL over the run-off period. This generally also leads to an underestimation of the capital reserves. Especially for a portfolio consisting of males in UK and US aged 65 years with old-age pension insurance the approximation seems to work well. For a portfolio with females insureds the capital reserves calculated using approximation A_3 lead to a large underestimation of the capital reserves. A good approximation would require that the SCR relative to BEL is indeed constant over the run-off period. Figure 1 displays the size of the buffer in the internal model relative to the best estimate of the liabilities over the run-off period. Note that the lines are not smooth, since we have not smoothed the mortality rates and the reduction factor, i.e., we did not smooth b_x in the Lee-Carter model. One of the effects of a change in the size of the buffer relative to the best estimate of the liabilities is the value of b_x. The values of b_x is plotted in Figure 4. In general, for males the value of b_x decreases with age. For females the value of b_x stays more or less constant until the age of 85 and after the age of 85 it decreases with age. A decrease of b_x with age leads to less uncertainty in the annual mortality improvement factor. This leads to a lower buffer, relative to the best estimate of the liabilities. Besides the effect of b_x, there are also two other important effects, namely:

1) *The time to future payments is decreasing.* When time elapses there is less parameter risk in a future payment. This leads to a decrease in the uncertainty of the market value of the liabilities in the succeeding year. Hence, this results in a lower size of the buffer relative to the best estimate of the liabilities over time.

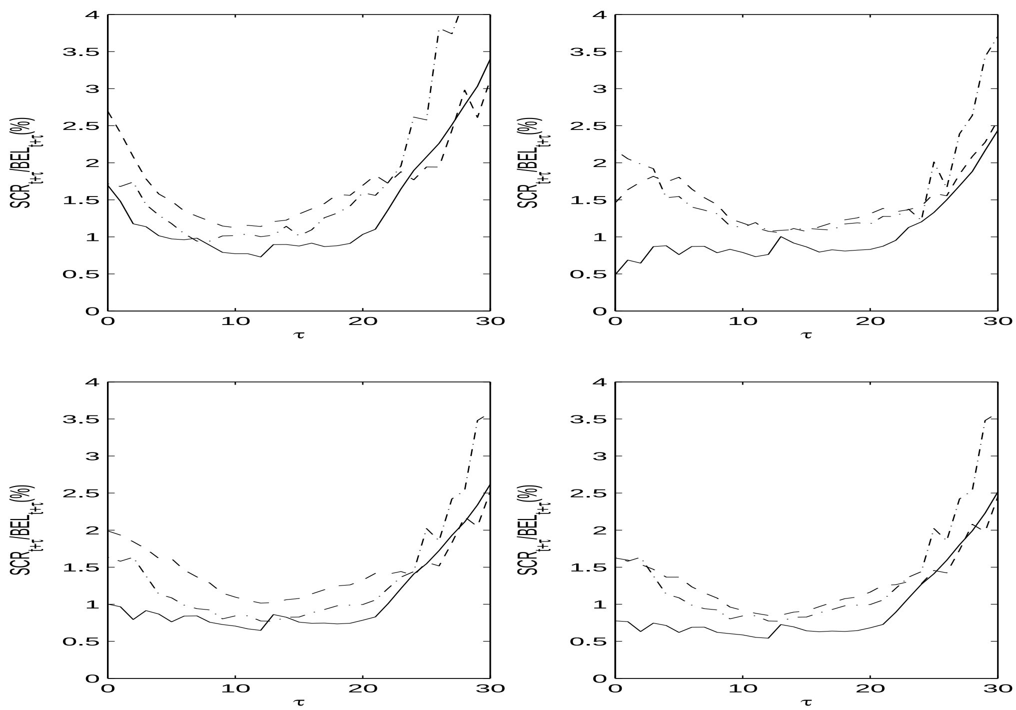
2) *Mortality probabilities are increasing with age*, leading to a higher size of the buffer relative to the best estimate of the liabilities. Because mortality probabilities for older are larger, a general change in mortality probabilities (i.e., a change in the parameter k_t in the Lee-Carter model) has more effect for mortality probabilities at higher ages.

From Figure 1 we observe that the buffer relative to the best estimate of the liabilities in general first decreases when time elapses, which is due to the first effect, and later increases, which is due to the second effect. The relatively good approximation for males insured of the MVM is due to the age of the insured. When the insureds are older the simplification would underestimate the MVM more, whereas when the insureds would be younger the MVM would overestimate the MVM. The size of the buffer relative to the best estimate is not constant, not even monotonic increasing or decreasing, in the run-off period, which makes it difficult to set a general rule for the MVM using only one SCR. The large underestimation for females is due to the shape of the value b_x. After the age of 65 the value of b_x is increasing with age until the age of approximately 80. This implies that the uncertainty in the mortality probabilities at age 65 is relatively low and therefore the approximation of a constant buffer relative to the best estimate of the liabilities leads to an underestimation of the value of the required future buffers.

6. Conclusions

This paper investigates the capital requirements for a portfolio of life insurance products. The capital requirements are set according to the funding ratio approach. The funding ratio approach is in line with the QIS4 Technical Specification of the Solvency II proposal. In order to calculate the capital requirements we derived a closed form approximation of the distribution of the discounted cash flows and the market value of the liabilities of a portfolio of life insurance products using the Cost of Capital approach. The results from closed form approximation are close to the results obtained from simulations, which indicates that the approximations does not lead to large estimation error of the capital requirements. The closed form approximation of the market value of the liabilities in the Cost of Capital approach allows us to calculate the capital requirements, whereas this would require too many simulations using simulations without making simplified

Figure 1. Buffer as percentage of best estimate of the liabilities in the run-off period



The figure displays the SCR as percentage of the best estimate of the liabilities over time for the run-off period for the US (solid curves), UK (dashed curves), and the Netherlands (dashed-dotted curves). The upper left panel displays the SCR as percentage of the best estimate of the liabilities for the portfolio $\delta(1)$, the upper right panel for the portfolio $\delta(2)$, the lower left panel displays for the portfolio $\delta(3)$, and the lower right panel for the portfolio $\delta(4)$.

assumptions to calculate the MVM of life insurance products. Using our approximation we calculated the capital requirements for different portfolios of life insurance products and compared them with the capital requirements using the Solvency II proposed standard formula. Our results suggest that the capital requirements using the simplified approach in the Solvency II proposal lead to an overestimation of the capital requirements, which occurs because the shock in mortality probabilities is larger than past data would suggest. Moreover, our results suggest that an one size fits all immediate and permanent decrease in mortality probabilities, as proposed in the Solvency II proposal, to calculate the capital requirements will lead to an overestimation of capital requirements with a short coverage duration and an underestimation of capital requirements with a long coverage duration. In addition, we have show that using the best estimate scenario instead of the expected value of future SCR to calculate the MVM leads to an underestimation of the MVM, but the underestimation of the MVM is not too large. Finally, our results suggests that the SCR are robust to the method used for calculating the market value of the life insurance products. The SCR is not only robust to the percentile used in the percentile approach and the Cost of Capital rate in the Cost of Capital approach, but also robust to choice of the method. Let us finally indicate an interesting direction for future research. In this paper we compared the capital requirements using the funding ratio approach with the capital requirements in the simplified approach of Solvency II. We have shown that using only an immediate and permanent shock in mortality probabilities will not reflect the true uncertainty in the future market value of the liabilities. We argued that a more appropriate, but not complex, simplified approach would include not only a immediate and persistent shock in mortality probabilities, but would also include a change in the forecasted trend of the reduction of mortality probabilities. It could be interesting to investigate the size of the both the shock in mortality probabilities and the change in the trend of future decline in mortality probabilities and it effects on capital requirements for different life insurance products.

