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On the relationship between Spearman's rho and Kendall's tau for pairs of continuous random variables

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Abstract

It has long been known that for many joint distributions exhibiting weak dependence, the sample value of Spearman's rho is about 50% larger than the sample value of Kendall's tau. We explain this behavior by showing that for the population analogs of these statistics, the ratio of rho to tau approaches 3/2 as the joint distribution approaches that of two independent random variables. We also find sufficient conditions for determining the direction of the inequality between three times tau and twice rho when the underlying joint distribution is absolutely continuous.

Keywords: Copula; Inequality; Kendall's tau; Spearman's rho

1. Introduction

The two most commonly used nonparametric measures of association for two random variables are Spearman's rho (ρ) and Kendall's tau (τ). For many joint distributions these two measures have different values, as they measure different aspects of the dependence structure. For example, if X and Y are random variables with marginal distribution functions F and G, respectively, then Spearman's ρ is the ordinary (Pearson) correlation coefficient of the transformed random variables $F(X)$ and $G(Y)$, while Kendall's τ is the difference between the probability of concordance $P[(X_1 - X_2)(Y_1 - Y_2) > 0]$ and the probability of discordance $P[(X_1 - X_2)(Y_1 - Y_2) < 0]$ for two independent pairs (X_1, Y_1) and (X_2, Y_2) of observations drawn from the distribution. In terms of dependence properties, Spearman's ρ is a measure of average quadrant dependence, while Kendall's τ is a measure of average likelihood ratio dependence (Nelsen, 1992).

However, in spite of these differences, there is often an observable pattern in the sample values. In comparing R and T (the sample values of ρ and τ) Gibbons (1976) writes 'For most degrees of association that occur in practice (that is, absolute values not too close to 1) R is about 50 percent greater than T in absolute value.' Kendall (1948) states 'T will be about two-thirds of the value of R when [the sample size] n is large.' In this paper we will examine relationships between the population versions of ρ and τ that lead to such observations.

The relationship between ρ and τ has received considerable attention in recent years. Hutchinson and Lai (1990) conjectured that $-1 + \sqrt{1 + 3\tau} \leqslant \rho \leqslant \min\{3\tau/2, 2\tau - \tau^2\}$ for stochastically increasing random variables; however, the bound $\rho \leqslant 3\tau/2$ was disproved in (Nelsen, 2006, Exercise 5.38). Capéraà and Genest (1993) have shown that $\rho \geqslant \tau \geqslant 0$

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whenever one of the random variables is simultaneously left-tail decreasing and right-tail increasing in the other (see Section 5). Hürlimann (2003) has shown that the entire Hutchinson and Lai conjecture holds for bivariate extreme value distributions. Schmitz (2004) conjectured that $\lim_{n\to\infty} \rho_n/\tau_n = \frac{3}{2}$, where ρ_n and τ_n denote Spearman's ρ and Kendall's τ for the extreme order statistics $X_{(1)} = \min\{X_1, \ldots, X_n\}$ and $X_{(n)} = \max\{X_1, \ldots, X_n\}$ of an i.i.d. sample $X_1, \ldots, X_n$; and this was recently proved by Li and Li (2007). Chen (2006) has established inequalities between ρ_n and τ_n.

The contribution of this paper is to prove (in Section 3) that, under mild regularity conditions, the limit of the ratio ρ/τ is $3/2$ as the joint distribution of the random variables approaches independence. (Durrleman et al., 2000 present integral conditions equivalent to this limit in the absolutely continuous case.) In Section 4 we give sufficient conditions (in the absolutely continuous case) for determining the direction of the inequality between 3τ and 2ρ, and in Section 5 we present a new proof of the above-mentioned result of Capéraà and Genest. We begin with some background material and two preliminary lemmas.

2. Preliminaries

Let X and Y be continuous random variables with joint distribution function H and marginal distribution functions F (of X) and G (of Y). The *copula* of X and Y is the unique function $C : \mathbf{I}^2 \to \mathbf{I} = [0, 1]$ defined implicitly by the relation $H(x, y) = C(F(x), G(y))$ for all real x and y. The population version of Spearman's ρ is expressible in terms of C as follows (Schweizer and Wolff, 1981):

$$\rho = 12\iint_{\mathbf{I}^2} C(u, v)\,\mathrm{d}u\,\mathrm{d}v - 3 = 12\iint_{\mathbf{I}^2} uv\,\mathrm{d}C(u, v) - 3, \tag{2.1}$$

where $\mathrm{d}C$ denotes the doubly stochastic measure induced on $\mathbf{I}^2$ by C (and equals $(\partial^2 C/\partial u\partial v)(u, v)\,\mathrm{d}u\,\mathrm{d}v$ when C is absolutely continuous). Similarly, the population version of Kendall's τ is also expressible in terms of C (Schweizer and Wolff, 1981; Nelsen, 2006):

$$\tau = 4\iint_{\mathbf{I}^2} C(u, v)\,\mathrm{d}C(u, v) - 1 = 1 - 4\iint_{\mathbf{I}^2} \frac{\partial C}{\partial u}(u, v)\frac{\partial C}{\partial v}(u, v)\,\mathrm{d}u\,\mathrm{d}v. \tag{2.2}$$

