ADVANCED MONTE CARLO TECHNIQUES: AN APPROACH FOR FOREIGN EXCHANGE DERIVATIVE PRICING

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Abstract

In this thesis, an investigation into foreign exchange rate option (commonly called currency option) pricing models is described. Using the example of cash currency options,the pricing of the options is sought as a more general case of other plain vanilla options.

By setting up a high-dimensional stochastic environment, selection of an appropriate mathematical implementation becomes crucial. In this thesis, advanced Monte Carlo techniques are presented and used intensively. For early-exercise currency options, an enhanced version of the basic Longstaff and Schwartz (2001) technique as proposed by Duck et al. (2005) is employed, which enables an option pricing speedup of 20 times. With this powerful tool, currency-option models can be easily extended to stochastic-interest-rates and stochastic-volatilities models. Having addressed the practical issue of pricing and hedging difficulties of one of the most heavy traded products in the foreign exchange market, discretely-monitored barrier options, the focus of this thesis then moves on to exploring a new area of the foreign exchange market to overcome the discontinuity of the Greeks of standard barrier options, with a new class of options, which we term quantile Parisian and ParAsian options. A number of other aspects, linked to currency exchange are also studied.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institution of learning.

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Chapter 1

Option Pricing Theory

A theory is a good theory if it satisfies two requirements: it must accurately describe a large class of observations on the basis of a model that contains only a few arbitrary elements, and it must make definite predictions about the results of future observations.

—— *Stephen Hawking*
A Brief History of Time

Modern quantitative finance sprang to life in the early 1970s with the development of the Nobel-prize winning Black-Scholes (1973) theory on how to price an option, a financial derivative security whose payoff is contingent on the behaviour of an underlying asset. Over the past three decades, there has been explosive growth in the trading of financial derivative securities and increasing sophistication in techniques used to value financial products. Therefore, the development of the financial market requires more accurate and more standard pricing models and more efficient techniques to obtain the solutions to the models. In terms of improving the accuracy is for the options with sophisticated features, such as high-dimensional American options. Whereas in terms of improving the pricing model itself is for the options lack of literature background, which implies that the instruments are not investigated in previous literature, such as Parisian or ParAsian options with extra barrier. Furthermore, more efficient numerical methods are always appealing in quantitative finance. Hence, the motivation of this thesis.

The major themes of this chapter are to introduce the financial market structure and the basic theories needed for option pricing generally. This chapter presents a history of the derivative securities development, and then introduces a typical instrument in derivative security markets, an option. Thereafter, the fundamentals of option pricing theory are introduced, including the classic Black-Scholes (1973) theory. Finally, the characteristics of several different implementations of option pricing models are described.

1.1 History of Derivative Securities

A derivative security is a financial contract whose value is derived from the value(s) of one or more underlying assets (such as a stock price) or quantities (such as interest rate); it is also known as a contingent claim (see Jarrow and Turnbull, 2000). The trading of financial derivatives on organised exchanges has exploded since the early 1970s and, furthermore, trading in over-the-counter markets has become very popular since the mid 1980s. According to the Bank for International Settlements semi-annual report[1] (hereafter, BIS), the global over-the-counter derivative market value from beginning of January 2006 to the end of June 2006 was over 10 trillion U.S. dollars.

In 1972, the Chicago Mercantile Exchange (hereafter, CME), responding to the collapse of the Bretton Woods system, formed the International Monetary Market, which hosted its first futures trades on foreign currencies (involving seven major currencies). These were the first derivative contracts that were not based on physical commodities[2]. Surprisingly, the CME did not pioneer the trading of currency options, rather the Philadelphia Stock Exchange (hereafter, PHLX) became the first exchange that traded options on currencies in 1982.

[1]BIS is an international organisation which fosters international monetary and financial cooperation and serves as a bank for central banks.

[2]For more detailed timeline of CME, see the official website of CME at `http://www.cme.com/about/ins/caag/history2801.html`.

In 1973, members of the Chicago Board of Trade (hereafter, CBOT) started the Chicago Board Options Exchange (hereafter, CBOE), the world's first stock options exchange[3]; only call options on just 16 issues were traded. As a consequence, the world class clearing organisation, the Options Clearing Corporation (hereafter, OCC) was founded the same year. Then the CBOT launched the first interest-rate futures contract in 1975, based on mortgages issued by the Government National Mortgage Association (also known as Ginnie Mae). However, trading failed to develop, even though the launch was initially successful.

1973 saw the publication of two seminal papers of Black and Scholes (1973) and Merton (1973), which revolutionised the investment world. This set up a mathematical framework that accompanied an explosive revolution in the use of derivatives.

In 1976, the CME proposed trading on 90-day U.S. Treasury Bill futures. This was the first successful pure interest-rate futures contract and over the next six years it became the CME's most actively traded product. In 1977, the CBOT launched the U.S. Treasury Bond futures contract, which went on to become the highest volume contract in the world for a time. In 1981, the CME created the Eurodollar contract, which has now surpassed the CBOT's Treasury Bond contract to become the most actively traded of all futures contracts.

1982 is regarded as a year of innovation for the financial derivatives market. On February 24th, the Kansas City Board of Trade launched the first stock index futures, a contract on the Value Line Index[4]. On April 21st, the CME quickly followed with their highly successful futures contract on the S&P 500 index, and options on the S&P 500 index were born nine months later[5]. The CBOT launched the first options on future contracts, namely options on U.S. Treasury Bond futures on October 1st[6].

[3] For more detailed timeline of CBOE, see the official website of CBOE at `http://www.cboe.com/AboutCBOE/History.aspx`.

[4] The index represents 1,700 companies from the New York and American Stock Exchanges and the over-the-counter market. The index is published by an independent investment research firm called Value Line.

[5] For more detailed timeline of CME, see the official website of CME at `http://www.cme.com/about/ins/caag/history2801.html`.

[6] For more detailed timeline of CBOT, see the official website of CBOT at `http://www.cbot.`

The mid-1970s marked the beginning of the era of over-the-counter (hereafter, OTC) derivatives. The OTC market is a non-regulated market, consisting mostly of large banks and institutional clients, where trades are conducted privately (not on the exchanges) and with the terms of the contract being customised to the specific needs of the parties. Although OTC options and forwards had previously existed, that generation of corporate financial managers of that decade was the first to graduate from business schools with exposure to derivatives. Soon, virtually every middle-large corporation was using derivatives to hedge, and in some cases, speculate on interest rates, exchange rates and commodity risks. New products were rapidly created to hedge the now-recognised wide variety of risks. The instruments became more complex and were sometimes even referred to as "exotic". Two types of derivative contracts were the most common: "swaps"[7] and "hybrids"[8].

In 1990, the CBOE introduced Long-term Equity AnticiPation Securities (LEAPS), which are long-term dated options and give investors more flexibility in using options in their portfolios (for more information, see Chance, 1995).

With the growth of the derivatives world, scandals appeared more and more frequently. In 1994, the derivatives world was hit with a series of large losses on derivatives trading announced by some well-known organisations, including Procter and Gamble and Metallgesellschaft. One of America's wealthiest localities, Orange County, California, publicly announced the loss of 1.5 billion U.S. dollars on municipal bonds, municipal bond funds, and bank stocks. England's Barings Bank declared bankruptcy due to speculative trading in futures contracts by 28-year-old Nick Leeson in its Singapore office. These and other large losses led to a huge outcry, sometimes against the instruments and sometimes against the firms that sold them. While some minor changes occurred in the way in which derivatives were sold, most firms simply instituted tighter controls and continued to use derivatives. These have

`com/cbot/pub/page/0,3181,942,00.html`.

[7]An agreement to exchange cashflows in the future according to a prearranged formula, see Hull (2002).

[8]Derivatives which combine features and risks from different markets, such as interest rates, equity and credit. See Graziano and Rogers (2006).

not involved what might be called best practice, but they have certainly brought derivatives into the public eye.

In the autumn of 1998 an American hedge fund, Long Term Capital Management, including amongst its founders the two Nobel Prize-winning economists, Myron Scholes and Robert Merton, was bailed out and then rescued by the Federal Reserve Bank of New York at a cost of 3.65 billion U.S. dollars because of worries that its total collapse would have severe repercussions for the world financial system.

In 1999, a group of traders calling themselves the Flaming Ferraris, including the son of a well-known British politician, at Credit-Suisse First Boston were sacked following allegations of illegal trades in an attempt to manipulate the Swedish stock market index. In 2001, Enron, the "America's Most Innovative Company"[9] and the world's leading energy company made extensive use of energy and credit derivatives but became the biggest bankruptcy in U.S. history after systematically attempting to conceal huge losses. In 2002, Ireland's biggest bank, Allied Irish Bank lost 750 million U.S. dollars. A currency trader John Rusnak had used fictitious options contracts to cover losses on spot and forward foreign exchange contracts, and the trading losses had gone unnoticed for over five years.

In January 2004, the National Australia Bank admitted losing 280 million U.S. dollars. Four foreign currency traders at the bank had conducted unauthorised trading in currency options. In August 2004, Citigroup traders led by Spiros Skordos made 15 million Euro by suddenly selling 11 billion Euro worth of European bonds and bond derivatives, and buying many of them back at a lower price. Citigroup's short sale cost the bank far more in reputation and legal headaches. Citigroup is now 14th among advisers on European privatisations, down from third, according to bloomberg.com. In December 2004, China Aviation Oil, which supplies almost all of China's jet fuel imports, lost about 550 million U.S. dollars in speculative trade. This loss was the largest amount a company in Singapore had lost by betting on derivatives since the case of Barings. Then in October 2005, Refco, one of the world's largest derivatives brokers was forced to freeze trades due to its chief executive officer

[9]Fortune Magazine awarded Enron this title for six consecutive years before its bankruptcy.

and chairman, Phillip R. Bennett hiding 430 million U.S. dollars in bad debts from the company's auditors and investors. The Economist magazine addressed this affair as "the latest scandal in America,"[10] and the New York Times commented "If Refco isn't scary, what is?"[11]

In spite of the ever-growing scandals, the derivative market continues to grow dramatically. By the end of last year, the year-end total volume exceeded 1.5 billion contracts at OCC. Moreover, the annual volume of trading hit new highs every year up to 2006[12]. On 22nd December 2006, total options contract surpassed two billion U.S. dollars contracts for the first time ever.

1.2 Introduction to Options

Derivatives markets are populated with a vast range of instruments, and amongst these options features are perhaps the most interesting, mathematically and financially, in terms of complexity and scope. A comprehensive review of options features is not given in this thesis (alternatively, see Hull, 2002); however a brief reminder of a few fundamentals will be presented for completeness.

There are two basic types of options: call options and put options. A call option gives its owner the right, but not the obligation, to buy the underlying asset(s) at a specified price on, and in some cases before, the date the option expires. The specified price is called the strike price (or exercise price) the date the option expires is called the maturity (or expiration date), and the premium (i.e. option price) is the price paid to acquire the option. A put option is similar, but with the right to sell the underlying asset(s).

Call and put options are further categorised in different ways according to their additional features (more detailed introduction, see Hull, 2002).

- Underlying asset

[10] The news was published on 14th October 2005 on `http://www.economist.com/agenda/displaystory.cfm?story_id=5039643`.

[11] The news was published on 16th October 2005 by Grethchen Morgenson, New York Times.

[12] For more detailed timeline of OCC, see the official website of OCC at `http://www.theocc.com/about/timeline.jsp`.

With the rapid growth of financial markets, options are becoming increasingly popular and are available on many assets. Currently, options are actively traded on stocks, commodities, indices, foreign exchange rates (of particular relevance to this thesis), futures, and even on weather and electricity.

- Exercise frequency

  Options that can be exercised only at maturity are called European options. Those that can be exercised at any time up to the maturity are called American options. Variants include Bermudan options, which can also be exercised before maturity, but only on a fixed number of pre-determined dates during the contract life.

- Payoff functions

  New types of options may be devised by changing the payoff function for the option. For example, binary options have payoffs of either a fixed amount or zero, rather than being linearly related to underlying asset value at the maturity. Whether the payoffs are achieved at all may be made to depend on the path followed by the underlying asset. A barrier option, for instance, may cause the payoff to be knocked out (alternatively, knocked in), dependent on path. Some new classes of options based on time triggers to determine the knock-in or knock-out are described in Chapter 7. The payoffs from lookback options depend on the maximum or minimum asset price reached during their contract life. For Asian options, the payoff depends on the average price of the underlying asset during the life of the contract.

- Moneyness (i.e. Intrinsic value)

  This classification is generally for the purpose of option price analysis. When the strike price is equal to the spot price of the underlying asset, the option is called an at-the-money option. When the strike price is greater than the spot price of the underlying asset, the option is called an out-of-the-money call option or an in-the-money put option. If the strike price is less than the spot price of the underlying asset, the option is called an in-the-money call option or an out-of-the-money put option.

There are many more types of options traded in the financial markets, such as compound options or basket options, or varieties of combinations (see Wilmott, 2001). In this thesis, two major classes will be studied in detail, that is American options (Bermudian options included) and barrier options (including two sub-classes, namely, Parisian and ParAsian options).

1.3 Option Pricing Fundamentals

Financial markets are driven by many complicated factors. A complete market means that any derivative (i.e. contingent claim) can be synthesised from other instruments (assets or quantities). Intuitively speaking, in such a circumstance, whenever the number of different ways to obtain payoffs equals the number of states, any payoff can be attained. According to the fundamental theorem of financial economics, risk-neutral probabilities are unique if and only if the market is complete (see Bailey, 2005). Whereas an incomplete market does not have this property.

Alternatively, in a complete financial market model, derivatives can be perfectly hedged by a dynamic trading strategy, and can be priced by taking expectations under a unique martingale measure (see Duffie, 1988).

Market completeness is one of the fundamental assumptions of the Black-Scholes (1973) framework. It is also one of the assumptions embedded in this thesis, regardless of whether in reality the market is actually incomplete.

1.3.1 Arbitrage Pricing Method

The concept of arbitrage is the essence of derivative pricing theory. Assuming a portfolio has value $Z_t(\theta)$ at time t, where θ denotes the components in the portfolio the formal definition of arbitrage is described as, during an investment time horizon

$[0, T]$ as

$$
\begin{aligned}
Z_0(\theta) &= 0, & (1.1)\\
\mathbb{P}(Z_T(\theta) \geq 0) &= 1, & (1.2)\\
\mathbb{P}(Z_T(\theta) > 0) &> 0. & (1.3)
\end{aligned}
$$

The no-arbitrage assumption implies no risk-free profit, which in turn, implies that a risk-less portfolio has no more than the risk-free rate of return. It also implies that two portfolios have the same present value if they are exposed to the same sources of risk; this is also known as the law of one price (see Björk, 2004).

No-arbitrage pricing requires that the market for the instruments in the replicating portfolio be complete. Also, the assumption is that these instruments can be traded continuously and without frictions (such as transaction costs or taxes). No-arbitrage pricing can be used only where the markets for the underlying assets are complete. Nevertheless, it is still possible to use no-arbitrage pricing for markets that are "nearly" complete, either by assuming away the incompleteness (useful for developed markets with very small transaction costs, for example) or by use of super-replicating portfolios[13]. No-arbitrage pricing has the benefit of not involving the investor's attitude towards risk (see Bailey, 2005).

1.3.2 Equilibrium Pricing Method

In the equilibrium pricing method, the lack of no-arbitrage pricing opportunities is part of the general equilibrium condition. The method is built on assumptions about how the economy works. It can be used to value an asset or derivative under a wide range of circumstances. Unfortunately, it is necessary to know information about the preferences of market participants (or agents), particularly their attitudes towards risk. This is mainly used for derivative pricing under incomplete markets, such as for cases where the market contains sources of untradeable risk or the assets

[13] A super-replicating portfolio is the portfolio consisting of units of underlying asset and risk-free bond in such a way that at the end of the investment time horizon, the portfolio is worth at least as much as the value of the derivative. See Duffie (1988).

are illiquid. In the incomplete market, derivatives carry intrinsic risks, and there is no canonical choice of a preference-free pricing mechanism. Thus, any reasonable valuation and any efficient hedging procedure should be based on criteria which take into account preferences towards risk. In terms of equilibrium, it can be seen that a no-arbitrage price is a unique ultra-stable equilibrium price (see Duffie, 1988).

1.4 Black-Scholes-Merton Theory

As mentioned in Section 1.1, Black and Scholes (1973) and simultaneously Merton (1973) presented a theory for option pricing. The influence of option pricing theory on finance practice has not been limited to plain options, however, the "Black-Scholes-Merton" methodology has played a fundamental role in supporting the development of new financial instruments around the globe. The derivation of the Black-Scholes-Merton pricing formula is based on the following assumptions:

i:) The market is perfect: there are no transaction costs or taxes. Trading takes place continuously. Borrowing and short-selling are with no restriction.

ii:) The underlying asset follows a stochastic differential equation in the form of a geometric Brownian motion,

$$\frac{dS_t}{S_t} = \alpha dt + \sigma dW_t,$$

where α is the expected rate of return on the asset which is constant, σ is the volatility which is a constant as well, and dW_t is the increments of a standard Brownian motion[14].

iii:) The risk-free rate of interest is a constant over time, denoted as r.

iv:) The option is "European" (defined in Section 1.2).

v:) The option price is assumed to be a twice-continuously differentiable function of underlying asset price S_t, and time t.

[14]The mathematical definition of volatility and Brownian motion will be introduced in Chapter 2.

Black and Scholes (1973) considered a hedging strategy which satisfies a self-financing condition, that is, there are no cashflows (in or out) during the investment time horizon $[0, T]$ for the portfolio adjustments. Under the risk-neutral measure, all assets yield the risk-free return. The Black-Scholes partial differential equation (hereafter, PDE) is then

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S} rS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 - rV = 0, \tag{1.4}$$

where $V(S_t, t)$ is the option price at time t for an underlying asset S_t (see also Wilmott, 2000a). Except for a few special cases, there is no general analytical solution to the Black-Scholes PDE, although prices for the European call option C and European put P can be derived (see Wilmott, 2000a):

$$C = S_0 N(d_1) - Ke^{-rT} N(d_2), \tag{1.5}$$

$$P = Ke^{-rT} N(-d_2) - S_0 N(-d_1), \tag{1.6}$$

where

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}},$$

$$d_2 = d_1 - \sigma\sqrt{T},$$

$N(\cdot)$ is the standard normal cumulative distribution function, S_0 is the underlying asset price at time 0, and K is the strike price.

Subsequent research in the field has broadly proceeded along three directions: applications of the methodology to other than financial options; empirical testing of the formula; attempts to weaken the assumptions (see Merton, 1997).

1.5 Option Pricing Implementation

The aim of option pricing analysis may be regarded as determining the “fair” option price. Depending on the assumptions of a model, different approaches can be chosen to obtain the most accurate value within a reasonable computational time, although there is no “perfect” algorithm suitable for all problems; the most efficient algorithm depends on the specific problem. For instance, some algorithms give very low accuracy but are readily implemented for high-dimensional problems. Some algorithms might be computational expensive but give a highly accurate result. More in-depth descriptions of algorithms will be given later in this section (see also Neftci, 2000).

1.5.1 Analytical Solutions

Analytical solutions of pricing problems may sometimes be obtained by applying the equivalent martingale measure (hereafter, EMM) or by solving the PDEs, which is, in general, based on the assumption of a continuous-time economy. An EMM is the calculation of an expectation with respect to a given probability measure, which is normally calibrated by a risk-free asset as numeraire. This way of pricing reflects an absence of arbitrage: if an EMM exists, and so there is no arbitrage; if the EMM is unique, then derivatives prices can be calculated, implying that the market is complete. PDEs can be derived by hedging the risk of the underlying asset, for example, the Black-Scholes PDE in Section 1.4. Also, using stochastic calculus, most famously the Feynman-Kac formula, PDEs can be derived in the formulation of the pricing problem, (see Wilmott, Dewynne and Howison, 1995; Kallianpur and Karandikar, 2000). Strictly speaking, closed-form solutions are possible for some simple cases of European options, but not for American options, as it is generally very difficult to find the solution for the optimal early-exercise criterion because of the inherent nonlinearity[15]. As a consequence, numerical methods often have to be employed.

1.5.2 Numerical Techniques

Numerical techniques are practical methods that are used by both academic researchers and market professionals. Depending on the problem at hand, one method

[15]However, there are analytic approximation applicable to simple American option prices by MacMillan (1986) and extended by Barone-Adesi and Whaley (1987), and Peskir (2005a).

may be more convenient or computationally cheaper to use than another. Some well-known numerical procedures are finite-difference methods, binomial and trinomial trees, quadrature methods, and Monte Carlo simulations (as used extensively in this thesis).

Finite-difference

The finite-difference method is a direct and generally efficient approach to the solution of PDEs and was introduced to finance problems by Brennan and Schwartz (1978). Pricing different types of options often only leads to a change in boundary conditions associated with the PDE (see Neftci, 2000). The method generally provides reliable results for low-dimensional problems (generally up to three dimensions). Higher-dimensional PDE for multi-factor models can be derived, however, even for those claiming sophistication, it is difficult to implement four or more factor models using these methods (the reason will be explained in detail in Section 4.5). Also, Hull (2002) pointed out that finite-difference methods (as well as tree methods — see below) are difficult to apply to non-Markovian driving processes (for example, path-dependent processes).

Trees

The most easily understood approach for discrete-time models is the family of tree methods (also known as lattice methods). Cox, Ross and Rubinstein (1979) proved that as the lattice is refined, these methods converge to the correct option values produced by a continuous-time model. Trees are convenient for more straightforward situations where analytic solutions are not available. For example a simple and effective but coarse approximation can delivered for one-factor American options with few extra features. The rates of convergence of basic trees are relatively poor[16]. The method also does not scale well to higher dimensions. One example is Amin and Bodurtha (1995), in which the algorithm is a tree method for three dimensions,

[16]But some literature has shown that it can be considerably improved. See Figlewski and Gao (1999), Widdicks et al. (2002).

restrict to just 12 nodes. A more detailed explanation is given in Section 3.4.1.

Quadrature (QUAD)

In mathematics, methods of approximate integration based on quadrature are historically the oldest of the integration techniques (Evans and Swartz, 2000). In finance, Andricopoulos et al. (2003, 2004) presented a quadrature method to evaluate option prices. The idea behind the technique is to approximate the integrals representing all possible future outcomes, in a manner which, in essence, involves approximating areas under curves. The method yields excellent results for discrete-time options. The insight is to recognise that boundary conditions such as the final payoff and intermediate early-exercise possibilities need to be dealt with, but that between these significant events only straightforward integration is required; consequently, convergence is exceptionally fast, and any standard mathematical technique for quadrature can be applied. Evans and Swartz suggest that quadrature is an effective technique in low-dimensional problems but not as effective in higher dimensions. Andricopoulos et al. (2006) presented results using QUAD for high dimensions which has largely resolved this problem.

