

# OPTIMAL STRATEGIES FOR HEDGING PORTFOLIOS OF UNIT-LINKED LIFE INSURANCE CONTRACTS WITH MINIMUM DEATH GUARANTEE

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## Abstract

In this paper, we are interested in hedging strategies which allow the insurer to reduce the risk to their portfolio of unit-linked life insurance contracts with minimum death guarantee. Hedging strategies are developed in Black and Scholes' model and in Merton's jump-diffusion model. According to the new frameworks (IFRS, Solvency II and MCEV), risk premium is integrated into our valuations. We will study the optimality of hedging strategies by comparing risk indicators (Expected loss, volatility, VaR and CTE) in relation to transaction costs and costs generated by the re-hedging error. We will analyze the robustness of hedging strategies by stress-testing the effect of a sharp rise in future mortality rates and a severe depreciation in the price of the underlying asset.

**KEYWORDS:** Unit-linked, Death guarantee, Hedging strategies, Transaction and error of re-hedging costs, risk indicators, stress-testing.

## Résumé

Dans ce papier, nous nous intéressons à la couverture des contrats en unités de compte avec garanties décès. Nous présentons des stratégies de couverture opérationnelles permettant de réduire de façon significative les coûts futurs liés à ce type de contrats. Suivant les recommandations des nouveaux référentiels (IFRS, Solvabilité 2 et MCEV), la prime de risque est introduite dans les évaluations. L'optimalité des stratégies est constatée au moyen de la comparaison des indicateurs de risque (Pertes espérée, écart type, VaR, CTE et perte Maximale) des stratégies dans le modèle standard de Black-Scholes et dans le modèle à sauts de Merton. Nous analysons la robustesse des stratégies à une hausse brutale de la mortalité future et à une forte dépréciation du prix de l'actif sous-jacent.

**MOTS-CLEFS :** Unités de comptes, Garanties décès, Stratégies de couverture, Coûts de transaction et erreur de couverture, indicateurs de risque, stress-testing.

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## 1. Introduction

The new frameworks (Accountant: IFRS/IAS, Prudential: Solvency II, and financial communication: Market Consistent Embedded Value) encourage insurance companies to adopt an economic approach when evaluating their liabilities (Thérond (2007)). On this subject, the concept of “Fair Value” is fundamental. Fair Value of an asset or a liability is the amount for which two interested and informed parties would exchange this asset or this liability. Fair values are usually taken to mean arbitrage-free values, or values consistent with pricing in efficient markets. The arbitrage-free valuation of an item is one which makes it impossible to guarantee riskless profits by buying or selling the item. This leads to the concept that if two portfolios have identical cash flows, and the portfolios can be priced in an efficient market, then the two portfolios will have the same price. Otherwise, an investor could sell one portfolio, buy the other and make free money. The Fair value is therefore the price that the market naturally assigns to any tradable asset.

Risk neutral valuation produces the fair value of any liability. As noted by Milliman Consultants and actuaries (2005) the main reason for using risk neutral or fair valuations is because they represent the objective market cost of purchasing a replicating portfolio in terms of the liability, thus ensuring that the company will have sufficient resources to meet the liability over all possible market movements. Risk-neutral valuation effectively translates the risky, market-dependent costs of the guarantee into a fixed cost item for the insurance company.

Thus, using the logic of fair valuation, purchasing a replicating portfolio is essential in the evaluation of liabilities. Accordingly, in the case of unit-linked life insurance for example, Frantz et al. (2003) showed that fair valuation is only valid if the underlying hedging is actually applied<sup>1</sup>. In such contracts, the return obtained by the policyholders on their savings is linked to some financial asset, and in this way it is the policyholder who supports the risk of the investment. The investment can be made on one asset or on a portfolio of assets, and various types of guarantees can be added to the pure unit-linked contract. In our study, we shall concentrate on the minimum death benefit guarantee. In this case, the insurer’s liability in the case of the death of the policyholder will be:  $\max(K, V) = V + [K - V]^+$ , where  $V$  is the value of the unit-linked contract and  $K$  is the guarantee. If  $V < K$  then the insurer will pay the additional amount  $K - V$ . It therefore stands that the risk related to these contracts is real. However, this risk is often underestimated by the insurance companies, which then expose themselves to massive losses connected to a market in strong decline.

Frantz et al. (2003) analyzed delta hedging within the framework of Black & Scholes’ model (1973). Black and Scholes’ model supposes that the returns process is continuous, distributed according to a normal distribution, and that its volatility is constant during time. However, the empirical reports show that none of these assumptions are always true when applied to the markets, as shown by the works of Cont (2001). Moreover, the classic valuation of unit-linked contracts supposes a perfect mutualisation of the deaths in the insurance portfolio. It therefore follows logically that we can wonder about the effectiveness of the insurer setting up a hedging strategy in order to protect from abnormally high death rates in the portfolio in the future.

Moreover, other hedging strategies exist. Hedging strategies which we can develop come primarily from the methods used for hedging derivatives. In practice, hedging a portfolio of

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<sup>1</sup> This is true even if the application of the strategy is not always desirable or even feasible in practice.

derivatives is typically done by matching different sensitivities between the given portfolio and the hedging portfolio. As an alternative, a hedging portfolio can be chosen to minimize a measure of the hedge risk for a given time horizon.

The object of this paper is to analyze the optimality of some hedging strategies being offered to the insurer to cover the risks related to unit-linked life insurance contracts with minimum death benefit guarantee. These contracts are subjected to two types of risk: financial risk and mortality risk.

The financial risk is represented by the possibility of a poor evolution in the underlying asset, whereas the mortality risk results from the possibility of a strong fluctuation in the sample. In this last case, if the future mortality of the insured parties in the portfolio is stronger than foreseen, this may be due to the non-validity of the assumption of mutualisation of the deaths retained during the evaluation of the contract.

## 2. Risk-neutral valuation

Except for the reasons already noted in the introduction, another reason for using risk neutral valuation comes from the microeconomic theory of the uncertain. Indeed, let us remind ourselves of two of the founder assumptions:

- Individuals strictly prefer more income to less income; (or the equivalent less loss rather than more loss);
- Individuals are risk adverse.

The consequences of these assumptions on the agent's choices are strong. Indeed, a risk adverse individual prefers to have the expectation of the random variable with probability 1 rather than having a random variable where the probability is unknown. This means that between two games with identical expectations of earnings, the agent will choose the least risky. However, they will be inclined to change their choice if an additional amount is proposed to them. This amount is the risk premium.

Fair value must integrate this risk premium as this is what reflects the risk adverse character of investors on the markets. The incorporation of this risk premium allows the passage from the initial environment to a risk neutral environment. The valuation of financial assets is generally made in this risk neutral framework. The passage to this universe is made through the formulae of a change in probability, as justified by the Theorem of Radon-Nikodym.

The theory of deflators is an alternative to the risk neutral valuation. Generally used in Assets-Liabilities Management (ALM), the deflators are stochastic factors of actualization which make it possible to predict the future flows of the liability. They allow us to obtain a "Market Consistent" valuation of projected flows i.e. to find the initial value of risky assets.

Essentially, the use of deflators and the risk neutral valuation are equivalent. Indeed, the deflator is nothing other than the density of the risk neutral measure according to the historic measure. The existence of this density results from the Radon-Nikodym Theorem.

The neutral density risk (or the deflator) depends on the nature of the studied risk. Within the framework of our study we shall be confronted with two risks as mentioned above: mortality risk and financial risk.

To begin with, we shall accept the collectively accepted assumption about the risk of mortality, namely: "the perfect mutualisation of the deaths"<sup>2</sup>. Accordingly, the mortality risk

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<sup>2</sup> We will reconsider this assumption in a later study.

“disappears”, in the sense that we can foresee with certainty the future number of deaths. Having said this, a mortality risk premium can be introduced by modeling mortality prudentially. In this case, the question of the level of prudence to adopt is open.

For the financial risk involved in managing the contracts, we will restrict ourselves to the Markovian models and apply them to an efficient environment. Thus, we make the assumption of an absence of arbitrage opportunity. Within this framework, one of the standard results of the financial theory is that all the assets are martingales under the risk neutral probability.

This specific character of assets under the risk neutral probability, besides simplifying the calculations, allows us to resolve the problem of actualizing generated future flows. To have Fair value it will be enough to generate the asset under the risk neutral probability and to actualize using the free-risk rate.

### 3. Insurance portfolio

We suppose that a portfolio is constituted by  $N$  policyholders who invest in a single financial asset. Policyholder  $i$  aged  $x_i$  invests in a single risky asset  $(S_t)_{t \geq 0}$ . The insurer gives a guarantee of  $K_i$  in case the insured  $i$  dies before retirement. In the case that policyholder  $i$  dies at time  $T$  the insurer will pay  $S_T + [K_i - S_T]^+$  to the beneficiaries of the insured  $i$ . Note that  $[K_i - S_T]^+$  is the payoff of a European put option with maturity  $T$  and strike price  $K_i$  on the underlying asset  $(S_t)_{0 \leq t \leq T}$ .

The engagement of the insurer with respect to the policyholder  $i$  is written:  $e^{-rT_{x_i}} [K_i - S_{T_{x_i}}]^+ 1_{T_{x_i} \leq \tau_i}$ . Where  $T_{x_i}$  is the time to death of the policyholder,  $\tau_i$  is the maturity of the contract and  $r$  is the risk-free interest rate<sup>3</sup>.

We note as  $P$  the physical probability measure and  $Q$  the risk-neutral measure. We assume that physical and risk-neutral measures are independent. We also assume that in the case of death within a certain period, the insurer makes the payments at the end of the corresponding period<sup>4</sup>.

Now, we can write the expression of single pure premium  $\Pi_i$  related to the contract as:

$$\Pi_i = E_{P \times Q} \left[ e^{-rT_{x_i}} [K_i - S_{T_{x_i}}]^+ 1_{T_{x_i} \leq \tau_i} \right].$$

Using the assumption of independence between physical measures and risk-neutral measures, and using the properties of conditional expectation<sup>5</sup>, we can write:

$$\Pi_i = \sum_{t \leq \tau_i} \Pr(x_i, t) \times E_Q \left[ e^{-rt} [K_i - S_t]^+ \right],$$

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<sup>3</sup> We suppose this to be constant.

<sup>4</sup> For example, for annual payments, the insurer pays the value of the Unit-linked contract with a guarantee on December 31 of each year.

<sup>5</sup> We are conditioning by the time to death.

with  $\Pr(x_i, t)$  as the probability, under the physical measure, that an individual aged  $x_i$  today dies exactly  $t$  years later.

The single pure premium  $\Pi$  for the portfolio is:  $\Pi = \sum_{i=1}^N \Pi_i$ .

$$\Pi = \sum_{i=1}^N \sum_{t \leq \tau_i} \Pr(x_i, t) \times E_Q \left[ e^{-rt} [K_i - S_t]^+ \right] \Leftrightarrow \Pi = \sum_{i=1}^N \sum_{t \leq \tau_i} \Pr(x_i, t) \times P_0(t, S_0, K_i), \quad (0.1)$$

where  $P_0(t, S_0, K_i)$  is the price at time 0 of a put on the underlying asset of strike price  $K_i$  (guarantee of policyholder  $i$ ) of maturity  $t$ .

#### 4. Hedging Strategies

In the following section, we investigate different kinds of hedging strategies allowing us to optimally hedge the risk related to our insurance portfolio. Optimal strategy consists of the insurer buying European put options on the market. In this case, the value of the hedging portfolio is equal, at all times, to the value of the insurance portfolio. Hedging is perfect.

The cost of this strategy is:

$$L_{Per} = \sum_{i=1}^N \sum_{t \leq \tau_i} \Pr(x_i, t) \times P_0(t, S_0, K_i) = \Pi.$$

Let us point out that  $\Pr(x_i, t)$  represents the probability, under the physical measure, that an individual aged  $x_i$  today dies exactly  $t$  years later. It is also the optimal quantity of the maturity option  $t$  to be held in the hedging portfolio. This optimal quantity is relevant only under the assumption of a perfect mutualisation of the deaths of the policyholders and with good anticipation of future mortality within the insurance portfolio.

If an insurer makes a good forecast of future mortality and the market has effectively all these options, this strategy allows the insurer to optimally hedge the risk to their portfolio. But this is not actually possible since the corresponding put options are hardly ever found, mainly due to the very long maturities involved. Moreover, the insurer is subjected to the risk relating to a bad estimation of future mortality. It's necessary to think about other strategies which take account of these aspects.

We are going to adapt the traditional strategies of hedging derivatives (matching sensitivities and risk minimisation) to hedge our portfolio. We will build hedging strategies using options and extend them to the insurance portfolio assuming a perfect mutualisation of the deaths.

We will compare the results with Carr & Wu's semi-static hedging strategy (2004). This strategy supposes that it is possible to hedge a long term option using a portfolio of options with short maturity.