(Note: Since C is Lipschitz, it is differentiable almost everywhere (a.e.), and hence $\partial C/\partial u$ and $\partial C/\partial v$ exist a.e. on $\mathbf{I}^2$.)

The following lemma provides an alternate form for the evaluation of Spearman's ρ that will be useful in the sequel.

Lemma 2.1. *Let X and Y be continuous random variables with copula C. Then the population version of Spearman's ρ for X and Y is given by*

$$\rho = 3 - 6\iint_{\mathbf{I}^2} \left(u\,\frac{\partial C}{\partial u}(u, v) + v\,\frac{\partial C}{\partial v}(u, v)\right)\mathrm{d}u\,\mathrm{d}v. \tag{2.3}$$

Proof. For any v in $\mathbf{I}$, $uC(u, v)$ is Lipschitz and hence an absolutely continuous function of u on $\mathbf{I}$, so we can apply the Fundamental Theorem of Calculus to conclude

$$\int_0^1 \frac{\partial}{\partial u}[uC(u, v)]\,\mathrm{d}u = [uC(u, v)]_{u=0}^{u=1} = v$$

so that

$$\iint_{\mathbf{I}^2} \left(C(u, v) + u\,\frac{\partial C}{\partial u}(u, v)\right)\mathrm{d}u\,\mathrm{d}v = \int_0^1\int_0^1 \frac{\partial}{\partial u}[uC(u, v)]\,\mathrm{d}u\,\mathrm{d}v = \int_0^1 v\,\mathrm{d}v = \frac{1}{2};$$

and similarly

$$\iint_{\mathbf{I}^2} \left(C(u, v) + v\,\frac{\partial C}{\partial v}(u, v)\right)\mathrm{d}u\,\mathrm{d}v = \frac{1}{2}.$$

It now follows from (2.1) that

$$\iint_{\mathbf{I}^2}\left(u\,\frac{\partial C}{\partial u}(u,v)+v\,\frac{\partial C}{\partial v}(u,v)\right)\mathrm{d}u\,\mathrm{d}v=1-2\iint_{\mathbf{I}^2}C(u,v)\,\mathrm{d}u\,\mathrm{d}v=1-\frac{\rho+3}{6},$$

which yields the desired result. □

The following technical lemma will be needed in Proof of Theorem 3.1 in the next section.

Lemma 2.2. *Let* $F(\theta)=\iint_{\mathbf{I}^2}f(\theta,u,v)\,\mathrm{d}u\,\mathrm{d}v$, *where* $f_\theta(u,v)=f(\theta,u,v)$ *is Lebesgue integrable on* $\mathbf{I}^2$ *for each* θ *in an interval* J *centered at* 0. *If* $\partial f/\partial\theta$ *is continuous on* $J\times\mathbf{I}^2$, *then* F *is differentiable on* J *and* $F'(\theta)=\iint_{\mathbf{I}^2}(\partial f/\partial\theta)(\theta,u,v)\,\mathrm{d}u\,\mathrm{d}v$ *for each* $\theta\in J$.

Proof. Fix $\theta_0\in J$ and choose a compact subinterval K of J with θ_0 in its interior. Let $\varepsilon>0$ be given. Since $K\times\mathbf{I}^2$ is compact, $\partial f/\partial\theta$ is uniformly continuous on $K\times\mathbf{I}^2$, so we choose $\delta>0$ so that $|(\partial f/\partial\theta)(\theta,u,v)-(\partial f/\partial\theta)(\theta',u',v')|<\varepsilon$ whenever $|(\theta,u,v)-(\theta',u',v')|<\delta$ and (θ,u,v), (θ',u',v') in $K\times\mathbf{I}^2$. For any θ in K with $0<|\theta-\theta_0|<\delta$ we have