Monte Carlo

The Monte Carlo method provides approximate solutions to a wide variety of mathematical problems, by performing statistical sampling experiments on a computer (see Goodman, 2005). In applications of the Monte Carlo method, the process can be simulated directly by random sampling. Many observations are then performed to obtain a large enough sample space, and the desired result is taken as an average over the number of observations. In order to obtain a reasonably accurate result, the number of observations may need to be several millions. It is possible to predict the statistical error (i.e. the "variance") in this average result, and therefore an estimate of the number of Monte Carlo simulations that are needed to achieve a given accuracy. To be more precise, Monte Carlo methods provide an algorithm which gives a numerical estimate of an integral together with an estimate of the error (see Higham, 2004).

Monte Carlo methods are used extensively to deal with multiple random factors, for instance options on multiple assets, asset processes with jumps, stochastic interest rates or stochastic volatilities. They are by far the most efficient numerical methods for high-dimensional problems (justified in Section 4.5). Furthermore, in the past 15 years Monte Carlo methods have been developed to solve problems with early exercise. These techniques are used extensively in this thesis. A detailed introduction to Monte Carlo methods will be presented in chapter 4.

1.6 layout of the thesis

In this thesis, the main focus is to set up the models with more relaxed assumptions and to apply these models to other financial instruments. Having given the financial introduction to option pricing theory, some mathematical preliminaries are presented in Chapter 2 in order to provide a better understanding of the mechanics of the option pricing models. This is followed in Chapter 3 by a comprehensive introduction to foreign exchange market and a literature review of currency option modelling. Chapter 4 is focused on the Monte Carlo techniques, discussing in depth the advantages of Monte Carlo method as a numerical method for high-dimensional models. Chapter 5 applies an advanced Monte Carlo technique to American currency-option pricing and investigates a more realistic framework for currency-option pricing models. Chapter 6 addresses a practical problem of pricing and hedging the discretely-monitored barrier currency-option. In the real world, options are hedged discontinuously, which makes the losses from mis-hedging substantial. To overcome the disadvantages of standard barrier options as well as to introduce a new class of options, Chapter 7 explores the pricing models for so called quantile Parisian and ParAsian options. And the difficulties of overcoming time-discretisation error leave us some scope for future research. Chapter 8 concludes this thesis by providing a more realistic model for currency options as well as introducing a new class of options, namely quantile Parisian and ParAsian options. Also future research in this area, such as data calibrations, is suggested in Chapter 8.

Chapter 2

Mathematical Preliminaries

The most important questions of life are, for the most part, really only problems of probability.

—— Pierre Simon Laplace (1749 - 1827)
Théorie Analytique des Probabilités, 1812

In this chapter, a brief summary of several concepts and theorems is given. An understanding of these will provide a foundation to construct the financial models employed in this thesis. Certain additional conditions applied for the completeness of theorems, such as existence and uniqueness, will be taken as understood without proof.

Sections 2.1 and Section 2.2 give very basic definitions in classic probability theory. Section 2.3 introduces several stochastic processes which are of great importance in the field of financial mathematics. Section 2.4 presents the fundamental convergence theory and Sections 2.5 to 2.8, give a more specific introduction to stochastic calculus embedded in option pricing theory. Section 2.9 concludes this chapter. A more advanced introduction to this area may be found in Peskir and Shiryaev (2006).

2.1 Probability Space

Suppose that Ω is a set. Then a collection of subsets of Ω, $\mathcal{F}$, is called a σ-algebra (or σ-field) if:

i:) $\emptyset \in \mathcal{F}$;

ii:) if $A \in \mathcal{F}$, then so is the complement of A (i.e. $A^c \in \mathcal{F}$);

iii:) if A_i for $i = 1, 2, \ldots$ is a family of subsets such that $A_i \in \mathcal{F}$, then

$$A = \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}.$$

A probability measure $\mathbb{P}$ is a real-valued function defined as:

i:) $0 \leq \mathbb{P}(A) \leq 1$, $\forall A \in \mathcal{F}$;

ii:) $\mathbb{P}(\Omega) = 1$, where Ω is a sample space;

iii:) if A_i for $i = 1, 2, \ldots$ is a family of subsets such that $A_i \in \mathcal{F}$, and $A_i \cap A_j = \emptyset$ for any $i \neq j$, then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

Then a probability space is a triplet $(\Omega, \mathcal{F}, \mathbb{P})$ such that

i:) Ω is a non-empty set (called sample space);

ii:) $\mathcal{F}$ is a family of subsets of Ω with the property of a σ–algebra (a set of "events");

iii:) $\mathbb{P}$ is a probability measure such that $\mathbb{P} : \mathcal{F} \to \mathbb{R}$.

2.2 Random Variables

2.2.1 Measurability

A random variable X is $\mathcal{F}$–measurable if the value of X is completely determined by the information in $\mathcal{F}$. Formally speaking:

A random variable $X : \Omega \to \mathbb{R}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called $\mathcal{F}$–measurable if

$$X^{-1}(U) = \{\omega \in \Omega : X(\omega) \in U\} \in \mathcal{F},$$

for all open sets $U \in \mathbb{R}$.

2.2.2 Conditional Expectation

The conditional expectation of X given σ–algebra $\mathcal{G} \subset \mathcal{F}$ is a random variable $\mathbb{E}[X|\mathcal{G}] : \Omega \to \mathbb{R}$ satisfying:

i:) $\mathbb{E}[X|\mathcal{G}]$ is $\mathcal{G}$–measurable;

ii:) $\mathbb{E}[\mathbb{E}[X|\mathcal{G}]\mathbf{1}_G] = \mathbb{E}[X\mathbf{1}_G]$, where $\mathbf{1}_{(\cdot)}$ is an indicator function[1].

Conditional expectation is the essence of option price modelling, especially for the options with early exercise features. Option prices are expectations conditioned on the information given at the present time (see Goodman, 2004).

2.2.3 Stopping Time

One of the important notions in the analysis of stochastic processes is the concept of stopping time. The theory of stopping times plays a key role in finance, notably in the determination of the optimal time at which to exercise an option prior to its maturity. The American option (to be introduced in Chapter 5) is a typical example.

It is natural to introduce the concept of filtration. A family of σ–algebra $\{\mathcal{F}_t\}, \mathcal{F}_t \subseteq \mathcal{F}$ is called a filtration if each $\mathcal{F}_t$ is represents the information known at time t. Formally, a filtration $\{\mathcal{F}_t\}$ is

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_t \subseteq \cdots \subseteq \mathcal{F}. \tag{2.1}$$

[1]Suppose Ω is a set with typical element ω, and let A be a subset of Ω. The indicator function of A, denoted by $\mathbf{1}_A$, is defined by

$$\mathbf{1}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases},$$

That is, $\mathbf{1}_A$ indicates the set A. See Murison (2000).

A stopping time is a random variable $\tau : \Omega \to [0, \infty]$ with respect to a filtration $\{\mathcal{F}_t\}$, such that

$$\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t, \quad \forall t \leq \infty.$$

A hitting time is a stochastic process defined on a set U as follows:

$$\tau_A(\omega) = \inf\{t > 0 : X_t(\omega) \in U\}.$$

Then, τ is called a hitting time of U for X.

Stopping times are only encountered in the context of hitting times. Given a criterion for stopping, enough information is known to determine whether to stop or not. Approximately speaking, the hitting time is the time that a process hits the fixed level a, whereas a stopping time is the first hitting time at which the criterion is satisfied (i.e. if the process is right-continuous at the hitting time, it is a stopping time, see Peskir and Shiryaev, 2006).

2.3 Stochastic Processes

A stochastic process is a family of random variables $\{X_t(\omega), t \in T\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with a set T which is called the index set of the process. Given any $t \in T$ fixed, the possible values of X_t are called the states of the process at t. Whereas, given $\omega \in \mathcal{F}$ fixed, $X(\omega)$ is called its sample path of the stochastic process, and the family of all sample paths is a path space. This path space is the probability space (see Doob, 1996).

If T is discrete, then the stochastic process is referred to as a discrete-time process, and it is sometimes called a "sequence". If T is an interval of $\mathbb{R}$, then the stochastic process is a continuous-time process. Note that continuous-time stochastic processes are more general than discrete-time stochastic processes. Therefore, in theoretical finance, continuous-time stochastic processes are widely used, and these processes are of practical importance. For instance, partial differential equations or stochastic differential equations may be built up on a continuous-time platform. The most well-known numerical approach which is applied on a discrete-time platform is tree method introduced in Section 1.5.2. Note that the properties presented in this chapter are for continuous-time processes by default, which are then applicable to discrete-time cases, in the limit of small time steps.

The stochastic processes are basic building blocks for financial models. Below, four fundamental processes are considered: Brownian motions, Poisson processes, Markov processes, and martingales.

2.3.1 Brownian Motions

A stochastic process $(W_t)_{t\geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a Brownian motion (i.e. a Wiener process) if:

i:) the random variables $\{(W_{t_i} - W_{t_{i-1}}), i = 1, 2, \ldots, n\}$ are independent for any given $0 \leq t_0 < t_1 < \ldots < t_n$ (independent increment);

ii:) $W_t - W_s \sim W_{t-s}$ for any $0 \leq s \leq t$ (stationary increment);

iii:) W_t is continuous in t with $\mathbb{P}$–a.s.[2];

iv:) $W_0 = 0$, $\mathbb{P}$–a.s.

With the above four conditions satisfied, a useful result can be obtained:

$$W_t \sim N(\mu t, \sigma^2 t), \quad \forall t > 0, \tag{2.2}$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$ are given and fixed constants. A standard Brownian motion is defined as $W_t \sim N(0, t)$ (i.e. $\mu = 0$, $\sigma^2 = 1$) where $N(\cdot)$ is defined in Section 1.4.

Brownian motion is the simplest example of a stochastic process. Many properties of more general stochastic processes appear explicitly in Brownian motions. In fact, most of the stochastic processes in financial models may be described in terms of Brownian motions (moreover, standard Brownian motions). Also, the solutions to many other mathematical problems, particularly various stochastic differential

[2] $\mathbb{P}$–a.s. is abbreviation of $\mathbb{P}$ almost surely, which means with probability one.

equations, may be expressed in terms of standard Brownian motions. For these reasons, Brownian motions are the central object to study. Some very important and fundamental properties of a standard Brownian motion are presented as follows:

Assume $(W_t)_{t\geq 0}$ is a standard Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Then,

- W_t is continuous in t, but it is nowhere differentiable with respect to t, $\mathbb{P}$–a.s.
- The law of large numbers (see Section 2.4.2) implies:

$$\lim_{t\to\infty} \frac{W_t}{t} = 0, \quad \mathbb{P}\text{–a.s.} \tag{2.3}$$

- Assume t_k^n is a doubly infinite subdivision of $[0,t]$, where $0 = t_0^n < t_1^n < \ldots < t_{N-1}^n < t_N^n = t$, such that:

$$\delta_n := \max_{1\leq k\leq N} \left(t_k^n - t_{k-1}^n\right) \longrightarrow 0, \quad \text{as} \quad n \to \infty.$$

  The quadratic variation of a standard Brownian motion is defined by means of

$$S_n = \sum_{k=1}^{N} \left(W_{t_k^n} - W_{t_{k-1}^n}\right)^2.$$

  Then,

$$S_n \longrightarrow t, \quad \text{in} \quad \mathbb{P}\text{–probability.}^3$$

  If the rate of convergence of δ_n is sufficiently fast to imply $\sum_{n=1}^{\infty} \delta_n < \infty$, then

$$S_n \longrightarrow t, \quad \mathbb{P}\text{–a.s.} \tag{2.4}$$

  Note that a standard Brownian motion is of unbounded variation, but it has a bounded quadratic variation, which moreover is equal to t. The quadratic variation is one of the most important concepts in stochastic calculus.

In a discrete-time version, a Brownian motion is also known as a simple symmetric random walk, which is commonly used in financial models (see Peskir, 2005b).

[3] It is a weak convergence. The formal definition of $\mathbb{P}$–probability is given in Section 2.4.1.

2.3.2 Poisson Processes

A process $(N_t)_{t\geq 0}$ with parameter $\lambda > 0$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a Poisson process if

i:) N_t is increasing in t and each jump is of unit size, $\mathbb{P}$–a.s.;

ii:) $\{(N_{t_i} - N_{t_{i-1}}), i = 1, 2, \dots, n\}$ are independent for given any $0 \leq t_0 < t_1 < \cdots < t_n$, (independent increment);

iii:) $N_t - N_s \sim N_{t-s}$, for any $0 \leq s \leq t$ (stationary increment);

iv.) $N_0 = 0$, $\mathbb{P}$–a.s.

The Poisson process is a discrete-distribution process. It is popular for modelling jump features in financial models, such as stock prices, firm values, company indices, exchange rates and interest rates (see Glasserman, 2003).

2.3.3 Markov Processes

Intuitively speaking, a Markov process is a stochastic process for which the future does not depend on the past, but only on the present. It is a general class into which many stochastic processes fall, such as the path-dependent stochastic process introduced in Chapter 7. A stochastic process $(X_t)_{t\geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called Markov process if

$$\mathbb{P}(X_t \leq x \mid X_u) = \mathbb{P}(X_t \leq x \mid X_s), \quad \text{for} \quad 0 \leq u \leq s \leq t. \tag{2.5}$$

Markov processes form a simple class of stochastic processes, which seem to represent a good level of abstraction and generality. For the discrete-time case, a sequence which has Markov property is called a Markov chain.

In finance, the term "diffusion" is frequently used. A diffusion process is a (strong) Markov process whose paths are continuous in time. It generalises Brownian motion, allowing a much wider variety of phenomena to be modelled and studied.

Note that Brownian motion is the quintessential example of a diffusion, and the Poisson process is an example of a Markov process that is not a diffusion. However, it is useful to clarify the term "jump-diffusion" process, since it is widely used in the financial world. A jump-diffusion process is a hybrid of a diffusion process and a jump process (see Rogers and Williams, 1994).

2.3.4 Martingales

Martingales are very important and useful in the study of stochastic processes. A martingale is a stochastic process whose future movements are always unpredictable — it is a model of a fair game. A formal definition is given below.

A process $(M_t)_{t\geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a martingale with respect to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, with $\mathcal{F}_t \subset \mathcal{F}$ if the following conditions are satisfied:

i:) M_t is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$, (i.e. M_t is $\mathcal{F}_t$-measurable for all t);

ii:) $\mathbb{E}[|M_t|] < \infty$ for all $t \geq 0$;

iii:) $\mathbb{E}[M_t|\mathcal{F}_s] = M_s$ for all $0 \leq s \leq t$, $\mathbb{P}$–a.s.

Intuitively, a martingale implies no prediction on the outcomes of the future events. Recall the definition of a Markov process, whereby history is irrelevant — a Markov process implies no history of past. Brownian motion is the most trivial example of both a martingale and a Markov process, and is one of the reasons that Brownian motion performs a key role in stochastic calculus and mathematical finance.

Some useful extensions of martingales are described below (see Hunt and Kennedy, 2005):

- super- and sub- martingale

  $(M_t)_{t\geq 0}$ is called a supermartingale if the condition (iii) is changed to $\mathbb{E}[M_t|\mathcal{F}_s] \leq M_s$ (i.e. the future value given information up to the present is no greater than the present value), whereas a submartingale if $\mathbb{E}[M_t|\mathcal{F}_s] \geq M_s$. Most of the properties held for martingales also hold for supermartingales.

- local martingale

  More generally, the class of local martingales M_{loc} is defined as an adapted process M null at zero such that for an increasing sequence of stopping times $\{\tau_n\}$, the stopped process M^{τ_n} is a martingale. Roughly speaking, a local martingale is a martingale with a finite time horizon. It is very useful in practical applications.

- semimartingale

  Semimartingales form the largest class of integrators for which the stochastic integral can be defined, which will be introduced later. A process X is called a semimartingale if X is adapted and can be decomposed as

$$X = A + M_{loc}, \tag{2.6}$$

  where A is an adapted right-continuous process of finite variation, and M_{loc} is a right-continuous local martingale.

2.4 Convergence and the Central Limit Theorem

Convergence theory is a core theory of stochastic simulation, and is the essence of Monte Carlo methodology. Some important definitions and theorems are provided below.

This section is reserved for the discrete-time case. It can be used likewise for the continuous-time case (see Peskir and Shiryaev, 2006).

2.4.1 Convergence

Three major types of convergence are introduced as follows:

- Convergence $\mathbb{P}$–a.s.

  Suppose that X and $\{X_n, n = 1, 2, \ldots\}$ are real-valued random variables. Then X_n converges to X $\mathbb{P}$–a.s if

$$\mathbb{P}(\lim_{n\to\infty} X_n = X) = 1.$$

This is the most common convergence notion in probability theory. It is also called strong convergence.

- Convergence $\mathbb{P}$–probability

  Suppose that X and $\{X_n, n = 1, 2, \ldots\}$ are real-valued random variables. Then X_n converges to X in probability for every $\epsilon > 0$ if

  $$\mathbb{P}(|X_n - X| > \epsilon) \to 0 \quad \text{as} \quad n \to \infty.$$

  Convergence in probability is much weaker than convergence $\mathbb{P}$–a.s. and thus, it is also called weak convergence.

- Convergence in distribution (or convergence in law)

  Suppose that X and $\{X_n, n = 1, 2, \ldots\}$ are real-valued random variables with distribution functions F and F_n, $n = 1, 2, \ldots$ respectively. Then X_n converges to X in distribution (denoted as $\tilde{\to}$) if

  $$F_n(x) \to F(x) \quad \text{as} \quad n \to \infty,$$

  for all $x \in \mathbb{R}$ at which F is continuous.

  Note that convergence in distribution only involves the distributions of the random variables. Thus, the random variables need not even be defined on the same probability space. This is the weakest convergence so that it is not often used as financial concepts (see Goodman, 2004).

2.4.2 The Law of Large Numbers

Suppose X_n, $n = 1, 2, \ldots$ is a sequence of independent and identically distributed random variables with mean μ. Then

$$\frac{1}{n}\left(\sum_{i=1}^{n} X_i\right) \to \mu \quad \text{as} \quad n \to \infty, \quad \mathbb{P}\text{–a.s.} \tag{2.7}$$

This is called the strong law of large numbers. It provides the theoretical basis for stochastic simulations, such as the Monte Carlo method. There is also a weak law of large numbers which is omitted from this thesis in the interests of brevity.

2.4.3 Central Limit Theorem

Suppose X_n, $n = 1, 2, \dots$ is a sequence of independent and identically distributed random variables with mean μ and variance σ^2. Then

$$\frac{\sum_{i=1}^{n}(X_i - \mu)}{\sigma\sqrt{n}} \tilde{\rightarrow} N(0,1) \quad \text{as} \quad n \to \infty, \quad \mathbb{P}\text{–a.s.} \tag{2.8}$$

Central limit theorem implies that no matter what distribution X_i has, the sum of X_i (properly normalised) has a normal distribution when n is large enough.

2.5 Stochastic Integration

A stochastic integral can be interpreted financially as the gain from trading which is expressed as:

$$I_t = \int_0^t H_s dX_s, \tag{2.9}$$

where X is the asset, and H_s is the quantity held at time s of this asset X. However because the asset X is no longer a process of finite variation, the randomness makes the classical integral fail. Itô's integral is introduced in the section below to resolve this difficulty.

2.5.1 Itô's Integral

Since a more sophisticated stochastic process can be reduced to a study of Brownian motion W_t, it is only necessary to discuss a stochastic integral with respect to W_t.

The process $(h_t)_{t\geq 0}$ is defined as a simple process, such that

$$h_t = \sum_{i\geq 0} b_i \mathbf{1}_{(t_i, t_{i+1}]},$$

where b_i is $\mathcal{F}_{t_i}$–measurable and $\mathbf{1}_{(\cdot)}$ is defined in Section 2.2.2. $(h_t)_{t\geq 0}$ is adapted to $\mathcal{F}_t$, where $\mathcal{F}_t$ is the filtration generated by the Brownian motion W_s, for $0 \leq s \leq t$.

Itô's integral (2.9) on $(0, t]$ with respect to W_t is:

$$\begin{aligned} \int_0^t H_s dW_s &= \lim_{n\to\infty} \int_0^t h_s^n dW_s \\ &= \sum_{t_{k+1}\leq t} b_k \left(W_{t_{k+1}} - W_{t_k} \right), \end{aligned} \tag{2.10}$$

where

$$\lim_{n\to\infty}\left(\mathbb{E}\left[\int_0^t |H_s - h_s^n|^2 ds\right]\right) = 0.$$

The idea of Itô's integration is to sum up the values $H_{t_i}(W_{t_{i+1}} - W_{t_i})$ with respect to a subdivision $(t_i, t_{i+1}]$. If H is not a simple process, there always exists a family of integrands $(h_t^n)_{t\geq 0}$ that are simple processes converging to H with $\mathbb{P}$–a.s.

Some properties of the stochastic integral are presented here: Assume X and Y are square-integrable processes on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to a filtration $\{\mathcal{F}_t\}$, $\mathcal{F}_t \subset \mathcal{F}$. Then

- Time additivity:

$$\int_0^t X_s dW_s = \int_0^u X_s dW_s + \int_u^t X_s dW_s, \quad \text{where} \quad 0 < u < t. \tag{2.11}$$

- Linearity:

$$\int_0^t (aX_s + bY_s) dW_s = a\int_0^t X_s dW_s + b\int_0^t Y_s dW_s, \tag{2.12}$$

  where a and b are constant.

- Martingale property:

$$\int_0^t X_s dW_s \text{ is a martingale.}$$

2.5.2 Itô's Isometry

Itô's isometry is useful in practical calculations.

Assume that H is an adapted (measurable) process satisfying $\mathbb{E}[\int_0^t H_s^2 ds] < \infty$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to a filtration $\{\mathcal{F}_t\}$, $\mathcal{F}_t \subset \mathcal{F}$. Itô's isometry is

$$\mathbb{E}\left|\int_0^t H_s dW_s\right|^2 = \mathbb{E}\left[\int_0^t H_s^2 ds\right]. \tag{2.13}$$

Note that using Itô's isometry, the stochastic integral can be calculated in terms of the expectation.

2.6 Stochastic Differential Equations

The theory of stochastic differential equations (hereafter, SDEs) is a framework for expressing dynamical models that include both random and deterministic components; the theory being based on the Itô integral. The solution to an SDE is a stochastic process which is expressed in terms of a stochastic integral with respect to Brownian motion.

Consider an SDE taking the form

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t. \tag{2.14}$$

A solution to (2.14) is an adapted process that has the form

$$X_t = X_0 + \int_0^t \mu(X_s, s)ds + \int_0^t \sigma(X_s, s)dW_s, \tag{2.15}$$

where the first integral on the right-hand-side is a Riemann integral and the second integral is an Itô integral.