We study the relevance of hedging strategies by analyzing characteristics of discounted future costs  $L$ . The investors support costs during each of their deals. These costs reduce the profitability of their operations by a considerable amount. The consideration of these costs is essential in the evaluation of the performance of the stock-exchange portfolios.

On the share market, the transaction costs are generally divided into two components: the implicit component and the explicit component. The explicit costs correspond to expenses, commissions and taxes supported during the passage of an order, while the implicit costs go back to the price ranges or to the impact of deals on the prices for large-sized transactions. The total cost of transaction appears as a sum of heterogeneous components which it is difficult to estimate. Deville (2001) showed that the costs vary according to the level of capitalization of the underlying asset. He also showed that, except during certain exceptional years, the total cost of transaction on the Paris Stock Exchange varies between 0.02 % and 1.20 % of exchanged value. In our study, we assume that transaction costs represent a proportion  $c$  of the exchanged value.

Let  $P_T^H$  be the terminal value of the hedging portfolio and  $P_0^H$  the initial value of the hedging portfolio. Let  $F$  be the total amount of friction i.e. the sum of the transaction costs and the costs of re-hedging errors. The expectation of discounted future costs is the expectation of future payments<sup>6</sup>  $\Pi^*$  minus the net value of the hedging portfolio ( $P_T^H - P_0^H$ ) corrected by the transaction costs and the cost of re-hedging errors  $F$  :

$$L = \Pi^* - (P_T^H - P_0^H - F). \quad (0.2)$$

#### 4.1. Delta hedging

We want to match the sensitivity of the underlying asset between the insurance portfolio and the hedging portfolio. This means that during an infinitely small time, a hedging portfolio constituted by matching sensitivities is risk free. Our hedging portfolio will be made up of the underlying asset and the risk-free asset. This approach is linear; it is easy to extend the hedging portfolio of an option to an insurance portfolio.

##### 4.1.1. Frequently rebalancing

Firstly, we are going to consider a strategy consisting of modifying the hedging portfolio in a periodic way (of period  $h$ <sup>7</sup>). This technique requires frequent buying/selling of the assets in a portfolio, and hence may incur significant transaction costs.

###### 4.1.1.1 Option hedging

Now consider the put option  $P(n, S_0, K_i)$  with maturity  $n$  and strike price  $K_i$ . Assuming that the value of the hedging portfolio at period  $t \leq n$  is written  $\alpha_t^{i,n} S_t + \beta_t^{i,n}$ .

Where:  $\alpha_t^{i,n}$  is the quantity of the underlying asset in the portfolio and  $\beta_t^{i,n}$  is the quantity of the risk-free asset in the portfolio, we can write the expression of the error related to the hedging:

$$W_t(i, n) = -P(n-t, S_t, K_i) + \alpha_t^{i,n} S_t + \beta_t^{i,n}.$$

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<sup>6</sup>  $\Pi$  is the expectation of  $\Pi^*$  under risk-neutral measure,  $\Pi = E_Q(\Pi^*)$

<sup>7</sup> Year, 6 months, 3 months, 1 month, week, day, hour...

We want to immunize the error against fluctuations in the underlying asset. The composition of the portfolio must be such that:

$$\frac{\partial W_t}{\partial S_t} = 0 \text{ And } W_t = 0 \Leftrightarrow \begin{cases} \alpha_t^{i,n} = \frac{\partial P(n-t, S_t, K_i)}{\partial S_t} \\ \beta_t^{i,n} = P(n-t, S_t, K_i) - \frac{\partial P(n-t, S_t, K_i)}{\partial S_t} S_t \end{cases} \quad (0.3)$$

This hedging strategy could result in high costs: costs associated with the transactions and the errors of hedging. We can write the values of the transaction costs and hedging errors as shown below.

#### 4.1.1.2 Errors of hedging

The error of hedging,  $W_k$  is the difference between the new portfolio made up at time  $k$  and the value of the portfolio made up in the previous period. This difference represents the amount exchanged at time  $k$ . It is also the cost of recombining the hedging portfolio.

$$\begin{aligned} W_k(i, n) &= P(n-k, S_k, K_i) - \alpha_{k-1}^{i,n} S_k - \beta_{k-1}^{i,n} e^{rh} \\ \Leftrightarrow W_k(i, n) &= (\alpha_k^{i,n} - \alpha_{k-1}^{i,n}) S_k + (\beta_k^{i,n} - \beta_{k-1}^{i,n} e^{rh}) \end{aligned}$$

#### 4.1.1.3 Transaction costs

We add the cost of transaction  $C_k$  to the total costs of the hedging strategy at time  $k$ , which constitutes proportion  $c$  of the exchanged value:

$$C_k(i, n) = c \left| \alpha_k^{i,n} - \alpha_{k-1}^{i,n} \right| S_k + c \left| \beta_k^{i,n} - \beta_{k-1}^{i,n} e^{rh} \right|.$$

#### 4.1.1.4 Total frictions

Frictions are the total transaction costs and costs of the error of re-hedging associated with option hedging portfolios. We notice  $W_{DY,1}$ <sup>8</sup> frictions for dynamic delta hedging with frequent rebalancing:

$$W_{DY,1}(i, n) = c \left( \alpha_0^{i,n} S_0 + \beta_0^{i,n} \right) + \sum_{k=1}^{\frac{n}{h}-1} (W_k(i, n) + C_k(i, n)) e^{-k \times h \times r}. \quad (0.4)$$

We can estimate the discounted future costs of this strategy for put  $P(n, S_0, K_i)$ . We make  $N_s$  simulations of trajectories of the underlying asset with maturity  $n$ . For simulation  $j$  we find friction  $W_{DY,1}^j$  using(0.4). Discounted future costs  $L_{DY,1}(i, n)$ , of dynamic delta hedging with frequent rebalancing of put  $P(n, S_0, K_i)$  are therefore written:

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<sup>8</sup> *DY* indicates dynamic delta hedging and 1 frequent rebalancing, 2 rebalancing according to the interval of error.

$$L_{DY,1}(i,n) = E \left( \left( \left[ K_i - S_n \right]^+ - (1-c) \left( \alpha_{n/h-1}^{i,n} S_n + \beta_{n/h-1}^{i,n} e^{hxr} \right) \right) e^{-rn} + W_{DY,1}(i,n) + (1+c) \left( \alpha_0^{i,n} S_0 + \beta_0^{i,n} \right) \right)$$

#### 4.1.1.5 Insurance portfolio hedging

Let us remind ourselves that the insurer makes payments periodically, according to the period  $h$ . Thus, his portfolio consists of  $N_d = \frac{T}{h}$  options of maturity  $1 \times h, \dots, i \times h, \dots, N_d \times h$ .

We can easily extend the preceding results to the whole of the insurance portfolio.

At time  $n$ , for the simulation  $j$  and the insurer  $i$ , the amount to be paid in time  $n$  is:

$$M_j^{i,n} = \left[ K_i - S_n^j \right]^+ 1_{T_{x_i}^j = n}.$$

To extend this result to the whole portfolio, we are going to assume, as we will subsequently, a perfect mutualisation of the deaths between the policy-holders. So, we can write an estimation of the discounted future costs of insurance portfolio  $L_{DY,1}$ :

$$\Rightarrow L_{DY,1} = \Pi_{Dyn,1} - \left( P_{Dyn,1}^H - P_{Dyn,1}^{H,0} - F_{Dyn,1} \right) \quad (0.5)$$

With:

$$\left\{ \begin{array}{l} \Pi_{Dyn,1} = \frac{1}{N_s} \sum_{i=1}^N \sum_{n=1}^{\tau_i} \sum_{j=1}^{N_s} M_j^{i,n} e^{-rn} \\ P_{Dyn,1}^H = \frac{1}{N_s} \sum_{i=1}^N \sum_{n=1}^{\tau_i} \sum_{j=1}^{N_s} (1-c) \Pr(x_i, n) \left( \alpha_{j,n/h-1}^{i,n} S_n^j + \beta_{j,n/h-1}^{i,n} e^{hxr} \right) e^{-rn} \\ P_{Dyn,1}^{H,0} = \frac{(1+c)}{N_s} \sum_{i=1}^N \sum_{n=1}^{\tau_i} \Pr(x_i, n) \left[ \alpha_0^{i,n} S_0 + \beta_0^{i,n} \right] \\ F_{Dyn,1} = \frac{1}{N_s} \sum_{i=1}^N \sum_{n=1}^{\tau_i} \sum_{j=1}^{N_s} \Pr(x_i, n) W_{DY,1}^j(i, n) \end{array} \right.$$

where:

- $N_s$  is the number of simulated trajectories of the underlying asset,
- $\Pi_{Dyn,1}$  is the expectation of future payments provided by dynamic delta hedging,
- $P_{Dyn,1}^H$  is the “final” value of the hedging portfolio provided by dynamic delta hedging,
- $P_{Dyn,1}^{H,0}$  is the initial value of the delta hedged portfolio,
- $F_{Dyn,1}$  is the transaction costs and the costs of re-hedging errors incurred by delta hedging.



#### 4.1.2. Rebalancing according to the hedging error

An alternative to the first strategy consists of rebalancing the hedging portfolio at the times  $k$  if the hedging error is higher than a given threshold.

##### 4.1.2.1 Hedging options

The idea is to rebuild the hedging portfolio when the hedging error in the period  $k$  goes out of the interval  $[a, b]$ . Originally, we built the hedging portfolio to match the sensitivities of the options and of the hedging portfolio at time  $k = 0$ :

$$\begin{cases} \alpha_0^{i,n} = \frac{\partial P(n, S_0, K_i)}{\partial S} \\ \beta_0^{i,n} = P(n, S_0, K_i) - \frac{\partial P(n, S_0, K_i)}{\partial S} S_0 \end{cases}$$

At time  $k$ , we make an estimation of the cost of re-hedging the portfolio

$$W_k^2(i, n) = P(n-k, S_k, K_i) - \alpha_0^{i,n} S_k - \beta_0^{i,n} e^{rhk}.$$

If  $W_k^2(i, n) \notin [a, b]$ , then we modify our hedging portfolio. Concretely, we build the meter  $(r_l)_{1 \leq l \leq l_{\max}}$  which identifies the moment to rebalance of our portfolio.

Then, if  $W_k^2(i, n) \notin [a, b]$ ,  $r_1 = k$  and

$$\begin{cases} \alpha_{r_1}^{i,n} = \frac{\partial P(n-r_1, S_{r_1}, K_i, r)}{\partial S} \\ \beta_{r_1}^{i,n} = P(n-r_1, S_{r_1}, K_i, r) - \frac{\partial P(n-r_1, S_{r_1}, K_i, r)}{\partial S} S_{r_1} \end{cases}.$$

In the same way, for  $k > r_1$ , we calculate

$$W_k^2(i, n) = P(n-k, S_k, K_i) - \alpha_{r_1}^{i,n} S_k - \beta_{r_1}^{i,n} e^{rh(k-r_1)},$$

$$\text{if } W_k^2(i, n, j) \notin [a, b], \text{ then } r_2 = k, \text{ and } \begin{cases} \alpha_{r_2}^{i,n} = \frac{\partial P(n-r_2, S_{r_2}, K_i)}{\partial S} \\ \beta_{r_2}^{i,n} = P(n-r_2, S_{r_2}, K_i) - \frac{\partial P(n-r_2, S_{r_2}, K_i)}{\partial S} S_{r_2} \end{cases}$$

We continue until the payment of the pay-off.

By analogy with the first strategy, we can write the total costs associated with the frictions, necessary to hedge put  $P(n, S_0, K_i)$  as:

$$W_{DY,2}(i, n) = c \left( \alpha_0^{i,n} S_0 + \beta_0^{i,n} \right) + \sum_{l=1}^{l_{\max}} \left( W_{r_l}^2(i, n) + C_{r_l}^2(i, n) \right) e^{-r \times h \times r_l},$$

where:

$$C_{\eta}^2(i, n) = c \left| \alpha_{\eta}^{i, n} - \alpha_{\eta-1}^{i, n} \right| S_{\eta} + c \left| \beta_{\eta}^{i, n} - \beta_{\eta-1}^{i, n} \right|,$$

and

$$W_{\eta}^2(i, n) = \left( \alpha_{\eta}^{i, n} - \alpha_{\eta-1}^{i, n} \right) S_{\eta} + \left( \beta_{\eta}^{i, n} - \beta_{\eta-1}^{i, n} \right).$$

Discounted future costs of this strategy for put  $P(n, S_0, K_i)$  are written:

$$L_{DY,1}(i, n) = E \left\{ \left( \left[ K_i - S_n \right]^+ - (1-c) \left( \alpha_{\eta_{\max}}^{i, n} S_n + \beta_{\eta_{\max}}^{i, n} e^{(\eta_{\max} - \eta_{\max-1})h \times r} \right) \right) e^{-rn} \right. \\ \left. + W_{DY,2}(i, n) + \alpha_0^{i, n} S_0 + \beta_0^{i, n} \right\}$$

In this study, we choose the threshold as a percentage of the maximum pay-off. For example, for one put option with a strike price of 100, the maximum pay-off is 100. This situation would result when the underlying asset was worth 0. A threshold with 1% of maximum pay-off would be worth 1, and the re-hedging interval would be  $[-1, 1]$ .