$$\begin{aligned}\left|\frac{F(\theta)-F(\theta_0)}{\theta-\theta_0}-\iint_{\mathbf{I}^2}\frac{\partial f}{\partial\theta}(\theta_0,u,v)\,\mathrm{d}u\,\mathrm{d}v\right|&\leqslant\iint_{\mathbf{I}^2}\left|\frac{f(\theta,u,v)-f(\theta_0,u,v)}{\theta-\theta_0}-\frac{\partial f}{\partial\theta}(\theta_0,u,v)\right|\mathrm{d}u\,\mathrm{d}v\\&=\iint_{\mathbf{I}^2}\left|\frac{\partial f}{\partial\theta}(\xi,u,v)-\frac{\partial f}{\partial\theta}(\theta_0,u,v)\right|\mathrm{d}u\,\mathrm{d}v,\end{aligned}$$

where, by the mean-value theorem, ξ is a function of (u,v) with values between θ and θ_0 and hence in K. The last expression is less than ε, and the result follows. □

Remarks. (1) One can conclude that F is differentiable at 0 and that $F'(0)=\iint_{\mathbf{I}^2}(\partial f/\partial\theta)(0,u,v)\,\mathrm{d}u\,\mathrm{d}v$ under the weaker assumption that the limits $\lim_{\theta\to 0}(\partial f/\partial\theta)(\theta,u,v)=(\partial f/\partial\theta)(0,u,v)$ exist and are uniform for almost all (u,v) in $\mathbf{I}^2$ (Rudin, 1976). (2) The one-sided versions of Lemma 2.2 and the preceding remark hold.

3. The limit theorem

We are now in a position to establish a relationship between Spearman's ρ and Kendall's τ in many families of distributions that include the independence case (recall that the copula for any independent pair of continuous random variables is $\Pi(u,v)=uv$).

Theorem 3.1. *Let* $\{C(\theta,u,v)\}$ *be a family of copulas in which the (real-valued) parameter* θ *belongs to an open interval containing* 0, *with* $C(0,u,v)=uv$. *Let* $\rho(\theta)$ *and* $\tau(\theta)$ *denote the population versions of Spearman's* ρ *and Kendall's* τ, *respectively, for the copula* $C(\theta,u,v)$. *If* (a) $\partial C/\partial\theta$, $\partial/\partial\theta(u\,(\partial C/\partial u)+v(\partial C/\partial v))$, *and* $\partial/\partial\theta((\partial C/\partial u)(\partial C/\partial v))$ *are continuous on* $J\times\mathbf{I}^2$ *for some interval* J *centered at* 0, *and* (b) $\iint_{\mathbf{I}^2}\partial C/\partial\theta(0,u,v)\,\mathrm{d}u\,\mathrm{d}v\neq 0$, *then* $\lim_{\theta\to 0}(\rho(\theta)/\tau(\theta))=\frac{3}{2}$.

Proof. Since $\rho(\theta)=12\iint_{\mathbf{I}^2}C(\theta,u,v)\,\mathrm{d}u\,\mathrm{d}v-3$, Lemma 2.2 yields $\rho'(0)=12\iint_{\mathbf{I}^2}(\partial C/\partial\theta)(0,u,v)\,\mathrm{d}u\,\mathrm{d}v\neq 0$. It follows from Lemmas 2.1 and 2.2 that

$$\rho'(0)=-6\iint_{\mathbf{I}^2}\left(u\,\frac{\partial^2 C}{\partial\theta\partial u}(0,u,v)+v\,\frac{\partial^2 C}{\partial\theta\partial v}(0,u,v)\right)\mathrm{d}u\,\mathrm{d}v.$$

Using Lemma 2.2 in the second form for $\tau(\theta)$ in (2.2) and noting that $(\partial C/\partial v)(0,u,v)=u$ and $(\partial C/\partial u)(0,u,v)=v$ yield

$$\tau'(0)=-4\iint_{\mathbf{I}^2}\left(\frac{\partial^2 C}{\partial\theta\partial u}(0,u,v)u+v\,\frac{\partial^2 C}{\partial\theta\partial v}(0,u,v)\right)\mathrm{d}u\,\mathrm{d}v=\frac{2}{3}\,\rho'(0)\neq 0.$$

Since $\rho(0)=\tau(0)=0$, we see that τ is nonvanishing in a deleted neighborhood of 0 and that

$$\lim_{\theta\to 0}\frac{\rho(\theta)}{\tau(\theta)}=\lim_{\theta\to 0}\frac{(\rho(\theta)-\rho(0))/\theta}{(\tau(\theta)-\tau(0))/\theta}=\frac{\rho'(0)}{\tau'(0)}=\frac{3}{2}. \qquad \square$$

Remarks. (1) Condition (b) in Theorem 3.1 can be replaced by the nonvanishing of either of the integral expressions for $\rho'(0)$ or $\tau'(0)$. In that case the continuity assumption on $\partial C/\partial\theta$ is unnecessary. (2) Since the preceding proof required derivatives only at 0, the result remains valid if (a) is replaced by the weaker condition that the θ limits at zero of each of the partial derivatives with respect to θ in (a) exists and is uniform for almost all (u, v) in $\mathbf{I}^2$. (3) The one-sided versions of Theorem 3.1 and the preceding remark hold.