The initial conditions X_0 are often specified, where X_0 may be a random variable. In the general SDE, μ is called drift, and σ is a diffusion coefficient; in finance, σ is called the volatility (introduced in Section 1.2).

Three extensively used SDEs in financial modelling will now be discussed.

2.6.1 Geometric Brownian Motion

A general Brownian motion is introduced in Section 2.3.1. Geometric Brownian motion is a rather more special case whose logarithm follows a general Brownian motion. A geometric Brownian motion is the most common process utilised in financial market modelling, and can be used to model the uncertain return of an asset, such as a stock. The SDE of a geometric Brownian motion is defined as (and mentioned in Section 1.4):

$$\frac{dX_t}{X_t} = \mu dt + \sigma dW_t, \tag{2.16}$$

with initial value $X_0 = x_0$ where $\mu \in \mathbb{R}$ and $\sigma > 0$. The solution is,

$$X_t = x_0 e^{(\mu - \sigma^2/2)t + \sigma W_t}. \tag{2.17}$$

It is clear that X_t has a lognormal distribution with expectation and variance, conditioned on $X_0 = x_0$ given by

$$\mathbb{E}[X_t|x_0] = x_0 e^{\mu t}, \tag{2.18}$$

$$\mathbb{V}\text{ar}[X_t|x_0] = x_0^2 e^{2\mu t}(1 - e^{\sigma^2 t}) \tag{2.19}$$

respectively. The derivation of Equation (2.17) will be introduced in Section 2.7.1. More importantly, Equation (2.16) is used to model the exchange-rate process in this thesis.

2.6.2 Ornstein-Uhlenbeck Process

A stochastic process is called an Ornstein-Uhlenbeck process if the SDE has the form

$$dX_t = \kappa(\theta - X_t)dt + \sigma dW_t, \tag{2.20}$$

with initial value $X_0 = x_0$. The solution is

$$X_t = \theta + (x_0 - \theta)e^{-\kappa t} + \sigma \int_0^t e^{-\theta(t-s)} dW_s. \tag{2.21}$$

Note that X_t has a normal distribution with expectation and variance, conditioned on $X_0 = x_0$ given by

$$\mathbb{E}[X_t|x_0] = \theta + (x_0 - \theta)e^{-\kappa t}, \tag{2.22}$$

$$\mathbb{V}\text{ar}[X_t|x_0] = \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa t}\right). \tag{2.23}$$

The processes which have all (marginal) distributions normal distributed are called Gaussian processes, which includes the Ornstein-Uhlenbeck process. In finance, the Ornstein-Uhlenbeck process is a widely used stochastic processes for the term structure model of interest rates, including the Vasicek (1977) model (see Section 3.3.3).

2.6.3 Square-root Process

A process

$$dX_t = \kappa(\theta - X_t)dt + \sigma\sqrt{X_t}dW_t, \tag{2.24}$$

with initial value $X_0 = x_0$ is classed as a square-root process.

The SDE has no explicit solution generally, although its transition density can be characterised as

$$\mathbb{E}[X_t|x_0] = x_0 e^{-\kappa t} + (1 - e^{-\kappa t})/\kappa, \tag{2.25}$$

$$\mathbb{V}\text{ar}[X_t|x_0] = \frac{\sigma^2}{\kappa}(1 - e^{-\kappa t})\left(x_0 e^{-\kappa t} + \frac{1}{2\kappa}(1 - e^{-\kappa t})\right). \tag{2.26}$$

This process is also known as a Cox, Ingersoll and Ross (1985) model in finance which is often used for interest rate, volatility, and other financial models because it has a non-negative mean-reverting feature which is introduced in Section 3.3.3, and is used throughout this thesis.

2.7 Itô's Lemma

Itô's lemma is one of the most useful tools in stochastic calculus. It gives a representation for functions with respect to SDEs.

Itô's lemma is a formula for the Itô differential, which in turn is defined using the Itô integral. It enables us to find the process followed by a known function of another process.

2.7.1 One-dimensional Case

Assume $f(X, t)$ is a twice continuously differentiable function and $(X_t)_{t\geq 0}$ is a stochastic process following the SDE

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t.$$

Then Itô's lemma gives the SDE for $f(X, t)$ as dt, $dX_t \to 0$,

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X}dX_t + \frac{1}{2}\frac{\partial f}{\partial X^2}dX_t^2 + \cdots . \tag{2.27}$$

It is straightforward to substitute the expression for dX_t into Equation (2.27) to obtain the simplest version of Itô's lemma using $dW_t^2 \to dt$, as $dt \to 0$:

$$df = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial X}\mu + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}\sigma^2\right)dt + \sigma\frac{\partial f}{\partial X}dW_t. \tag{2.28}$$

For instance, to derive Equation (2.17) from Equation (2.16), we can assume $f = \ln X_t$. Therefore

$$\frac{\partial f}{\partial t} = 0, \tag{2.29}$$

$$\frac{\partial f}{\partial X} = \frac{1}{X}, \tag{2.30}$$

$$\frac{\partial^2 f}{\partial X^2} = -\frac{1}{X^2}. \tag{2.31}$$

By applying Itô's lemma (2.28), we have

$$df = (\mu - \sigma^2/2)dt + \sigma dW_t. \tag{2.32}$$

Then, using Itô's integral,

$$f_t = f_0 + (\mu - \sigma^2/2)t + \sigma W_t, \tag{2.33}$$

and converting f_t back to $\ln X_t$, the equation above then becomes

$$X_t = x_0 e^{(\mu - \sigma^2/2)t + \sigma W_t}, \tag{2.34}$$

corresponding to Equation (2.17).

2.7.2 Multi-dimensional Case

Itô's lemma for multi-dimensional cases is very useful (and has implications for this thesis). Assume that the process $X = (X_1, X_2, \ldots X_n)$ is a continuous semimartingale, i.e. each process $\{X_i,\ i = 1, 2, \ldots, n\}$ is a continuous semimartingale. Then for $f(X, t)$,

$$df = \frac{\partial f}{\partial t}dt + \sum_{i=1}^{n} \frac{\partial f}{\partial X_i}dX_i + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial X_i X_j}dX_i dX_j, \tag{2.35}$$

where $f = f(X, t)$ is twice continuously differentiable.

Consider the case when $X = (X_1, X_2, \ldots, X_n)$ is a diffusion, and each process $\{X_i,\ i = 1, 2, \ldots, n\}$ is a diffusion solving the SDE

$$dX_i = \mu_i dt + \sigma_i dW_i\ ,\ i = 1, 2, \ldots, n, \tag{2.36}$$

where W_i for $i = 1, 2, \ldots, n$ are correlated Brownian motions with

$$\mathbb{E}[dW_i dW_j] = \rho_{ij} dt, \quad \rho_{ii} = 1.$$

Then Itô's lemma with respect to the n–dimensional Brownian motion is

$$df = \left(\frac{\partial f}{\partial t} + \sum_{i=1}^{n} \frac{\partial f}{\partial X_i} \mu_i + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial X_i \partial X_j} \rho_{ij} \sigma_i \sigma_j \right) dt + \sum_{i=1}^{n} \frac{\partial f}{\partial X_i} \sigma_i dW_i. \tag{2.37}$$

2.8 Change of Measure

The idea of changing probability measure is of central importance in derivative pricing theory. As mentioned in Section 1.1, a derivative is contingent on one or several underlying assets whose uncertainties do not affect the price of the derivative. Therefore, changing the probability measure makes derivative pricing easier. These changes of measure have many other applications, for instance, "importance sampling" in the Monte Carlo method introduced in Section 4.4.3.

2.8.1 Radon-Nikodým Derivative

Probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\Omega, \mathcal{F}, \mathbb{Q})$ are called equivalent if

$$\mathbb{P}(A) = 0 \Longleftrightarrow \mathbb{Q}(A) = 0, \quad \forall A \in \mathcal{F}.$$

This is often written as $\mathbb{P} \sim \mathbb{Q}$.

Suppose $\mathbb{P} \sim \mathbb{Q}$ on space $(\Omega, \mathcal{F})$. The random variable R defined on $(\Omega, \mathcal{F})$ is called the Radon-Nikodým derivative of $\mathbb{P}$ with respect to $\mathbb{Q}$ if

i:) R is strictly positive;

ii:) R is unique with $\mathbb{P}$–a.s.;

iii:) $\mathbb{Q}(A) = \mathbb{E}_{\mathbb{P}}[R\mathbf{1}_A]$, $\forall A \in \mathcal{F}$.

It is customary to write $R = \frac{d\mathbb{P}}{d\mathbb{Q}}\big|_{\mathcal{F}}$, which is defined as the Radon-Nikodým derivative of $\mathbb{P}$ with respect to $\mathbb{Q}$; it is also called a numeraire in the financial world.

This generalises the concept of numeraire, as mentioned in Section 1.5.1, the numeraires are normally risk-free assets, however, when the model setup becomes more sophisticated, the numeraire can also be a stochastic process, which is sometimes called the stochastic discount factor (see Benninga, Björk and Wiener, 2002).

2.8.2 Girsanov's Formula

Girsanov's theorem establishes a link between two probability measures and can be extended to (continuous) semimartingales. However, Girsanov's formula, in the context of a Brownian motion is sufficient for this thesis.

Assume $(W_t)_{t\geq 0}$ is a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to filtration $\{\mathcal{F}_t\}$, $\mathcal{F}_t \subset \mathcal{F}$. Then define

$$d\tilde{W} = \alpha dt + dW_t, \tag{2.38}$$

and

$$R_t = \exp\left(-\frac{1}{2}\int_0^t \alpha^2 du - \int_0^t \alpha dW_u\right), \tag{2.39}$$

where $\alpha = \alpha(t)$ is adapted to $\{\mathcal{F}_t\}$. Assume that a new probability measure is defined by $\mathbb{Q}(F) = \int_A R_t d\mathbb{P}$ for all $A \in \mathcal{F}$, then under $\mathbb{Q}$, the process $(\tilde{W}_t)_{t\geq 0}$ is a Brownian motion. Basically, Girsanov's theorem implies that a Brownian motion process with any drift can be converted to another Brownian motion process with the same variance but with different drift.

2.8.3 Equivalent Martingale Measure

Assume M is a continuous martingale defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to a filtration $\{\mathcal{F}_t\}$, $\mathcal{F}_t \subset \mathcal{F}$. The set of equivalent martingale measures for M is the set of probability measures $\mathbb{Q}$ satisfying:

i:) $\mathbb{P} \sim \mathbb{Q}$ with respect to $\mathcal{F}$;

ii:) $\mathbb{P}$ and $\mathbb{Q}$ agree on $\mathcal{F}_0$;

iii:) M is a $\mathcal{F}_t$–measurable martingale on the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$.

One of the most important concepts in finance is the risk-neutral measure (also known as a martingale measure), which is any probability measure, equivalent to the market measure (i.e. the real world measure), which makes all discounted asset prices martingales. This property of the risk-neutral measure makes it more desirable in option pricing, as the risk-neutral measure does not require investors' preference towards the risk which is very difficult to quantify (see Hunt and Kennedy, 2005), and it is the essence of arbitrage pricing theory mentioned in Section 1.3.1.

2.9 Summary

This chapter can be summarised perfectly with a quote by one of the greatest probabilists of the 20th century, William Feller (1906–1970), from "An Introduction to Probability Theory and its Applications":

> *All possible definitions of probability fall short of the actual practice.*

Chapter 3

Introduction to Foreign Exchange Markets

If I have been able to see further, it was only because I stood on the shoulders of giants.

—— *Isaac Newton (1642–1727)*

This chapter focuses on foreign exchange markets and one of the most heavily traded derivatives in the market, foreign exchange options (i.e. currency option). Section 3.1 explores the structure of the foreign exchange market. Section 3.2 focuses on an introduction to currency options, and Section 3.3 exams the characteristics of currency options, from the modelling point of view. Section 3.4 reviews the literatures on currency-option pricing models.

3.1 Overview of the Foreign Exchange Market

Currency is the creation of a circulating medium of exchange based on a store of value. It evolved from two basic innovations: the use of counters to assure that shipments arrived with the same goods that were sent, and the use of silver ingots to represent value; both of these developments had occurred by 2000BC. Foreign exchange refers to money denominated in the currency of another nation. A foreign exchange rate is therefore a price (see Cross, 1998).

3.1.1 Foreign Exchange Market Structure

The foreign exchange market is the oldest, largest and most extensive financial market in the world. It can be roughly viewed as a global, largely OTC market. The market is basically unregulated and is handled by banks in different locations via telephones, faxes, and computer networks 24 hours a day. The OTC market offers a vast range of foreign exchange products, from spot exchange rates to exotic exchange-rate derivatives.

In the 1990s, when the trading of currency options was introduced to the interbank foreign exchange market, option trading exploded in volume. Virtually every large financial institution offers currency options trading. It is worth mentioning that there is still about 10 percent of foreign exchange market activity is traded through the organised exchanges. Although, exchange-traded products are limited to currency futures and certain currency options. The instruments that are traded on established exchanges are generally more standardised and more liquid than those traded on the OTC market. Activities on the exchange-traded instruments are monitored by independent associations, such as clearing houses and the financial integrity of futures and options markets has withstood some rigorous tests. The rapid growth of the OTC market has been the subject of numerous studies by central banks and regulatory authorities. Much of this work has critically examined derivative transactions privately negotiated in the OTC market (see Henigan, 2006).

3.1.2 Participants in Foreign Exchange Market

The main participants in the foreign exchange market are dealers, brokers, central banks, and customers. According to the 76th annual report (for the financial year which began on 1st April, 2005 and ended on 31st March, 2006) of foreign exchange and derivatives market from BIS, over half of daily foreign exchange transactions take place between bank dealers[1]. A substantial percentage of reporting dealers are commercial banks; others are investment banks and insurance firms. A market maker is a dealer who makes a two-sided market regularly for customers, whereas a broker is more an agent for one or both parties in the transaction. In principle, the broker does not commit capital, but relies on commission for services provided. Central banks play two roles in the foreign exchange market. They intervene in the market by buying or selling foreign currencies, and they also may be in the market as agents for other central banks. The range of customers includes small commercial banks and investment banks, firms and corporations, managers of money funds, mutual funds, hedge funds, and individuals (see Cross, 1998).

3.2 Introduction of Currency Options

It was not until 4000 years after the appearance of currency that options on foreign exchange were devised. These can be used by corporations to hedge foreign-exchange nature exposures or hedge against extreme events that threaten the business, and are heavily traded in financial markets. The option gives the holder the right to buy or sell one currency against another currency at a specified price on or before the date the option expires.

The most far-reaching innovation in the development of financial derivative markets during the twentieth century was the start of trading of exchange traded currency options at the PHLX in 1982, as mentioned in Section 1.1. By 1988, currency options were trading in volumes as high as four billion U.S. dollars per day in underlying value. Currency options brought trading interest internationally, from America, Europe, the Pacific Rim to the Far East. Furthermore, currency option trading hours are far longer than other open outcry auction marketplaces. Currently, many major stock exchanges offer trading options on seven major currencies: U.S. dollars, Australian dollars, British pounds, Canadian dollars, Euros, Japanese yen, and Swiss francs. Some customised currency options which are on any two currently

[1] The full article can be found at `http://www.bis.org/publ/arpdf/ar2006e.pdf`.

approved currencies can be traded in some exchanges[2].

Why are currency options so attractive? Currency options are widely used to hedge foreign exchange risk for a future date. They provide foreign exchange risk managers, investors and traders with a wide array of capabilities for controlling the risks inherent in foreign exchange exposure, and for participating in market movements and implementing investment research decisions related to exchange rate fluctuations. In academic research, currency options are important in measuring the value of other international financial instruments, such as currency option bonds (also known as a quanto[3]), currency future options, currency option forwards and so on. As international financial markets further develop, currency options will play an increasingly important role as a major international financial instrument.

Among a vast range of instruments of exchange rate, American options and barrier options are of great importance. If a European currency-option is a standard cover for foreign exchange exposure, an American option can be viewed as a "premium" cover. As American options offer great freedom for the owners to exercise anytime they think more appropriate. Whereas barrier options are the "cheap" cover compared with European options, as they are normally customised for the request of buyers, depending on the buyers' view of the market. To avoid paying for the unnecessary protection, barrier options generally offer protection in a narrower bound, therefore making them less expensive for the cost. The existing disadvantages of these two types of options in terms of implementation will be introduced later in this chapter.

Recalling the definition of a currency option, there are four key terminologies (mentioned in Section 1.4) that will be incorporated into the mathematical models and will occur repeatedly in the following chapters:

i:) $\max\{\text{payoff}, 0\}$, the payoff function is a **right** not an obligation;

[2]See PHLX official website for user's guide to currency options: `http://www.phlx.com/products/currency/cug.pdf`

[3]Quanto is an option which has a payoff defined with respect to an asset or an index or an interest rate in one country, but the payoff is converted to another currency for payment with the contractual exchange rate. See Wilmott (2000a).

ii:) K, the specified price, is called the **strike price**;

iii:) T, the date the option expires is called the **expiry date** or maturity date;

iv:) V, the **premium** (i.e. option price) is the price paid to acquire the option.

3.3 The Structure of Currency Option Models

In addition to the key words which were introduced above, currency option pricing models are dependent on the following key factors:

- Exchange rate;
- Interest rate;
- Expected volatility of the exchange rates and/or interest rates.

All these factors will directly affect the models' performance. Therefore, this section is focused on assessments of these three factors.

3.3.1 Exchange Rate Models

Many economic factors affect exchange rate movements, such as the merchandise trade balance, the flow of funds, the interest rate differences and inflations[4]. Due to the complicated nature of exchange rate dynamics, there is no widely accepted explanation for exchange rate movements. Past theoretical research on exchange rate models may be classified into three categories:

- Models relating exchange rates to macroeconomic fundamentals
  Many models suggest that exchange rates should be jointly determined with macroeconomic fundamentals such as target zones, purchasing power parities (PPP), or uncovered interest parities (UIP). However, with these macroeconomic fundamentals, the models for exchange rate processes can be too specific for individual countries to be generalised for currency option pricing.

[4]For more detailed introduction, see "Economic Factors in Forex" at `http://www.cambridgefx.com/currency-exchange/exchange-rates-news.html`.

- Lognormal distribution models

  Exchange rates are in close relation to bonds, and bond prices are lognormally distributed (following general geometric Brownian motions, introduced in Section 2.6.1), therefore lognormal distribution models for exchange rate dynamics are very convenient and widely used in theoretical research. Most of the literature mentioned in this thesis has been based on geometric Brownian motion for exchange rate dynamics (see Section 3.4).

- Time-series models

  Time-series models are the favourite of economists. These models are built up with specified data. Empirical evidence shows that exchange rates fluctuate around a moving average. Therefore, the most popular model for the exchange rates is GARCH (Generalised Autoregressive Conditional Heteroscedasticity) model, in which past observations of the variance and variance forecast are used to forecast future variances (see Duan and Wei, 1999). However, time-series models do not provide any information about the dynamics of the system, which implies that the terms and variables chosen for the models do not normally have financial interpretations.

3.3.2 Volatility Models

Volatility has always been both fundamental and problematic, because it can have a substantial influence on the option price. Clark, Tamirisa and Wei (2004) pointed out that the liberalisation of capital flows in the last two decades and the enormous increase in the scale of cross-border financial transactions have increased exchange rate movements. Currency crises in emerging market economies are particular examples of high exchange rate volatility. In addition, the transition to a market-based system in Central and Eastern Europe has often involved major adjustments in the international value of these economies' currencies. Stochastic volatility models can be useful in this respect, because they can help to explain why options with different strike prices and maturities have different implied volatilities and volatility smiles.

However, volatility is not directly observable for the future. To date, there is no fully successful stochastic volatility model for foreign exchange dynamics. Empirically, volatility has similar characteristics to interest rates and so models originally developed for interest rate models are applied to stochastic volatility models.

Melino and Turnbull (1990) considered the existence of stochastic volatility in currency option pricing; they took the logarithm of the volatility to be an Ornstein-Uhlenbeck process. The first analytic currency option pricing formula was developed by Heston (1993), in which he used a mean-reverting square-root process for volatility of exchange rate; much other research extended Heston's plausible model. Recently, a large amount of literature has emerged using time-series models because of the difficulties with estimating variables for the theoretical volatility models. Duan and Wei (1999) obtained currency-option prices using GARCH model for the volatility. For a more detailed survey on stochastic volatility, see Hobson (1998).

3.3.3 Interest Rate Models

Interest rates are intrinsic to the time value of money, which is one of the crucial components in derivative pricing, in particular currency options. Thus, it is necessary to explore the features of the term structure of interest rates.

Bonds are central to the theory of term structure of interest rates. A bond is a fundamental instrument in financial markets, which may pay a regular stream of coupons (typically every six months or annually) until its maturity, when it pays its face value in addition to the final coupon. Bonds without coupon payments are called zero-coupon bonds, also known as pure discount bonds. In modern finance theory, the zero-coupon bond is used to calculate the time value of money, which is one of the basic concepts in the analysis of many financial instruments. The yield-to-maturity of a bond is the discount rate which relates the present value of its payments to the price paid for the bond. Note that the yield-to-maturity is equal to the spot rate (i.e. the discount rate) only for zero-coupon bonds (see Wilmott, 2000b).

Term structure is a series of interest rates corresponding to the yields on comparable instruments of different maturities, such as a set of zero-coupon government bonds. It is also known as a yield curve which expresses the interest rate as a function of time to maturity. These might be yields derived from the prices of zero-coupon bonds, or the fixed leg of swaps, or any number of other rates of practical concern. Term structure theory falls into three classes: short-rate term structure models, forward-rate term structure models, and market term structure models (which are tailored to fit specific interest rate products for practitioners).

Short-rate Models

The short rate is a crucial interest rate in all models. It is fundamental to pricing theory and it is the key variable in the first generation of term-structure models. A short-rate model is a model of term structure of spot interest rates, where the spot rate is referred to as the (continuously compounded) yield of the discount bond. The choice of short-rate model arises from a combination of mathematical convenience and tractability, or numerical ease of implementation. The most widely used of these models are generally one-factor models, in which the entire yield curve is specified by a single stochastic state variable; popular examples of these include the models of Vasicek (1977), and Cox, Ingersoll and Ross (1985).