Another possibility consists of choosing the threshold according to the total frictions. The threshold can be variable, related to the price of the underlying asset.

#### 4.1.2.2 Insurance portfolio hedging

Using the analogy of frequent rebalancing, an estimation of discounted future costs  $L_{DY2}$  for the insurance portfolio is:

$$L_{DY,2} = \Pi_{Dyn,2} - \left( P_{Dyn,2}^H - P_{Dyn,2}^{H,0} - F_{Dyn,2} \right), \quad (0.6)$$

where:

$$\left\{ \begin{array}{l} \Pi_{Dyn,2} = \frac{1}{N_s} \sum_{i=1}^N \sum_{n=1}^{\tau_i} \sum_{j=1}^{N_s} M_j^{i,n} e^{-rn} \\ P_{Dyn,2}^H = \frac{1}{N_s} \sum_{i=1}^N \sum_{n=1}^{\tau_i} \sum_{j=1}^{N_s} \left[ (1-c) \Pr(x_i, n) \left( \alpha_{j, r_{k \max}}^{i, n} S_n^j + \beta_{j, r_{k \max}}^{i, n} e^{(r_{k \max} - r_{k \max-1})h \times r} \right) \right] e^{-rn} \\ P_{Dyn,2}^{H,0} = \frac{(1+c)}{N_s} \sum_{i=1}^N \sum_{n=1}^{\tau_i} \Pr(x_i, n) \left[ \alpha_0^{i, n} S_0 + \beta_0^{i, n} \right] \\ F_{Dyn,2} = \frac{1}{N_s} \sum_{i=1}^N \sum_{n=1}^{\tau_i} \sum_{j=1}^{N_s} \Pr(x_i, n) W_{DY,2}^j(i, n) \end{array} \right.$$

## 4.2. Risk minimisation strategies

The risk-minimizing hedging strategy consists of optimizing hedging portfolios by checking the residual error. The main idea here is to build a portfolio in which the risk

between portfolio hedging and the engagements of the insurer are minimal. In this approach, we also use the underlying asset and risk free asset to form a hedged portfolio.

In a risk-neutral environment, this means minimizing the economic value of the residual risk of cover. The economic indicators retained are numerous, the most common being utility and variance. The inconvenience of this strategy is that it corresponds to a non-linear rule of pricing and hedging. We cannot easily extend it to our insurance portfolio because it is not adapted. The second strategy is called ‘quadratic minimisation’, despite the fact that it penalizes the profits and the losses in the same way, it gives a linear ratio of cover. It can therefore be easily extended to the insurance portfolio.

#### 4.2.1. Static hedging

Firstly, we will find our static portfolio. This consists of building a hedging portfolio at the beginning of a period and not modifying it until it reaches maturity.

##### 4.2.1.1 Option hedging

We assume that the hedging portfolio of put  $P(n, S_0, K_i)$  is constituted<sup>9</sup> by  $a$  ( $a \in \mathbb{N}^*$ ) assets, and that its value at  $n$  is:  $V_n(\beta^{i,n}, a) = \sum_{l=1}^a \beta_0^{i,n,l} A_n^l$ , with  $\beta_0^{i,n,l}$  as the quantity of assets  $A_n^l$  in the portfolio. The error of this strategy is written:  $W_n(i, n) = -[K_i - S_n]^+ + V_n(\beta^{i,n}, a)$

If we only consider the risk-free asset, the underlying asset and the put options in this hedging portfolio, then we can write:

$$W_n(i, n) = -[K_i - S_n]^+ + \beta^{i,n,0} + \beta^{i,n,1} \times S_n + \sum_{l=2}^a \beta_0^{i,n,l} \times [K^l - S_n]^+ \quad (0.7)$$

$$W_n(i, n) = -[K_i - S_n]^+ + {}^t(\beta^{i,n})P_n$$

Where  $\beta^{i,n} = \begin{bmatrix} \beta^{i,n,0} \\ \beta^{i,n,1} \\ \beta^{i,n,2} \\ \cdot \\ \cdot \\ \beta^{i,n,a} \end{bmatrix}$  and value at maturity  $n$  of our hedging portfolio  $P_n = \begin{bmatrix} 1 \\ S_n \\ [K^2 - S_n]^+ \\ \cdot \\ \cdot \\ [K^a - S_n]^+ \end{bmatrix}$

The optimal composition of the static portfolio hedging put is defined by:

$$\min_{\beta^{i,n}} Risk(W_n(i, n)) \quad (0.8)$$

If we assume that the measure of risk is the variance, then (0.8) is equivalent to

<sup>9</sup> Note: we have the same assets in the portfolio for all options.

$$\begin{cases} \min_{\beta^{i,n}} Risk(W_n(i,n)) \\ sc \ E^Q(W_n(i,n)) = 0 \end{cases} \quad (0.9)$$

The constraint  $E^Q(W_n(i,n)) = 0$  means that the hedging error must be null on average.

If  $H(i,n) = [K_i - S_n]^+$ , then it is equivalent to making a regression of  $H(i,n)$  on  $P_n$ . If we simulate  $N_S$   $\bar{H}(i,n) = (H^j(i,n))_{1 \leq j \leq N_S}$  and  $\bar{P}_n = (P_n^j)_{1 \leq j \leq N_S}$ .

$$\begin{aligned} H^j(i,n) &= {}^t(\beta^{i,n}) \times P_n^j + e^j, \quad j=1, \dots, N_S \\ \Leftrightarrow \bar{H}(i,n) &= {}^t\bar{P}_n \times \beta^{i,n} + E, \quad \text{Where, } E = [e^j]_{j=1, \dots, N_S} \\ \Rightarrow (\beta^{i,n})^* &= \left( (\bar{P}_n \times {}^t\bar{P}_n)^{-1} \times \bar{P}_n \right) \times \bar{H}(i,n) \end{aligned} \quad (0.10)$$

$W_n(i,n) = e$ , where  $e$  is a white noise.

Discounted future costs of this strategy for put  $P(n, S_0, K_i)$  are written:

$$L(i,n) = \frac{1}{N_S} \sum_{j=1}^{N_S} \left( \left( [K_i - S_n^j]^+ - (1-c)(\beta^{i,n})^* \times P_n^j \right) e^{-rn} + (1+c)(\beta^{i,n})^* \times P_0 \right). \quad (0.11)$$

#### 4.2.1.2 Insurance portfolio hedging

If we now include the insurance portfolio, we can write the value of the hedging portfolio at time 0:

$$\begin{aligned} P_{st1}^{H,0} &= \sum_{i=1}^N \sum_{n=1}^{\tau_i} \Pr(x_i, n) {}^t(\beta^{i,n}) P \\ &= \sum_{i=1}^N \sum_{n=1}^{\tau_i} \left( \Pr(x_i, n) \times \beta^{i,n,0} \right) e^{-nr} + \sum_{i=1}^N \sum_{n=1}^{\tau_i} \left( \Pr(x_i, n) \times \beta^{i,n,1} \right) S_0 \\ &\quad + \sum_{i=1}^N \sum_{n=1}^{\tau_i} \Pr(x_i, n) \sum_{l=2}^A \beta_0^{i,n,l} \times P_0(n, S_0, K^l) \\ &= B_0 + B_1 \times S_0 + \sum_{l=2}^A \sum_{i=1}^N \sum_{n=1}^{\tau_i} B^{i,n,l} \times P_0(n, S_0, K^l) \end{aligned} \quad (0.12)$$

where:

$$\left\{ \begin{array}{l} B_0 = \sum_{i=1}^N \sum_{n=1}^{\tau_i} \left( \Pr(x_i, n) \times \beta^{i,n,0} \right) e^{-nr} \\ B_1 = \sum_{i=1}^N \sum_{n=1}^{\tau_i} \left( \Pr(x_i, n) \times \beta^{i,n,1} \right) \\ B^{i,n,l} = \Pr(x_i, n) \times \beta_0^{i,n,l} \end{array} \right.$$

$B_0$  (resp.  $B_1$ ) is the quantity of risk-free asset (resp. underlying asset) necessary to cover the risk of the insurance portfolio. Our hedging portfolio will only be constituted of the underlying asset and the risk-free asset.

We can write an estimation of the discounted future costs of the insurance portfolio  $L_{ST}$  :

$$L_{ST} = \Pi_{Stat} - \left( P_{Stat}^H - P_{Stat}^{H,0} \right), \quad (0.13)$$

where:

$$\left\{ \begin{array}{l} \Pi_{Stat} = \frac{1}{N_s} \sum_{i=1}^N \sum_{n=1}^{\tau_i} \sum_{j=1}^{N_s} M_j^{i,n} e^{-rn} \\ P_{Stat}^H = \frac{1}{N_s} \sum_{i=1}^N \sum_{n=1}^{\tau_i} \sum_{j=1}^{N_s} (1-c) \Pr(x_i, n) \times \left( \beta^{i,n,0} e^{nr} + \beta^{i,n,1} S_n^j \right) \\ P_{Stat}^{H,0} = (1+c) P_{st}^{H,0} \end{array} \right.$$

Static hedging does not make it possible for the manager to integrate future extra information provided by the market. Dynamic hedging overcomes this shortfall because dynamic minimisation involves continuously rebalancing the hedging portfolio.

#### 4.2.2. Dynamic minimisation

We will now improve upon the previous strategy. Instead of maintaining the hedging portfolio unchanged until maturity, we will modify it frequently.

We write the value at  $t$  of the hedging portfolio as  $P_{Min,Dyn}^{H,t}(i, n)$ .

$$P_{Min,Dyn}^{H,t}(i, n) = Cap + \beta_t^{i,n} e^{-r(n-t)} + \alpha_t^{i,n} S_t, \quad (0.14)$$

where  $Cap$  is the initial capital.

Our portfolio is self-financing if we have:

$$\begin{aligned}
P_{Min,Dyn}^{H,t}(i,n) &= Cap + \beta_t^{i,n} e^{-r(n-t)} + \alpha_t^{i,n} S_t \\
&= Cap + \int_0^t \beta_s^{i,n} d\left(e^{-r(n-s)}\right) + \int_0^t \alpha_s^{i,n} d\left(S_s\right) \\
\Leftrightarrow P_{Min,Dyn}^{H,t}(i,n) &= Cap + \int_0^t \alpha_s^{i,n} dS_s. \text{ With } \beta_t^{0,i,n} = P_{Min,Dyn}^{H,t}(i,n) - Cap e^{-rt} - \alpha_t^{i,n} S_s
\end{aligned}$$

Valuation is made under the risk-neutral measure. The hedging error is written:

$$W_n(i,n) = Cap + \int_0^n \alpha_s^{i,n} dS_s - e^{-rn} H(i,n), \text{ where } H(i,n) = [K_i - S_n]^+.$$

The optimal portfolio is given using the program of variance minimization:

$$\left(\beta_t^{i,n}, \alpha_t^{i,n}\right)_{0 \leq t \leq n} = \text{Arg} \left( \min E^Q \left( W_n(i,n) \right)^2 \right). \quad (0.15)$$

This approach makes it possible to obtain the optimal composition of the portfolio at each period. Rebalancing is frequently. It is similar to the delta neutral approach with frequent rebalancing, but the composition of the hedging portfolio changes according to the model of the underlying asset (see section 5)<sup>10</sup>.

By analogy with dynamic delta hedging, we can write an estimation of the discounted future costs:

$$\Rightarrow L_{Min,dyn} = \Pi_{Min,dyn} - \left( P_{Min,dyn}^H - P_{Min,dyn}^{H,0} - F_{Min,dyn} \right), \quad (0.16)$$

with:

$$\left\{ \begin{aligned}
\Pi_{Min,dyn} &= \frac{1}{N_s} \sum_{i=1}^N \sum_{n=1}^{\tau_i} \sum_{j=1}^{N_s} M_j^{i,n} e^{-rn} \\
P_{Min,dyn}^H &= \frac{1}{N_s} \sum_{i=1}^N \sum_{n=1}^{\tau_i} \sum_{j=1}^{N_s} (1-c) \Pr(x_i, n) \left( \alpha_{j,n/h-1}^{i,n} S_n^j + \beta_{j,n/h-1}^{i,n} e^{h \times r} \right) e^{-rn} \\
P_{Min,dyn}^{H,0} &= \frac{(1+c)}{N_s} \sum_{i=1}^N \sum_{n=1}^{\tau_i} \Pr(x_i, n) \left[ \alpha_0^{i,n} S_0 + \beta_0^{i,n} \right] \\
F_{Min,dyn} &= \frac{1}{N_s} \sum_{i=1}^N \sum_{n=1}^{\tau_i} \sum_{j=1}^{N_s} \Pr(x_i, n) W_{DY,1}(i, n, j)
\end{aligned} \right.$$

<sup>10</sup> The optimal hedging ratio integrates the jumps risk, to see the comparison between the delta and the optimal ratio; we can see Tankov (2008).