Example 3.1. The requirement in part (b) of Theorem 3.1 that $\iint_{\mathbf{I}^2}(\partial C/\partial\theta)(0,u,v)\,\mathrm{d}u\,\mathrm{d}v\neq 0$ (or equivalently, that $\rho'(0)\neq 0$) is necessary. Consider the copulas $C(\theta,u,v)=uv+\theta uv(1-u)(1-v)(u-v)$ for θ in $[-1,1]$. Since C is a polynomial in θ, u, v, part (a) of Theorem 3.1 holds on $(-1,1)\times\mathbf{I}^2$. Simple calculations using (2.1) and (2.2) yield $\rho(\theta)\equiv 0$ and $\tau(\theta)=\theta^2/450$, so that $\lim_{\theta\to 0}\rho(\theta)/\tau(\theta)=0$ rather than $\frac{3}{2}$.

For many families of distributions the conclusion of Theorem 3.1 can be established without appealing to copula properties by using the computational forms for ρ and τ. The classic example is the bivariate normal with (Pearson) correlation coefficient θ, for which $\rho(\theta)=(6/\pi)\arcsin(\theta/2)$ and $\tau(\theta)=(2/\pi)\arcsin\theta$ (Kruskal, 1958). In this case, elementary calculus yields $\lim_{\theta\to 0}\rho(\theta)/\tau(\theta)=\frac{3}{2}$. We now examine four families of distributions in which this is not the case. In each example, (a) and (b) refer to the two conditions in the hypothesis of Theorem 3.1; and in Example 3.4 we recall (Nelsen, 2006) that survival copulas (i.e., copulas which couple univariate survival functions to form joint survival functions) can be used in place of copulas in the computation of ρ and τ.

Example 3.2. The *Ali-Mikhail-Haq* family of copulas are given by $C(\theta,u,v)=uv/[1-\theta(1-u)(1-v)]$ for θ in $[-1,1]$. Since C is a rational function of θ, u, v defined at each point of $(-1,1)\times\mathbf{I}^2$, each partial derivative in (a) is continuous on $(-1,1)\times\mathbf{I}^2$; and since $(\partial C/\partial\theta)(0,u,v)=uv(1-u)(1-v)$, it is clear that (b) holds. Hence $\lim_{\theta\to 0}\rho(\theta)/\tau(\theta)=\frac{3}{2}$.

Example 3.3. Let $C(\theta,u,v)=\theta\min\{u,v\}+(1-\theta)uv$, θ in $[0,1]$. The partial derivatives in (a) are continuous on $[0,1)\times\mathbf{I}^2$ since C is a polynomial in θ, u, v in both the triangle $0\leqslant u\leqslant v\leqslant 1$ and its complement, and the partial derivatives agree on $u=v$. Since $\rho'(0)=1$, (b) holds. Hence $\lim_{\theta\to 0}\rho(\theta)/\tau(\theta)=\frac{3}{2}$. Note that the only absolutely continuous member of this family is $C(0,u,v)=uv$.

Example 3.4. The survival copulas for *Gumbel's bivariate exponential distributions* are given by $C(\theta,u,v)=uv\exp(-\theta\ln u\ln v)$ for θ in $[0,1]$ (extended by continuity to $C(\theta,u,0)=C(\theta,0,v)=0$). Now $(\partial C/\partial\theta)(\theta,u,v)=-uv\ln u\ln v\exp(-\theta\ln u\ln v)$ if $uv\neq 0$, and $(\partial C/\partial\theta)(\theta,u,v)$ vanishes if $uv=0$, and thus $\partial C/\partial\theta$ is continuous on $[0,1)\times\mathbf{I}^2$. The other partial derivatives in (a) are similarly continuous on $[0,1)\times\mathbf{I}^2$ and (b) clearly holds, so $\lim_{\theta\to 0^+}\rho(\theta)/\tau(\theta)=\frac{3}{2}$. This result can also be established by using the expressions for $\rho(\theta)$ and $\tau(\theta)$ in terms of the exponential integral $\mathrm{Ei}(x)=\int_{-\infty}^{x}(\mathrm{e}^t/t)\,\mathrm{d}t$:

$$\tau(\theta)=\mathrm{e}^{2/\theta}\mathrm{Ei}(-2/\theta)\quad\text{and}\quad\rho(\theta)=-3-[12\mathrm{e}^{4/\theta}\mathrm{Ei}(-4/\theta)]/\theta.$$