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7. Appendix

7.1. Approximations for the Distribution of the Discounted Cash Flows

In this section we derive the closed form approximation of the sum of the discounted cash cash flows of a portfolio of life insurance products. Introduce the index set

$$\mathcal{I} = \{(x,t,g) \mid x \in \{1,...,MA\}, T \in \mathbb{R}, t \in \{1,...,MA-x\}, g \in \{M,F\}\}$$

with x representing the age class, t the time period under consideration, and g the gender. Define the vector ℓ, with components $\ell(i)$, for $i = (x,T,t,g) \in \mathcal{I}$, by,

$$\ell(i) = \log\left(\mu^g_{x+t,T+t}\right) - \log\left(\mu^g_{x+t,T}\right). \tag{24}$$

Using (21)—(22), we find for $i = (x,T,t,g) \in \mathcal{I}$,

$$\begin{aligned}\ell(i) = &\left(a^g_{x+t} + tb^g_{x+t}c^g + b^g_{x+t}\sum_{s=1}^{t} e^g_{T+s} + \epsilon^g_{x+t,T+t}\right) - \left(a^g_{x+t} + \epsilon^g_{x+1,T}\right)\\ &= tb^g_{x+t}c^g + b^g_{x+t}\sum_{s=1}^{t} e^g_{T+s} = \left(\epsilon^g_{x+t,T+t} - \epsilon^g_{x+t,T}\right).\end{aligned}$$

Straightforward calculations result in the following lemma.

Lemma 1

$$\ell \sim N(\mu_\ell, \Sigma_\ell) \tag{25}$$

with μ_ℓ the mean vector with components

$$\mu_\ell(i) = tb^g_{x+t}\mu^g_c,$$

for $i = (x,T,t,g) \in \mathcal{I}$, and with Σ_ℓ the covariance matrix, with components

$$\Sigma_\ell(i,j) = b^g_{x+t}b^h_{y+\tau}f(i,j) + 2\Sigma^{g,h}_\epsilon(x+t, y+\tau),$$

for $i = (x,T,t,g)$, $j = (y,T,\tau,h) \in \mathcal{I}$, where

$$f(i,j) = \left(1_{(g=h)} + 1_{(g\neq h)}\rho_e\right)\times\sigma^g_e\sigma^h_e\min(t,\tau) + \left(1_{(g=h)} + 1_{(g\neq h)}\rho_c\right)\times\sigma^g_c\sigma^h_c t\tau.$$

Next, we introduce the vector of Reduction Factors rf, with components $rf(i)$, for $i \in \mathcal{I}$, as follows

$$rf(i) = \exp(\ell(i)). \tag{26}$$

Let $X \sim N(\mu, \Sigma)$, and $Y = \exp(X)$. Then $Y \sim \log N(\mu, \Sigma)$. Using Lemma 1, we have $mf \sim \log N(\mu_\ell, \Sigma_\ell)$. We define the vector $\widetilde{\mu}$, with components $\widetilde{\mu}(i)$, for $i = (x, t, g) \in \mathcal{I}$, by

$$\widetilde{\mu}(i) = rf(i) \times \widehat{\mu}^g_{x+t,T}, \tag{27}$$

with $\widehat{\mu}^g_{x+t,T}$ the actually observed $\mu^g_{x+t,T}$, used as "starting value." We define the vector ℓ_T, with components $\ell_T(i)$, for $i = (x, T, t, g) \in \mathcal{I}$, by $\ell_T(i) = \log\left(\widehat{\mu}^g_{x+t,T}\right)$. Then we have $\widetilde{\mu} \sim \log N(\ell_T + \mu_\ell, \Sigma_\ell)$

First Approximation. We use as first approximation $\widetilde{\mu} \sim N(\mu_\mu, \Sigma_\mu)$, with the vector μ_μ, with components $\mu_\mu(i)$, for $i \in \mathcal{I}$, given by

$$\mu_\mu(i) \equiv \exp\left(\ell_T(i) + \mu_\ell(i) + \frac{1}{2}\Sigma_\ell(i,i)\right),$$

and with the matrix Σ_μ, with components $\Sigma_\mu(i,j)$ for $i, j \in \mathcal{I}$, given by

$$\begin{aligned}\Sigma_\mu(i,j) \equiv & \left(\exp\left(\Sigma_\ell(i,j)\right) - 1\right) \\ & \times \exp\left(\ell_T(i) + \mu_\ell(i) + \ell_T(j) + \mu_\ell(j) + \frac{1}{2}\Sigma_\ell(i,i) + \frac{1}{2}\Sigma_\ell(j,j)\right).\end{aligned}$$

□

The parameters in the first approximation are chosen just that the first two moments of the distributions match. In Figure 2 and Table 4 the accuracy of the approximation is displayed. The upper left panel of Figure 2 corresponds with the parameters of the distribution of the one-year probability of surviving of a male individual age 100 in 35 years from now in the Netherlands. The upper right panel corresponds with the parameters of the distribution of the one-year probability of surviving of a male individual individual age 100 in 75 years from now. The lower left panel displays the the effect of an increase in μ_X for the parameters of the upper right panel. The lower right panel displays the the effect of an increase in σ_X for the parameters of the upper right panel.

We define the vector $\widetilde{p}$, with components $\widetilde{p}(i)$, for $i \in \mathcal{I}$ by