### 4.3. Semi-Static hedging using short-term options

This hedging strategy finds its relevance in the results of Breeden & Litzenberger (1978), who showed that risk-neutral density relates to the second strike derivative of the put pricing function by:

$$f(S,t,K,T) = e^{r(T-t)} \frac{\partial^2 P}{\partial K^2}(S,t,K,T). \quad (0.17)$$

Using this result Carr P. & Wu L. (2004) stated the following theorem:

**Theorem:** Under no arbitrage and the Markovian assumption, the time- $t$  value of a European put option maturing at a fixed time  $T \geq t$  relates to the time- $t$  value of a continuum of European put options at a shorter maturity  $u \in [t, T]$  by:

$$P(S,t,K,T) = \int_0^\infty w(k) P(S,t,k,u) dk, \quad u \in [t, T] \quad (0.18)$$

For all possible nonnegative values of  $S$  and at all time  $t \leq u$ , the weighting function  $w(k)$  is

given by  $w(k) = \frac{\partial^2}{\partial k^2} P(k,u,K,T)$  (for proof see Carr & Wu (2004)).

This Theorem states the spanning relation in terms of put options. The spanning relation holds for all possible values of the spot price  $S$  and at all times up to the expiry of the options in the spanning portfolio. The option weights  $w(k)$  are independent of  $S$  and  $t$ . This property characterizes the static nature of the spanning relation. Under no arbitrage, once we form the spanning portfolio, no rebalancing is necessary until the options within reach maturity.

In practice, investors can neither rebalance a portfolio continuously, nor can they form a static portfolio involving a continuum of securities. Both strategies involve an infinite number of transactions. In the presence of discrete transaction costs, both would lead to financial ruin. The number of put options used in the portfolio is chosen to balance the cost from the hedging error with the transaction costs from these options. We approximate the spanning integral in the equation by a weighted sum of a finite number  $a$  of put options at strike  $K_j$ ,  $j = 1, 2, \dots, a$ .

$$\int_0^\infty w(k) P(S,t,k,u) dk \approx \sum_{j=1}^a w(K_j) P(S,t,K_j,u), \quad (0.19)$$

where we choose the strike points  $K_j$  and their corresponding weights based on the Gauss-Hermite quadrature rule<sup>11</sup>. To apply the quadrature rules, we need to map the quadrature nodes and weights  $\{x_j, w_j\}_{j=1}^a$  to our choice of option strikes and to the portfolio weights  $W_j$ . One reasonable choice of a mapping function between the strikes and the quadrature nodes is given by:

$$\kappa(x) = \text{Ke}^{\left( x\sigma\sqrt{2(T-u)} - \left( r + \frac{\sigma^2}{2} \right) (T-u) \right)},$$

<sup>11</sup>See appendix D.

where  $\sigma^2$  denotes the annualized variance of the log asset return. This choice is motivated by the gamma weighting function under the Black & Scholes model, which is given by:

$$q(\kappa) = \frac{\partial^2 P(\kappa, u, K, T)}{\partial \kappa^2} = \frac{\varphi(d_2)}{\kappa \sigma \sqrt{T-u}},$$

where  $\varphi(\cdot)$  denotes the probability density of a standard normal and the standardized variable  $d_1$  is defined as:

$$d_2 = \frac{\ln(\kappa/K) + \left(r - \frac{\sigma^2}{2}\right)(T-u)}{\sigma \sqrt{T-u}}.$$

We can then obtain the mapping in (9) by replacing  $d_1$  with  $\sqrt{2}x$ , which can also be regarded as a standard normal variable. Thus, given the Gauss-Hermite quadrature  $\{x_j, w_j\}_{j=1}^a$ , we choose the strike points as

$$K_j = \text{Ke}^{\left(x_j \sigma \sqrt{2(T-u)} - \left(r + \frac{\sigma^2}{2}\right)(T-u)\right)}$$

The portfolio weights are then given by

$$q_j = \frac{q(K_j) K_j'(x_j)}{e^{-x_j}} w_j = \frac{q(K_j) K_j \sigma \sqrt{2(t-u)}}{e^{-x_j}} w_j.$$

Conceivably, we can use different methods for the finite approximation. The Gauss-Hermite quadrature method chooses both the optimal strike levels and the optimal weight under each strike. This method is applicable to a market where options at many different strikes prices are available. Carr and Lu note that the nearer the maturity of the options of cover is to that which is covered, the more the cover is effective. The error of cover decreases with the number of hedging options selected. Our hedging portfolio will be covered by options of maturity lower than  $u$  years. Discounted future costs of this strategy for put  $P(n, S_0, K_i)$ <sup>12</sup> are written:

$$L_{Carr}(i, n) = \frac{1}{N_s} \sum_{j=1}^{N_s} \left( \left( [K_i - S_n^j]^+ e^{-m} - (1-cs) \sum_{j=1}^a q_j^{i,n} [K_j^{i,n} - S_u]^+ e^{-ru} \right) + (1+cs) \sum_{j=1}^a q_j^{i,n} P(u, S_0, K_j^{i,n}) \right)$$

Now we can write our hedging portfolio:

$$P_{Carr}^{H,0} = \sum_{i=1}^N \left( \sum_{n=1}^u \text{Pr}(x_i, n) \times P_0(n, S_0, K_i) + \sum_{n=u+1}^{\tau_i} \left( \text{Pr}(x_i, n) \times \sum_{j=1}^a q_j^{i,n} P(u, S_0, K_j^{i,n}) \right) \right), \quad (0.20)$$

where  $q_j^{i,n}$  and  $K_j^{i,n}$  represent weights and strike points necessary to hedge put  $P_0(n, S_0, K_i)$  with  $n > u$ . We can write an estimation of the discounted future costs of the insurance portfolio  $L_{Carr}$ :

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<sup>12</sup> where  $n > u$ .



$$\Rightarrow L_{Car} = \Pi_{Carr} - (P_{Carr}^H - P_{Carr}^{H,0}), \quad (0.21)$$

where:

$$\left\{ \begin{array}{l} \Pi_{Carr} = \frac{1}{N_s} \sum_{i=1}^N \sum_{n=1}^{\tau_i} \sum_{j=1}^{N_s} M_j^{i,n} e^{-rn} \\ P_{Carr}^H = \frac{1}{N_s} \left\{ \sum_{i=1}^N \sum_{n=1}^u \sum_{j=1}^{N_s} (1-cs) \Pr(x_i, n) [K_i - S_n]^+ e^{-rn} \right. \\ \quad \left. + \sum_{i=1}^N \sum_{n=u+1}^{\tau_i} \sum_{j=1}^{N_s} (1-cs) \Pr(x_i, n) \times \sum_{j=1}^a q_j^{i,n} [K_j^{i,n} - S_u]^+ e^{-ru} \right\} \\ P_{Carr}^{H,0} = \frac{(1+cs)}{N_s} P_{Carr}^{H,0} \end{array} \right.$$

## 5. Implementation of hedging

Now, we will implement the hedging strategies which we have just developed. For that, we will model the return of the underlying asset and the risk of mortality in relation to our insurance portfolio.

### 5.1. Mortality risk

Now we want to estimate the probability density function  $\Pr(x, n)$ . It is a standard problem in the insurance world. A standard solution is to estimate this probability using mortality tables. Indeed, the risk of mortality is generally modeled in this way. These tables can be built starting from the data on the mortality of the policyholders within the portfolio or with regulated mortality tables. The approach adopted here is adapted for either of these methods.

In this case, we can write  $\Pr(x, n) = q_{x+n} \times {}_n p_x$ , with:

$$\left\{ \begin{array}{l} q_{x+n} = \frac{L_{x+n} - L_{x+n+1}}{L_{x+n}}, \text{ the instantaneous rate of mortality} \\ {}_n p_x = \frac{L_{x+n+1}}{L_{x+n}}, \text{ the probability of survival until the age } x+n, \text{ for a policyholder aged } x \end{array} \right.$$

Where  $(L_x)_{0 \leq x \leq X_{end}}$  is the mortality table which describes the mortality of the portfolio population.  $L_x$  is the number of survivors to each age.

We assume that the evaluation is made at the end of the period of the death and that the company makes the payment this time. Here, we made an implicit approximation on the age of the policyholder. Indeed, the date of the evaluation cannot correspond to the birthday of the policyholder. But Planchet et al (2005) affirm that the precision brought by this adjustment is modest taking into consideration the inaccuracy of modelling the underlying asset or the imperfect mutualisation of mortality.

To begin with, we retain the mortality table TH 00-02. This table was drawn up on the basis of observations made by the INSEE of the French population from 2000 to 2002 and has since been smoothed. It is also recommended for the evaluation of contracts with a death guarantee.

## 5.2. Modelling financial risk: Black and Scholes' model

The standard modelling of the underlying asset is based on the model of Black & Scholes. Assumptions of the Black and Scholes' Model are:

- The stock pays no dividends during the option's life,
- Markets are efficient,
- No commissions are charged,
- Returns are log-normally distributed,
- Interest rates remain constant and known,
- European exercise terms are used.

We will reconsider these assumptions further.

$$\text{Dynamic of risk-free asset: } \begin{cases} S_0^0 = 1 \\ \frac{dS_t^0}{S_t^0} = rdt \end{cases}, \text{ where } r, \text{ the risk-free interest rate, is constant.}$$

In this model, the underlying asset is assumed following classical geometric Brownian motion, described by the following stochastic differential equation (SDE) under the physical probability measure:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t.$$

The solution of this SDE is:

$$S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t\right). \quad (0.22)$$

Under the risk neutral probability measure and with the price of the European put at  $t$ , with strike price  $K$  and maturity  $T$  is:

$$P^{BS}(T-t, S_0, K, \sigma) = Ke^{-r(T-t)}\Phi(-d_2) - S_0\Phi(-d_1), \quad (0.23)$$

where:

$$\begin{cases} d_1(S_0, K, \sigma, T-t) = d_2 + \sigma\sqrt{T-t} \\ d_2(S_0, K, \sigma, T-t) = \frac{\log\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \end{cases} \quad (0.24)$$

The delta is given by:  $\Delta^{BS}(T-t, S, K, \sigma) = \frac{\partial P^{BS}(T-t, S, K, \sigma)}{\partial S} = -\Phi(-d_1)$ .

In Black and Scholes' environment, the variance-minimizing hedging strategy is identical to delta neutral hedging (See Appendix D).

Given the Gauss-Hermite quadrature  $\{x_j, w_j\}_{j=1}^a$ , the static portfolio strikes and weights, deduced by the Carr static hedging, are given by,

$$\begin{cases} K_j = \text{Ke}^{\left(x_j \sigma \sqrt{2(T-u)} - \left(r + \frac{\sigma^2}{2}\right)(T-u)\right)} \\ q_j = \frac{q(K_j) K_j \sigma \sqrt{2(t-u)}}{e^{-x_j}} w_j = \frac{1}{\sqrt{\pi}} w_j \end{cases} \quad j = 1, \dots, a$$

### 5.3. New underlying asset model: Merton's jump model

Black and Scholes assume ideal conditions in the market for the underlying assets and options. They suppose that the underlying asset return process belongs to the subfamily of the diffusion processes, which rely on the Brownian movement. The problem with this family of processes is that it supposes continuity in price trajectories, which does not seem very realistic when the reality of the markets is observed. Thus, the observation of the asset prices reveals the presence of jumps, which can be seen as discontinuities in the price trajectories. Other empirical reports made using data from the markets show that, contrary to what is envisaged by this model, implicit volatility is not constant. Its curve even has, in several cases, a convexity compared to the strike, a phenomenon known classically under the name of a "volatility smile". In addition, further empirical studies show that the distribution of return presents an asymmetry on the left and tails of distribution heavier than those of a normal distribution, as demonstrated by the work of Cont (2001).

For all of these reasons, like Merton (1976), we are going to introduce jumps into the Black and Scholes model. The resulting model, known as the Merton jump-diffusion model, is not perfect since it does not make it possible to reproduce the asymmetrical character of the underlying asset's distribution, but this new model provides closed semi formulas and is less difficult to calibrate than the Kou model (see Kou & Wang (2004)) or its extension (see Randrianarivony (2006)). We will now analyze what occurs within the framework of the jumps models. More precisely, we will be interested in the risk indicators within the framework of this model.