Vasicek (1977) proposed the first no-arbitrage model for the term structure of interest rates. Vasicek assumed that the instantaneous rate of interest, $r(t)$, is described by an SDE

$$dr = \mu(r,t)dt + \sigma(r,t)dW_t, \tag{3.1}$$

where $\mu(r,t)$ is the instantaneous drift, $\sigma(r,t)$ is the instantaneous volatility, and dW_t is the increments of a standard Brownian motion. Vasicek replicated a portfolio to obtain the corresponding PDE. The derivation procedure is similar to Black-Scholes-Merton methodology (mentioned in Section 1.4). Let $P(t,T)$ denote the price of a zero-coupon bond with maturity T at time t, where $0 \le t \le T$. The bond is normalised to have unit face value, i.e. $P(T,T) = 1$. The PDE is then:

$$\frac{\partial P}{\partial t} + (\mu - \sigma\lambda)\frac{\partial P}{\partial r} + \frac{\sigma^2}{2}\frac{\partial^2 P}{\partial r^2} - rP = 0, \tag{3.2}$$

where $\lambda = \lambda(t,r)$ denotes the market price of interest-rate risk. Vasicek restricted his general model by assuming that the market price of interest rate risk, λ, is constant, and that the spot rate follows an Ornstein-Uhlenbeck process (introduced in Section 2.6.2):

$$dr = \kappa(\theta - r)dt + \sigma dW_t; \tag{3.3}$$

where

$r \equiv$ the short term interest rate;

$\kappa \equiv$ the mean reversion parameter;

$\theta \equiv$ the long-run mean spot interest rate;

$\sigma \equiv$ the instantaneous volatility of the process;

$dW_t \equiv$ the increments of a standard Brownian motion.

Vasicek obtained a closed-form solution for the zero-coupon bond price, namely

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}; \tag{3.4}$$

where

$$B(t,T) = \frac{1 - e^{-\kappa(T-t)}}{\kappa}; \tag{3.5}$$

$$A(t,T) = \exp\left(\frac{(B(t,T) - (T-t))(\kappa^2\theta - \sigma^2/2)}{\kappa^2} - \frac{\sigma^2 B(t,T)^2}{4\kappa}\right). \tag{3.6}$$

One serious shortcoming of Vasicek's model, however, is that it admits negative interest rates.

The model introduced by Cox, Ingersoll and Ross (1985) (hereafter,CIR) may preclude negative interest rates by assuming the volatility σ is proportional to the square root of the spot rate. The characteristics of interest rate movements offered by the CIR model include mean reversion toward a long-term rate, and non-negative interest rates (mentioned in Section 2.6.3). The CIR process is expressed as

$$dr = \kappa(\theta - r)dt + \sigma\sqrt{r}dW_t; \tag{3.7}$$

where

$r \equiv$ the short term interest rate;

$\kappa \equiv$ the mean reversion parameter;

$\theta \equiv$ the long-run mean spot interest rate;

$\sigma \equiv$ the instantaneous volatility of the process;

$dW_t \equiv$ the increments of a standard Brownian motion.

Feller (1951) presented the result that the square-root diffusion (i.e. CIR process) will remain positive in a continuous time if

$$\frac{2\kappa\theta}{\sigma^2} > 1 \quad \text{with} \quad r(0) \geq 0;$$

this is sometimes called the "Feller condition". However, in discrete time case (i.e. with finite Δt), even with the Feller condition satisfied, the CIR process can still go negative (see Higham and Mao, 2005; Johnson, 2006).

Again, let $P(r,t)$ denote the price of a zero-coupon bond at time t with maturity T , with $0 \leq t \leq T$, and normalise the bond to have unit face value, i.e. $P(T,T) = 1$. Cox, Ingersoll and Ross (1985) obtained the PDE

$$\frac{\partial P}{\partial t} + (\kappa(\theta - r) - \sigma\lambda)\frac{\partial P}{\partial r} + \frac{r\sigma^2}{2}\frac{\partial^2 P}{\partial r^2} - rP = 0, \tag{3.8}$$

where $\lambda = \lambda_0\sqrt{r}$, which denotes the market price of interest rate risk. They derived in closed form the zero-coupon bond price, as well as providing formulae for European options on zero-coupon bonds. In their model, bond prices have the same general form as in Vasicek's (1977) model, that is,

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}, \tag{3.9}$$

where $A(t,T)$ and $B(t,T)$ are as follows:

$$B(t,T) = \frac{2(e^{\gamma(T-t)}-1)}{(\gamma+\kappa)(e^{\gamma(T-t)}-1)+2\gamma}; \tag{3.10}$$

$$A(t,T) = \left(\frac{2\gamma e^{(\gamma+\kappa)(T-t)/2}}{(\gamma+\kappa)(e^{\gamma(T-t)}-1)+2\gamma}\right)^{2\kappa\theta/\sigma^2}; \tag{3.11}$$

where

$$\gamma = \sqrt{\kappa^2+2\sigma^2}.$$

There are a number of other short-rate models that have been very popular with practitioners, such as the Hull and White (1994a) model, the Black and Karasinski (1991) model and the Black, Derman and Toy (1990) model (which is a special case of Black and Karasinski model). Also, a number of other short-rate models involve multi-factors such as the Brennan and Schwartz (1982) model, the Longstaff and Schwartz (1992) model, and the Hull and White (1994b) model, but these are rarely used for derivative pricing in practice due to high computational demand (see Hunt and Kennedy, 2005), and consequently are not discussed in this thesis.

Short-rate models can be tractable and amendable to numerical methods. In practice, short-rate models are often used as a complement to more sophisticated models. Even though they are not as "broad" as forward-rate models which are introduced below nor as "fit" as market models, they can be useful for pricing derivatives quickly and flexibly.

Forward-rate Models

Forward-rate models are also referred to as whole yield curve models, and are specified generally in terms of the instantaneous forward rate process. The forward rate is the implied rate of return between two future dates, derived from the rates currently available via two bonds already issued and maturing at the future dates. The forward rates can be viewed as expectations of future spot rates, and therefore there is an explicit relation to transform a forward-rate model into a short-rate model.

Ho and Lee (1986) originally presented a discrete-time forward-rate model. Using a binomial model, they allowed the model parameters to be deterministic functions of time, calibrated to fit today's forward-rate curve. Models that incorporate the idea of starting with the prices of zero-coupon bonds of various maturities and proceeding to build a model that admits no arbitrage possibilities, then modelling how bond prices and interest rates evolve through time are sometimes referred to as no-arbitrage models. However, the time-varying drift in the Ho and Lee model makes the long-term rate unbounded (see James and Webber, 2004). Note that Ho and Lee's model assumes that interest rates are normally distributed with instantaneous volatility constant; it also falls into the category of a short-rate model (see Wilmott, 2000b; Hull, 2002).

Heath, Jarrow and Morton (1992) (hereafter, HJM) generalised the Ho and Lee (1986) model to a continuous-time economy with multiple factors. The key concept in the HJM model is that the entire yield curve is modelled as a state variable, not just the short end or only two factors. The HJM methodology uses the instantaneous forward rate as the driving stochastic variable. Given the initial forward curve, HJM describes the evolution of the forward curve by a family of SDEs that are generally path-dependent; this family of processes is under the no-arbitrage condition. Accordingly, it can (in principle) price many derivatives whose values are particularly sensitive to the term structure of interest rates, such as currency warrants and cross-rate swaps.

It is assumed that there is a zero-coupon bond $P(t,T)$ maturing at time T, with $P(T,T)=1$. The instantaneous forward rate at time t, $f(t,T)$ is then defined by

$$f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T}, \quad \text{for all} \quad 0 \leq t \leq T. \tag{3.12}$$

Solving Equation (3.12) yields

$$P(t,T) = \exp\left(-\int_t^T f(t,s)ds\right), \quad \text{for all} \quad 0 \leq t \leq T. \tag{3.13}$$

Since the curve is closely associated with bond prices, bond price dynamics can be inferred from it, and it is easy to deduce the spot rate with respect to the forward rate, namely $r(t) = f(t,t)$, which shows that any short-rate model is also included within a sub-set of the HJM model. In fact, any interest rate model that satisfies the principles of arbitrage-free bond dynamics must be within the HJM framework (see Lee, 2000).

The advantage of the forward-rate models over the short-rate models is that they achieve an automatic fit to the yield curve, whereas the short-rate models require some extra computation. However, the HJM model is path-dependent, in general, and consequently the PDE approach can be difficult to implement. As an extra dimension is required to accommodate the path-dependent feature. Non-recombining ("bushy") trees should also be abandoned in favour of Monte Carlo methods, for the computation reasons. Moreover, the calibration of the model presents a serious problem, as HJM model allows high dimensional diffusion coefficients, every single diffusion variable in the HJM model has to be calibrated to data, which is not realistic in practice.

Market Models

Market models were only introduced to the interest rate market in the late 1990s to overcome calibration problems. They form a class of models within the HJM framework that describe variables directly observed in the market, such as LIBOR and swap rates. Models in this latest generation create an environment to make calibration of market data relatively straightforward. Brace, Gatarek and Musiela (1997) as well as Miltersen, Sandmann and Sondermann (1997) presented a no-arbitrage interest rate model using specific parts of the forward curve from LIBOR market rate. Due to the feature of lognormal distribution, they produced Black's (1976) formula for caps/floors (also known as LIBOR Market Models). A similar model for swap rates and swap rate derivatives was developed by Jamshidian (1997), and so-called Swap Market Model leads to the Black formula for swaptions. However, these models are not compatible. The market models are tailor-made for the specific products. For instance, LIBOR market models cannot be applied to swap market models, and quarterly LIBOR models cannot be applied to semiannual LIBOR models, and so on.

The motivation for the development of market models arose from the fact that although the HJM framework is appealing theoretically, its standard formulation is based on a continuous spectrum of rates and is therefore fundamentally different from actual forward LIBOR and swap rates as traded in the market (see Lee, 2000). The lognormal HJM model was also well known to exhibit unbounded behaviour (producing infinite values) in contrast with the use of lognormal LIBOR distribution in Black's (1976) formula for caplets.

Given the building blocks of currency-option pricing model, the following chapters will be extensively using these concepts to establish more sophisticated pricing models.

3.4 Literature Review

This section will present a timeline of the development of currency option pricing theory from two perspectives: on the modelling of currency options and on numerical methods for different types of option implementation.

3.4.1 Review of Currency Option Modelling

Early Work

An unsurprising early treatment of currency options was to convert earlier work for options on dividend-paying stocks of Merton (1973) formula, preserving the essential mathematics. This was undertaken by Garman and Kohlhagen (1983), who applied Merton's formula to European currency option pricing. Mathematically, the Garman and Kohlhagen formula is identical to Merton's formula with dividend payments, and consequently relies on the same modelling assumptions, in particular, constant risk-free interest rates and constant volatilities. The term f, which represents a stock's dividend yield in Merton's model, is translated into the foreign currency's continuously compounded risk-free rate in the Garman and Kohlhagen formula, in order to derive the formula. Garman and Kohlhagen assumed the domestic interest rate, the foreign interest rate and the implied volatility are constants, furthermore the underlying exchange rate price follows a geometric Brownian motion introduced in Section 2.6.1.

The value for a European call price C at time t is then:

$$C = S_t e^{-f(T-t)} N(d_1) - K e^{-r(T-t)} N(d_2), \tag{3.14}$$

where

$$\begin{aligned} d_1 &= \frac{\ln(S_t/K) + (r - f + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \\ d_2 &= d_1 - \sigma\sqrt{T-t}, \end{aligned}$$

$$\begin{aligned} S_t &\equiv \text{the spot exchange rate;} \\ K &\equiv \text{the strike exchange rate;} \\ r &\equiv \text{the continuously compounded domestic risk free interest rate;} \\ f &\equiv \text{the continuously compounded foreign risk free interest rate;} \\ T &\equiv \text{the time in years of the expiration of the option;} \\ \sigma &\equiv \text{the implied volatility for the underlying exchange rate;} \\ N(\cdot) &\equiv \text{the standard normal cumulative distribution function.} \end{aligned}$$

As with the Black-Scholes (1973) model, the Garman and Kohlhagen formula has been a popular practical choice for currency option pricing over the years, despite the fact that interest rate and volatility are not constant in practice.

Biger and Hull (1983) again used the "Black-Scholes (1973) methodology" including dividends, and obtained comparable results based on the same assumptions as employed by Garman and Kohlhagen (1983). In the same year, Grabbe (1983) presented a model for European options, which relaxes the assumption of constant interest rates. He assumed that the processes of interest rates in the domestic currency and the foreign currency are deterministic functions of time, using an arbitrage-free approach to obtain a PDE and consequently the European call price.

However, the model is not supported by the empirical evidence of Adams and Wyatt(1987a), who showed that the interest rate risk is an important element in the valuation of currency options.

Although currency option pricing has become a very topical subject in academic research more recently, American-style currency-option pricing remained unstudied until the end of the 1980s.

A currency option is closely related to cashflows in the domestic and the foreign economy. Adams and Wyatt (1987b) assumed that in a risk-neutral world, the relationship between the prices of domestic and foreign bonds is:

$$\mathbb{E}\left[e^{-rT}\right] = \mathbb{E}\left[e^{-fT}\frac{S_T}{S_0}\right], \tag{3.15}$$

where $\mathbb{E}[\cdot]$ is the expectation operator; r is the domestic interest rate; f is the foreign interest rate; S_0 is the spot exchange rate at time $t = 0$; S_T is the forward exchange rate. Again, it is assumed that both interest rates are non-stochastic, consequently, for an American put, the option value is obtained by rearranging Equation (3.15) to the following:

$$S_0 = e^{(f-r)T}\mathbb{E}[S_T]. \tag{3.16}$$

A quadratic approximation was used to develop a method of estimating the early exercise premium, and of determining when early exercise is optimal.

Shastri and Tandon (1987) presented an analytical approximation for the valuation of American options on foreign currencies. The pricing formula uses the techniques developed in the influential paper by Geske and Johnson (1984) to price American options on foreign currencies as a sequence of compound options, which has been mentioned in Section 1.5.1. Unfortunately, the assumptions in Shastri and Tandon's model are also restricted to constant interest rates and constant volatility of the exchange rate process.

Bodurtha and Courtadon (1987) considered the limitation of constant volatility. They presented empirical tests on the ability of the American option pricing model to explain the pricing of foreign currency options traded on the PHLX from February

28, 1983 to March 26, 1985. The results show that the model underprices out-of-the-money options relative to at-the-money options and in-the-money options. However, Bodurtha and Courtadon's basic assumptions of this empirical testing model are identical to the assumptions made by Garman and Kohlhagen (1983), which implies that the conclusion they drew in the paper was based on somewhat unrealistic assumptions.

Stochastic Environments for Currency Options

Melino and Turnbull (1990) investigated the consequences of stochastic volatility for currency option pricing. They assumed the spot exchange rate satisfies a general form process

$$dS_t = (a + bS_t)dt + \upsilon S_t^{\frac{\beta}{2}} dW_t, \quad \text{where} \quad \beta = 0, 1, 2. \tag{3.17}$$

Note that different values of β give different probability distributions of the underlying asset. These are normal distributions for $\beta = 0$, chi-square distributions for $\beta = 1$ and lognormal distributions for $\beta = 2$. According to their empirical work on the market data, Melino and Turnbull also assumed the stochastic volatility was described by the following SDE:

$$d \ln \upsilon = (\alpha + \theta \ln \upsilon)dt + \gamma dZ_t. \tag{3.18}$$

However, they held both the domestic interest rate and foreign interest rate constant. They argued that neither the lognormal probability distribution for exchange rates nor constant volatility fit empirical data. By simply setting the interest rates as constant, Melino and Turnbull used Equation (3.18) with historical data, then used this SDE for the volatility of the exchange rate process. They did find that making volatility stochastic gave a much better fit to the Canada-U.S. exchange rate distribution and more accurate predictions of observed option prices.

Hilliard, Madura and Tucker (1991) proposed a simple approach to price European currency options under stochastic interest rates, assuming that domestic and foreign bond prices have local variances depending only on time and not on other state variables. By constructing a delta-hedging strategy following Grabbe (1983), invoking the risk-neutrality argument of Cox and Ross (1976), and by identifying Vasicek's (1977) term structure model as the appropriate bond pricing model, Hilliard, Madura and Tucker derived a closed-form European currency-option pricing model under stochastic interest rates. Unfortunately, as noted in Section 3.3.3, the Vasicek model allows the occurrence of negative interest rates, which is unreasonable, and moreover, these models cannot be extended to American option pricing analytically. Hilliard, Madura and Tucker's model is competitive for currencies with highly volatile interest rates and for long-lived options. Hilliard, Madura and Tucker, and Amin and Bodurtha (1995) indicated that allowing for stochastic interest rates leads to a more accurate valuation of currency options with longer maturities than the constant interest rates alternative.

Tucker (1991) suggested that foreign exchange rates follow a jump-diffusion process, and that an option pricing model, that takes this process into account, is likely to be more accurate. Many other authors also demonstrate that these large jumps exist in foreign exchange price movements, which are responsible for the leptokurtic distribution on price returns (i.e. the inflation of peak and tails as the result of the occurrence of more frequent small and large price changes than normal). Unfortunately, Tucker also assumed interest rates are non-stochastic.

Amin and Jarrow (1991) introduced a general framework for valuing a European option on a foreign currency with stochastic interest rates. Their model allows domestic and foreign term structures of interest rates to follow the stochastic processes of the HJM (1992) structure. Amin and Jarrow also obtained closed-form solutions for European options by assuming the market is complete and the volatility functions governing the term structure are deterministic. The Amin and Jarrow model is a multi-factor model in which there are at least three different sources of uncertainty, those associated with the domestic interest rate, the foreign interest rate, and the exchange rate. To find a risk-free equivalent martingale measure, they choose the domestic current account (i.e. the cash account) as the numeraire and calibrated the domestic bond, the foreign bond and the exchange rate into the new equivalent martingale measure, and obtained an analytic solution for European-style options. However, the complexity of the assumptions of the Amin and Jarrow model make it applicable only to European-type options. Even using numerical approaches, it is difficult to implement the HJM framework (mentioned in Section 3.3.3).

It has been widely documented in the literature that the price volatility of many financial assets follows a stochastic process. Heston (1993) pioneered the development of stochastic volatility in currency option models. As volatility is an unobservable yet very important parameter, Heston applied a mean-reverting feature to the volatility process. Thus the model is composed of a stochastic domestic zero-coupon bond, a foreign zero-coupon bond, an exchange rate, and the volatility of the exchange rate process. Heston obtained a PDE by delta-hedging two different maturity portfolios in the model, and then obtained an analytic solution by invoking the Black-Scholes (1973) formula. Although Heston's model has been influential, he assumed interest rates are non-stochastic, which is somewhat inconsistent with the stochastic bond prices in the model. Also it is heuristic since he separated Garmen and Kohlhagen's formula (3.14) into two probability parts and assumed these satisfy the PDE independently in order to obtain the option price.

Amin and Bodurtha (1995) produced the first highly stochastic currency-option model allowing American-style (i.e. early exercise) feature. They considered an arbitrage-free discrete time implementation of the Amin and Jarrow (1991) framework, using a multinomial version of the lattice technique of Cox, Ross and Rubinstein (1979). They derived a path-dependent model with specific interest-rate functions. A property of path dependence is that the outcome of a process depends on its past history, which implies the tree cannot recombine. Therefore it is difficult to obtain an accurate result due to the vast computational cost. In Amin and Bodurtha's model, a simplified HJM model is adopted. They assumed the volatilities of both the domestic and the foreign forward rates are constant (i.e. the Ho and Lee, 1986 model). Moreover, they only managed to obtain up to 12 time steps for a five-year American put option value. The multinomial tree with only 12 time steps applied to a three-factor model is rather unsatisfactory, although it is stated in their paper: "... for option maturities up to five years, path-dependent models with fewer than 12 steps can still yield option values that are accurate to within one or two percent of their continuous-time limiting values." Since there is no benchmark for the American option price with stochastic interest rates, they adopted a path-independent model in their paper, with a recombining tree. However, the accuracy of the recombine tree is rather poor, there remains much scope for improvement in early-exercise currency option pricing which subsequent work has not adequately addressed.

Bakshi, Cao and Chen (1997) stated that many assumptions can be made concerning the distribution of the underlying asset, the interest rates components and the market price of risk. They used a generalised least-squares technique to estimate the parameters, essentially minimising an error term each day of the sample. This approach, although straightforward to implement, is somewhat contrary to the assumptions of the model, as it allows the parameters to take on a different value every day. Moreover, obtaining parameters using cross-sectional information may result in an excellent fit at the current date, but does not provide any information about the dynamics of the system (disadvantages of time series models have already been mentioned in Section 3.3.1).

Chang (2001) extended Geske and Johnson's (1984) approach to a stochastic interest-rate economy. He used only the values of once and twice exercisable options and described how stochastic interest rates affect the option value. He built a tree of forward exchange rate process $\frac{S_t B_f}{B_d}$, where S_t is the spot exchange rate at time t, B_f is a foreign bond and B_d is a domestic bond. However, Chang's paper suffers the disadvantage of tree methods, as mentioned for Amin and Bodurtha's (1995) model, which is computationally inefficient and is difficult to apply to American options.

Choi and Marcozzi (2001) enhanced Amin and Bodurtha's (1995) model numerically. They used a radial basis function (RBF) methodology to approximate the

PDE for currency options[5]. Choi and Marcozzi transformed the HJM framework into a short-rate version to obtain a PDE and established a risk-neutral measure by using the domestic interest rate as a risk-free numeraire. However, they presented numerical results for a one-year option with merely five steps for both the foreign interest rate and the domestic interest rate processes, 31 steps for the exchange rate and 360 time steps for a one-year option.

Choi and Marcozzi (2003) managed to obtain an analytic solution for European currency-options. They considered the state variables to be the short rates of interest and the exchange rate, as opposed to the forward rates as proposed in Amin and Jarrow (1991) and utilised in Amin and Bodurtha (1995), in which case the associative diffusion representing the global economy possesses a coercive diffusion matrix.

Chesney and Jeanblanc (2004) focused on the exchange-rate process with jump diffusion. They obtained a PDE for a European option, then claimed on page 216: "If the American and European option values satisfied the same linear PDE (in the continuation region), their difference ΔC, the American premium, must also satisfy this PDE in the same region." Unfortunately, the sign of the jump size significantly affects the pricing model. Only negative-jump processes can be priced and furthermore, PDEs describing American options are inherently nonlinear and, as a consequence, the quantity ΔC cannot satisfy the same PDE.

Substantial evidence has been cited in the literature that volatility in the currency market is stochastic; Low and Zhang's (2005) paper contributed in this area. They used a large database of daily volatility quotes on at-the-money delta-neutral straddles in the OTC currency option market. Some significant observations were found. First, risk premium was found to be negative, which means the buyers pay the premium to compensate for bearing the risk. Secondly, the short-term volatility has higher variability than long-term volatility, implying that the volatility of the

[5]The RBF method is originally an approximate solution to different types of interpolation problem. As distinct from the finite-difference method, the PBF method is meshless, which means it is not difficult for high-dimensional problems as its most important geometrical property is the pairwise distance between points.

short-term volatility is higher than the long-term volatility.