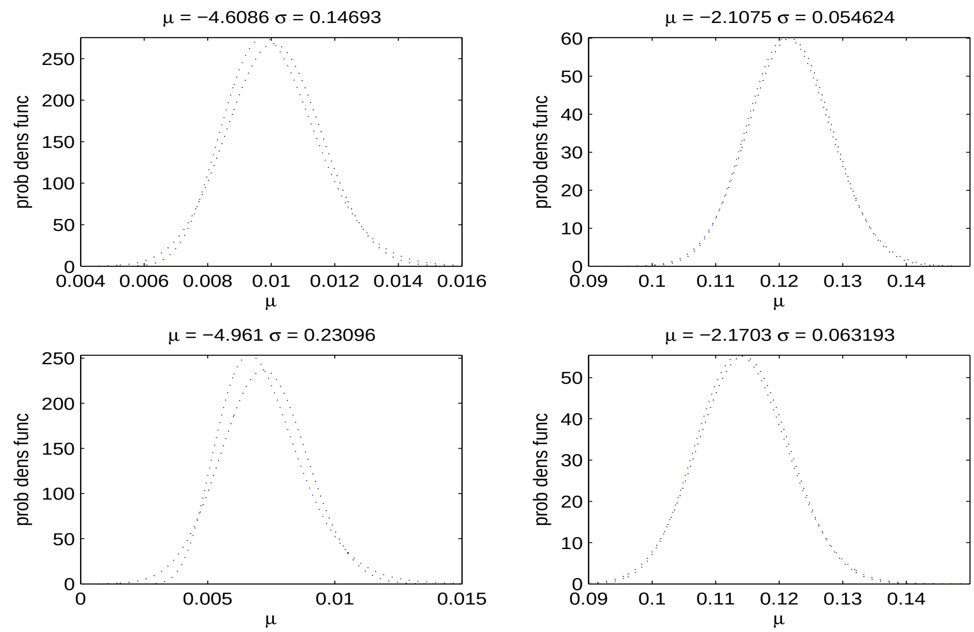
$$\widetilde{p}(i) \equiv {}_1p^g_{x+t,T+t} = \exp\left(-\widetilde{\mu}(i)\right), \tag{28}$$

and we define the vector $\widetilde{S}$, with components $\widetilde{S}(i)$, for $i = (x, T, t, g) \in \mathcal{I}$, by

$$\widetilde{S}(i) \equiv {}_tp^g_{x,T+t} = \prod_{s=1}^{t} \widetilde{p}(x, s, g). \tag{29}$$

The vector $\widetilde{p}$ represents the one-year survivor probabilities of an $x+t$ year old individual in year $T+t$ with gender g (${}_1p^g_{x,s}$), and the vector $\widetilde{S}$ represents

Figure 2. Comparison Log Normal - Normal



This figure displays the probability density function of $X \sim \log N\left(\mu_X, \sigma_X^2\right)$ and probability density function of the approximation $Y \sim N\left(\mu_Y, \sigma_Y^2\right)$ for different values of $\mu \equiv \mu_X$ and $\sigma \equiv \sigma_X$. The parameters μ_Y and σ_Y^2 are set such that they match the first two moments of those of X: $\mu_Y = \exp\left(\mu_X + \frac{1}{2}\sigma_X^2\right)$ and $\sigma_Y^2 = \left(\exp\left(\sigma_X^2\right) - 1\right)\exp\left(2\mu_X + \sigma_X^2\right)$.

Table 4
Comparison Log Normal-Normal

	μ_X = -3.9553; $\sigma_X = 0.14693$		μ_X = -1.9579; $\sigma_X = 0.054624$		μ_X = -4.3077; $\sigma_X = 0.23096$		μ_X = -2.0208; $\sigma_X = 0.063193$	
Q	X	Y	X	Y	X	Y	X	Y
0.01	0.014	0.013	0.124	0.123	0.008	0.006	0.114	0.113
0.025	0.014	0.014	0.127	0.126	0.009	0.007	0.117	0.116
0.05	0.015	0.015	0.129	0.129	0.009	0.009	0.119	0.119
0.1	0.016	0.016	0.132	0.131	0.010	0.001	0.122	0.122
0.25	0.017	0.017	0.136	0.136	0.012	0.012	0.127	0.127
0.5	0.019	0.019	0.141	0.141	0.013	0.014	0.133	0.133
0.75	0.021	0.021	0.146	0.147	0.016	0.016	0.138	0.138
0.9	0.023	0.023	0.151	0.151	0.018	0.018	0.144	0.144
0.95	0.024	0.024	0.154	0.154	0.020	0.019	0.147	0.147
0.975	0.026	0.025	0.157	0.157	0.021	0.020	0.150	0.149
0.99	0.027	0.026	0.160	0.159	0.023	0.021	0.154	0.152

This table displays the quantiles of the function $X \sim \log N\left(\mu_X, \sigma_X^2\right)$ and the corresponding quantiles of the approximating distribution $Y \sim N\left(\mu_Y, \sigma_Y^2\right)$. The parameters of Y are set such that the first two moments of Y match the first two moments of X: $\mu_Y = \exp\left(\mu_X + \frac{1}{2}\sigma_X^2\right)$ and $\sigma_Y^2 = \left(\exp\left(\sigma_X^2\right) - 1\right)\exp\left(2\mu_X + \sigma_X^2\right)$. The column with heading Q displays the quantiles. The other columns show the quantiles of X and Y for different values of μ_X and σ_X, where the subcolumns with heading X present the quantiles of the distribution of X, and the subcolumns with heading Y the quantiles of the corresponding distribution of Y.

the probability that an individual with age x and gender g in year T will survive another t years $\left({}_{\tau}p^{g}_{x,s}\right)$. Straightforward calculations result in the following lemma.

Lemma 2 *Given the approximation $\widetilde{\mu} \sim N(\mu_\mu, \Sigma_\mu)$, we have*

$$\begin{pmatrix} \widetilde{p} \\ \widetilde{S} \end{pmatrix} \sim \log N\left(\begin{pmatrix} \mu_{\ell p} \\ \mu_{\ell S} \end{pmatrix}, \begin{pmatrix} \Sigma_{\ell p} & \Sigma_{\ell pS} \\ \Sigma_{\ell Sp} & \Sigma_{\ell S} \end{pmatrix} \right),$$

with $\mu_{\ell p} = -\mu_\mu$, $\Sigma_{\ell p} = \Sigma_\mu$, and where, for $i = (x, T, t, g) \in \mathcal{I}$,

$$\mu_{\ell S}(i) = \sum_{s=1}^{t} \left(-\mu_\mu (x, T, s, g)\right),$$

and for $i = (x, T, t, g), j = (y, T, \tau, h) \in \mathcal{I}$,

$$\Sigma_{\ell S}(i, j) = \sum_{s_1=1}^{t} \sum_{s_2=1}^{\tau} \Sigma_\mu \left((x, T, s_1, g), (y, T, s_2, h)\right),$$

and

$$\Sigma_{\ell Sp}(j, i) = \Sigma_{\ell pS}(i, j) = \sum_{s=1}^{\tau} \Sigma_\mu \left((x, T, t, g), (y, T, s, h)\right).$$

Let $X \sim \log N(\mu_{\ell X}, \Sigma_{\ell X})$, with $X = \widetilde{p}$ or $X = \widetilde{S}$. Then for $i \in \mathcal{I}$,

$$\mu_X(i) \equiv E(X(i)) = \exp\left(\mu_{\ell X}(i) + \frac{1}{2}\Sigma_{\ell X}(i, i)\right), \tag{30}$$

and for $i, j \in \mathcal{I}$,

$$\begin{aligned} \Sigma_X(i, j) \equiv Cov(X(i), X(j)) &= \left(\exp\left(\Sigma_{\ell X}(i, j)\right) - 1\right) \\ &\times \exp\left(\mu_{\ell X}(i) + \mu_{\ell X}(j) + \frac{1}{2}\Sigma_{\ell X}(i, i) + \frac{1}{2}\Sigma_{\ell X}(j, j)\right). \end{aligned} \tag{31}$$