The Merton jump-diffusion model assumes the following risk-neutral dynamics for the underlying security price movement,

$$\frac{dS_t}{S_t} = dZ_t, \quad \text{with } Z_t = rt + \sigma W_t + \sum_{j=0}^{N_t} L_j, \quad (0.25)$$

where  $\sum_{j=0}^{N_t} L_j$  is a compound Poisson characterized by processes:  $(N_t)_{t>0}$  a Poisson process

which has intensity  $\lambda$  and  $L_j$  is distributed according to a log-normal law of probability.  $\ln(1+L) \sim N(m, \sigma_u)$ . It is also assumed that  $N$ ,  $L$  and  $W$  are independent.

The jump risk is an idiosyncratic risk; it is the risk associated with individual assets. This specific risk can be reduced through diversification within a portfolio of assets (Markowitz (1952)).

Under risk neutral probability measure the solution of this stochastic differential equation is (Merton (1976)):

$$S_t = S_0 \exp \left\{ \left( r - \frac{\sigma^2}{2} \right) t + \sigma W_t + \sum_{j=1}^{N_t} \text{Log} (1 + L_j) \right\}. \quad (0.26)$$

And the price of a European put option at  $t$ , with price strike  $K$  and maturity  $T$  is written:

$$P^M (T-t, S_0, K, \sigma) = \sum_{n \geq 0} \frac{e^{-\lambda(T-t)} (\lambda(T-t))^n}{n!} P^{BS} (T-t, S_n, K, \sigma_n), \quad (0.27)$$

$$\text{where: } \begin{cases} -S_n = S_0 e^{nm + \frac{n\sigma_u^2}{2} - \lambda(T-t) \exp \left( m + \frac{\sigma_u^2}{2} \right) + \lambda(T-t)} \\ -\sigma_n = \sqrt{\sigma^2 + \frac{n\sigma_u^2}{2}} \end{cases}$$

### 5.3.1. Delta Hedging

The delta of this option is given by following expression:

$$\Delta^M = \frac{\partial P^M (T-t, S, K, \sigma)}{\partial S} = \sum_{n \geq 0} \frac{e^{-\lambda(T-t)} (\lambda(T-t))^n}{n!} \frac{S_n}{S} \Delta^{BS} (T-t, S_n, K, \sigma_n). \quad (0.28)$$

### 5.3.2. Dynamic minimisation

In the framework of Merton, an optimally hedged portfolio is given by (See Tankov & Voltchkova (2006) or Gabriel & Surlas (2006)):  $(\beta_t, \alpha_t)$  where

$$\begin{cases} \hat{c} = E_Q (e^{-rT} H) \\ \alpha_t = \frac{\sigma^2 \Delta^M + \frac{\lambda}{S} \left[ \left( e^{m + \frac{\sigma_u^2}{2}} P^M \left( t, S_t e^{m + \frac{3\sigma_u^2}{2}}, \sqrt{\sigma^2 + \frac{\sigma_u^2}{T-t}} \right) - P^M \left( t, S_t e^{m + \frac{\sigma_u^2}{2}}, \sqrt{\sigma^2 + \frac{\sigma_u^2}{T-t}} \right) \right) - P^M (t, S_t) \left( e^{m + \frac{\sigma_u^2}{2}} - 1 \right) \right]}{\sigma^2 + \lambda \left[ e^{2m + 2\sigma_u^2} - 2e^{m + \frac{\sigma_u^2}{2}} + 1 \right]} \\ \beta_t = e^{-rt} \left( P_{Min, Dyn}^{H, t} (t, T) - \hat{c} - \alpha_t S_t \right) \end{cases}$$

### 5.3.3. Carr hedging

The strike weighting function is given by:

$$\begin{aligned}
q(\kappa) &= \frac{\partial^2 P^M(T-u, S, \kappa, \sigma)}{\partial \kappa^2} = \sum_{n \geq 0} \frac{e^{-\lambda(T-u)} (\lambda(T-u))^n}{n!} \frac{\partial^2 P^{BS}(T-u, S_n, \kappa, \sigma_n)}{\partial \kappa^2} \\
&= \sum_{n \geq 0} \frac{e^{-\lambda(T-u)} (\lambda(T-u))^n}{n!} \frac{e^{-r(T-u)} \varphi(d_2(T-u, S_n, \kappa, \sigma_n))}{\kappa \sigma_n \sqrt{T-u}}
\end{aligned}$$

We define the strike price points based on the Gauss-Hermite quadrature  $\{x_j, w_j\}_{j=1}^a$  as follows:

$$K_j = Ke^{x_j \sqrt{2\nu(T-u)} - \left(r + \frac{\nu}{2}\right)(T-u)}, \quad (0.29)$$

where  $\nu = \sigma^2 + \lambda(m^2 + \sigma_u^2)$  is the annualized variance of the asset return under the risk neutral measure. The portfolio weights are then given by

$$q_j = \frac{q(K_j) K_j \sqrt{2\nu(T-u)}}{e^{-x_j^2}} w_j. \quad (0.30)$$

## 6. Results in Black-Scholes' framework

The results presented below were obtained starting from the following parameters:

Number of simulations	10,000
Maturity of insurance portfolio	15 years
Volatility of underlying asset	25 %
Drift of underlying asset	8.5%
Risk-free interest rate	5%
Guarantee	100
Initial value of underlying asset	100
Frequency of observation of portfolio	Monthly
Transaction costs	1%
Number of short-term options used in Carr hedging	2
Maturity of short-term options used in Carr hedging	1 year
Insurance portfolio	1,000 insured aged 45
Interval of re-hedging in corrected delta hedging	[-1,1]

### 6.1. Results

The results obtained are summarized in Table 1.

**Table 1 - Risk indicators of costs related of Hedging strategies in Black-Scholes' framework**

	Expected	Median	Standard deviation	VaR 99,75%	VaR 99%	CTE 99,75%	CTE 99%	maximum
No Hedging	950	400	1 177	5 006	4 459	5 007	4 463	6 124
Static minimisation	980	660	923	4 487	3 841	4 488	3 845	5 902
Delta Hedge	1 406	1 398	260	2 158	2 040	2 158	2 041	2 581
Delta2	1 140	1 137	328	2 179	1 965	2 179	1 966	2 605
Semi-Static Hedging	984	807	1 093	4 803	4 102	4 804	4 106	5 398

We remark that:

1. taking hedging strategies into account will generate future costs which will be on average higher than the liabilities of the insurer;
2. all hedging strategies reduce the volatility of the insurance portfolio costs. This result consolidates the analyze of Frantz et al (2003);
3. hedging strategies reduce extreme losses (Value-at-risk: VaR and Conditional Tail Expectation: CTE). But Static minimisation generates a maximum future cost that is higher than if hedging was not used;
4. delta hedging, with frequent rebalancing (DFR), costs the insurer more. It provides the lowest volatility of the future costs;
5. delta hedging with rebalancing of the portfolio when the hedging error is strong (Delta2), generates future costs which are on average weaker than in the case of frequent rebalancing, but it also generates a greater volatility. However, the extreme losses are reduced.

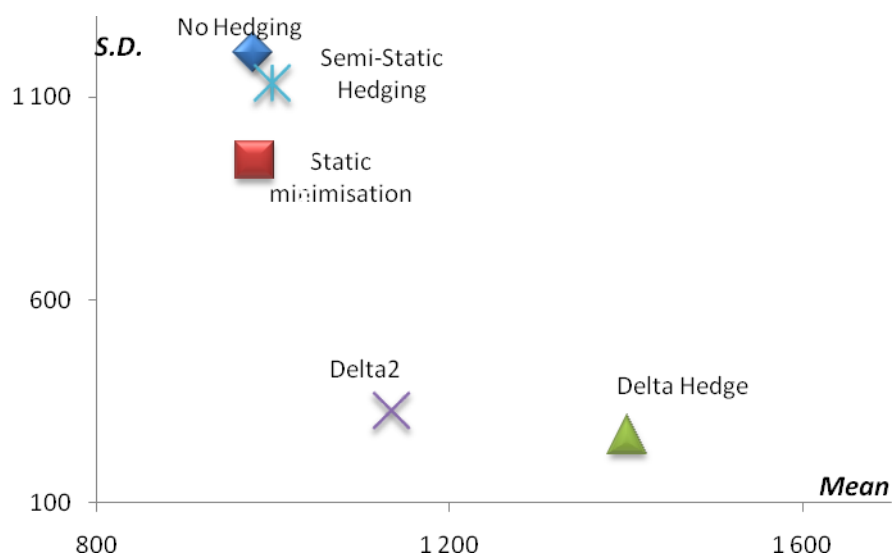
In conclusion, the delta strategies provide better results than the other strategies. The delta2 strategy reduces the costs and the indicators of extreme risks (the VAR and CTE), but the potential maximal loss is higher.

Figure 1 compares hedging strategies using Mean-Variance Analysis<sup>13</sup>.

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<sup>13</sup> S.D. : Standard Deviation

**Figure 1 – Representation of the hedging strategies using a Mean-Variance graph**



According to the mean-variance criterion, we note that all the strategies are preferable to not hedging at all. We can also note that DFR (Delta hedge) provides the lowest volatility of the future costs, whereas the semi-static strategy provides the weakest expected costs.

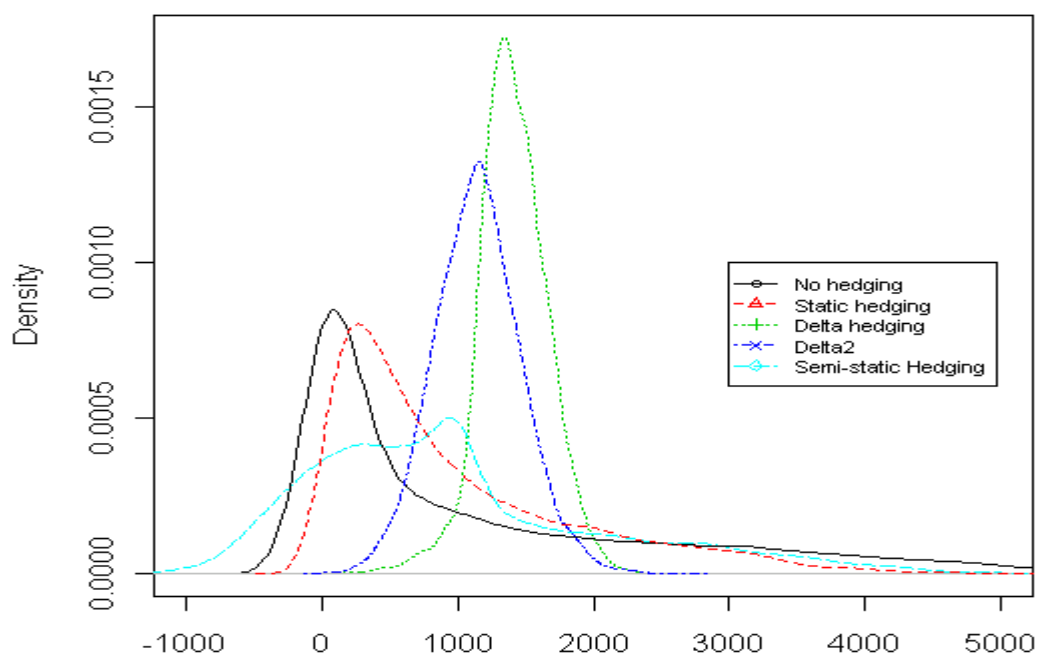
The delta strategies are always preferred to the static strategy and semi-static hedging.

If we are satisfied only with the mean-variance criterion, the choice between DFR and delta2 depends on the preference between a stronger volatility or a weaker expectation of costs.

If we are interested in the tails of distribution, the delta2 strategy has the least thick tail of distribution, which is shown by the fact it has the weakest indicators of extreme risk.

The following Figure 2 shows the cost density of hedging strategies.

**Figure 2 – Density of discounted future costs of hedging strategies**



## 6.2. Robustness

Now, we are going to analyse the risk indicators in the case of a modification of the parameters, and in case of shocks on the mortality rates or for the underlying asset.

### 6.2.1. Impact of the choice of parameters

Firstly, we will compare hedging strategies in the case of an error in the choice of the parameters.

#### 6.2.1.1 Period of reallocation

Table 2 shows the variation in risk indicators between annual reallocation and monthly reallocation.