Example 3.5. The *Plackett* family of copulas is given by

$$C(\theta,u,v)=\frac{[1+\theta(u+v)]-\sqrt{[1+\theta(u+v)]^2-4\theta(\theta+1)uv}}{2\theta}$$

for $\theta>-1$, $\theta\neq 0$, and $C(0,u,v)=uv$. Note that C is continuous on $(-1,\infty)\times\mathbf{I}^2$. Since $C(\theta,u,v)$ is defined implicitly (for all $\theta>-1$) by the equation

$$\theta C^2-[1+\theta(u+v)]C+(\theta+1)uv=0,$$

we see that $\partial C/\partial\theta = (C^2 - (u+v)C + uv)/(1+\theta(u+v) - 2\theta C)$. Since the denominator of $\partial C/\partial\theta$ is 1 on $\{0\}\times\mathbf{I}^2$, it is nonvanishing on $J\times\mathbf{I}^2$ for some interval J centered at 0 by continuity. Hence $\partial C/\partial\theta$ is continuous on $J\times\mathbf{I}^2$. The other partial derivatives in (a) are also continuous on $J\times\mathbf{I}^2$ and $\iint_{\mathbf{I}^2}(\partial C/\partial\theta)(0,u,v)\,\mathrm{d}u\,\mathrm{d}v = \frac{1}{36}$, so (b) holds, and hence $\lim_{\theta\to 0}\rho(\theta)/\tau(\theta) = \frac{3}{2}$. Although $\rho(\theta) = [2\theta + \theta^2 - 2(\theta+1)\ln(\theta+1)]/\theta^2$ for $\theta \neq 0$, there does not appear to be a closed form expression for $\tau(\theta)$.

4. The inequality between 3τ and 2ρ

While Theorem 3.1 establishes the limiting behavior for the ratio $\rho(\theta)/\tau(\theta)$, it does not tell us if the limit of $\frac{3}{2}$ is approached from below or above, i.e., if $3\tau(\theta)\geqslant 2\rho(\theta)$ or $3\tau(\theta)\leqslant 2\rho(\theta)$. In this section we establish two sufficient conditions for determining the direction of the inequality between $3\tau(\theta)$ and $2\rho(\theta)$ for an absolutely continuous family of copulas.

Assume throughout this section that C is an absolutely continuous copula. Using various forms for ρ and τ from (2.1)–(2.3), we have (for simplicity we have suppressed the arguments of C and its partial derivatives, and of ρ and τ)

$$\iint_{\mathbf{I}^2}\left[C\,\frac{\partial^2 C}{\partial u\partial v} - \frac{\partial C}{\partial u}\,\frac{\partial C}{\partial v}\right]\mathrm{d}u\,\mathrm{d}v = \frac{1+\tau}{4} - \frac{1-\tau}{4} = \frac{\tau}{2} \tag{4.1}$$

and

$$\iint_{\mathbf{I}^2}\left[uv\,\frac{\partial^2 C}{\partial u\partial v} - u\,\frac{\partial C}{\partial u} - v\,\frac{\partial C}{\partial v} + C\right]\mathrm{d}u\,\mathrm{d}v = \frac{3+\rho}{12} - \frac{3-\rho}{6} + \frac{3+\rho}{12} = \frac{\rho}{3}. \tag{4.2}$$

Hence whenever the difference of the two integrands is nonnegative, the difference of the two integrals yields $\tau/2 - \rho/3\geqslant 0$, and thus $3\tau\geqslant 2\rho$. At points (u,v) where $C = uv$, both integrands are 0, and at points (u,v) where $C\neq uv$ we have

$$\begin{aligned} &C\,\frac{\partial^2 C}{\partial u\partial v} - \frac{\partial C}{\partial u}\,\frac{\partial C}{\partial v} - \left[uv\,\frac{\partial^2 C}{\partial u\partial v} - u\,\frac{\partial C}{\partial u} - v\,\frac{\partial C}{\partial v} + C\right] \\ &\quad = (C-uv)\left(\frac{\partial^2 C}{\partial u\partial v} - 1\right) - \left(\frac{\partial C}{\partial u} - v\right)\left(\frac{\partial C}{\partial v} - u\right) = (C-uv)^2\,\frac{\partial^2}{\partial u\partial v}\,\ln|C-uv|, \end{aligned}$$

so that the difference of the integrands is nonnegative if and only if $(\partial^2/\partial u\partial v)\ln|C-uv|$ is nonnegative. Hence we have proved.