Introduce $\widetilde{q} = 1 - \widetilde{p}$ and $\widetilde{D} = 1 - \widetilde{S}$. The vector $\widetilde{q}$ represents the one-year mortality probabilities of an $x + t$ year old individual in year $T + t$ with gender g, the vector $\widetilde{D}$ denotes the probability that an x year old in year T with gender g will not survive another t years. Next, we define H, with components $H(k, i)$, for $k \in \{1, 2, 3, 4\}$ and $i \in \mathcal{I}$, by $H(1, i) = \widetilde{p}(i)$, $H(2, i) = \widetilde{q}(i)$, $H(3, i) = \widetilde{S}(i)$, and $H(4, i) = \widetilde{D}(i)$. Note that the vectors of the matrix H corresponds to the four different probabilities which reflects survivor probabilities (i.e., one-year survivor probabilities, τ years survivor probabilities, one-year death probabilities, and the probabilities of dying within τ years). Using lemma 2, we can easily calculate $\mu_H \equiv E(H)$ and $\Sigma_H \equiv Cov(H)$.

Second Approximation. As our second approximation, we take $H \sim \log N(\mu_{\ell H}, \Sigma_{\ell H})$, with for $u = (k, i)$, $k \in \{1, 2, 3, 4\}$, and $i \in \mathcal{I}$,

$$\mu_{\ell H}(u) = \log(\mu_H(u)) - \frac{1}{2}\Sigma_H(u, u),$$

and for $u_1 = (k_1, i_1), u_2 = (k_2, i_2)$, with $k_1, k_2 \in \{1, 2, 3, 4\}$, and $i_1, i_2 \in \mathcal{I}$,,

$$\Sigma_{\ell H}(u_1, u_2) = \log\left(1 + \frac{\Sigma_H(u_1, u_2)}{\mu_H(u_1)\mu_H(u_2)}\right).$$

□

The parameters in the second approximation are chosen such that the first two moments of the distributions match. Figure 3 and Table 5 display the accuracy of the approach. The upper left panel of Figure 3 corresponds with the parameters of the distribution of the probability of dying for a Dutch male individual currently aged 65 in 1 year from now. The upper right panel corresponds with the parameters of the distribution of the probability of dying for a Dutch male individual currently aged 65 in 32 year from now. The lower left panel corresponds with the parameters of the distribution of the probability of dying for a Dutch male individual currently aged 25 in 41 year from now. The lower right panel corresponds with the parameters of the distribution of the probability of dying for a Dutch male individual currently aged 25 in 72 year from now.

Introduce the index set

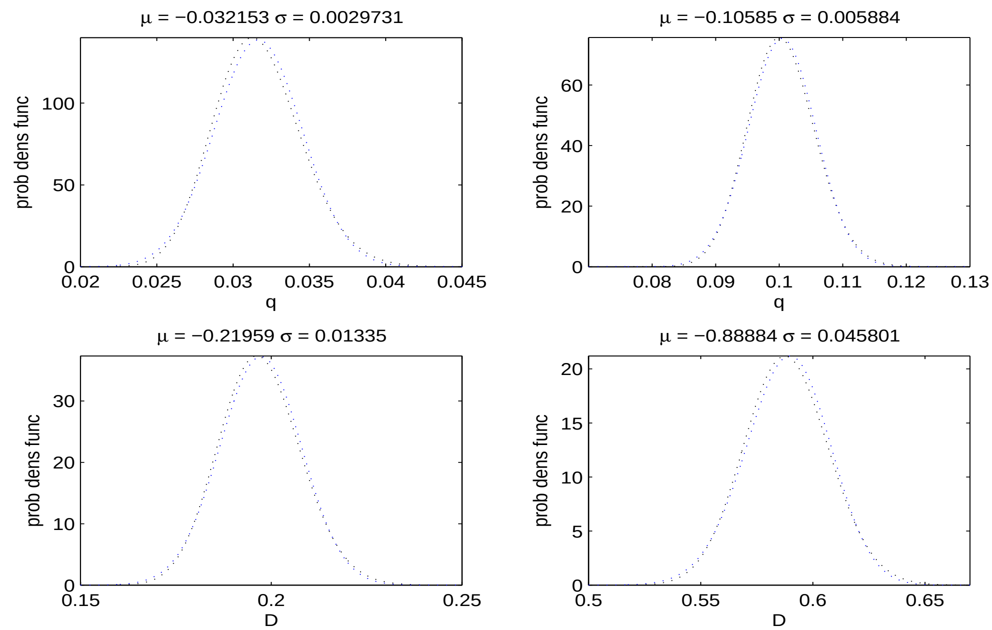
$$\begin{aligned}\mathcal{J} = &\{(a, T, x, g, p = 0) \mid a \in \mathcal{A}, T \in \mathbb{R}, x \in \{1, ..., MA\}, g \in \{M, F\}\} \\ &\bigcup \{(a, T, x, g, p = 1, y, g\prime) \mid a \in \mathcal{A}, T \in \mathbb{R}, x, y \in \{1, ..., MA\}, g, g\prime \in \{M, F\}\}\end{aligned}$$

with a representing the pension product from the product class $\mathcal{A}$, T representing the base year, x, y representing the age class of respectively the insured and partner of the insured if present, $g, g\prime$ the gender of respectively the insured and partner of the insured if present, and p be the partner indicator, with $p = 1$ in case a partner is present, and $p = 0$ otherwise. For $j = (a, T, x, g)$, let $r_j = max\{65 - x, 1\}$, the number of years until the first old-age retirement payment. Let $P_T^{(t)}$ denote the current market value of one unit to be paid at time $T + t$, i.e., the market value of a zero coupon bond maturing at time $T + t \geq T$. For $j = (a, T, x, g) \in \mathcal{J}$, define the vector L with the sum of the discounted payments of each pension product:

1. *Old-age pension:*

$$L(OA, T, x, g) = \sum_{t=r_i}^{MA-x} \left[\widetilde{S}(x, T, t, g) \times P_T^{(t)}\right].$$

Figure 3. Comparison 1 – Log Normal - Log Normal



This figure displays the probability density function of $Y = 1 - X$, with $X \sim \log N(\mu_X, \sigma_X^2)$ and the probability density function of the approximation $Z \sim \log N(\mu_Z, \sigma_Z^2)$ for different values of $\mu \equiv \mu_X$ and $\sigma \equiv \sigma_X$. The parameters of Z are set such that the first two moments of Z match the first two moments of X: $\mu_Z = \log\left(1 - \exp\left(\mu_X + \sigma_X^2/2\right)\right) - \sigma_Z^2/2$ and $\sigma_Z^2 = \log\left(1 + \frac{\left(\exp\left(\sigma_X^2\right)-1\right)\exp\left(2\mu_X+\sigma_X^2\right)}{\left(1-\exp\left(\mu_X+\sigma_X^2/2\right)\right)^2}\right)$.