Dupoyet (2006) undertook an empirical investigation into Japanese Yen/U.S. Dollar currency-options traded on the Philadelphia stock exchange during March 29th, 1996 to December 31st, 1999, with the aim of determining the information content of European option prices (for which analytic solutions are available). The models tested were Black-Scholes (1973) and three others, using stochastic volatility, stochastic interest rates and stochastic volatility with jumps. In order to increase the sample size, both European calls and American calls were studied, in a stochastic interest-rate environment where American call values could be safely approximated by corresponding European call values (with the Japanese interest rate much lower than U.S. interest rate). The greatest improvement over Black-Scholes in pricing and hedging was found by using stochastic volatility. Stochastic interest rates improved pricing only for in-the-money long-term options, with insignificant effect on hedging; including jumps improved pricing and the volatility smile, but again contributed little to hedging. Dupoyet's empirical work provides useful parameters for this thesis.

The number of option pricing models that can be derived is virtually unlimited because of the many combinations of assumptions that are possible. A number of models have attempted to capture many processes simultaneously (see Nawalkha and Chambers, 1995). Such as, models include the stochastic volatility and stochastic interest rate models of Amin and Ng (1993), Bakshi and Chen (1997), the stochastic volatility and jump models of Bates (1996), and the stochastic interest rates and jump-diffusion models of Doffou and Hilliard (2001).

This literature review provides a background to the development of the currency option pricing models. New models will be developed in later chapters.

3.4.2 Review of Numerical Techniques

American-style Options

As mentioned in Section 1.5.2, the Monte Carlo method is the most effective method for pricing high-dimensional problems with forward time trajectories but intractable when applied to backward time dynamic programs until quite recently. The first researcher who introduced Monte Carlo methods to American options was Tilley (1993). His approach attempted to record every possible early-exercise time for every sample path, then determine the optimal stopping time. However, it requires a huge storage, and consequently it is computationally inefficient.

Carriere (1996) proposed a non-parametric regression technique for pricing options with early-exercise feature and it can be seen as the precursor to the Longstaff and Schwartz (2001) approach. Tsitsiklis and Van Roy (1999, 2001) also proposed the use of regression to estimate continuation values from simulated paths and to price American-style options. They introduced a variant of value iteration, adapted to the parametric setting. Tsitsiklis and Van Roy (2001) focused on high-dimensional American-style option pricing.

Longstaff and Schwartz (2001) used least-squares regression to approximate the conditional expectations for the American-style option payoff at each point in time if not exercised. This approach will be introduced in detail in Section 4.6 and form the basis for the techniques used in this thesis for American currency-option pricing in Chapter 5.

Carriere (1996), Longstaff and Schwartz (2001), Tsitsiklis and Van Roy (1999, 2001) are categorised as regression-based methods[6].

There are two typical examples of a state-space partitioning approach to evaluate American-style options. Barraquand and Martineau (1995) considered the problem of pricing an American option with several sources of uncertainty. They partitioned the space of underlying assets (the state space) into a tractable number of cells,

[6]For more detailed reviews on early-exercise feature pricing by simulation, see Kind (2005) or Glasserman (2003).

and then computed an approximate early exercise strategy that is constant over these cells. This can be viewed as a version of approximate value iteration involving piecewise constant approximations, which tend to be somewhat restrictive. Bally, Pages and Printems (2005) presented a quantisation method which is adapted for the pricing and hedging of American options on a basket of assets. Numerical tests were performed up to 10 dimensions with American-style exchange options.

The random tree method of Broadie and Glasserman (1997) is an alternative to option pricing with early-exercise feature. The concept of a random tree is a generalisation of the traditional tree concept; the random tree is non-recombining and in addition has some inbuilt random irregularity that comes from the use of the Monte Carlo simulation to generate the states. The stochastic mesh method by Broadie and Glasserman (2004) can be regarded as a recombining random tree, which is best suited for high-dimensional cases. General building procedures based on moment fitting are developed, which are applicable to most commonly used multi-dimensional models.

The duality approach provides option pricing modelling from a new perspective. Since it is rather difficult to obtain an accurate option value based on simulated paths, the duality approach obtains a range of values for option prices. Rogers (2002), and Haugh and Kogan (2004) , and also Jamshidian (2006) sought for an option with an American feature a band of value, arguing that the option price is within the band and converging to the upper bound. However, the complexity of multi-dimensional cases makes the probability theorems proof very difficult.

Barrier Options

For what was then an exotic option, barrier option pricing was first introduced by Merton (1973), using the same basic assumptions as for pricing plain European-style options. By modifying the boundary conditions, closed-form solutions may be obtained. However, these solutions are limited to some very special cases, as mentioned in Section 1.5.1. More relaxed assumptions or more complicated barrier options do not admit analytical solutions.

Numerical techniques generally involve discretisation[7], and therefore in practice, the continuous barrier options are normally transformed into discretely monitored barrier options. Broadie, Glasserman and Kou (1997) demonstrated the quantitative difference between continuous barrier options and discretely monitored barrier options. Under the Black-Scholes (1973) assumptions, they proposed using the Merton (1973) formula as an approximation to the discretely monitored barrier option price. The barrier must be replaced by a factor of $\exp(\beta\sigma\sqrt{T/m})$ for an up-and-in or up-and-out option (by a factor of $\exp(-\beta\sigma\sqrt{\Delta t})$ for an down-and-in or down-and-out option), where $\beta = -\zeta(0.5)/\sqrt{2\pi} \approx 0.5826$ with ζ the Riemann zeta function, σ is the volatility and $\sqrt{\Delta t}$ is the interval between two monitoring time. The correction term, β can be seen as a barrier adjustment term. As when the discrete-time process of the underlying asset hits the barrier, it overshoots it. $\beta\sigma\sqrt{\Delta t}$ is an approximation to the overshoot in the logarithm of the underlying asset price.

Parisian-style Options

In 1994, an important innovation in option markets was the idea of options with the number of time units as a variable in valuation of barrier options (see Rich, 1994); in this way, Parisian options were first introduced to the financial market. The definition of Parisian options is similar to the barrier option. However, the options are not knocked out (or knocked in) unless the **consecutive** time that the underlying asset price spends beyond the barrier reaches the predetermined time in the option contract. Chesney, Jeanblanc-Picque and Yor (1997) presented an analytical solution for the simple Parisian option price based on Brownian excursions theory (see also Cornwall et al., 1997). Avellaneda and Wu (1999) developed a lattice approach for the PDE of Parisian option models. Costabile (2002) provided a random tree approach to evaluate Parisian options with either a constant barrier or with an exponential boundary. Schröder (2003) addressed the extensions to Chesney,

[7] Discretisation is an approximation of a continuous dimension with a finite set of points. As a computer can not represent a continuous function, nor can it represent infinity.

Jeanblanc and Yor model for the case that the excursion has not yet lasted long enough for action to be taken. Bernard, Le Courtois and Quittard-Pinon (2005) also applied an inverse Laplace transform to evaluate Parisian options and their Greeks.

ParAsian options are an extension of Parisian options (referred to as "cumulative Parisian options" in Chesney, Jeanblanc-Picque and Yor, 1997; "delayed barrier options" in Linetsky, 1999; and "cumulative barrier options" in Hugonnier, 1999). ParAsian options are not knocked out (or knocked in) unless the **total** time that the underlying asset price spends beyond the barrier reaches the predetermined time in the option contract. The terminology used in this thesis is based on that of Haber, Schonbucher and Wilmott (1999).

In previous literature on this class of options, the ParAsian option pricing model was introduced by Chesney, Jeanblanc-Picque and Yor (1997), who provided the analytical expression to ParAsian options as well as Parisian options mentioned before. Hugonnier (1999) obtained a closed-form formula for ParAsian options evaluated by means of quadratures, whilst Haber, Schonbucher and Wilmott (1999) used PDEs to derive formulations of both Parisian and ParAsian option prices, which were solved with a finite-difference method. Kwok and Lau (2001) used so-called forward shooting grid approach to obtain the similar numerical results for a class of exotic barrier options[8]. Moraux (2002) corrected one of the Hugonnier (1999) propositions, then provided a closed-form solutions for the ParAsian option values. Note that these option models are all within the usual Black-Scholes (1973) framework.

As for Parisian and ParAsian options, although the Black-Scholes (1973) analysis remains relevant in all cases, the more complicated models, such as stochastic interest rate or stochastic volatility features, do not admit analytical expressions for the value (same as for standard barrier options). Therefore, accurate and numerical solutions are desirable. Linetsky (1999) stated on page 79: "Effective numerical

[8]The forward shooting grid methodology is characterised by the augmentation of an auxiliary state vector at each grid node on a lattice tree that simulates the discrete underlying asset price process.

schemes need to be developed to price discrete occupation time derivatives with time-dependent interest rates, discrete dividends, and time- and state-dependent volatility." Amongst all the numerical methods, the Monte Carlo method can be a very productive tool for this class of options, allowing modifications of standard Parisian and ParAsian options to accommodate more exotic features. This will be undertaken in the following chapters of this thesis.

Chapter 4

Advanced Monte Carlo Methods

Anyone who considers arithmetical methods of producing random digits is, of course, in a state of sin.

—— John Von Neumann (1903–1957)

Various Techniques Used in Connection with Random Digits

A comprehensive introduction to the Monte Carlo method will be given in this chapter. And a more detailed justification for choosing Monte Carlo methods as the numerical implementation will be given. Later in this chapter, the focus will move on to the least-squares Monte Carlo method, which is one of the central techniques employed in this thesis, utilised to treat options allowing early exercise.

4.1 Introduction

The Monte Carlo method is a statistical simulation method, which is defined in quite general terms, using random numbers to perform simulation calculations. In finance, very often the basic problem is to calculate an expectation of a function given a distribution density, which can be regarded as the probability weighted average. In particular, the Monte Carlo method is commonly used in estimating multi-dimensional integrations because of its advantage when dealing with high-dimensional problems, including options on multiple assets, asset processes with jumps, stochastic interest rates or stochastic volatilities. In many applications of the method, simulations are straightforward. However the desired result is taken as an average over a large number of observations. This highlights a weakness of the Monte Carlo method, namely the low convergence rate, though different variance reduction techniques can help mitigate the problem, although considerably increases computational cost.

This chapter is organised as follows: in Section 4.2, the basic Monte Carlo integration method is introduced. Section 4.3 describes basic Monte Carlo simulation for simple option pricing. In Section 4.4, several variance reduction techniques are discussed, whilst Section 4.5 introduces the high-dimensional problems for numerical calculus, and finally Section 4.6 is devoted to the least-squares Monte Carlo method, which is a tool used for pricing options with early-exercise features.

4.2 Monte Carlo Integration

For integrations which cannot be performed analytically, approximations take on great importance. In chapter 2, the probability theory of stochastic integration was briefly introduced; now the approximation of integration using the Monte Carlo method is developed.

Let I denote an integral of a function $f(X)$ over a domain Ω,

$$I = \int_{\Omega} f(X) DX, \tag{4.1}$$

where $f(X)$ is assumed square integrable[1].

The Monte Carlo estimate for the integral is given as:

$$I_N = \frac{1}{N} \sum_{i=1}^{N} f(X_i), \tag{4.2}$$

[1] A function $f(x)$ is said to be square integrable if

$$\int |f(x)|^2 dx$$

is finite.

where X_i are independent samples distributed in the domain Ω. Here, it is useful to point out that the domain Ω may be multi-dimensional without affecting the basic procedure. The expectation of I_N is therefore

$$\mathbb{E}[I_N] = \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N} f(X_i)\right]. \tag{4.3}$$

The law of large numbers (see Section 2.4.2) ensures that the Monte Carlo estimate converges to the true value of the integral:

$$\lim_{N\to\infty} I_N = I. \tag{4.4}$$

For finite N, the estimate error can be expressed as the variance of the estimator I_N, that is

$$\begin{aligned} \mathbb{V}\text{ar}[I_N] &= \mathbb{E}[(I_N - \mathbb{E}[I_N])^2] \\ &= \mathbb{E}\left[\left(\frac{1}{N}\sum_{i=1}^{N} f(X_i)\right) - I^2\right] \\ &= \frac{\sigma^2(I)}{N}, \end{aligned} \tag{4.5}$$

where $\sigma(I)$ is the standard deviation of I, and N is the number of the samples. Equation (4.5) implies that the standard error of I_N is $\sigma(I)/\sqrt{N}$.

By the central limit theorem introduced in Section 2.4.3, the set of all possible sums over different $\{X_i, i = 1, 2, \ldots\}$ has a normal distribution. The standard deviation $\sigma(I)$ of the different values of I is therefore a measure of the uncertainty in the value of the integral.

Monte Carlo integration offers a tool for numerical evaluation of integrals including those involving high dimensions, since the integration error scales as $1/\sqrt{N}$, independent of the number of dimensions. This implies that Monte Carlo methods provide the opportunity to price financial instruments with sophisticated process dynamics, as well as complex payoff functions. Furthermore, Monte Carlo integration is applicable to both smooth integrands and integrands with discontinuities, thus allowing an easy application to problems with complex integration boundaries (see Higham, 2004).

4.3 Basic Monte Carlo Simulation

As a first illustration of a Monte Carlo method, the computation for the price of a European vanilla option is demonstrated. To clarify the algorithm, the notation has to be modified slightly from that previously introduced (notably in chapter 2); thus $S(t) = S_t(\omega)$, where ω will be interpreted as a large number of sample observations.

Let $S(t)$ denote the price of the underlying asset at time t, whose process under risk-neutral probability measure is a generalised Brownian motion:

$$dS(t) = r(S(t), t)dt + \sigma(S(t), t)dW_t, \tag{4.6}$$

where $r(S(t), t)$ is the risk-free interest rate and $\sigma(S(t), t)$ is the volatility, without lose of generality, both of these quantities may be dependent on both S and t, and dW_t denotes the increments of a standard Brownian motion.

Consider a call option with the strike price K at expiry time T in the future; the current time is $t = 0$. The payoff of a call option at time T is thus[2]

$$V(S(T)) = \max\{S(T) - K, 0\}. \tag{4.7}$$

The discounted payoff (i.e. the option value) $V(S(0))$ is $V(S(T))$ multiplied by a discount factor, namely

$$e^{\int_0^T r(S(u),u)du}$$

with $r(S(t), t)$ the dynamics of the interest rate.

To obtain $S(t)$ from current time 0 up to expiry date T, an Euler approximation to the SDE (4.6) is applied. A sample path can be found by generating a sequence $dW_1, dW_2, \ldots$ of independent normal random variables distributed with mean 0 and variance 1. The simulation must be repeated a large number of times to reflect accurately the distribution of the payoff $V(S(T))$.

This (general) algorithm is presented with the following stages:

[2]For a put option, the payoff function is expressed as

$$V(S(T)) = \max\{K - S(T), 0\}.$$

1. Divide the time period $[0, T]$ into M steps. Set $\Delta t = T/M$, and therefore $t_i = i\Delta t$, for $i = 0, 1, 2, \ldots, M$. This is called discretisation of time. This discretisation is the basis of many numerical procedures, as explained in Section 3.4.2. Note that a divided time step is not necessary for European options with constant interest rates, the underlying asset processes can be simulated directly, since there is no need to consider the intermediate states by the definition of European option prices. Furthermore, variable time steps Δt can be used if more appropriate. This is another advantage of Monte Carlo methods.

2. Sample N independent paths of underlying asset $S_k(t_i)$ for $k = 1, 2, \ldots, N$. Set the current value $S_k(t_0) = S_0$. At each time step $S_k(t_{i+1})$ is determined from:

$$S_k(t_{i+1}) = S_k(t_i) + r(S_k(t_i), t_i)\Delta t + \sigma(S_k(t_i), t_i)\epsilon_i\sqrt{\Delta t}, \tag{4.8}$$

   where $\epsilon_i \sim N(0, 1)$, is a sequence of independent standard normal variables. Note that the increment of a standard Brownian motion $dW_{t_i} = \epsilon_i\sqrt{\Delta t}$.

3. Obtain the value of payoff $V(S_k(T))$ at expiry date $T = t_M$, for $k = 1, 2, \ldots, N$.

4. Discount $V(S_k(T))$, $k = 1, 2 \ldots, N$ back to time $t = 0$ with a discount factor, that is

$$V(S_k(0)) = e^{\int_0^T r(S_k(u),u)du} V(S_k(T)), \quad k = 1, 2 \ldots, N. \tag{4.9}$$

5. Compute the average result of $V(S(0))$,

$$\bar{V} = \frac{1}{N}\sum_{k=1}^{N} V(S_k(0)). \tag{4.10}$$

   Here, $\bar{V}$ is consistent due to the law of large numbers (see Section 2.4.2)

$$\bar{V} \to V(S(0)), \tag{4.11}$$

   $\bar{V}$ is unbiased as

$$E[\bar{V}] = V(S(0)). \tag{4.12}$$

   Therefore, we may say $\bar{V}$ is a good estimator.

Note that if the processes of both r and σ are stochastic, the value for them can be obtained by invoking steps 2 for the simulation. More stochastic factors can be added into the model, such as jumps. The concept of Scholes factorisation will be introduced when the stochastic factors are correlated (see Section 4.5). Therefore, high-dimensional models can be easily dealt with (see James and Webber, 2004).

4.4 Variance Reduction Techniques

Variance reduction techniques can be very helpful to improve the efficiency for the Monte Carlo computations (see Hammerless and Handsome, 1964). Clearly simulations can be as accurate as required by increasing the number of samples, however more samples require more computation time. As mentioned in Section 4.2, the error in the estimator is proportional to $1/\sqrt{N}$, implying that it is computational expensive to improve the efficiency of the estimator simply by increasing the number of samples. An alternative approach to improve efficiency is to use variance reduction techniques, including classical variance reduction techniques and several combinations of methods.

There are four classical variance reduction techniques which are widely used in Monte Carlo applications, as described by Hammerless and Handsome (1964): control variates, antithetic variates, importance sampling, and stratified sampling. Also Bramley, Fox and Sciage (1987) and Law and Keaton (1991) give a detailed introduction on variance reduction techniques. These techniques can be effective in financial applications and are described briefly below.

4.4.1 Control Variates

The control variates technique is based on the idea of using a correlated random variable whose expectation is known to minimise the variance. The paper of Lavenders and Welch (1981) provided a complete and rigorous exposition of control variates.

For instance, we wish to calculate the expectation of X, i.e. $\mathbb{E}[X]$. There is a correlated random variable Y with a known expectation $\mathbb{E}[Y]$. Then the new random variable $Z = X + \beta(\mathbb{E}[Y] - Y)$ satisfies

$$\begin{aligned} \mathbb{E}[Z] &= \mathbb{E}[X + \beta(\mathbb{E}[Y] - Y)] \\ &= \mathbb{E}[X], \end{aligned} \tag{4.13}$$

$$\begin{aligned} \mathbb{V}\text{ar}[Z] &= \mathbb{V}\text{ar}[X + \beta(\mathbb{E}[Y] - Y)] \\ &= \mathbb{V}\text{ar}[X] - 2\beta\mathbb{C}\text{ov}[X, Y] + \beta^2\mathbb{V}\text{ar}[Y], \end{aligned} \tag{4.14}$$

where Y is called control variate, and β is a scale to adjust the variance.

Consider the optimal case when

$$\hat{\beta} = \frac{\mathbb{C}\text{ov}[X, Y]}{\mathbb{V}\text{ar}[Y]}, \tag{4.15}$$

it can be shown that (see Glasserman, 2003)

$$\mathbb{V}\text{ar}[Z] < \mathbb{V}\text{ar}[X] \iff 0 \leq |\beta| \leq |\hat{\beta}|. \tag{4.16}$$

Note that for the financial applications, the control variate Y may not be a true financial instrument; however, to increase the efficiency substantially, the control variate Y must be a function of the same underlying process with a known expectation and be highly correlated with the instrument that is to be evaluated.

4.4.2 Antithetic Variates

The intuition of antithetic variates is rather simple. As the estimator works better when the simulated variables are distributed as closely as possible to the true distribution, then mirroring the samples will give a better spread in sample space, and most importantly, antithetic variates guarantee the simulated variables symmetrically distributed about their means. A simple example is given as an illustration.

Assume the expectation of X, i.e. $\mathbb{E}[X]$ is unknown. Unlike the control variate method, we seek another estimator Y with the same expectation as X, but with a negative correlation with X. It is easy to see that the new random variable

$Z = \frac{1}{2}(X + Y)$ has

$$\begin{aligned} \mathbb{E}[Z] &= \mathbb{E}[\tfrac{1}{2}(X+Y)] \\ &= \mathbb{E}[X], \end{aligned} \tag{4.17}$$

$$\begin{aligned} \mathbb{V}\mathrm{ar}[Z] &= \mathbb{V}\mathrm{ar}[\tfrac{1}{2}(X+Y)] \\ &< \tfrac{1}{2}\mathbb{V}\mathrm{ar}[X] \quad \text{if} \quad \mathbb{C}\mathrm{ov}[X,Y] < 0. \end{aligned} \tag{4.18}$$

This idea of antithetic variates was first presented by Hammerless and Morton (1956), and is straightforward to implement into a Monte Carlo algorithm.

Some results are shown in Table 4.1. The values are the errors of a European put option comparing with the results given by the Black-Scholes (1973) formula (in Section 1.4) with parameters $S_0 = 36$, $K = 40$, $r = 0.06$, $\sigma = 0.20$ and $T = 1$. Table 4.1 shows the efficiency of antithetic variates. Generally, antithetic variates

Path	Basic MC	Antithetic MC
10,000	-0.0354	0.0057
100,000	-0.0032	-0.0013
1,000,000	-0.0025	-0.0007

Table 4.1: Comparison of Basic Monte Carlo method and Antithetic Monte Carlo method.

improve the estimate, however increasing the number of sample paths does not always improve the efficiency significantly compared with the basic Monte Carlo method. It is very clear shown that when the sample space is large enough, the errors of both Monte Carlo methods become reasonably small, and the antithetic technique loses its “shine”.

4.4.3 Importance Sampling

The concept of importance sampling is to reduce variance by changing the probability measure, focusing on the distribution of the samples in the regions that are numerically most significant. Importance sampling works particularly well in estimating probabilities of rare events, for instance deep out-of-the-money or deep-in-the-money options (see Glynn and Iglehart, 1989).