Theorem 4.1. *Let C be an absolutely continuous copula. If $(\partial^2/\partial u\partial v)\ln|C-uv|\geqslant 0$ whenever $C\neq uv$, then $3\tau\geqslant 2\rho$; and if $(\partial^2/\partial u\partial v)\ln|C-uv|\leqslant 0$ whenever $C\neq uv$, then $3\tau\leqslant 2\rho$.*

(*Note*: The hypothesis of absolute continuity is necessary, as the singular copula $M(u,v) = \min\{u,v\}$ satisfies $M\neq uv$ on $(0,1)^2$ and $(\partial^2/\partial u\partial v)\ln|M-uv|\leqslant 0$ a.e. in $(0,1)^2$, yet $3 = 3\tau_M > 2\rho_M = 2$.)

Example 4.1. The *Kotz and Johnson iterated Farlie-Gumbel-Morgenstern family* of absolutely continuous copulas (Drouet Mari and Kotz, 2001) is given by $C_{\alpha,\beta}\,(u,v) = uv + uv(1-u)(1-v)(\alpha+\beta uv)$, for $|\alpha|\leqslant 1$, $-\alpha-1\leqslant\beta\leqslant(3-\alpha+\sqrt{9-6\alpha-3\alpha^2})/2$. Here $(\partial^2/\partial u\partial v)\ln|C_{\alpha,\beta} - uv| = \alpha\beta/(\alpha+\beta uv)^2$ for (u,v) in $(0,1)^2$ for which $C_{\alpha,\beta}\neq uv$. Thus, $(\partial^2/\partial u\partial v)\ln|C_{\alpha,\beta}-uv|\geqslant 0$ (and consequently $3\tau_{\alpha,\beta}\geqslant 2\rho_{\alpha,\beta}$) if and only if $\alpha\beta\geqslant 0$. This is confirmed by evaluating $\rho_{\alpha,\beta}$ and $\tau_{\alpha,\beta}$: $\rho_{\alpha,\beta} = \alpha/3 + \beta/12$; $\tau_{\alpha,\beta} = 2\alpha/9 + \beta/18 + \alpha\beta/450$, so that $\rho_{\alpha,\beta} = 3\tau_{\alpha,\beta}/2 - \alpha\beta/300$. Note that for this two-parameter family, $\lim_{(\alpha,\beta)\to(0,0)}\rho_{\alpha,\beta}/\tau_{\alpha,\beta}$ does not exist, since $\lim_{\alpha\to 0}\rho_{\alpha,0}/\tau_{\alpha,0} = 3/2$ while $\lim_{\alpha\to 0}\rho_{\alpha,-4\alpha}/\tau_{\alpha,-4\alpha} = 0$.

Recall that a pair (X,Y) of continuous random variables with copula C is positively quadrant dependent (PQD) if $C(u,v)\geqslant uv$ on $\mathbf{I}^2$. The above example shows that the direction of the inequality between 3τ and 2ρ is not a consequence of positive quadrant dependence, since $C_{\alpha,\beta}$ is PQD if and only if $\alpha\geqslant 0$ and $\beta\geqslant 0$.

Remark. One can construct a family of copulas for which $3\tau = 2\rho$ by solving the partial differential equation $(\partial^2/\partial u \partial v) \ln |C - uv| = 0$. Solutions have the form $C(u, v) = uv + f(u)g(v)$ for appropriate functions f and g. These copulas are studied in detail in (Rodríguez Lallena and Úbeda Flores, 2004).

The integrals in (4.1) and (4.2) can be re-written as

$$\frac{\tau}{2} = \iint_{\mathbf{I}^2} C^2 \frac{\partial^2}{\partial u \partial v} \ln C \, du \, dv \quad \text{and} \quad \frac{\rho}{3} = \iint_{\mathbf{I}^2} (uv)^2 \frac{\partial^2}{\partial u \partial v} \left(\frac{C}{uv}\right) du \, dv,$$

and hence $3\tau \geqslant 2\rho$ whenever

$$C^2 \frac{\partial^2}{\partial u \partial v} \ln C \geqslant (uv)^2 \frac{\partial^2}{\partial u \partial v} \left(\frac{C}{uv}\right) \quad \text{on } \mathbf{I}^2.$$