Table 5
Comparison 1 – Log Normal- Log Normal

	μ_X = -0.052786; $\sigma_X = 0.0048811$		μ_X = -0.31694; $\sigma_X = 0.022088$		μ_X = -0.38612; $\sigma_X = 0.023094$		μ_X = -3.5138; $\sigma_X = 0.064043$	
Q	X	Z	X	Z	X	Z	X	Z
0.01	0.041	0.042	0.233	0.236	0.283	0.285	0.965	0.966
0.025	0.042	0.043	0.239	0.241	0.289	0.290	0.966	0.966
0.05	0.044	0.044	0.245	0.246	0.294	0.295	0.967	0.967
0.1	0.045	0.046	0.251	0.251	0.300	0.300	0.968	0.968
0.25	0.048	0.048	0.261	0.260	0.310	0.309	0.969	0.969
0.5	0.051	0.051	0.272	0.271	0.320	0.320	0.970	0.970
0.75	0.055	0.054	0.282	0.282	0.331	0.330	0.971	0.971
0.9	0.057	0.057	0.292	0.292	0.340	0.340	0.973	0.973
0.95	0.059	0.059	0.298	0.299	0.346	0.347	0.973	0.973
0.975	0.060	0.061	0.302	0.304	0.350	0.352	0.974	0.974
0.99	0.062	0.063	0.308	0.311	0.356	0.358	0.974	0.975

This table displays the quantiles of the distribution $Y = 1 - X$, with $X \sim \log N(\mu_X, \sigma_x^2)$ and the quantiles of the approximating distribution $Z \sim \log N(\mu_Z, \sigma_Z^2)$. The parameters of Z are set such that the first two moments of Z match the first two moments of X: $\mu_Z = \log\left(1 - \exp\left(\mu_X + \sigma_X^2/2\right)\right) - \sigma_Z^2/2$ and $\sigma_Z^2 = \log\left(1 + \frac{\left(\exp\left(\sigma_X^2\right)-1\right)\exp\left(2\mu_X+\sigma_X^2\right)}{\left(1-\exp\left(\mu_X+\sigma_X^2/2\right)\right)^2}\right)$. The first column with heading Q displays the quantiles. The other columns show the quantiles of X and Z for different values of μ_X and σ_X, where the subcolumns with heading X present the quantiles of the distribution of X, and the subcolumns with heading Z the quantiles of the corresponding distribution of Z.

2. *Partner pension:*

$$L\left(PP,T,x,g,1,y,g\prime\right)=\sum_{t=1}^{MA-y}\left[\widetilde{D}\left(x,T,t,g\right)\times\widetilde{S}\left(y,T,t,g\prime\right)\times P_T^{(t)}\right].$$

Notice that each component $L\left(j\right)$, $j\in\mathcal{J}$, is of the form

$$L\left(j\right)=\sum_{v\in V_j}\left(\prod_{u\in U_{(v,j)}}H\left(u\right)\right)c_{(v,j)}, \tag{32}$$

with constants $c_{(v,j)}$ of the form $P_T^{(t)}$ for appropriate t.

Introduce the vector $\widetilde{L}$ with components $\widetilde{L}\left(w\right)$, for $w=(v,j)$, with $v\in V_j$ and $j\in\mathcal{J}$, defined by $\widetilde{L}\left(w\right)\equiv\prod_{u\in U_{(w)}}H\left(u\right)$. Given $H\sim\log N\left(\mu_{\ell H},\Sigma_{\ell H}\right)$, we have $\widetilde{L}\sim\log N\left(\mu_{\ell\widetilde{L}},\Sigma_{\ell\widetilde{L}}\right)$, with for $w=(v,j)$, $v\in V_j$, and $j\in\mathcal{J}$,

$$\mu_{\ell\widetilde{L}}(w)=\sum_{u\in U_w}\mu_{\ell H}\left(u\right), \tag{33}$$

and for $w_1=(v_1,j_1),w_2=(v_2,j_2)$, with $v_1\in V_1$, $v_2\in V_2$, and $j_1,j_2\in\mathcal{J}$,

$$\Sigma_{\ell\widetilde{L}}\left(w_1,w_2\right)=\sum_{u_1\in U_{w_1}}\sum_{u_2\in U_{w_2}}\Sigma_{\ell H}\left(u_1,u_2\right). \tag{34}$$

Define for $w=(v,j)$, $v\in V_j$, and $j\in\mathcal{J}$,

$$\mu_{\widetilde{L}}\left(w\right)\equiv E\left(\widetilde{L}\left(w\right)\right)=\exp\left(\mu_{\ell\widetilde{L}}\left(w\right)+\frac{1}{2}\Sigma_{\ell\widetilde{L}}\left(w,w\right)\right), \tag{35}$$

and for $w_1=(v_1,j_1),w_2=(v_2,j_2)$, with $v_1\in V_1$, $v_2\in V_2$, and $j_1,j_2\in\mathcal{J}$,

$$\begin{aligned}\Sigma_{\widetilde{L}}\left(w_1,w_2\right)&\equiv Cov\left(\widetilde{L}\left(w_1\right),\widetilde{L}\left(w_2\right)\right)\\&=\left(\exp\left(\Sigma_{\ell\widetilde{L}}\left(w_1,w_2\right)\right)-1\right)\\&\times\exp\left(\mu_{\ell\widetilde{L}}\left(w_1\right)+\mu_{\ell\widetilde{L}}\left(w_2\right)+\frac{1}{2}\Sigma_{\ell\widetilde{L}}\left(w_1,w_1\right)+\frac{1}{2}\Sigma_{\ell\widetilde{L}}\left(w_2,w_2\right)\right).\end{aligned} \tag{36}$$

Notice that L is just a linear transformation of $\widetilde{L}$, i.e., we can write $L=B\widetilde{L}$, with B a matrix with components $B\left(j,w\right)$, for $j\in\mathcal{J}$ and $w=(v,j)$, $v\in V_j$, and $j\in\mathcal{J}$, given by $B\left(j,w\right)=c_w$ in case $w=(v,j)$ is such that $v\in V_j$, and $B\left(j,w\right)=0$ otherwise. Then we define μ_L by $\mu_L\equiv B\mu_{\widetilde{L}}$, and Σ_L by $\Sigma_L\equiv B\Sigma_{\widetilde{L}}B'$. Next, let δ, with components $\delta\left(j\right)$, $j\in\mathcal{J}$, represent a portfolio of pension liabilities, where $\delta\left(j\right)$ denotes the weight of liability $L\left(j\right)$. More generally, let $\delta'\equiv\left(\delta_1,...,\delta_K\right)$ denote a matrix with portfolios of pension liabilities $\delta_1, ..., \delta_K$. Then δL is the K-dimensional vector with as components $\delta_1' L, ..., \delta_K' L$, the liabilities of the corresponding portfolios. Define μ_δ by $\mu_\delta \equiv \delta\mu_L$, and Σ_δ by $\Sigma_\delta \equiv \delta\Sigma_L\delta'$.