Suppose the random variable X has the probability density function $p(x)$, then the integral (4.1) can be written as

$$I = \int_{\Omega} f(x)p(x)dx = \int_{\Omega} \frac{f(y)p(y)}{g(y)}dy, \quad \text{where} \quad X = \int_0^x g(y)dy. \tag{4.19}$$

Here $p(x)/g(x)$ can be viewed as a new Monte Carlo estimator, written as Z. By restricting g to be positive such that

$$X(\Omega) = \int_{\Omega} g(y)dy, \tag{4.20}$$

it is clear that

$$\mathbb{E}[f(Z)] = \mathbb{E}[f(X)], \tag{4.21}$$

$$\mathbb{V}\text{ar}[f(Z)] - \mathbb{V}\text{ar}[f(X)] = \int_{\Omega} f^2(y)(1 - Z)dy. \tag{4.22}$$

It is clear that $\mathbb{V}\text{ar}[Z]$ can be small if Z is as close to one as possible. It is clear that a choice of $g(x)$ that follows most closely the shape of $p(x)$ is a good importance sampling function. However, it should be pointed out that while $g(x)$ might be approximately the same shape as $p(x)$, serious difficulties arise if $g(x)$ decreases much faster than $p(x)$ in the tails in distribution (see Anderson, 1999). Note that the $g(x)$ is called the score function in Monte Carlo methods, the likelihood ratio in statistics, and the Radon-Nikodým derivative in financial mathematics.

4.4.4 Stratified Sampling

In stratified sampling, the sampling domain is subdivided into smaller areas so that the estimate can be carried out with smaller domains, then spread out in sample space to yield a better approximation.

The concept of stratified sampling is similar to adapted lattice methods. The sampling can be more focused in certain sub-domains which are highly variant. However, it is rather computationally expensive, especially in the case of high-dimensional integrations, since partitioning each coordinate into N strata produces

N^d strata for a d–dimensional integrations. For more details on this technique, see Glasserman (2003).

4.5 The Multi-dimensional Simulation

Contingent claims on multiple state variables are common in most financial institutions as well as academia, for instance, options with stochastic interest rates or stochastic volatilities, or options on multi-assets (i.e. basket options). Multi-dimensional models are commonly used by both practitioners and academics alike. Analytic solutions for such problems are available only in a few special cases, therefore numerical methods are of great advantage (see Jäckel, 2002), especially when the interdependence between the various factors (or underlying assets) is taken into account. The problem of how to specify a correlation matrix occurs in several important areas of finance.

Usually, numerical techniques in finance suffer from the "curse of dimensionality"[3]. The classical integration rules are considered as an iteration of one-dimensional integrals, so that there is a dependence on the dimension. The error bound is established as $O(N^{-1/d})$. This means that increasing the dimension d, the required computational effort increases exponentially. Therefore, in the numerical techniques described in Section 1.5.2, the inefficiency of multi-dimensional integrals has always been a disadvantage. However Monte Carlo integration has an error scaling as $1/\sqrt{N}$ (as explained in Section 4.2), independent of the number of dimensions, which means it does not suffer from the "curse of dimensionality". This has made Monte Carlo integration the preferred method for integrals in high dimensions. In finance, the technique was first employed by Boyle (1977).

Evans and Swartz (2000) argue that multiple quadrature methods cannot replace the need for Monte Carlo methods, but a pure Monte Carlo method that fails to recognise and take advantage of the efficiency improvements available with multiple

[3]It is the minimal cost of computing an approximation using deterministic algorithms depends exponentially on the dimension. See Traub and Werschulz (1998).

quadrature is not appropriate either. For low-dimensional problems (fewer than four dimensions) well-known classical discretisation techniques can be an obvious choice for solving the partial differential equations with methods from numerical mathematics, methods which are relatively fast and accurate. For higher dimensions, Monte Carlo simulations are in principle adequate, although relatively slow and may be very inefficient. There is currently no numerical method that copes well with such a problem. Notice that, without advanced numerical techniques, an option on five state variables, for example, with 32 points in each dimension may give rise to 33 million computational points at each time step (see Oosterlee, 2003). The computational work is therefore extremely large for higher-dimensional problems.

The Monte Carlo method is relatively straightforward for high-dimensional models with correlation. We will demonstrate the case of generating correlated stochastic processes, in terms of the standard Brownian motions, for models which require more than one stochastic factor.

As mentioned in Section 4.3, when we sample the independent paths of underlying asset, $\epsilon_i \sim N(0,1)$, is a sequence of independent standard normal variables. The problem of generating correlated stochastic processes can therefore be simplified to generate correlated random variables. Suppose we wish to generate random variables $\{Z_i, i = 1, \ldots, n\}$ with a correlation matrix C, given correlation coefficients

$$c_{ij} = c_{ji} \quad \text{and} \quad c_{ii} = 1.$$

As C is a positive symmetric matrix, there always exists a lower triangular matrix A with $AA^{\mathsf{T}} = C$, where A^{T} is the transpose of A, and choose independent random variables $\{\epsilon_i, i = 1, \ldots, n\}$ take $A\epsilon$. It is easy to show that $Z = A\epsilon$. The procedure used to obtain the A is called Cholesky factorisation (see Van Loan, 2000) .

Cholesky factorisation basically decomposes a symmetric and positive definite matrix into a lower and an upper triangular matrix i.e. $C = AA^{\mathsf{T}}$, A is a lower triangular matrix with positive diagonal elements. A is also called the Cholesky triangle.

To derive $C = AA^{\mathsf{T}}$, we simply equate coefficients on both sides of the equation:

$$\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{12} & \cdots & c_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{12} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ 0 & a_{12} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \tag{4.23}$$

Solving for the unknowns (the nonzero a_{ij}'s), for $j = 1, \ldots, n$ and $i = j+1, \ldots, n$, we obtain:

$$a_{jj} = \sqrt{\left(c_{jj} - \sum_{k=1}^{j-1} a_{jk}^2\right)}, \tag{4.24}$$

$$a_{ij} = \left(c_{ij} - \sum_{k=1}^{j-1} a_{ik}a_{jk}\right) / a_{jj}. \tag{4.25}$$

For example, the interest rate can be stochastic as well as the underlying asset, and its process is obtained by invoking steps 2 in Section 4.3 for the simulation of an interest rate process. Assume interest rate dynamics

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dZ_t. \tag{4.26}$$

with dZ_t the increments of a standard Brownian Motion. $\mu(t, r(t))$ and $\sigma(t, r(t))$ are functions of $r(t)$. Whereas the underlying asset SDE,

$$dS(t) = a(S(t), t)dt + \sigma(S(t), t)AdW_t, \tag{4.27}$$

with the dW_t being the increments standard Brownian motions and W_t and Z_t are correlated with $E[dW_t dZ_t] = \rho dt$. Using Cholesky factorisation, Equation (4.26) can be written as

$$r_k(t_{i+1}) = r_k(t_i) + \mu(r_k(t_i), t)\Delta t + \sigma(r_k(t_i), t)\sqrt{\Delta t}(\rho\epsilon_i + \sqrt{1-\rho^2}\epsilon_i'), \tag{4.28}$$

where ϵ_i and ϵ_i' are independent standard normal variables. More stochastic factors can be added into the model, such as stochastic volatilities. The concept of Cholesky factorisation will be extensively used when multiple stochastic factors are correlated, and more sophisticated models can be quite easily dealt with. It is the flexibility that gives the Monte Carlo technique its appeal.

For more detailed introductions on the pricing of multi-dimensional option models, see Stulz (1982) , Johnson (1987), Boyle, Evnine and Gibbs (1989), Boyle and Tse (1990) and Wilmott (2000a).

4.6 Least-squares Monte Carlo Method

The difficulty in option pricing involving early exercise used to be one of the drawbacks of Monte Carlo methods, which has been addressed in Section 3.4.2. The procedure of necessity runs simulations forwards in time, whilst, for an American option, valuation includes some pattern of early-exercise to predict when it is optimal for the option holder to exercise the option, which is typically performed backwards. The history of using Monte Carlo methods to solve American options has been quite short. It was a common belief that Monte Carlo methods could not be applied to American-style options, until Tilley (1993) tackled this problem. Since then, Monte Carlo methods for early exercise feature have been developed from several different perspectives. A more detailed literature review on those approaches to American-style option pricing has been introduced in Section 3.4.2. Here, we only focus on one of the most commonly used methods: least-squares Monte Carlo method.

4.6.1 Least-squares Fitting

The exercise boundary in the case of American-style option pricing is not fixed, which means an American option has the value function V_A at current time $t = 0$, that satisfies the equation under the risk-neutral measure $\mathbb{Q}$,

$$V_A = \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{Q}} \left[e^{-r\tau} V(S(\tau), \tau) \right], \tag{4.29}$$

where $V(\cdot)$ is the payoff function, r is the risk-free interest rate, T is the expiry date, and $\mathcal{T}$ is the set of all possible stopping times with respect to the underlying asset S.

Since any stopping time can be expressed as a set of discrete stopping times, the optimal stopping time is therefore the object to achieve the supremum in Equation (4.29), denoted as τ^*, satisfying

$$\tau^* = \inf\{t \geq 0 \,:\, S(t) = S^*\}, \tag{4.30}$$

where S^* is the optimal exercise boundary. The option price can therefore be evaluated numerically as the maximum of the immediate payoff if the option is exercised, which implies that to compare the intrinsic value

$$V_A(S(t_{i-1}), t_{i-1})$$

and continuation value

$$\mathbb{E}^{\mathbb{Q}}\left[V_A(S(t_i), t_i) | S(t_{i-1}) = S^*\right]$$

at every exercisable time.

Therefore, (4.29) can be rewritten as:

$$V_A(S(T), T) = V(S(T), T), \tag{4.31}$$

$$V_A(S(t_{i-1}), t_{i-1}) = \max\{V(S(t_{i-1}), t_{i-1}), \mathbb{E}^{\mathbb{Q}}\left[V(S(t_i), t_i) | S(t_{i-1}) = S^*\right]\}, \tag{4.32}$$

where $0 \leq \ldots \leq t_{i-1} < t_i \leq \ldots \leq T$. Here, Equations (4.31) and (4.32) represent the essence of the dynamic programming recursion. It is also called Bellman equation.

In Monte Carlo methods, the continuation value is not tractable. Thus, regression-based models have been developed to estimate continuation values from simulated paths and to price the option values. Least-squares fitting provides a simple yet accurate approximation to the conditional expectation of continuation value.

The term "least-squares" comes from the idea of squared deviation. Given a set of data, the aim is to find numerical values for the parameters that minimise the sum of the squared deviations between the data and the functional portion of the model (see Daniel and Woods, 1980). It can be shown that the estimates based on least-squares are the maximum likelihood estimates, and they are also the minimum-variance unbiased estimates (see Draper and Smith, 1998).

Suppose N sample data $\{(x_k, y_k),\ k = 1, 2, \ldots, N\}$ need to be fitted to the linear (or nonlinear) function of exercise boundary. Choose J linear independent basis functions $L_i(X),\ i = 1, 2, \ldots, J$. Then a linear combination of $L(X)$ is defined as

$$f(X) = \sum_{i=1}^{J} a_i L_i(X), \tag{4.33}$$

where $X = (X_1, X_2, \ldots, X_N)a_i$ are adjustable coefficients, which are not yet known. The least-squares fitting is to evaluate the function $f(X)$ at each of the N sample data and to minimise the error

$$\sum_{k=1}^{N} (y_k - f(x_k))^2 .$$

Note that this minimisation treats all the x_k equally, and that it penalises large deviations dramatically.

With least-squares fitting, the continuation value with respect to the given information at time t can be well approximated. This is the key to Monte Carlo method overcoming the early exercise problem. An illustration of implementation is given next.

4.6.2 LSM Approach for American/Bermudan Options

The least-squares Monte Carlo (LSM) approach was proposed by Longstaff and Schwartz (2001). The core to this approach is to use least-squares fitting to estimate the conditional expected payoff to the option holder from continuation. In this section, a brief illustration of the LSM approach is presented.

The algorithm starts with an American put option on an underlying asset, $S(t)$, which expires at time T, and which the option holder can exercise at any time up to T. Numerically, it can be implemented by choosing an M so that the time interval $[0, T]$ is divided into M sub-intervals whose length is $\Delta t = T/M$. As mentioned before, this is actually the same as a Bermudan option, which approaches the American value in the limit of an infinite number of exercise times. An approximate value

$V_k(S(0), 0)$ of the kth sample path is performed by rolling-back on the underlying asset paths.

The objective of the LSM algorithm is to find the optimal exercise time with respect to the underlying asset S. Under the risk-neutral probability measure, recalling the American put option pricing problem is to find

$$V_A = \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{Q}} \left[e^{-r\tau} V(S(\tau), \tau) \right], \tag{4.34}$$

over all stopping times $\mathcal{T}$. Here $S(\tau)$ is the underlying asset price at time τ, V is the payoff function, and r is the risk-free interest rate. Note that, theoretically a European call option always has the same value as an American call if no dividends are paid.

Given the valuation problem in the previous section, for an American put on $S(t)$ expiring at T, an approximation of the value is obtained by generating N sample paths of the stochastic process $S(t)$. To avoid confusion, we re-denote $S_k(t)$ as the value of the process at time t along the kth path and τ_k the stopping time with respect to the information generated by $S_k(t)$ in the discrete set of dates where the state variables dynamics are generated.

The algorithm is to find the optimal exercise time restricted to the set of dates $t_0 = 0, t_1 = \Delta t, t_2 = 2\Delta t, \ldots, t_M = M\Delta t = T$. The determination of continuation value works backwards, and so if at time t_i, along the kth path, the option has not been exercised (i.e. the stopping time along the kth path, as determined in previous time steps of the algorithm, is greater than t_i), the optimal decision is made by comparing the payoff $V_k(t_i)$ with $F_k(t_i)$, where $F_k(t_i)$ is the conditional expected value with respect to the time t_i. If $F_k(t_i) \leq V_k(t_i)$, then $\tau_k = t_i$, for the kth path. The intuition behind this procedure is that the stopping time satisfies the following condition:

$$\tau = \inf\{t \geq 0 \,:\, F_k(t) = V_k(t)\}, \tag{4.35}$$

which is the first time the value of the option is equal to the payoff from exercise. Unfortunately, $F_k(t_i)$ is not available at this step of the procedure. A resolution of this is offered by the Bellman equation (see Equation (4.31) and (4.32)) of the optimal stopping problem in discrete time:

$$F_k(t_i) = \max\{V_k(t_i), e^{-r(t_i)\Delta t}\mathbb{E}_{t_i}[F_k(t_{i+1})]\}. \tag{4.36}$$

Using this equation, the optimal policy can be determined, restricted to the given dates, by comparing the continuation values,

$$\Pi_k(t_i) = e^{-r(t_i)\Delta t}\mathbb{E}[F_k(t_{i+1})|\mathcal{F}_{t_i}], \tag{4.37}$$

with the payoff $V_k(t_i)$. So the decision rule at time step t_i along the kth path is:

$$\text{if} \quad \Pi_k(t_i) \leq V_k(t_i) \quad \text{then} \quad \tau_k = t_i. \tag{4.38}$$

At $t_i = T$, since the option expires, $\Pi_k(t_i) = 0$, and the rule is to exercise the option if the payoff is positive. At any t_i the optimal stopping time is found by applying the decision rule in (4.38), from $t_i = T$ back to t_i. If one of the optimal stopping times has been determined, at some previous step of this procedure, $\tau_k > t_i$ for the kth path, and condition (4.38) holds at the current step, then the stopping time along the kth path is updated to $\tau_k = t_i$. The optimal stopping times along all paths are determined at $t_i = 0$. Consequently the value of the American/Bermudan put is estimated by averaging all the sample path values.

The key problem is to find the continuation value at t_i, in order to apply the decision rule. The intuition behind LSM is that if at t_i the option is still available, the continuation value is the expectation conditional on the information available at that date, of future optimal payoffs from the option. Denote $V'_k(t)$ as the cashflow from the option optimally exercised at time τ^* with respect to the stopping time τ_k, conditional on not being exercised at $t < s$, along the kth path. Therefore,

$$V'_k(t) = \begin{cases} V_k(t) & \text{if } \tau^* = \tau_k, \\ 0 & \text{if } \tau^* \neq \tau_k. \end{cases} \tag{4.39}$$

The dependence of this cashflow on t_i is due to the fact that when the decision rule is applied in Equation (4.38), the stopping time along the kth path can change step by step. The continuation value at t_i is the present value with respect to the risk neutral probability of all future expected cashflows from the option $\Pi_k(t_i)$. Then the continuation value can be interpreted as

$$\Pi_k = \sum_{j=1}^{\infty} a_j(t) L_j, \tag{4.40}$$

where L_j is the jth element in the basis function. Following Longstaff and Schwartz (2001), the estimated continuation value $\pi_j(t_i)$ can be determined by applying

$$\Pi_k^J(t) = \sum_{j=1}^{J} a_j(t) L_j(t). \tag{4.41}$$

Now, $\pi_j(t)$ can be estimated by a linear least squares regression of $\Pi_k^J(t)$ onto the basis L_{k_j}. The estimated continuation value is then used to apply recursively the decision rule in (4.38).

The accuracy of the estimates of the value of the American option can be improved by increasing the number of time steps M, the number of simulated paths N, and (up to a point) the degree of basis function J. Since the regression will become a simulation of the data curve if J is increased infinitely. For finite N, an optimal J exists (see Glasserman, 2003).

The building blocks of the least-squares Monte Carlo method have been introduced. In the following chapter, an American-style option pricing model is developed, using an enhanced least-squares Monte Carlo method (i.e. Duck et al., 2005, which will be formally introduced in Chapter 5), and also incorporating Cholesky factorisation for the co-movement between the stochastic factors in the model.

Chapter 5

American Currency-Options

In mathematics you don't understand things. You just get used to them.

—— *John Von Neumann (1903–1957)*

5.1 Introduction

A currency option can be viewed as an option to exchange a domestic bond with a foreign bond. Several variables may be included in a currency-option pricing model (as mentioned in Section 3.3): the exchange rate, two interest rates (domestic and foreign) and the volatilities of these quantities, all of which are open to modelling as stochastic processes. It is quite straightforward to implement a European-style option, where closed-form solution may be available, whereas for an American-style option which accommodates early exercise features, numerical procedures are necessary. However, American options with more than three stochastic factors are challenging for numerical methods, since most suffer "the curse of dimensionality", mentioned in Section 4.5. The Monte Carlo method is computationally advantageous since it can be implemented easily for dimensions as high as ten or even more. However, the early exercise feature does significantly complicate matters. Longstaff and Schwartz's (2001) least-squares technique allows Monte Carlo to be used in such cases.

American options can be priced numerically using lattice and grid methods (binomial and trinomial trees, finite-difference techniques), which work backwards in time and allow naturally for early exercise. The convergence/accuracy of these has been improved by various modifications of technique, though ultimately they suffer from the curse of dimensionality (addressed in Section 4.5). Of course, as available computing power has increased the practical cut-off point for number of factors has risen and it is dangerous to take textual quotes from older (even recent) literature concerning the difficulties of computation (also mentioned in Section 1.5.2). Monte Carlo simulation has obvious appeal, being intuitive, simple to implement and, though initially computationally intensive, possessing the feature that computational effort increases only linearly with the number of stochastic factors.

It was formerly the case that the Monte Carlo method could not readily handle early exercise, but this difficulty has been overcome in several alternative ways, as mentioned in Section 3.4.2, the one with the greatest impact in the literature being that of Longstaff and Schwartz (2001), which itself has been the subject of several enhancements, including that of Duck et al. (2005), which is adopted in this thesis, giving speed improvements, in general, of around twenty times the basic Longstaff and Schwartz original.

Given the clear importance of volatility (stated in Section 3.3.2), it will ultimately be considered in this model, but first a perfect market is constructed, having no transaction costs, no differential taxes, no long or short restrictions, and trading is continuous.

The remainder of this chapter is organised as follows. Section 5.2 considers the basic Amin and Bodurtha (1995) currency-option pricing models (based on Ho and Lee, 1986), which are treated using a Monte Carlo method, based on the enhanced Longstaff and Schwartz (2001) method as proposed by Duck et al. (2005). Section 5.3 extends the forward-rate model of Section 5.2 to a short-rate model (a mean-reverting diffusion process introduced by Cox, Ingersoll and Ross, 1985). Section 5.4 further refines the work, when stochastic mean-reverting volatilities are taken into account, assuming that the exchange rate, domestic interest rate and foreign interest rate all have stochastic volatilities. To aid sensible analysis of the numerical results, the chapter employs Treepongkaruna and Gray's (2003) empirical parameters for both interest rates, and Dupoyet's (2006) empirical parameters for stochastic exchange rate and the corresponding volatility. Section 5.5 presents some concluding remarks.

5.2 The Amin and Bodurtha Model

5.2.1 Assumptions

In order to develop a new model step by step and for later comparison, we begin with the basic Amin and Bodurtha (1995) three-factor model. To this we will apply Monte Carlo methods in place of the (limited 12-time step) multinomial tree. This will then form the basis for treating the enhanced models.

The exchange rates are assumed to follow a geometric Brownian motion process which is consistent with a bond price process, in line with the Amin and Bodurtha (1995) model. The HJM framework is adopted for both domestic and foreign interest rates. To be consistent with the Amin and Bodurtha model, the assumptions are based on real world data (without changing measure), the volatilities of interest rates are kept constant, and the diffusion is one dimensional, i.e. the interest-rate model is described by the Ho and Lee (1986) model. The volatility of the exchange rate is kept constant. For a practical investigation of foreign exchange rate volatility, see Chowdhury and Sarno (2004).

Consider now the assumptions in the model: the stochastic processes take the form

$$\frac{dx_t}{x_t} = (r_d - r_f)dt + \sigma_x dW_x, \tag{5.1}$$

$$df_d(t,T) = \alpha_d(t,T)dt + \sigma_d(t,T)dW_d, \tag{5.2}$$

$$df_f(t,T) = \alpha_f(t,T)dt + \sigma_f(t,T)dW_f, \tag{5.3}$$

where

$$x_t \equiv \text{the exchange rate,}$$

$$f_d \equiv \text{the domestic forward interest rate,}$$

$$f_f \equiv \text{the foreign forward interest rate,}$$

$$r_d \equiv \text{the domestic short rate,}$$

$$r_f \equiv \text{the foreign short rate.}$$

In the above, the exchange-rate process has a drift with a component of $r_d - r_f$ which is justified in Appendix B. α_d and α_f are the drift of f_d and f_f respectively, the σ_x, σ_d, and σ_f are the volatilities of x_t, f_d and f_f respectively, dW_x, dW_d and dW_f are the increments of one-dimensional standard Brownian motions; these three random processes are correlated as

$$\mathbb{E}[dW_i dW_j] = \rho_{ij}dt, \quad \text{where} \quad i,j = d,f,x; \quad \rho_{ij} = \rho_{ji},\ \rho_{ii} = 1.$$

The parameters α_d, α_f, σ_x, σ_r, σ_f and ρ_{ij} are all assumed constant in the first instance.