But since

$$\frac{\partial^2}{\partial u \partial v} \ln C = \frac{\partial^2}{\partial u \partial v} \ln \left(\frac{C}{uv}\right),$$

we have:

Theorem 4.2. *Let C be an absolutely continuous copula. If*

$$\left(\frac{C}{uv}\right)^2 \frac{\partial^2}{\partial u \partial v} \ln \left(\frac{C}{uv}\right) - \frac{\partial^2}{\partial u \partial v} \left(\frac{C}{uv}\right) \geqslant 0 \quad \textit{whenever } C \neq uv,$$

then $3\tau \geqslant 2\rho$; *and if*

$$\left(\frac{C}{uv}\right)^2 \frac{\partial^2}{\partial u \partial v} \ln \left(\frac{C}{uv}\right) - \frac{\partial^2}{\partial u \partial v} \left(\frac{C}{uv}\right) \leqslant 0 \quad \textit{whenever } C \neq uv,$$

then $3\tau \leqslant 2\rho$.

This sufficient condition for the direction of the inequality is advantageous when C has uv as a factor, as the following example illustrates.

Example 4.2. The family of survival copulas C_θ for Gumbel's bivariate exponential distribution from Example 3.4 are absolutely continuous and, for θ in (0, 1], $C_\theta \neq uv$ on $(0, 1)^2$ and

$$\left(\frac{C_\theta}{uv}\right)^2 \frac{\partial^2}{\partial u \partial v} \ln \left(\frac{C_\theta}{uv}\right) - \frac{\partial^2}{\partial u \partial v} \left(\frac{C_\theta}{uv}\right) = \frac{-\theta C_\theta(u, v)}{(uv)^2} [\exp(-\theta \ln u \ln v) - 1 + \theta \ln u \ln v] \leqslant 0$$

for (u, v) in $(0, 1)^2$. Hence $3\tau(\theta) \leqslant 2\rho(\theta)$ for this family.

5. The inequality between τ and ρ

As another application of Lemma 2.1, note that

$$\iint_{\mathbf{I}^2} \left(C - u \frac{\partial C}{\partial u} - v \frac{\partial C}{\partial v} + \frac{\partial C}{\partial u} \frac{\partial C}{\partial v}\right) du \, dv = \frac{\rho + 3}{12} - \frac{3 - \rho}{6} + \frac{1 - \tau}{4} = \frac{1}{4}(\rho - \tau), \tag{5.1}$$

which enables us to give a simple proof of the following result of Capéraà and Genest (1993). Recall (Lehmann, 1966; Capéraà and Genest, 1993) that the random variable Y is said to be *left-tail decreasing* in X, denoted $\mathrm{LTD}(Y|X)$, if $\Pr(Y \leqslant y | X \leqslant x)$ is nonincreasing in x for all y. Similarly, Y is said to be *right-tail increasing* in X, denoted $\mathrm{RTI}(Y|X)$, if $\Pr(Y > y | X > x)$ is nondecreasing in x for all y. The concepts left-tail increasing (LTI) and right-tail decreasing (RTD) are defined analogously.

Theorem 5.1. *Let X and Y be continuous random variables with copula C. If* $\mathrm{LTD}(Y|X)$ *and* $\mathrm{RTI}(Y|X)$ *both hold, then* $\rho \geqslant \tau \geqslant 0$. *Similarly, if* $\mathrm{LTI}(Y|X)$ *and* $\mathrm{RTD}(Y|X)$ *both hold, then* $\rho \leqslant \tau \leqslant 0$. (*In both statements, the roles of X and Y can be reversed.*)

Proof. We prove only the first statement. Since $\mathrm{LTD}(Y|X)$ and $\mathrm{RTI}(Y|X)$ each imply that X and Y are PQD, $\tau \geqslant 0$ (Lehmann, 1966). To establish that $\rho \geqslant \tau$, we need only show that the integrand in (5.1) is nonnegative a.e. in $\mathbf{I}^2$. Recall (Nelsen, 2006) that in terms of the copula C of X and Y, $\mathrm{LTD}(Y|X)$ if and only if for any v in $\mathbf{I}$, $C(u,v)/u$ is nonincreasing in u on (0, 1]. Since nonincreasing functions are differentiable a.e. and their derivatives are nonpositive at each point of differentiability, we see that $C - u \cdot \partial C/\partial u \geqslant 0$ a.e. on $\mathbf{I}^2$. Similarly, $\mathrm{RTI}(Y|X)$ if and only if for any v in $\mathbf{I}$, $(v - C(u,v))/(1-u)$ is nonincreasing in u on [0, 1), which implies that $\partial C/\partial u - v \geqslant u \cdot \partial C/\partial u - C$ a.e. on $\mathbf{I}^2$. Because $0 \leqslant \partial C/\partial v \leqslant 1$ a.e. in $\mathbf{I}^2$,