Third Approximation. As our third approximation, we take $\delta L \sim \log N\left(\mu_{\ell\delta}, \Sigma_{\ell\delta}\right)$, with for $j \in \mathcal{J}$,

$$\mu_{\ell\delta}(j) = \log\left(\mu_\delta\left(j\right)\right) - \frac{1}{2}\Sigma_\delta\left(j, j\right),$$

and for $j_1, j_2 \in \mathcal{I}$,,

$$\Sigma_{\ell\delta}\left(j_1, j_2\right) = \log\left(1 + \frac{\Sigma_\delta\left(j_1, j_2\right)}{\mu_\delta\left(j_1\right)\mu_\delta\left(j_2\right)}\right).$$

$\square$

In Table 6 the 95% confidence intervals of the simulation uncertainty for three quantiles of the distribution of the discounted cash flows are given. The three quantiles are the 75%, 90%, and the 99.5% quantile, the number of simulations are 10,000 and 100,000. From table 6 we observe that that the closed form approximation is quite accurate.

Table 6
95 percent confidence intervals of quantiles of the distributions

N	10,000	100,000	Model
	Q(0.75)		
oa m	[11.6048 - 11.6126]	[11.6066 - 11.6093]	11.6059
oa f	[13.2760 - 13.2872]	[13.2826 - 13.2865]	13.2824
pp m	[3.3288 - 3.3373]	[3.3293 - 3.3317]	3.3313
pp f	[1.6605 - 1.6658]	[1.6642 - 1.6658]	1.6666
	Q(0.90)		
oa m	[11.6959 - 11.7068]	[11.7007 - 11.7039]	11.7038
oa f	[13.4047 - 13.4171]	[13.4166 - 13.4212]	13.4256
pp m	[3.4154 - 3.4245]	[3.4182 - 3.4216]	3.4245
pp f	[1.7197 - 1.7257]	[1.7221 - 1.7241]	1.7254
	Q(0.995)		
oa m	[11.8873 - 11.9240]	[11.8918 - 11.9018]	11.9152
oa f	[13.6767 - 13.7078]	[13.6861 - 13.6982]	13.7361
pp m	[3.5967 - 3.6266]	[3.6084 - 3.6175]	3.6319
pp f	[1.8451 - 1.8636]	[1.8543 - 1.8593]	1.8577

7.2. Approximations for the Cost of Capital approach

Denote $\widetilde{L}(t+T) \equiv \widetilde{L}(w)$ for $w = (v, j)$, $v = s$, and $j \in \mathcal{J}$ for the pension payment in year $t+T$. Using the third approximation with $B(j, w) = 1$

in case $w = (v, j)$ is such that $v \in V_j, j = t$, and $B(j, w) = 0$ otherwise, we obtain that the distribution of the pension payments of a portfolio of pension liabilities in year $t+T$ i.e., $\delta\widetilde{L}(t+T)$ is LogNormally distributed. Hence, we denote:

$$\delta\widetilde{L}(t+T) \sim \log N\left(\mu_{\delta\widetilde{L}_{t+T}}, \Sigma_{\delta\widetilde{L}_{t+T}}\right).$$

The Cost of Capital is given by a percentage in the excess of the risk-free rate, i.e., we define $CoC_{t+T} = CoC + \frac{1}{P_{t+T}^{(t+T+1)}} - 1$. And in order to prevent overnotation, let $S(t+T) \equiv \widetilde{S}(x, T, t, g)$, recall that we have:

$$\begin{pmatrix} S(t+T) \\ S(t+T-1) \end{pmatrix} \sim \log N\left(\begin{pmatrix} \mu_{S_{t+T}} \\ \mu_{S_{t+T-1}} \end{pmatrix}, \begin{pmatrix} \Sigma_{S_{t+T,t+T}} & \Sigma_{S_{t+T,t+T-1}} \\ \Sigma_{S_{t+T-1,t+T}} & \Sigma_{S_{t+T-1,t+T-1}} \end{pmatrix}\right). \tag{37}$$

Define $s_{t,T} \equiv \log(S(t+T))$, then $s_{t,T}|\mathcal{F}_{t+T-1} \sim N(\mu_{s_{t+T}}, \Sigma_{s_{t+T}})$, where

$$\begin{aligned}
\mu_{s_{t+T}} \equiv& \mu_{S_{t+T}} + \Sigma_{S_{t+T,t+T-1}}\Sigma^{-1}_{S_{t+T-1,t+T-1}} \cdot (s_{t,T-1} - \mu_{S_{t+T}}), \\
\Sigma_{s_{t+T}} \equiv& \Sigma_{S_{t+T,t+T}} - \Sigma_{S_{t+T,t+T-1}} \cdot \Sigma^{-1}_{S_{t+T-1,t+T-1}} \cdot \Sigma_{S_{t+T-1,t+T}}.
\end{aligned}$$

Using the distribution of $s_{t,T}$ and $\delta\widetilde{L}(t+T)$ given information at time $t+T$ i.e., given $\mathcal{F}_{t+T}$, we can derive the fourth approximation.

Fourth approximation. *The market value of life insurance liabilities using the CoC-approach.*
Given the approximation $\delta\widetilde{L}(t+T) \sim \log N\left(\mu_{\delta\widetilde{L}_{t+T}}, \Sigma_{\delta\widetilde{L}_{t+T}}\right)$ and assume that the market value of the liabilities given $\mathcal{F}_{t+T-1}$ is LogNormally distributed $\delta L(t+T) \sim \log N(\overline{\mu}_{t+T}, \overline{\Sigma}_{t+T})$, we approximately have:

$$\begin{aligned}
\delta L(t+T-1)|\mathcal{F}_{t+T-2} =& \exp(d_{t,T-1} + a_{t,T-1} + b_{t,T-1}s_{t,T-1}) \\
&\sim \log N\left(d_{t,T-1} + a_{t,T-1} + b_{t,T-1}\mu_{s_{t+T-1}}, b^T_{t,T-1}\Sigma_{s_{t+T-1}}b\prime_{t,T-1}\right)
\end{aligned}$$

where

$$\begin{aligned}
a_{t,T-1} =& a_{t,T} + b_{t,T} \cdot (\mu_{S_{t+T}} + \Sigma_{S_{t+T,t+T-1}} \cdot \Sigma^{-1}_{S_{t+T-1,t+T-1}} \cdot \mu_{S_{t+T-1}}) \\
b_{t,T-1} =& b_{t,T} \cdot (\Sigma_{S_{t+T,t+T-1}} \cdot \Sigma^{-1}_{S_{t+T,t+T-1}}) \\
d_{t,T-1} =& \log[(1 - CoC_{t+T}) \cdot P^{(t+T)}_{t+T-1} \cdot \exp(\Sigma_{FW}(t,T)/2) + CoC_{t+T} \cdot P^{(t+T)}_{t+T-1} \\
& \cdot \exp(\Phi^{-1}(\alpha)\sqrt{\Sigma_{FW}(t,T)})]
\end{aligned}$$