**Table 2 – Variation of Risk indicators of costs between annual and monthly reallocation**

	Expected	Median	Standard deviation	VaR 99,75%	VaR 1%	CTE 99,75%	CTE 99%	maximum
No Hedging	3%	8%	4%	4%	3%	4%	3%	4%
Static minimisation	2%	0%	5%	2%	3%	2%	3%	48%
Delta Hedge	-22%	-25%	80%	30%	17%	30%	17%	45%
Delta2 Hedging	-2%	-5%	66%	39%	30%	39%	30%	35%
Semi-Static Hedging	1%	-2%	4%	0%	5%	0%	5%	11%

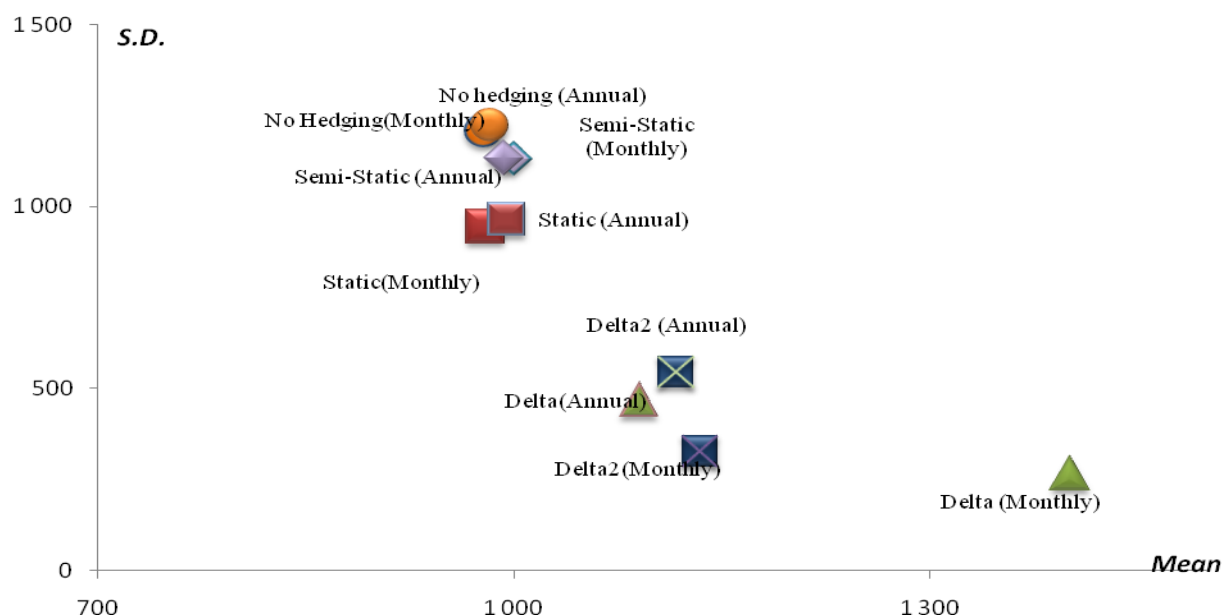
We can note that annual management of an insurance portfolio is slightly more expensive than monthly management. This is illustrated by a small rise in the expectation of future costs (No Hedging), as well as in the volatility.

We can see that changing to annual management increases the risk indicators for all strategies except the delta strategies, where we expect a reduction in the discounted future costs.

Figure 3 shows a graph of the mean-variance evolution of hedging strategies with two kinds of reallocation: annual and monthly.



**Figure 3 – Impact of period of reallocation on the Mean-Variance graph**



We also see that delta strategies are better than the other strategies.

#### 6.2.1.2 Transaction costs

We can see in Table 3 the impact of an increase in transaction costs on risk indicators.

**Table 3 – Variation of Risk indicators of costs related of 1% increase of transaction costs**

	Expected	Median	Standard deviation	VaR 99,75%	VaR 99%	CTE 99,75%	CTE 99%	maximum
No Hedging	0%	0%	0%	0%	0%	0%	0%	0%
Static minimisation	4%	5%	1%	3%	0%	3%	0%	-35%
Delta Hedge	31%	31%	32%	29%	30%	29%	30%	26%
Delta2 Hedging	15%	15%	5%	14%	12%	14%	12%	12%
Semi-Static Hedging	1%	-1%	0%	-4%	-2%	-4%	-2%	2%

We note that DFR is more sensitive to a rise in transaction costs.

Table 4 shows the risk indicators in the situation of a rise in transaction costs from 1% to 2%.

**Table 4 – New Risk indicators of COSTS with 2% transaction costs**

	Expected	Median	Standard deviation	VaR 99,75%	VaR 99%	CTE 99,75%	CTE 99%	maximum
No Hedging	950	400	1 177	5 006	4 459	5 007	4 463	6 124
Static minimisation	1 018	684	954	4 649	3 988	4 651	3 994	9 007
Delta Hedge	1 840	1 832	352	2 814	2 639	2 814	2 640	3 088
Delta2 Hedging	1 310	1 295	342	2 453	2 191	2 454	2 192	2 909
Semi-Static Hedging	1 009	805	1 128	4 845	4 261	4 846	4 265	6 082

We can see that the delta2 strategy gives the lowest volatility and the most extreme risk indicators.

### 6.2.1.3 Delta 2 hedging: the impact of changing the interval for re-hedging

Here, we analyze the impact of re-hedging at different intervals on the risk indicators of the delta2 strategy. In Table 5 we have the results of simulations for re-hedging at intervals of  $[-1,1]$ ,  $[-2,2]$  and  $[-3,3]$ .

**Table 5 - Risk indicators of costs related of Hedging strategies in Black-Scholes' framework**

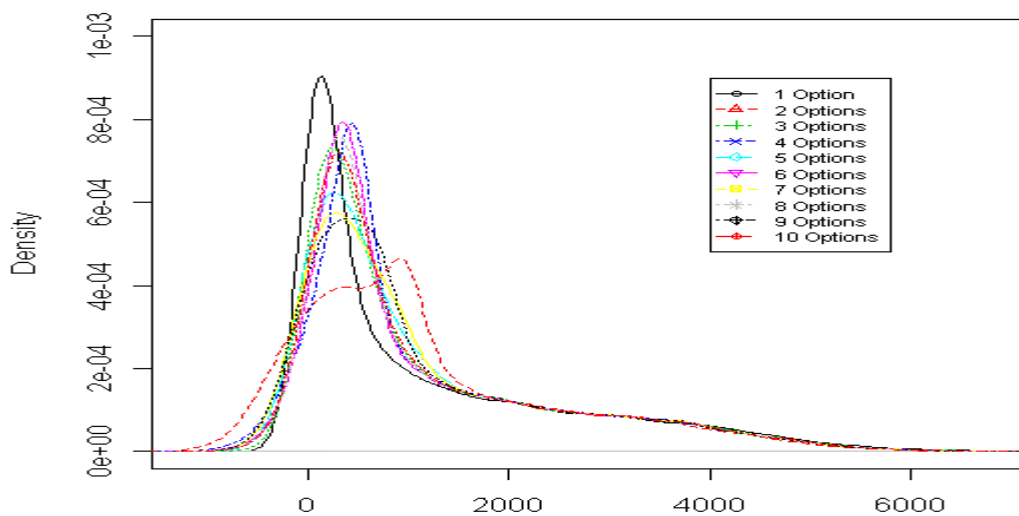
	Expected	Median	Standard deviation	VaR 99,75%	VaR 99%	CTE 99,75%	CTE 99%	maximum
No Hedging	950	400	1 177	5 006	4 459	5 007	4 463	6 124
Static minimisation	980	660	923	4 487	3 841	4 488	3 845	5 902
Delta Hedge	1 406	1 398	260	2 158	2 040	2 158	2 041	2 581
Delta2 $[-1,1]$	1 140	1 137	328	2 179	1 965	2 179	1 966	2 605
Delta2 $[-2,2]$	1 102	1 100	396	2 279	2 060	2 279	2 062	2 869
Delta2 $[-3,3]$	1 082	1 084	457	2 420	2 158	2 420	2 160	3 103
Semi-Static Hedging	984	807	1 093	4 803	4 102	4 804	4 106	5 398

Globally, we note that increasing the re-hedging interval decreases the volatility and the tail risk indicators (VaR and CTE). However, we also note a light decrease in the expectation of future costs.

### 6.2.1.4 Semi static hedging

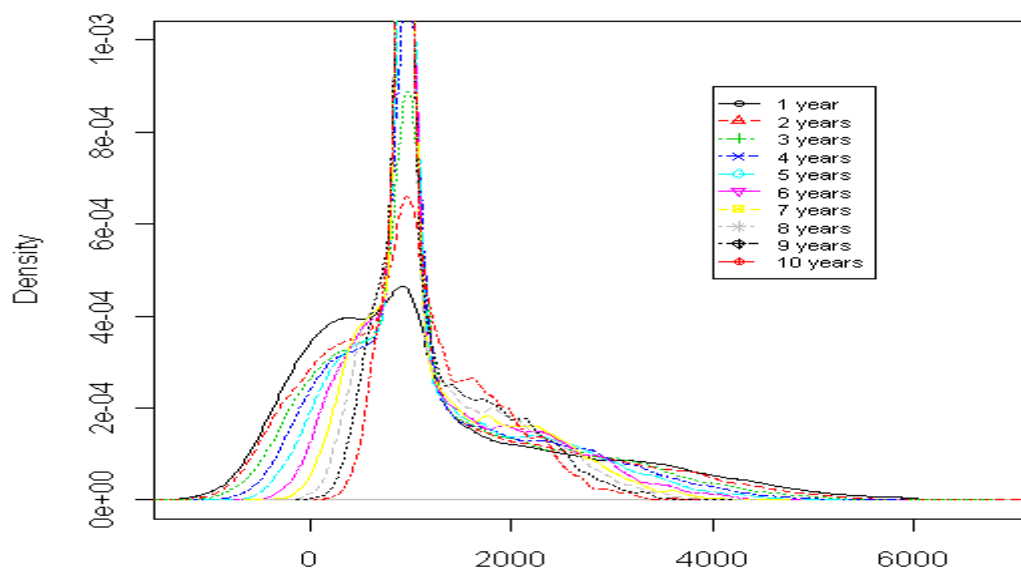
Here we analyse the impact of the number and of the maturity of short-terms options on the costs in semi-static hedging. Figure 4 and Figure 5 show the evolution of empirical density in relation to a rise in the number and the maturity of short term options.

**Figure 4 – Impact of number of shorter-options**



We can see that the impact of the number of short term options on density distribution is not clear.

**Figure 5 – Impact of maturity of shorter-options**

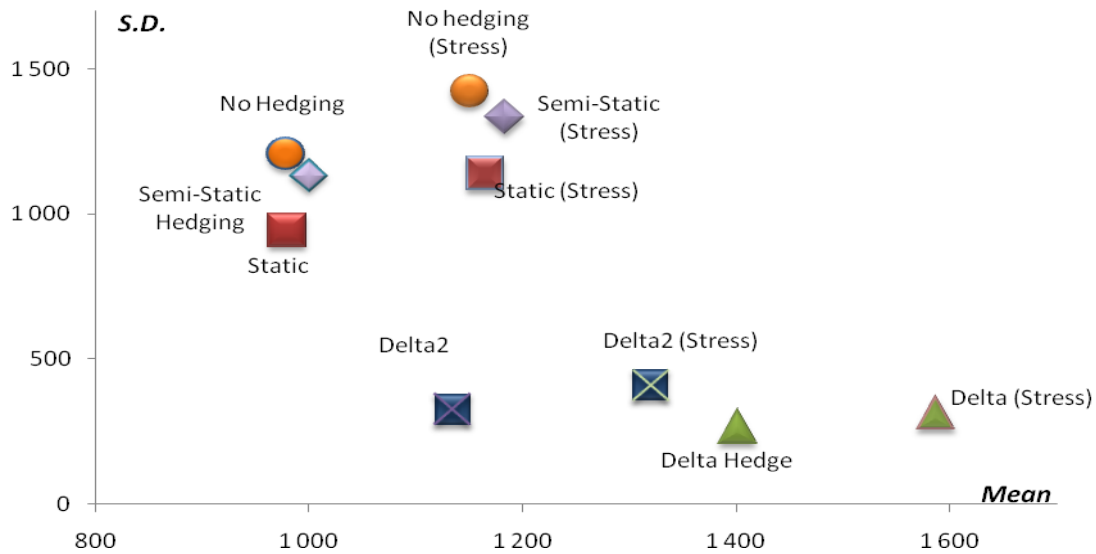


It is noted that the density of distribution condenses with an increase in the maturity of the hedging options. This represents a reduction in the volatility of the future costs. We also note that the tail of distribution has narrowed.

We can therefore conclude that the risk indicators decrease with the maturity of the hedging options, but that the impact of the number of options is not clear.

### **6.2.2. Impact of future mortality**

In the following section, we will stress-test the hedging strategies against a rise in future mortality rates. Figure 6 presents the evolution of Mean-variance on the graph in the case of a 20% increase in future mortality.

**Figure 6 – Evolution of Mean-Variance for a 20% increase in future mortality**

We can notice that, as expected, a rise in future mortality increased the expectation of future costs for all strategies. However, we note a fall in standard deviation for Delta and Delta2 strategies.