$$\begin{aligned} C - u\frac{\partial C}{\partial u} - v\frac{\partial C}{\partial v} + \frac{\partial C}{\partial u}\frac{\partial C}{\partial v} &= C - u\frac{\partial C}{\partial u} + \frac{\partial C}{\partial v}\left(\frac{\partial C}{\partial u} - v\right) \\ &\geqslant C - u\frac{\partial C}{\partial u} + \frac{\partial C}{\partial v}\left(u\frac{\partial C}{\partial u} - C\right) = \left(C - u\frac{\partial C}{\partial u}\right)\left(1 - \frac{\partial C}{\partial v}\right) \geqslant 0 \end{aligned}$$

a.e. on $\mathbf{I}^2$. The result follows from (5.1). □

Example 5.1. The Ali-Mikhail-Haq family of copulas C_θ from Example 3.2 are absolutely continuous and, for θ in $[-1,1]\backslash\{0\}$, $C_\theta \neq uv$ on $(0,1)^2$ and $(\partial^2/\partial u \partial v)\ln|C_\theta - uv| = \theta/[1-\theta(1-u)(1-v)]^2$ for (u,v) in $(0,1)^2$, so the sign of θ determines the direction of the inequality between $3\tau(\theta)$ and $2\rho(\theta)$ by Theorem 4.1. But C_θ is LTD and RTI for $\theta \geqslant 0$ and LTI and RTD for $\theta \leqslant 0$ (all for both $Y|X$ and $X|Y$, since C_θ is symmetric in its arguments), so the sign of θ also determines the signs of $\rho(\theta)$ and $\tau(\theta)$, and the direction of the inequality between $\rho(\theta)$ and $\tau(\theta)$ by Theorem 5.1. Hence $0 \leqslant \tau(\theta) \leqslant \rho(\theta) \leqslant 3\tau(\theta)/2$ for $\theta \geqslant 0$; and $0 \geqslant \tau(\theta) \geqslant \rho(\theta) \geqslant 3\tau(\theta)/2$ for $\theta \leqslant 0$. All of these results (and those in Example 3.2) can be obtained from the explicit expressions for $\tau(\theta)$ and $\rho(\theta)$, but with much more labor: $\tau(0) = \rho(0) = 0$ and for $\theta \neq 0$,

$$\tau(\theta) = 1 - \frac{2}{3\theta} - \frac{2(1-\theta)^2}{3\theta^2}\ln(1-\theta)$$

and

$$\rho(\theta) = \frac{12(1+\theta)}{\theta^2}\operatorname{dilog}(1-\theta) - \frac{24(1-\theta)}{\theta^2}\ln(1-\theta) - \frac{3(\theta+12)}{\theta},$$

where $\operatorname{dilog}(x)$ is the dilogarithm function defined by $\operatorname{dilog}(x) = \int_1^x (\ln t/(1-t))\,\mathrm{d}t$.

6. Concluding remarks

Another inequality between ρ and τ is due to Daniels (1950): $-1 \leqslant 3\tau - 2\rho \leqslant 1$ for any copula C. Daniels' probabilistic proof is involved and difficult to follow, thus it would be advantageous to have a copula-based proof. Note that

$$\left(u - \frac{\partial C}{\partial v}\right)\left(v - \frac{\partial C}{\partial u}\right) = uv - u\frac{\partial C}{\partial u} - v\frac{\partial C}{\partial v} + \frac{\partial C}{\partial u}\frac{\partial C}{\partial v},$$

and hence from (2.2) and (2.3),

$$\iint_{\mathbf{I}^2}\left(u - \frac{\partial C}{\partial v}\right)\left(v - \frac{\partial C}{\partial u}\right)\mathrm{d}u\,\mathrm{d}v = \frac{1}{4} - \frac{3-\rho}{6} + \frac{1-\tau}{4} = \frac{1}{12}(2\rho - 3\tau).$$

Hence an alternate proof of Daniels' inequality could be obtained if one could show that for any copula C,

$$-\frac{1}{12} \leqslant \iint_{\mathbf{I}^2}\left(u - \frac{\partial C}{\partial v}\right)\left(v - \frac{\partial C}{\partial u}\right)\mathrm{d}u\,\mathrm{d}v \leqslant \frac{1}{12}.$$

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