Derivation of fourth approximation:
Notice that the liabilities in year $t+T$ are given by the liabilities in year

$t+T+1$ and the payments between year $t+T$ and $t+T+1$. Let us denote $\delta\widehat{L}(t+T) \equiv \delta L(t+T) + \delta\widetilde{L}(t+T)$ for the value of the liabilities just before the payment of the life insurance contracts is made. Given that the market value of the liabilities in year $t+T+1$ is LogNormally distributed we have:

$$\begin{aligned}
\delta L(t+T-1|\mathcal{F}_{t+T-1}) =&(1-CoC_{t+T})\cdot\mathbb{E}[\widehat{L}(t+T)\cdot P^{(t+T)}_{t+T-1}|\mathcal{F}_{t+T-1}] \\
&+CoC_{t+T}\cdot\mathbb{Q}_{1-\alpha}[\widehat{L}(t+T)\cdot P^{(t+T)}_{t+T-1}|\mathcal{F}_{t+T-1}] \\
=&\exp(a_{t,T}+b_{t,T}\cdot(\mu_{S_{t+T}}+\Sigma_{S_{t+T,t+T-1}}\Sigma^{-1}_{S_{t+T-1,t+T-1}}(s_{t-1,T}-\mu_{S_{t+T-1}})) \\
&+\Sigma_{FW}(T,t)/2)\cdot(1-CoC)\cdot P^{(T+t)}_{T+t-1}+CoC_{T+t}\cdot P^{(T+t)}_{T+t-1} \qquad (38) \\
&\exp(a_{t,T}+b_{t,T}\cdot(\mu_{S_{t+T}}+\Sigma_{S_{t+T,t+T-1}}\Sigma^{-1}_{S_{t+T-1,t+T-1}}(s_{t,T-1}-\mu_{S_{t+T-1}})) \\
&+\Phi^{-1}(1-\alpha)\sqrt{\Sigma_{FW}(t,T)}) \\
=&\exp(d_{t,T-1}+a_{t,T-1}+b_{t,T-1}\cdot s_{t,T-1}) \qquad (39)
\end{aligned}$$

Using the Fenton and Wilkonson approximation for the sum of LogNormally distributed variables, we obtain:

$$\delta\widehat{L}(t+T)|\mathcal{F}_{t+T-1}\sim\log N(\widehat{\mu}_{t,T},\widehat{\Sigma}_{t,T}) \qquad (40)$$

Hence, approximately the value of the parameters are:

$$\widehat{\mu}_{t,T-1}=\log(\exp(\overline{\mu}_{t+T-1}+\overline{\Sigma}_{t+T-1}/2)+\exp(\mu_{\delta\widetilde{L}_{t+T-1}}+\Sigma_{\delta\widetilde{L}_{t+T-1}}/2))-\widehat{\Sigma}_{t+T-1}/2 \qquad (41)$$

$$\begin{aligned}
\widehat{\Sigma}_{t,T-1} =&\Sigma_{FW}(t-1,T) \qquad (42) \\
=&\frac{\exp\left(2\cdot\overline{\mu}_{t+T-1}+2\cdot\overline{\Sigma}_{t+T-1}+2\cdot\mu_{\delta\widetilde{L}_{t+T-1}}+2\cdot\Sigma_{\delta\widetilde{L}_{t+T-1}}\right)}{\left(\exp\left(2\cdot\overline{\mu}_{t+T-1}+\overline{\Sigma}_{t+T-1}/2\right)+\exp\left(2\cdot\mu_{\delta\widetilde{L}_{t+T-1}}+\Sigma_{\delta\widetilde{L}_{t+T-1}}/2\right)\right)^2} \\
&+\frac{2\cdot\exp\left(\overline{\mu}_{t+T-1}+\cdot\overline{\Sigma}_{t+T-1}/2+\mu_{\delta\widetilde{L}_{t+T-1}}+\Sigma_{\delta\widetilde{L}_{t+T-1}}/2+2\cdot\rho\cdot\sqrt{\overline{\Sigma}_{t+T-1}}\cdot\sqrt{\Sigma_{\delta\widetilde{L}_{t+T-1}}}\right)}{\left(\exp\left(2\cdot\overline{\mu}_{t+T-1}+\overline{\Sigma}_{t+T-1}/2\right)+\exp\left(2\cdot\mu_{\delta\widetilde{L}_{t+T-1}}+\Sigma_{\delta\widetilde{L}_{t+T-1}}/2\right)\right)^2}
\end{aligned}$$

Fifth approximation. *Taylor expansion*:
The Taylor expansion of a function is given by:

$$f(s^M_{t-1}-\mu^M_{t-1},s^F_{t-1}-\mu^F_{t-1})=\sum_{n=0}^{\infty}\left(\frac{1}{n!}\sum_{k=0}^{n}\frac{\partial^n f}{\partial(s^M_{t-1})^{n-k}\partial(s^F_{t-1})^k}|_{\mu^M_{t-1},\mu^F_{t-1}}\cdot(s^M_{t-1}-\mu^M_{t-1})^{n-k}\cdot(s^F_{t-1}-\right.$$

Hence, using the first two terms of the Taylor expansion we can rewrite equation (41) to:

$$\widehat{\mu}_{t,T}=a_{t,T}+b_{t,T}\cdot s_{t,T},$$

where

$$\begin{aligned}
a_{t,T} =& f(\mu_{S_{t+T-1}}) - f'(\mu_{S_{t+T-1}}) \cdot \mu_{S_{t+T-1}},\\
b_{t,T} =& f'(\mu_{S_{t+T-1}}),
\end{aligned}$$

and using equation (42) and the first term of the Taylor expansion we obtain $\Sigma_{FW}(t,T)$.

Notice that the market value of the liabilities after the last payment is made is equal to zero. Hence, let $t+\tau$ be the time the last payment is made, then the market value of the liabilities at time $t+\tau$ is given by:

$$\delta L(t+\tau) = 0$$

hence, we have that

$$\begin{aligned}
\delta L(t+\tau)|\mathcal{F}_{t+\tau-1} =& 0\\
\delta L(t+\tau-1)|\mathcal{F}_{t+\tau-2} =& (1 - CoC_{t+\tau-1}) \cdot \exp(\mu_{\delta\widetilde{L}_{t+\tau}} + \Sigma_{\delta\widetilde{L}_{t+\tau}}/2)\\
& + CoC_{t+\tau-1} \cdot \exp(\mu_{\delta\widetilde{L}_{t+\tau}} + \Phi^{-1}(1-\alpha) \cdot \sqrt{\Sigma_{\delta\widetilde{L}_{t+\tau}}})
\end{aligned}$$

which is indeed LogNormally distributed when $\overline{\mu}_{t+\tau-1} = \mu_{\delta\widetilde{L}_{t+\tau}}$ given the information $\mathcal{F}_{t+\tau-2}$ is normally distributed and $\overline{\Sigma}_{t+\tau-1} = \Sigma_{\delta\widetilde{L}_{t+\tau}}$ given the information $\mathcal{F}_{t+\tau-2}$ is constant. Hence the assumption made in the fourth approximation holds for $t = t+\tau, t+\tau-1$. ■