Since the general HJM model is non-Markovian, the SDEs describe only instantaneous forward rates, which are not appropriate for the exchange-rate process. Implementation is therefore not so straightforward as simply using Euler discretisation to simulate the instantaneous short rate of interest. Therefore, it is necessary to transform the forward rate processes into short rate processes by following Duffie (1996), to obtain short-rate values at each time step using the simulated forward rates as follows:

$$r_d(t) = f_d(0,t) + \int_0^t \sigma_d(v,t) \int_v^t \sigma_d(v,u)' du dv + \int_0^t \sigma_d(v,t) dW_d, \tag{5.4}$$

$$r_f(t) = f_f(0,t) + \int_0^t \sigma_f(v,t) \int_v^t \sigma_f(v,u)' du dv + \int_0^t \sigma_f(v,t) dW_f. \tag{5.5}$$

The equations above are general form of the conversion. Consequently Equations (5.2) and (5.3) can be transformed into the Ho and Lee (1986) model. Namely,

$$r_d(t) = f_d(0,t) + \frac{1}{2}\sigma_d t^2 + \sigma_d W_d, \quad (5.6)$$

$$r_f(t) = f_f(0,t) + \frac{1}{2}\sigma_f t^2 + \sigma_f W_f, \quad (5.7)$$

where $f_d(0,t)$ and $f_f(0,t)$ are the instantaneous forward rates of domestic and foreign interest respectively, at $t = 0$ for time horizon $[0,t]$. Note that $f_d(0,t)$ and $f_f(0,t)$ are required in order to obtain $r_d(t)$ and $r_f(t)$. Referring to Wilmott (2001),

$$f_d(0,T)T = f_d(0,t)t + f_d(t,T)(T-t), \quad (5.8)$$

$$f_f(0,T)T = f_f(0,t)t + f_f(t,T)(T-t). \quad (5.9)$$

$f_d(0,t)$ and $f_f(0,t)$ can be easily obtained.

5.2.2 Numerical Scheme

The least-squares Monte Carlo approach of Longstaff and Schwartz (2001) (as modified by Duck et al., 2005) for the evaluation of American options is applied. The Longstaff and Schwartz approach appealed to academics and practitioners alike, since it set about solving the problem of early exercise in Monte Carlo simulations by combining financial intuition (an expected value) with a least-squares fitting technique, using the latter to estimate the conditional expected payoff to the option holder from continuation.

The algorithm adopted for American/Bermudan[1] put options is as follows (again the notations are changed, namely $x(t) = x_t(\omega)$, where ω will be interpreted as a large number of sample observations):

1. Divide the time period $[0,T]$ into M steps (i.e. M the exercise dates). Set $\Delta t = T/M$, and therefore $t_i = i\Delta t$, for $i = 0,1,2,\ldots,M$.

2. Sample N independent paths of exchange rate $x_k(t_i)$, the domestic forward rate $f_{dk}(t_i)$ and the foreign forward rate $f_{fk}(t_i)$ (for $k = 1,2,\ldots,N$) using Euler

[1] As mentioned in Section 4.3, due to the unavoidable discretisation of numerical solution, an American option can only be exercised in a discrete time, which is actually a Bermudan option.

discretisation. As mentioned before, these three processes are correlated; the correlation matrix is a 3×3 symmetric matrix

$$\Sigma = \begin{pmatrix} 1 & \rho_{xd} & \rho_{xf} \\ \rho_{dx} & 1 & \rho_{df} \\ \rho_{fx} & \rho_{fd} & 1 \end{pmatrix}.$$

It is necessary to transform the correlation matrix above into a modified form which gives the independent standard normal variables the equivalent correlation coefficients. Take $W = A\epsilon$ so that the correlated Brownian motions W can be replaced by $A\epsilon$, where A is the Cholesky factorisation of Σ (described in Section 4.5).

Set $x_k(0) = x_0$, the current value of $x(t)$. $x_k(t_{i+1})$ is determined by:

$$x_k(t_{i+1}) = x_k(t_i)\exp\left(\left[r_{dk}(t_i) - r_{fk}(t_i) - \frac{1}{2}\sigma_x^2\right]\Delta t + A_{1,1}\sigma_x\epsilon_{xi}\sqrt{\Delta t}\right). \quad (5.10)$$

The instantaneous forward interest rates are:

$$\begin{aligned} f_{dk}(t_{i+1}, T) &= f_{dk}(t_i, T) + \alpha_d\Delta t \\ &+ (A_{2,1}\epsilon_{xi} + A_{2,2}\epsilon_{ri})\sigma_d\sqrt{\Delta t}, \quad (5.11) \\ f_{fk}(t_{i+1}, T) &= f_{fk}(t_i, T) + \alpha_f\Delta t \\ &+ (A_{3,1}\epsilon_{xi} + A_{3,2}\epsilon_{ri} + A_{3,3}\epsilon_{fi})\sigma_f\sqrt{\Delta t}, \quad (5.12) \end{aligned}$$

where ϵ_{xi}, ϵ_{ri} and $\epsilon_{fi} \sim N(0,1)$ are the sequences of independent standard normal variables at the ith time step.

3. Obtain the initial forward rates up to every time step t_i using Equations (5.8) and (5.9):

$$\begin{aligned} f_d(0, t_i) &= \frac{f_d(0,T)T - f_d(t_i,T)(T - i\Delta t)}{i\Delta t}, \quad (5.13) \\ f_f(0, t_i) &= \frac{f_f(0,T)T - f_f(t_i,T)(T - i\Delta t)}{i\Delta t}. \quad (5.14) \end{aligned}$$

Consequently the following SDEs can be obtained from Equations (5.6) and (5.7):

$$\begin{aligned} r_d(t_i) &= f_d(0,t_i) + \frac{1}{2}\sigma_d(i\Delta t)^2 \\ &+ \left(A_{2,1}\sum_{n=1}^{i}\epsilon_{xn} + A_{2,2}\sum_{n=1}^{i}\epsilon_{rn}\right)\sigma_d\sqrt{\Delta t}, \end{aligned} \tag{5.15}$$

$$\begin{aligned} r_f(t_i) &= f_f(0,t_i) + \frac{1}{2}\sigma_f(i\Delta t)^2 \\ &+ \left(A_{3,1}\sum_{n=1}^{i}\epsilon_{xn} + A_{3,2}\sum_{n=1}^{i}\epsilon_{rn} + A_{3,3}\sum_{n=1}^{i}\epsilon_{fn}\right)\sigma_f\sqrt{\Delta t}. \end{aligned} \tag{5.16}$$

4. The value of the payoff function $V_k(x(T)) = \max\{K - x_k(T), 0\}$ is obtained.

5. From the expiry date T to the current time $t = 0$, at each time step t_i, the option holder optimally compares the intrinsic value with the continuation value, which can be expressed as the conditional expectation of discounted payoff. The conditional expectation function can be estimated by invoking the least-squares basis representation

$$\min = \|Y_j - \sum_{\ell=0}^{J} a_\ell L_\ell(x_j)\|, \tag{5.17}$$

where $L(\cdot)$ is a set of basis functions, J is the number of basis functions, X_j is the value of the underlying asset, Y_j is the discounted payoff

$$Y_j = \exp\left(-\int_{t_i}^{T} r_{dk}(t)dt\right) V_k(x_k(T)), \tag{5.18}$$

j is the index of in-the-money paths at time t_i, and a_ℓ are the estimated coefficients (obtained from the least-squares fit).

6. Given the a_ℓ, it is straightforward to compute the value of continuation. Denoted as Y_j', it is obtained by simply re-invoking Equation (5.17) as

$$Y_j' = \sum_{\ell=0}^{J} a_\ell L_\ell(x_j). \tag{5.19}$$

Compare the intrinsic value $K - x_j$ with Y_j'; if the intrinsic value is greater than continuation value, the option is exercised and the optimal stopping time $\tau_k = t_i$ along the kth sample path is set.

7. Repeat Step 6 to obtain the set of optimal stopping time τ for all sample paths.

8. Discount the payoff with optimal stopping time to find V_A. That is

$$V_A = \sum_{k=1}^{N} \left[\exp\left(-\int_0^{\tau_k} r_{dk}(t)dt \right) \max\{K - x_k(T), 0\} \right]. \tag{5.20}$$

9. The entire procedure is replicated over a (large) number of runs, with values for the option thus obtained by averaging over all previous (and the current) runs.

Recall that different choices and numbers of the basis functions used in the least-squares fitting will influence the option price. The accuracy of the estimates of the value of the American contingent claim can be increased by increasing the number of time steps, M, the number of simulated paths, N, and the number of basis functions, J, in all cases. Note that only sample paths which are in-the-money are considered for the least-squares fit, in order to reduce the necessary number of basis functions, and consequently reduce the computational cost[2]. Simply increasing the number of basis functions is not necessarily advantageous. Glasserman and Yu (2004) showed that the minimum number of paths required for convergence on a worst-case basis grows exponentially with the number of basis functions; therefore for finite N, an optimal J exists (mentioned in Section 4.6.2). Having experimented with different choices of J for the case of the present model, the value seven was selected for J. In the Longstaff and Schwartz (2001) paper, the basis functions $L_\cdot$ are suggested to be either Hermite, Chebyshev, or Laguerre polynomials or also powers of polynomial. Atkinson (1989) suggested Chebyshev polynomials are the best choice for polynomial fits (see also Caporale and Cerrato, 2005), and these were consequently the choice of polynomials employed in the present study.

Considerable literature on the bias of least-squares Monte Carlo methods has been published. The obvious importance of understanding the sources of bias affect

[2]Longstaff and Schwartz (2001) investigated the least-squares fit using all the sample paths. However it only returns the same results with much higher computational cost.

the methods for pricing American-style options by simulations. Prior to the publication of Longstaff and Schwartz (2001), Carriere (1996) addressed the problem of bias in the estimators for the American option prices. The bias is caused by two main streams, low bias and high bias. The low bias results from the approximation of the optimal stopping strategy. Recalling the valuation formula for American-style options, a supremum of the upper bound for the boundary makes the valuation always underestimated. The high bias is caused by the so called "foresight effect", meaning the use of knowledge about the entire life of the option. Mathematically, the simulated sample paths are used for both option valuation and optimal strategy, therefore the estimator a gives higher values than the true option price. With a standard LSM method, it can be observed that the dominant bias is the high bias(see Glasserman, 2003 and Fries, 2005). To overcome the "foresight effect", using a separate set of simulated sample paths for the optimal strategy can somewhat reduce the high bias. However, it can be computational expensive. Duck et al. (2005) exploited the interesting observation of bias to find the relation between the sample paths and the convergence to the true option value.

It has been mentioned in Section 4.2 that the convergence rate of standard Monte Carlo methods is proportional to $1/\sqrt{\text{sample paths}}$. Very usefully, as will be seen later, as the number of independent sample paths N (and the number of runs) increases, the option value is always found to tend monotonically towards the exact value from above (as with Duck et al., 2005). Consequently, Duck et al. proposed the following form to describe this convergence (by analogy with the standard Monte Carlo rate of convergence)

$$V_N = V_{\text{ext}} + \frac{\alpha_1}{\sqrt{N}} + \frac{\alpha_2}{N} + O(N^{-\frac{3}{2}}), \tag{5.21}$$

where V_{ext} is a more accurate (extrapolated) value of the option price, V. Therefore, by choosing three values of sample paths N, we can obtain estimates for the values of α_1 and α_2, and especially V_{ext}. The resultant technique is simple and easy to comprehend and implement, yet efficiently reduces the computational time and cost.

Invoking Equation (5.21), with the values from three different sample paths yields, a better estimate of the value of the option. Later, results with $N = 4000, 8000, 16000, 32000$ are shown. For each choice of N, up to 10,000 runs are performed and results averaged over all previous runs.

5.2.3 Numerical Results

Amin and Bodurtha (1995) provided most of the parameters that this model requires and so to show an accurate comparison with Amin and Bodurtha's model, the same parameter choices are used wherever they are applicable. However Amin and Bodurtha (1995) did not give explicit values of initial interest rates, the initial forward rate in U.S. dollar used here is given by historical statistics at the U.S. Federal Reserve Board[3], whereas the initial forward rate in Japanese Yen is provided by historical statistics released at the Bank of Japan website[4]. The other parameters in Table 5.1 correspond to those of Amin and Bodurtha. Seven basis functions and 50 exercise opportunities (i.e. time steps) are chosen as recommended by Duck et al. (2005).

Table 5.1: American/Bermudan currency-option valuation parameters I

Expiry date	T	1 year
Initial value of exchange rate	$x(0)$	0.0079101
Initial value of domestic forward rate	$f_d(0)$	0.0856
Initial value of foreign forward rate	$f_f(0)$	0.024
Strike price	K	$x(0)$, $0.95x(0)$
Drift of domestic interest	μ_d	0.01
Drift of foreign interest	μ_f	0.005
Volatility of domestic interest	σ_d	0.01481
Volatility of foreign interest	σ_f	0.01525
Exchange rate volatility	σ_x	0.1236
Correlation between $x(t)$ and $f_d(t)$	ρ_{xd}	-0.013
Correlation between $x(t)$ and $f_f(t)$	ρ_{xf}	0.0628
Correlation between $f_d(t)$ and $f_f(t)$	ρ_{df}	- 0.0821
Contract size		10,000
Time step	M	50
Number of basis functions	J	7

[3]U.S. Federal Reserve Board: `http://www.federalreserve.gov/releases/h15/data.htm`.
[4]Bank of Japan: `http://www.boj.or.jp/en/type/stat/dlong/index.htm`.

For European options, two sets of comparisons are shown in Table 5.2. The accurate values are from Amin and Jarrow's (1991) closed-form solution, whereas the column of option price (MC) is the corresponding option price using Monte Carlo simulations were performed with 10 million observations. From Table 5.2, the

Table 5.2: European put prices (comparison with the analytical solution)

	Option price (MC)	Accurate value
In-the-money	3.7364	3.73
At-the-money	2.0811	2.08

numerical solutions by Monte Carlo simulations converged to these values (quoted to two decimal places by Amin and Bodurtha, 1995). Thus, we can have some confidence that the parameters we collected from the government websites are comparable to those used in Amin and Bodurtha.

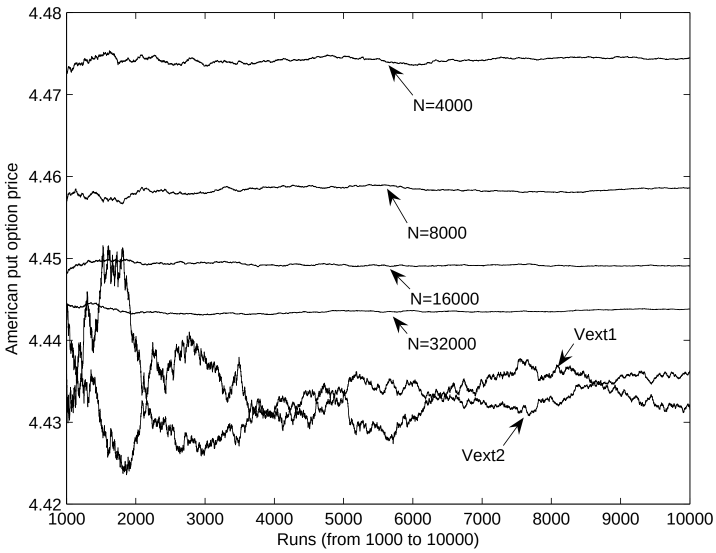


Figure 5.1: Amin and Bodurtha model for an at-the-money American put ($K = x(0)$) with 4000, 8000, 16000, 32000 sample paths

In order to compare the accuracy of our numerical method with the results of Amin and Bodurtha (1995), two sets of parameters are shown. Figure 5.1 corresponds to an at-the-money put option and Figure 5.2 to an out-of-the-money put

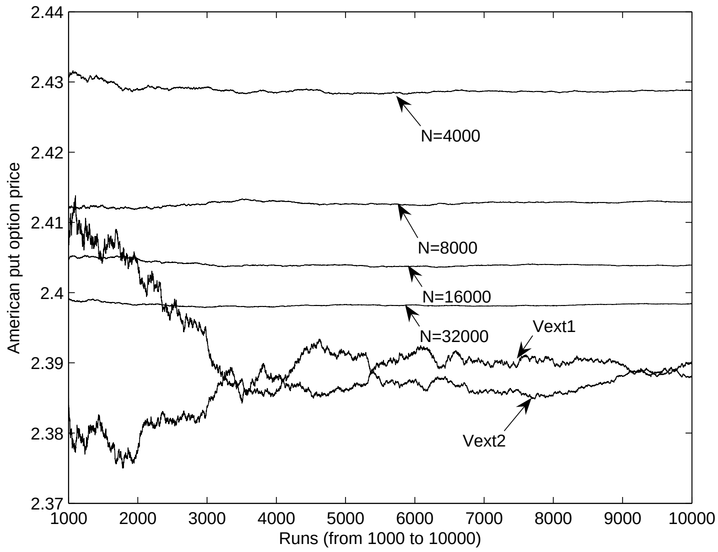


Figure 5.2: Amin and Bodurtha model for an out-of-the-money American put ($K = 0.95x(0)$) with 4000, 8000, 16000, 32000 sample paths

option. As anticipated, during the early runs, the averaged results are seen to fluctuate considerably due to the small number of averaged samples. Therefore, results for the first 1,000 runs are not shown in order to give a clearer picture of the processes. V_{ext1} is the extrapolated value using $N = 4000,\ 8000,\ 16000$, whereas V_{ext2} is the extrapolated value using $N = 8000,\ 16000,\ 32000$. Figures 5.1 and 5.2 indicate that when the estimator has sufficient samples, the estimated value will tend to an accurate reliable value (to within a penny accuracy). From these figures, it is clear that V_{ext1} and V_{ext2} are numerically close. In general, it is not always necessary to run 10,000 simulations to obtain an accurate extrapolated value. It should be borne in mind that the model is built up of a forward-rate model and converts this to a short-rate model, therefore the values fluctuate more than models which are based on a short-rate model per se. In the following two sections, results are found to be reliable using just 5000 runs. To further compare this numerical method with Amin and Bodurtha's (1995) tree method, two sets of results are presented in Table 5.3, one set is an at-the-money put ($K = x(0)$), the other an out-of-the-money put ($K = 0.95x(0)$). In each set, results are illustrated for two choices of time steps, namely 12 and 50. When 12 time steps are used (which is the same number as used by Amin and Bodurtha, 1995), the values are lower than those with 50 time steps, reflecting the fact that the more exercise opportunity, the more expensive the option is. However, 12-time-step values from this model are still higher than the Amin and Bodurtha results. This is presumably due to the error in the tree method (since 12 branch trees are likely to give very coarse results), and also, as mentioned before, our choices of initial interest rates are likely not the same as those of Amin and Bodurtha.

Table 5.3: Comparison of tree method and enhanced LSM method for the Amin and Bodurtha Model

		Number of time steps		Amin and Bodurtha
		12	50	
At-the-money	4000	4.458	4.474	4.31
	8000	4.449	4.459	
	16000	4.443	4.449	
	Ext	4.427	4.436	
Out-of-the-money	4000	2.413	2.429	2.32
	8000	2.404	2.413	
	16000	2.398	2.404	
	Ext	2.384	2.390	

5.3 Improved Interest-rate Modelling

5.3.1 Assumptions

In this section, a first modification of the pricing model is presented. Despite the notable advances in theoretical research and the apparent flexibility of the HJM model, it is difficult to calibrate data with this high-dimensional nonlinear model. Amin and Bodurtha (1995) were only able to apply the Ho and Lee (1986) model using constant volatility. In contrast, short-rate models have the advantage of flexibility in numerical implementation. It is relatively straightforward to extend one-factor short-rate models to multi-factor models, even with stochastic volatilities, when required. Consequently, based on the risk-neutral measure, the exchange-rate process is assumed to follow the SDE (5.22) below, whereas interest rates follow the CIR model, instead of the Ho and Lee model used in the previous section, namely

$$\begin{aligned} \text{Exchange Rate: } \frac{dx_t}{x_t} &= (r-f)dt + \sigma_x dW_x, & (5.22) \\ \text{Domestic Interest Rate: } dr &= \kappa_r(\theta_r - r)dt + \sigma_r\sqrt{r}dW_r, & (5.23) \\ \text{Foreign Interest Rate: } df &= \kappa_f(\theta_f - f)dt + \sigma_f\sqrt{f}dW_f, & (5.24) \end{aligned}$$

where κ_r and κ_f are the mean-reverting speed of interest rates of r and f respectively, θ_r and θ_f are the long-run mean of the interest rates r and f respectively, dW_x, dW_r and dW_f are the increments of standard Brownian motions, the σ_x, σ_r, and σ_f are the volatilities of x, r and f respectively, and again, the random processes are correlated as

$$\mathbb{E}[dW_i dW_j] = \rho_{ij}dt, \quad \text{where} \quad i,j = r,f,x.$$

Note again that $\rho_{ij} = \rho_{ji}$, $\rho_{ii} = 1$ and so the correlation matrix is a 3×3 symmetric matrix

$$\Sigma = \begin{pmatrix} 1 & \rho_{xr} & \rho_{xf} \\ \rho_{rx} & 1 & \rho_{rf} \\ \rho_{fx} & \rho_{fr} & 1 \end{pmatrix}.$$

The parameters κ_r, κ_f, θ_r, θ_f, σ_x, σ_r, σ_f and ρ_{ij} are all taken to be constant.

5.3.2 Numerical Scheme

The procedure is similar to that in the previous section, but note the following changes are made in Step 2:

2. Sample N independent paths of exchange rate $x_k(t_i)$ for $k = 1, 2, \ldots, N$. Set $x_k(0) = x_0$, the current value of $x(t)$. $x_k(t_{i+1})$ is determined by:

$$x_k(t_{i+1}) = x_k(t_i)\exp\left(\left[r_k(t_i) - f_k(t_i) - \frac{1}{2}\sigma_x^2\right]\Delta t + A_{1,1}\sigma_x\epsilon_{xi}\sqrt{\Delta t}\right); \quad (5.25)$$

similarly, the SDEs (5.23) and (5.24) are approximated as follows:

$$\begin{aligned} r_k(t_{i+1}) &= r_k(t_i) + \kappa_r(\theta_r - r_k(t_i))\Delta t \\ &+ (A_{2,1}\epsilon_{xi} + A_{2,2}\epsilon_{ri})\sigma_r\sqrt{r_k(t_i)\Delta t}, \qquad (5.26) \\ f_k(t_{i+1}) &= f_k(t_i) + \kappa_f(\theta_f - f_k(t_i))\Delta t \\ &+ (A_{3,1}\epsilon_{xi} + A_{3,2}\epsilon_{ri} + A_{3,3}\epsilon_{fi})\sigma_f\sqrt{f_k(t_i)\Delta t}, \qquad (5.27) \end{aligned}$$

where ϵ_{xi}, ϵ_{ri} and $\epsilon_{fi} \sim N(0,1)$ are the sequences of independent standard normal variables at the ith time step.