If we are interested in extreme risk indicators, (Table 6), DFR and delta2 strategies always have weaker VaR, CTE and Maximum losses than the other strategies.

**Table 6 –Risk indicators related to 20% increase in future mortality**

	Expected	Median	Standard deviation	VaR 99,75%	VaR 99%	CTE 99,75%	CTE 99%	maximum
No Hedging	1 150	486	1 424	6 124	5 433	6 125	5 438	7 712
Static minimisation	1 165	742	1 140	5 375	4 724	5 376	4 730	13 822
Delta Hedge	1 587	1 548	313	2 761	2 465	2 762	2 467	3 330
Delta2 Hedging	1 320	1 269	408	2 931	2 524	2 932	2 526	3 634
Semi-Static Hedging	1 183	886	1 333	5 734	5 160	5 734	5 164	6 877

But, in the case of an inaccurate estimation of future mortality, delta hedging with frequent rebalancing of the hedging portfolio provides the lowest extreme risk indicators.

### 6.2.3. What about the current financial crisis?

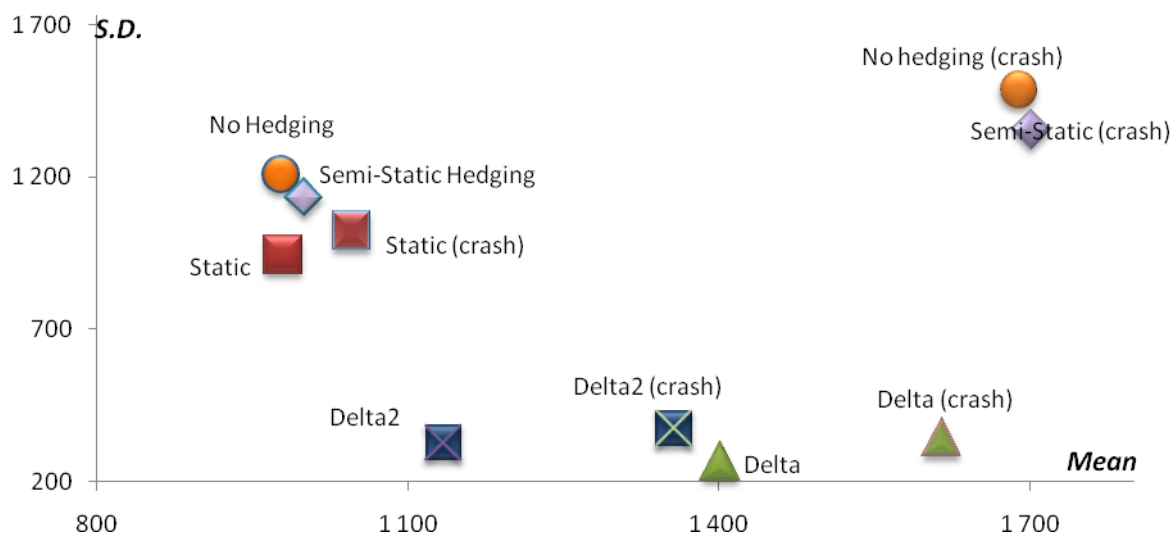
Since the beginning of the Subprime crisis in September, 2007, we have been waiting for a collapse in the main world markets. This crisis was highlighted in September, 2008, with the bankruptcy of one of Wall Street's first investment banks and the American state's repurchase of the largest world insurer.

The fear that this could spread to the whole economy led the main governments of industrial nations to take unprecedented measures to stop the crisis. This had a devastating impact on the shares of the main financial institutions. The CaC40, for example, lost more than 30% of its value during this period.

In this sub-section, we are going to try to show the impact of this crisis on our insurance portfolio. For that purpose, we will create a shock in the form of a 30 % decline in the price of the underlying asset.

The first results are resumed in Figure 7.

**Figure 7 – Impact on Mean-Variance of a 30% decline in the value of the Underlying asset**



Firstly, we note that if no hedging was used then the shock has had a very violent impact on the liabilities of the insurer. We note a strong increase in the mean and in the volatility of future costs.

Secondly, we note that the semi-static strategy is highly vulnerable to a violent shock in the share market. Indeed, as the portfolio of the insurer is being hedged using short-term options, the long-term payments are completely at the mercy of an unforeseen fluctuation in the price of the underlying asset.

Thirdly, the impact is fairly minimal on the static hedging strategy. Indeed, a strong fall in the price of the underlying asset increases the liabilities of the insurer but also increases the value of the hedging portfolio. Moreover, the hedging portfolio is not frequently modified; the costs of re-hedging are thus reduced. In fact, the impact of the fall of the asset price is less violent than if no hedging or semi-static hedging were used.

As for the delta strategies, we note an increase in hedging costs and in volatility. However, the delta2 strategy generates a low cost while DFR generates the lowest volatility.

According to the mean-variance criterion, it is difficult to make a choice between static hedging and delta hedging because the costs generated by static hedging are weaker than for delta hedging but with a stronger volatility.

However, if the indicators of extreme risk are analyzed (see Table 7), then the delta strategies provide the weakest protection.

**Table 7 – Risk indicators related to a 30% decline in the price of the underlying asset**

	Expected	Median	Standard deviation	VaR 99,75%	VaR 99%	CTE 99,75%	CTE 99%	maximum
No Hedging	1 689	1 320	1 486	5 838	5 258	5 839	5 262	6 586
Static minimisation	1 046	789	1 024	4 815	3 787	4 821	3 797	12 750
Delta Hedge	1 614	1 640	343	2 495	2 352	2 495	2 353	2 932
Delta2	1 357	1 347	374	2 486	2 297	2 486	2 298	3 044
Semi-Static	1 701	1 362	1 354	5 471	4 972	5 471	4 976	6 500

The delta2 hedging strategy gives the weakest VaR and CTE, but the DFR gives the weakest maximum Loss.

## 7. Results with Merton's jump-diffusion framework

The results presented below were obtained starting from the following parameters:

Number of simulations	1000
Maturity of insurance portfolio	15 years
Volatility of underlying asset	20 %
Intensity of poisson process	1
Volatility of discontinuous part (Jump volatility)	15%
Mean of amplitude of jump	0%
Drift of underlying asset	8.5%
Risk-free interest rate	5%
Guarantee	100
Initial value of underlying asset	100
Frequency of observation of portfolio	Monthly
Transaction costs	1%
Number of short-term options used in Carr hedging	2
Maturity of short-term options used in Carr hedging	1 year
Insurance portfolio	1,000 insured aged 45
Interval of re-hedging in Delta 2 strategy	[-1,1]

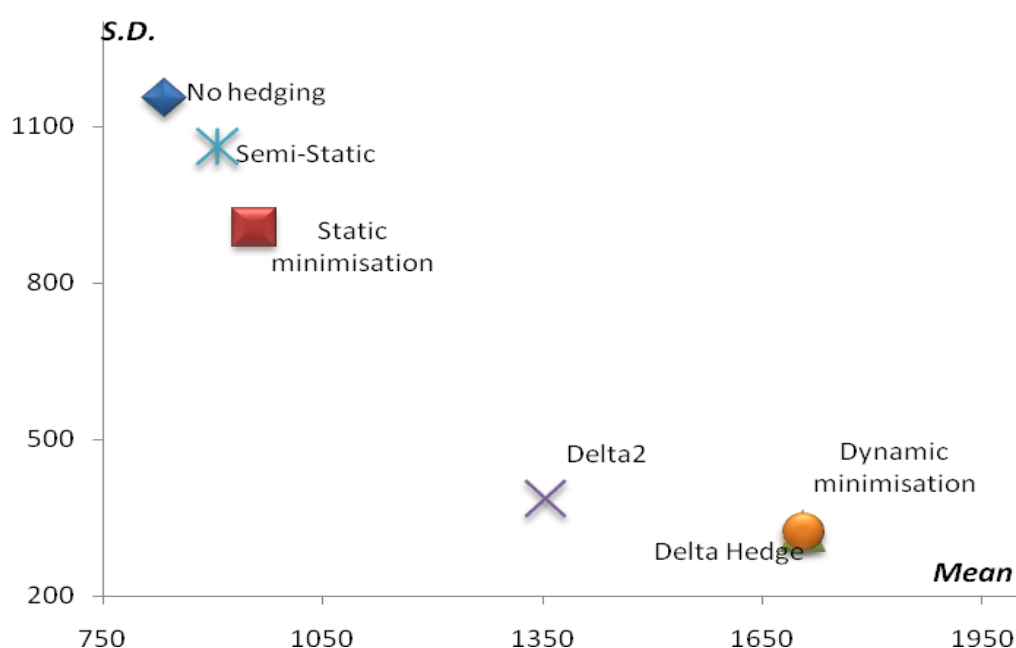
The results are summarized in Table 8.

**Table 8 - Risk indicators of costs relating to Hedging strategies in Merton's framework**

	Expected	Median	Standard deviation	VaR 99,75%	VaR 99%	CTE 99,75%	CTE 99%	Maximum
No Hedging	834	259	1154	5167	4587	5168	4591	5841
Static minimisation	958	629	906	4381	3895	4382	3899	5142
Dynamic minimisation	1706	1652	322	3016	2605	3017	2607	3421
Delta Hedge	1706	1652	322	3016	2605	3017	2607	3421
Delta2 Hedging	1355	1338	385	2748	2397	2748	2399	3011
Semi-Static	907	743	1060	4458	3851	4459	3857	5730

We can note that, as in Black and Scholes' framework, the implementation of hedging strategies will generate future costs that are, on average, higher than the insurer's liabilities. But hedging strategies reduce the standard deviation and tail distribution of liabilities. We see that dynamic minimisation and DFR provide the same risk indicators. They give the weakest volatility of costs. Delta2 strategies provide the weakest tail risk indicators.

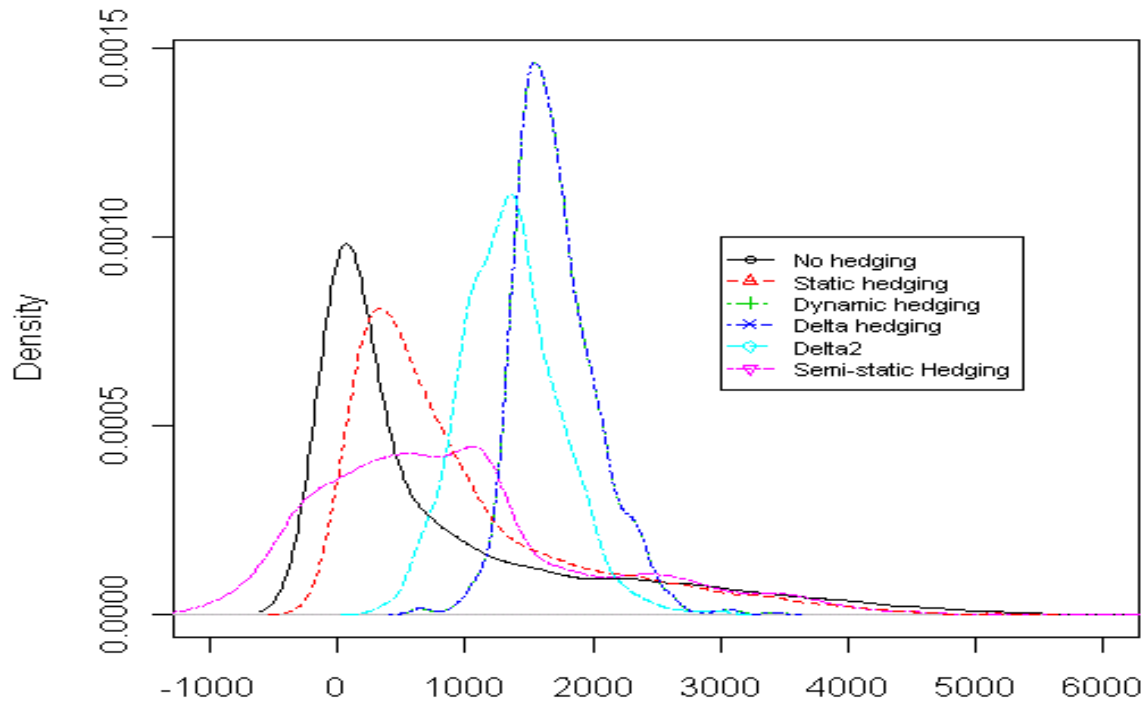
We can also compare hedging strategies using Mean-Variance (Figure 8).

**Figure 8 – Representation of the Mean-Variance of the hedging strategies in Merton's framework**

As in the Black-Scholes model, the Mean-Variance criterion show that all of the hedging strategies are more beneficial than not hedging at all. Static hedging is better than semi-static hedging with short-term options.

Delta hedging and dynamic minimisation provide lower volatility but higher cost expectations than static and semi-static hedging. Delta2 hedging gives a weaker expectation and higher volatility of costs than dynamic minimisation or DFR.