In order to quantify the effect of the approximations to calculate the MVL ideally one would compare the MVL using simulations and using the approximations. However, it requires too many simulations to calculate the MVL for life insurance products. Therefore, Table 7 displays the difference in the discounted sum of the SCR multiplied with the cost of capital rate, i.e., the MVM, in the best estimate scenario, using the approximation and using simulations. The difference in between the approximations and the simulations in year $t+\tau$ is obtained by the following steps:

i) Given the best estimate of the survival probabilities until time $t+\tau$, using equation (37) we simulate the distribution of the survival probabilities at time $t+\tau+1$ and the corresponding payments at time $t+\tau+1$.

ii) Using the simulated distribution of the survival probabilities, the market value of the liabilities at time $t+\tau+1$ is obtained using the approximations, i.e. using equation (39).

iii) The current MVL in the simulations is obtained by the mean and the 99.5% percentile of the distribution of the payments (obtained in i)) plus the market value of the labilities in the succeeding year (obtained in ii)), discounted.

iv) The difference between the simulation and the approximations in year $t + \tau$ is obtained by the MVL in the best estimate scenario in year $t + \tau$ using the approximations, as given in equation (39), minus the MVL using the simulations as obtained in iii).

Table 7 displays the aggregated sum of the differences, i.e. repeating steps i)–iv) for $\tau = 0, \cdots, T$ and taking the discounted sum for the four different portfolios as described in Section 5.

Table 7
Table with approximation error.

Difference	$\delta(1)$	$\delta(2)$	$\delta(3)$	$\delta(4)$
	US			
$CoC = 0.06$	1.25%	0.51%	0.81%	0.86%
$CoC = 0.04$	1.31%	0.60%	0.68%	0.80%
	UK			
$CoC = 0.06$	1.25%	0.51%	0.81%	0.86%
$CoC = 0.04$	1.31%	0.60%	0.68%	0.80%
	NL			
$CoC = 0.06$	1.25%	0.51%	0.81%	0.86%
$CoC = 0.04$	1.31%	0.60%	0.68%	0.80%

This table displays the aggregated discounted difference between simulations and model relative to MVM_t.

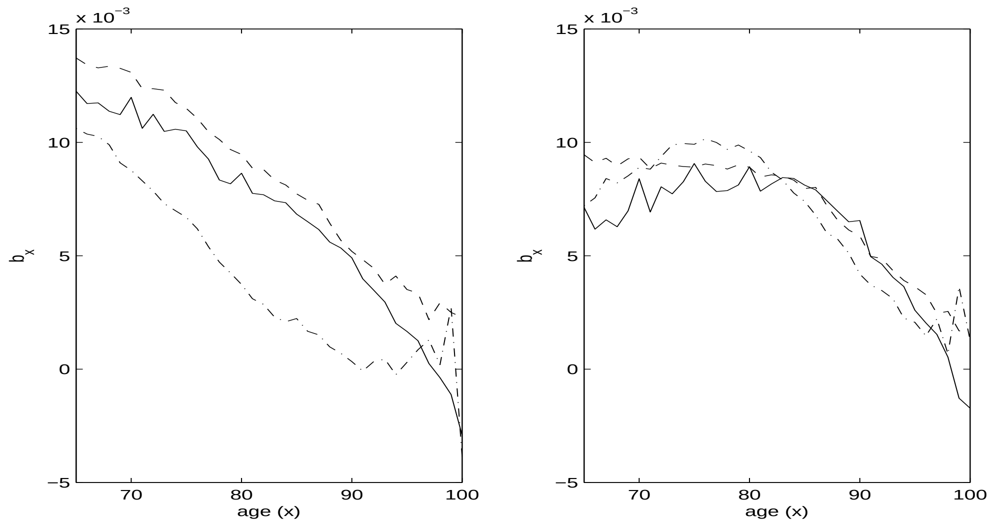
7.3. Tables and figures

Table 8
Table with capital requirements using the CoC-approach.

	Internal model			A_2	A_3
	$\frac{A_t^\star \text{ - BEL}_t}{\text{BEL}_t}$	$\frac{\text{MVM}_t}{\text{BEL}_t}$	$\frac{\text{SCR}_t}{\text{BEL}_t}$	$\frac{\text{MVM}_t}{\text{BEL}_t}$	$\frac{\text{MVM}_t}{\text{BEL}_t}$
	US				
$\delta(1)$	2.47%	0.87%	1.61%	0.81%	0.84%
$\delta(2)$	1.17%	0.67%	0.51%	0.63%	0.29%
$\delta(3)$	1.63%	0.68%	0.96%	0.65%	0.52%
$\delta(4)$	1.35%	0.61%	0.74%	0.58%	0.35%
	UK				
$\delta(1)$	4.25%	1.36%	2.91%	1.26%	1.48%
$\delta(2)$	2.73%	1.25%	1.49%	1.18%	0.82%
$\delta(3)$	3.34%	1.25%	2.11%	1.17%	1.12%
$\delta(4)$	2.82%	1.11%	1.72%	1.06%	0.98%
	NL				
$\delta(1)$	1.86%	0.86%	1.03%	0.80%	0.51%
$\delta(2)$	4.18%	1.59%	2.53%	1.52%	1.45%
$\delta(3)$	2.47%	0.93%	1.55%	0.90%	0.81%
$\delta(4)$	2.41%	0.93%	1.49%	0.91%	0.84%

This table displays the capital reserve in the internal model, as percentage of the best estimate of the liabilities, for four different portfolios of life insurance products for the US, UK, and the Netherlands. The market value of the liabilities is set according the internal model with a Cost of Capital rate of 4%. The last two columns display the MVM_t using the simplifications. The column A_2 refers to the second simplification as given in equation (15), and the column A_3 refers to the third simplification as given in equation (16).

Figure 4. Parameter b_x



The figure displays the estimate of b_x in the US (solid curve), UK (dashed curve), and NL (dashed-dotted curve). The left panel is for males, the right panel is for females. The parameter is estimated using the mortality probabilities from 1970 till 2006.

Table 9
Parameter estimates of the Lee-Carter model

parameter	US M	US F	UK M	UK F	NL M	NL F
c^g	-1.5875	-1.3652	-1.6429	-1.5975	-1.8741	-1.5525
σ_c^g	0.2097	0.2320	0.2357	0.3568	0.3753	0.5344
σ_e^g	1.2583	1.3920	1.4141	2.1406	2.2520	3.2063
ρ_c		0.7991		0.7661		0.3448

The table displays the estimate of the Lee-Carter model in the US, UK, and the Netherlands. The parameters are estimated using the mortality probabilities from 1970 till 2006.