In Figure 9, we represent the empirical density of the distribution of hedging strategies.

**Figure 9 – Density of costs of hedging strategies in Merton’s framework**

## 8. Conclusion

In our study, we analyzed the optimality of some of the available hedging strategies which allow the insurer to reduce the risk related to a portfolio of unit-linked life insurance contracts with minimum death guarantee. We noted that all of the hedging strategies developed generated costs which were on average higher than if hedging was not used, but that all of these strategies reduced the volatility of the future costs and the indicators of extreme risks (VaR and CTE).

We were interested in 3 types of hedging strategies: the delta neutral strategies, which consist of matching the sensitivities of the hedging portfolio and the liabilities of the insurer; the strategies that minimize the variance of the hedging error; and the semi-static strategy of hedging using short term options.

In the Black-Scholes and Merton environments, we noticed that the optimal strategies were the delta neutral strategies and the variance-minimizing dynamic strategy which provided the same results (If we content ourselves with a portfolio constituted of only the underlying asset and the free-risk asset).

The static strategy reduced volatility, the VaR and the CTE, but in the BS model it can generate a maximum loss that is higher than if hedging was not used. This strategy is not particularly sensitive to an increase in transaction costs, to the periodicity of compensation for the policyholders, nor to a strong fall in the price of the underlying asset. On the other hand, in case of an abnormally high death rate in the future, its costs and its volatility progressed in much the same way as when there was no hedging cover.

Semi-static hedging is not very sensitive to an increase in transaction costs but it, along with not hedging and static hedging, is at the mercy of the risk of increased future mortality. Although the increase in the maturity of the short-term options reduces the risk for the



portfolio, this strategy is not satisfactory. Indeed, semi-static hedging is based on a very strong assumption; the assumption that a continuum of options of maturity  $u$  exists. For example, our semi-static hedging portfolio is composed of 420 options with a maturity of one year. The availability of enough short-term options on the market with the required maturity is not guaranteed. Moreover, the results provided by this strategy are no better than for the other hedging strategies.

Certainly, the delta strategies are more expensive for the insurer, but they provide the best risk indicators. It is also clear that these strategies are sensitive to an increase in transaction costs, but in the case of an increase in future mortality rates, or a sharp fall in the price of the underlying asset, the impact on the risk indicators is less severe than for the other hedging strategies.

Insofar as we are satisfied with a hedging portfolio made up only of the risky asset and the risk-free asset, it seems logical that the DFR strategy and the dynamic strategy by minimization give the same results (see Gabriel & Surlas (2006)). However, the framework of Merton's model is an incomplete market; the underlying asset is not enough to hedge the risk of the portfolio. The insurer is subjected to the risk of a jump. The introduction of a second instrument of cover allows us to counteract this insufficiency. Let us also note that the options in the semi-static hedging portfolio are exerted in their maturity and that the profit thus made is then reinvested in the acquisition of the underlying asset and the risk-free asset. An alternative to this strategy would be to reinvest this amount in the purchase of a new hedging portfolio of short term options and to reproduce the operation. These aspects are not treated here. We will, however, return to them in a later study.

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## Appendix A. The theorem of CARR and Wu [2004]

### A1. Risk-neutral density

We assume that the random variable  $S_T$  for the maturity  $T$  admits a density under the risk-neutral measure  $\mathcal{Q}$ . We note this density as  $f_{S,T,t}(k) = f(S,t,k,T)$ . We also assume that  $P_K(k) = P(S,t,k,T)$  is  $C^2$  function. We can write:

$$\begin{aligned} P_K(k) &= P(S,t,k,T) = e^{-r(T-t)} E\left(\left[k - S_T\right]^+\right) \\ &= e^{-r(T-t)} \int_0^\infty [k-s]^+ f_{S,T,t}(s) ds \\ P_K(k) &= e^{-r(T-t)} \int_0^k (k-s) f_{S,T,t}(s) ds \\ &= e^{-r(T-t)} \left( k \times \int_0^k f_{S,T,t}(s) ds - \int_0^k s \times f_{S,T,t}(s) ds \right) \end{aligned}$$

If  $F_1(s)$  is the primitive<sup>1</sup> of  $f_{S,T,t}(s)$  and  $F_2$  is the primitive of  $s \times f_{S,T,t}(s)$ , we can write

$$P_K(k) = e^{-r(T-t)} \left[ k(F_1(k) - F_1(0)) - (F_2(k) - F_2(0)) \right]$$

We derive the two members of this equation by the strike  $k$ , we have:

$$\frac{\partial P_K(k)}{\partial k} = e^{-r(T-t)} \left[ (F_1(k) - F_1(0)) + k \times f_{S,T,t}(k) - k \times f_{S,T,t}(k) \right] = e^{-r(T-t)} \left[ F_1(k) - F_1(0) \right]$$

We derive once again and obtain the formula of Breeden & Litzenberger:

$$\frac{\partial^2 P_K(k)}{\partial k^2} = e^{-r(T-t)} f_{S,T,t}(k) = e^{-r(T-t)} f(S,t,k,T)$$

$$\Leftrightarrow f(S,t,k,T) = e^{r(T-t)} \frac{\partial^2 P(S,t,k,T)}{\partial k^2} \quad \square$$

### A2. Proof of the theorem of CARR and Wu [2004]

We assume frictionless markets and no arbitrage. No arbitrage implies that there exists a risk-neutral probability measure  $\mathcal{Q}$  defined on a probability space  $(\Omega, \mathcal{F}, \mathcal{Q})$  such that this instantaneous expected rate of return on every asset equals the instantaneous risk free rate  $r$ .

Analysis is restricted to the class of models in which the risk-neutral evolution of the stock price is Markov in the stock price  $S$  and the calendar time  $t$ .

We use  $P_t(K,T)$  to denote the time- $t$  price of a European put with strike  $K$  and maturity  $T$ . Our assumption that the state is fully described by the stock price and time implies that there exists a put pricing function  $P(S, t; K, T)$  such that:  $P_t(K,T) = P(S_t, t; K, T)$ ,  $t \in [0, T]$  and  $K \geq 0$

---

<sup>1</sup>  $\frac{\partial F_1(s)}{\partial s} = f_{S,T,t}(s)$  and  $\frac{\partial F_2(s)}{\partial s} = s \times f_{S,T,t}(s)$

Using the standard argument of financial theory,  $\bar{P}(S,t,K,T) = e^{-r(T-t)}P(S,t,K,T)$  is martingale, under the risk neutral measure. Then for all  $u \in [t, T]$  we have:

$\bar{P}(S,t,K,T) = E_Q(\bar{P}(S,u,K,T)|F_t)$  where  $(F_t)_{t \geq 0}$  is the filtration associated with the probability space  $(\Omega, F, Q)$ .

$$\begin{aligned} \bar{P}(S,t,K,T) &= E_Q(\bar{P}(S,u,K,T)|F_t) \\ &\Leftrightarrow P(S,t,K,T) = e^{-r(u-t)}E_Q(\bar{P}(S,u,K,T)|F_t) \\ P(S,t,K,T) &= e^{-r(u-t)} \int_0^\infty f(S,t,k,u)P(k,u,K,T)dk \\ &= e^{-r(u-t)} \int_0^\infty e^{r(u-t)} \frac{\partial^2 P}{\partial K^2}(S,t,k,u)P(k,u,K,T)dk, \text{ Using formula of Breeden \& Litzenberger.} \\ &= \int_0^\infty \frac{\partial^2 P(S,t,k,u)}{\partial k^2} P(k,u,K,T)dk \end{aligned}$$

We integrate that equation by parts twice and obtain:

$$\begin{aligned} P(S,t,K,T) &= \int_0^\infty \frac{\partial^2 P(S,t,k,u)}{\partial k^2} P(k,u,K,T)dk \\ &= \left[ P_u(K,T) \times \frac{\partial P(S,t,k,u)}{\partial k} \right]_{k=0}^{k=\infty} - \left[ P_t(k,u) \times \frac{\partial P(k,u,K,T)}{\partial k} \right]_{k=0}^{k=\infty} \\ &\quad + \int_0^\infty \frac{\partial^2 P(k,u,K,T)}{\partial k^2} P(S,t,k,u)dk \end{aligned}$$

Using the following boundary conditions

$$\left. \frac{\partial P(S,t,k,u)}{\partial k} \right|_{k \rightarrow \infty} = e^{-r(u-t)}, \left. \frac{\partial P(k,u,K,T)}{\partial k} \right|_{k \rightarrow \infty} = 0, P(0,u,K,T) = e^{-r(T-u)}K,$$

we obtain the final result:  $P(S,t,K,T) = \int_0^\infty \frac{\partial^2 P(k,u,K,T)}{\partial k^2} P(S,t,k,u)dk$ .

### A3. Proof of the theorem of CARR and Wu (2004) using no arbitrage arguments

Assume that at time  $t$ , the market price of a put option with strike  $K$  and maturity  $T$  exceeds the price of a gamma weighted portfolio of put options for some earlier maturity  $u$ , then, conditional on the validity of the Markovian assumption, the arbitrage is to sell the put option of strike  $K$  and maturity  $T$ , and to buy the gamma weighted portfolio of all put options maturing at the earlier date  $u$ . The cash received from selling the  $T$  maturity call exceeds the

cash spent buying the portfolio of nearer dated calls. At time  $u$ , the portfolio of expiring puts pay offs are  $\int_0^{\infty} \frac{\partial^2 P(k, u, K, T)}{\partial k^2} (k - S_u)^+ dk$ .

But we have:

$$\int_0^{\infty} \frac{\partial^2 P(k, u, K, T)}{\partial k^2} (k - S_u)^+ dk = \int_{S_u}^{\infty} \frac{\partial^2 P(k, u, K, T)}{\partial k^2} (k - S_u) dk .$$

Integrating this pay off by parts twice, we obtain:

$$\begin{aligned} \int_{S_u}^{\infty} \frac{\partial^2 P(k, u, K, T)}{\partial k^2} (k - S_u) dk &= \left[ (k - S_u) \times \frac{\partial P(k, u, K, T)}{\partial k} \right]_{S_u}^{\infty} - \int_{S_u}^{\infty} \frac{\partial P(k, u, K, T)}{\partial k} dk \\ &= \left[ (k - S_u) \times \frac{\partial P(k, u, K, T)}{\partial k} \right]_{S_u}^{\infty} - [P(k, u, K, T)]_{S_u}^{\infty} \\ &= P(S_u, u, K, T) \end{aligned}$$

If the price of a gamma weighted portfolio of put options for some earlier maturity  $u$  exceeds the price of a put option with strike  $K$  and maturity  $T$ , the arbitrage is to sell the gamma weighted portfolio of all put options maturing at the earlier date  $u$ , and to buy the put option of strike  $K$  and maturity  $T$ .

Under the assumption of no arbitrage opportunity, we have necessarily:

$$P(S, t, K, T) = \int_0^{\infty} \frac{\partial^2 P(k, u, K, T)}{\partial k^2} P(S, t, k, u) dk$$

## Appendix B. Gauss-hermite Quadrature

The Gauss-Hermite quadrature rule is designed to approximate an integral of the form

$$\int_{-\infty}^{+\infty} f(x) e^{-\frac{x^2}{2}} dx ,$$

where  $f(x)$  is an arbitrary smooth function. After some rescaling, the integral can be regarded as an expectation of  $f(x)$  where  $x$  is a normally distributed random variable with zero mean and variance of two.

For a given target function  $f(x)$ , the Gauss-Hermite quadrature rule generates a set of weights  $w_i$  and nodes  $x_i$ ,  $i = 1, 2, \dots, a$ . The abscissas for quadrature order  $a$  are given by the roots  $x_i$  of the hermite polynomials  $H_a(x)$ , which occur symmetrically around 0. The weights are

$$w_i = \frac{A_{a+1}\phi_a}{A_a H'_a(x_i) H_{a+1}(x_i)}$$

$$= \frac{A_a}{A_{a-1}} \frac{\phi_a}{H'_a(x_i) H_{a-1}(x_i)}$$

with  $A_a = 2^a$ ;  $\frac{A_{a+1}}{A_a} = 2$ ; and  $\phi_a = \sqrt{\pi} 2^a a!$

$$H'_a(x) = 2aH_{a-1}(x) = 2aH_a(x) - H_{a+1}(x)$$

that are defined by

$$\int_{-\infty}^{+\infty} f(x) e^{-\frac{x^2}{2}} dx = \sum_{j=1}^a w_j f(x_j) + \frac{a! \sqrt{\pi}}{2^a} \frac{f^{(2a)}(\xi)}{(2a)!} \text{ where } \xi \in (-\infty, +\infty).$$

The approximation error vanishes if the integrand  $f(x)$  is a polynomial of degree equal or less than  $2a-1$ .

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