As mentioned in Section 3.3.3, the absolute value of interest rate at any time t is necessary to avoid the scheme breaking-down numerically if the negative values occur (although for the parameters chosen, this is a rare event).

5.3.3 Numerical Results

Some sample results for the currency-option price are presented in Figures 5.3, 5.4 and 5.5. The choice of parameters is important for a newly built model, and therefore previously referenced parameters have been used wherever possible. The interest rate parameters in Table 5.4 correspond to those of Treepongkaruna and Gray's (2003) estimation where applicable, whilst other parameters are kept consistent with the Amin and Bodurtha (1995) model, which are shown in Table 5.1.

Table 5.4: American/Bermudan currency-option valuation parameters II

Initial value of domestic interest rate	$r(0)$	0.0585
Initial value of foreign interest rate	$f(0)$	0.00704
Mean-reversion rate of domestic interest	κ_r	0.3334
Mean-reversion rate of foreign interest	κ_f	0.1279
Long term growth rate of domestic interest	θ_r	0.0585
Long term growth rate of foreign interest	θ_f	0.00704
Volatility of domestic interest	σ_r	0.0161
Volatility of foreign interest	σ_f	0.0571

In Figures 5.3, 5.4 and 5.5, results are shown for an American put option obtained using 4000, 8000, 16000, 32000 sample paths. By averaging over the current and preceding values, a converging estimate of the option price is obtained. Again, two extrapolated processes for $N = 4000, 8000, 16000$ and $N = 8000, 16000, 32000$ are obtained using Equation (5.21), which are shown denoted as V_{ext1} and V_{ext2} respectively (again, to illustrate the figures more clearly, the values for the first 1,000 runs are omitted). Three types of options are presented, namely an in-the-money option (Figure 5.3), an at-the-money option (Figure 5.4), and an out-of-the-money option (Figure 5.5).

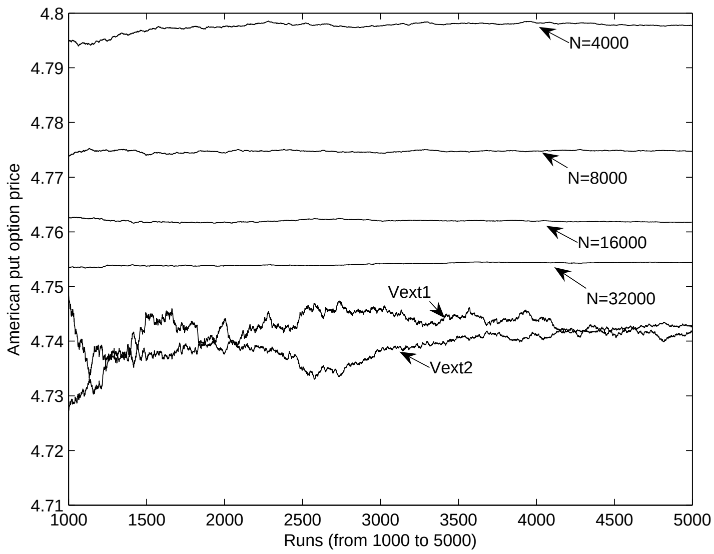


Figure 5.3: Extended model for an in-the-money American put ($K = 1.05x(0)$) with 4000, 8000, 16000, 32000 sample paths

Note again, the processes of extrapolated value (i.e. V_{ext1} and V_{ext2}) are initially more erratic than the original processes, but after 5000 runs, V_{ext1} and V_{ext2} have settled down and differ very little (giving better than one penny accuracy). The result is an interim model which we next extend in order to produce the final version which takes into account the full set of stochastic parameters.

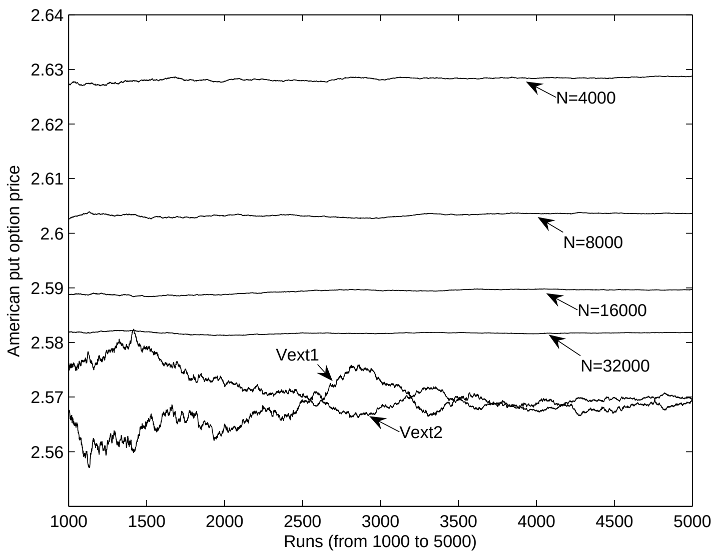


Figure 5.4: Extended model for an at-the-money American put ($K = x(0)$) with 4000, 8000, 16000, 32000 sample paths

5.4 Further Improved Stochastic Volatility Modelling

5.4.1 Assumptions

In this section, stochastic volatilities will be incorporated to complete the model. Heston's (1993) model of volatilities is included in the exchange-rate process and both interest rates processes, leading to the system following six stochastic processes.

$$\begin{aligned}
\text{Exchange Rate: } \frac{dx_t}{x_t} &= (r-f)dt + \sigma_x\sqrt{\upsilon_1}dW_x, && (5.28)\\
\text{Volatility of } x\text{: } d\upsilon_1 &= \kappa_1(\theta_1-\upsilon_1)dt + \sigma_1\sqrt{\upsilon_1}dW_1, && (5.29)\\
\text{Domestic Interest Rate: } dr &= \kappa_r(\theta_r - r)dt + \sigma_r\sqrt{r\upsilon_2}dW_r, && (5.30)\\
\text{Stochastic Volatility of } r\text{: } d\upsilon_2 &= \kappa_2(\theta_2-\upsilon_2)dt + \sigma_2\sqrt{\upsilon_2}dW_2, && (5.31)\\
\text{Foreign Interest Rate: } df &= \kappa_f(\theta_f - f)dt + \sigma_f\sqrt{f\upsilon_3}dW_f, && (5.32)\\
\text{Stochastic Volatility of } f\text{: } d\upsilon_3 &= \kappa_3(\theta_3-\upsilon_3)dt + \sigma_3\sqrt{\upsilon_3}dW_3, && (5.33)
\end{aligned}$$

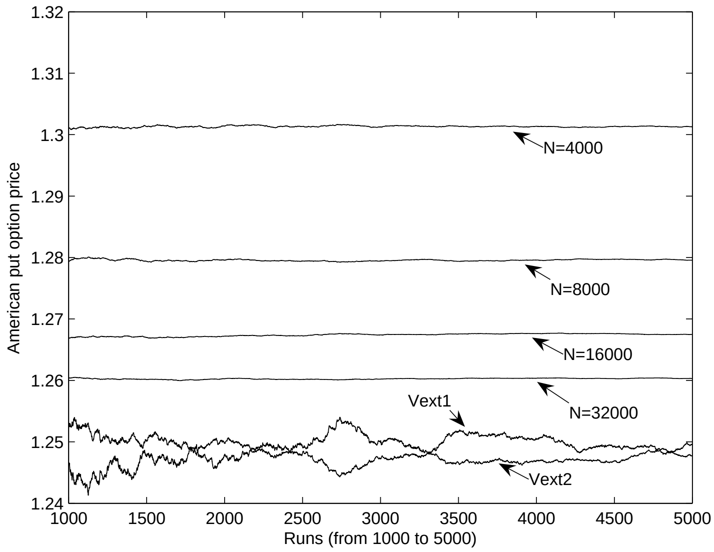


Figure 5.5: Extended model for an out-of-the-money American put ($K = 0.95x(0)$) with 4000, 8000, 16000, 32000 sample paths

where the κ's are the mean-reverting speed, the θ's are the long-run mean, the σ's are the volatility of volatility, dW's are increments of standard Brownian motions, and the ith and jth Brownian motion processes are correlated as follows

$$\mathbb{E}[dW_i dW_j] = \rho_{ij} dt \quad \text{where} \quad i, j = r, f, x, 1, 2, 3 \quad \rho_{ij} = \rho_{ji},\ \rho_{ii} = 1.$$

The correlation matrix clearly becomes a 6×6 symmetric matrix

$$\Sigma = \begin{pmatrix} 1 & \rho_{x1} & \rho_{xr} & \rho_{x2} & \rho_{xf} & \rho_{x3} \\ \rho_{1x} & 1 & \rho_{1r} & \rho_{12} & \rho_{1f} & \rho_{13} \\ \rho_{rx} & \rho_{r1} & 1 & \rho_{r2} & \rho_{rf} & \rho_{r3} \\ \rho_{2x} & \rho_{21} & \rho_{2r} & 1 & \rho_{2f} & \rho_{23} \\ \rho_{fx} & \rho_{f1} & \rho_{fr} & \rho_{f2} & 1 & \rho_{f3} \\ \rho_{3x} & \rho_{31} & \rho_{3r} & \rho_{32} & \rho_{3f} & 1 \end{pmatrix},$$

where the parameters κ's, θ's, σ's and ρ's are assumed to be constant.

5.4.2 Numerical Scheme

The main difference with the technique implemented in the previous section is that the sample paths of the volatilities must also be generated at Step 2:

2. $x_k(t_{i+1})$, $r_k(t_{i+1})$, $f_k(t_{i+1})$ and the volatilities of theirs can be determined by:

$$x_k(t_{i+1}) = x_k(t_i)\exp\left(\left[r_k(t_i) - f_k(t_i) - \frac{1}{2}\sigma_x^2\right]\Delta t + A_{1,1}\sigma_x\epsilon_{xi}\sqrt{\upsilon_{1k}(t_i)\Delta t}\right), \tag{5.34}$$

$$\begin{aligned}\upsilon_{1k}(t_{i+1}) &= \upsilon_{1k}(t_i) + \kappa_1\left[\theta_1 - \upsilon_{1k}(t_i)\right]\Delta t \\ &+ (A_{2,1}\epsilon_{xi} + A_{2,2}\epsilon_{\upsilon_1 i})\sigma_1\sqrt{\upsilon_{1k}(t_i)\Delta t},\end{aligned} \tag{5.35}$$

$$\begin{aligned}r_k(t_{i+1}) &= r_k(t_i) + \kappa_r[\theta_r - r_k(t_i)]\Delta t \\ &+ (A_{3,1}\epsilon_{xi} + A_{3,2}\epsilon_{\upsilon_1 i} + A_{3,3}\epsilon_{ri})\sigma_r\sqrt{r_k(t_i)\upsilon_{2k}(t_i)\Delta t},\end{aligned} \tag{5.36}$$

$$\begin{aligned}\upsilon_{2k}(t_{i+1}) &= \upsilon_{2k}(t_i) + \kappa_2[\theta_2 - \upsilon_{2k}(t_i)]\Delta t \\ &+ (A_{4,1}\epsilon_{xi} + A_{4,2}\epsilon_{\upsilon_1 i} + A_{4,3}\epsilon_{ri} + A_{4,4}\epsilon_{\upsilon_2 i})\sigma_2\sqrt{\upsilon_{2k}(t_i)\Delta t},\end{aligned} \tag{5.37}$$

$$\begin{aligned}f_k(t_{i+1}) &= f_k(t_i) + \kappa_f(\theta_f - f_k(t_i))\Delta t \\ &+ (A_{5,1}\epsilon_{xi} + A_{5,2}\epsilon_{\upsilon_1 i} + A_{5,3}\epsilon_{ri} + A_{5,4}\epsilon_{\upsilon_2 i} + A_{5,5}\epsilon_{fi})\sigma_f\sqrt{f_k(t_i)\upsilon_{3k}(t_i)\Delta t},\end{aligned} \tag{5.38}$$

$$\begin{aligned}\upsilon_{3k}(t_{i+1}) &= \upsilon_{3k}(t_i) + \kappa_3[\theta_3 - \upsilon_{3k}(t_i)]\Delta t \\ &+ (A_{6,1}\epsilon_{xi} + A_{6,2}\epsilon_{\upsilon_1 i} + A_{6,3}\epsilon_{ri} + A_{6,4}\epsilon_{\upsilon_2 i} + A_{6,5}\epsilon_{fi} + A_{6,6}\epsilon_{\upsilon_3 i})\sigma_3\sqrt{\upsilon_{3k}(t_i)\Delta t},\end{aligned} \tag{5.39}$$

where ϵ_{xi}, ϵ_{ri}, ϵ_{fi}, $\epsilon_{\upsilon_1 i}$, $\epsilon_{\upsilon_2 i}$ and $\epsilon_{\upsilon_3 i} \sim N(0,1)$ are the sequences of independent standard normal variables at the ith time step.

5.4.3 Numerical Results

There has been some empirical work, albeit focussed on options without early exercise, which provides useful parameters for this section of the thesis. As mentioned in Section 3.4.2, Dupoyet (2006) provided an empirical investigation into Japanese Yen/U.S. dollar currency-options, which are applicable in this complete model. The parameters of Table 5.5 correspond to those of Dupoyet for the exchange rate volatility (the upper portion of Table 5.5), whilst other parameters have been chosen by the author (the lower portion of Table 5.5).

Table 5.5: American/Bermudan currency-option valuation parameters III

Mean-reversion rate of exchange rate volatility	κ_1	6.17
Long term growth rate of exchange rate volatility	θ_1	0.0097
Volatility of the exchange rate volatility	σ_1	0.21
Correlation between $x(t)$ and $\upsilon_1(t)$	ρ_{x1}	-0.13
Initial value of exchange rate volatility	$\upsilon_1(0)$	0.1236
Initial value of domestic interest rate volatility	$\upsilon_2(0)$	0.0161
Initial value of foreign interest rate volatility	$\upsilon_3(0)$	0.0571
Mean-reversion rate of domestic volatility	κ_2	2.5
Mean-reversion rate of foreign volatility	κ_3	2.0
Long term growth rate of domestic volatility	θ_2	0.01
Long term growth rate of foreign volatility	θ_3	0.02
Volatility of domestic interest rate volatility	σ_2	0.1
Volatility of foreign interest rate volatility	σ_3	0.1
Correlation between $x(t)$ and $\upsilon_2(t)$	ρ_{x2}	-0.008
Correlation between $x(t)$ and $\upsilon_3(t)$	ρ_{x3}	0.007
Correlation between $r(t)$ and $\upsilon_1(t)$	ρ_{1r}	-0.008
Correlation between $\upsilon_1(t)$ and $\upsilon_2(t)$	ρ_{12}	-0.006
Correlation between $f(t)$ and $\upsilon_1(t)$	ρ_{1f}	0.008
Correlation between $\upsilon_1(t)$ and $\upsilon_3(t)$	ρ_{13}	0.005
Correlation between $r(t)$ and $\upsilon_2(t)$	ρ_{r2}	0.02
Correlation between $r(t)$ and $\upsilon_3(t)$	ρ_{r3}	0.003
Correlation between $f(t)$ and $\upsilon_2(t)$	ρ_{2f}	0.008
Correlation between $\upsilon_2(t)$ and $\upsilon_3(t)$	ρ_{23}	0.002
Correlation between $f(t)$ and $\upsilon_3(t)$	ρ_{f3}	0.01

Results for currency-option prices with stochastic interest rates and volatilities are shown in Figures 5.6, 5.7, 5.8, namely an in-the-money option, an at-the-money option, and an out-of-the-money option respectively. It can be seen that with the same degree of moneyness, the option with stochastic volatilities gives a higher value. For example, the price for the at-the-money option in Section 5.3 is about 2.57, whilst the option with stochastic volatilities is about 4.05. This, no doubt, is because with stochastic volatilities, the option has more potential for positive payoff, and so the value of the option is likely to be higher. Note that interest rates generally have small volatilities compared with exchange rate volatility, therefore the stochastic volatilities of interest rates do not influence the option value as significantly as the

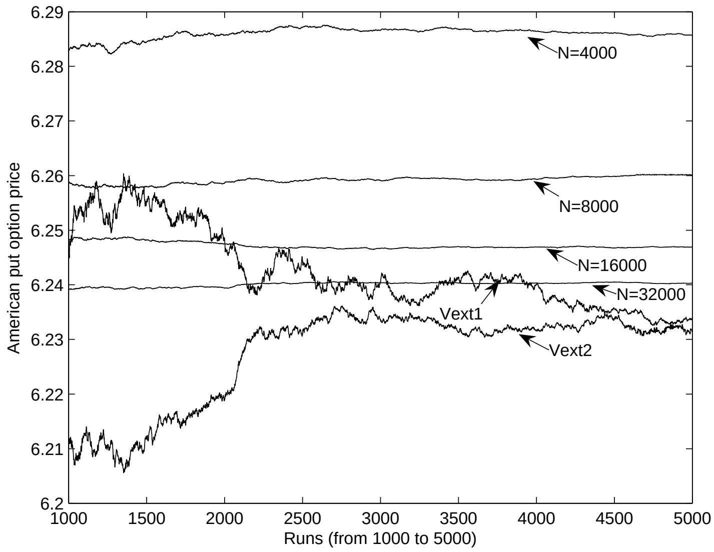


Figure 5.6: Stochastic volatility model for in-the-money American put ($K = 1.05x(0)$) with 4000, 8000, 16000, 32000 sample paths

exchange-rate volatility.

Figures 5.9, 5.10 and 5.11 show the influence of parameter changes on the correlations (i.e. the ρ_{ij}) between these factors (these are notoriously difficult to measure using real-world data). Here the values of the ρ_{ij} $(i \neq j)$ have been increased by a factor of 10, compared with those used in Figure 5.6 (an in-the-money case), Figure 5.7 (an at-the-money case), and Figure 5.8 (an out-of-the-money case) respectively. In Figures 5.9, 5.10 and 5.11, the line denoted as "Original V_{ext}" is the extrapolated value (using $N = 8000$, 16000, 32000) of the original correlation parameters, and V_{ext} is the extrapolated value (using $N = 8000$, 16000, 32000) with the larger correlation parameters. The two extrapolated processes follow somewhat the same trend, but over all and importantly these show that the correlation factors do not affect the option value significantly. This may be regarded as a very positive feature of the model, given the difficulty in estimating parameters in all multi-factor models. Further, when the correlation between the stochastic factors are larger, the movements of the processes are more likely to be bounded with each other. This implies that the less random the processes are. Consequently, the price range of V_{ext} is

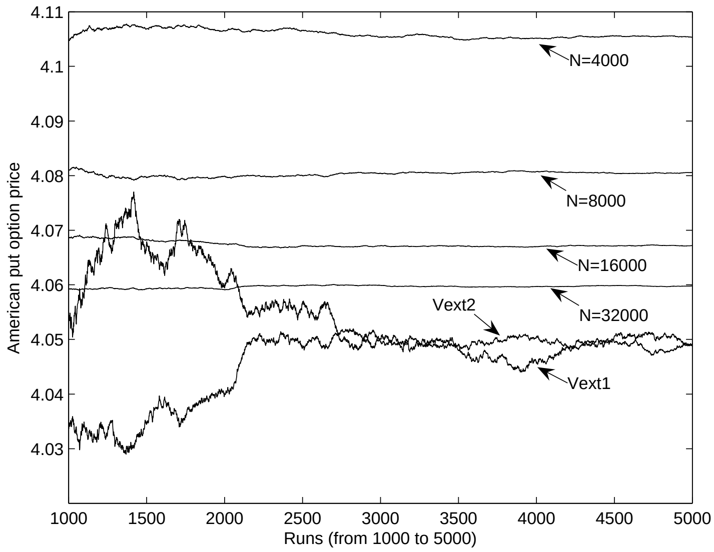


Figure 5.7: Stochastic volatility model for at-the-money American put ($K = x(0)$) with 4000, 8000, 16000, 32000 sample paths

less erratic than that of original V_{ext}, implying that the in-the-money V_{ext} is less expensive than the original in-the-money option, whereas an out-of-the-money V_{ext} is more expensive than the original out-of-the-money option.

5.5 Summary

Foreign exchange is the largest of the global financial markets, with daily trading volume now measured in trillions of U.S. dollars. Associated with this are exchange traded options and a very active OTC market in currency options. As noted by Carr and Wu (2007), OTC quotes are based on Garman and Kohlhagen (1983) implied volatilities, and there remains a tendency to favour analytic solutions for lack of suitable numerical approaches to richer models.

Until just over a decade ago, only European currency-option pricing was feasible. Amin and Bodurtha (1995) achieved partial success with early-exercise feature, using just a 12 step tree, and since then the pricing of American currency-options has

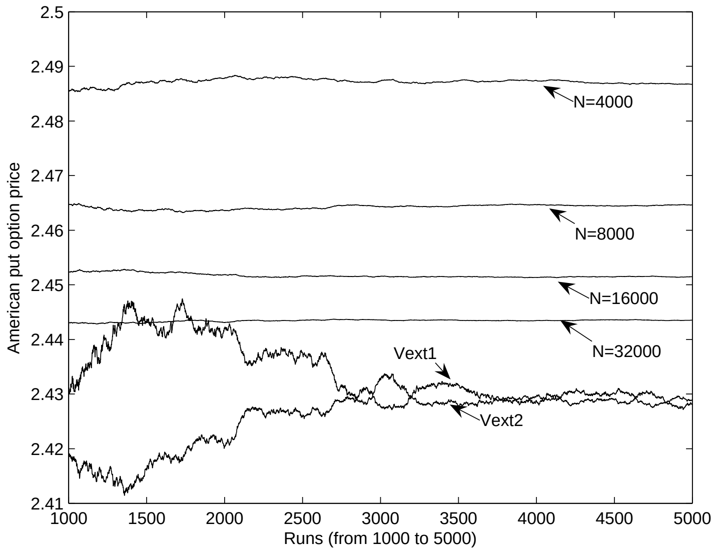


Figure 5.8: Stochastic volatility model for out-of-the-money American put ($K = 0.95x(0)$) with 4000, 8000, 16000, 32000 sample paths

remained limited. By applying a new, fast, enhanced accuracy Monte Carlo technique and using parameters derived from earlier empirical work, we have developed a more realistic but easily implementable model for American currency-options in a complex stochastic environment. The resulting model employs up to six stochastic processes, with early exercise, but remains tractable. Tests with empirical data and parameter sensitivity show the stochastic volatilities to have notable effects on option values, exchange rate volatility having greater influence than interest rate volatilities. Usefully, values have been shown to be relatively insensitive to correlations between factors.

This is not only a practical model for currency-option evaluation but also a promising multi-dimensional option pricing technique which includes early exercise. Therefore this methodology has the potential for use in many other areas, such as credit spread option pricing and quanto options.

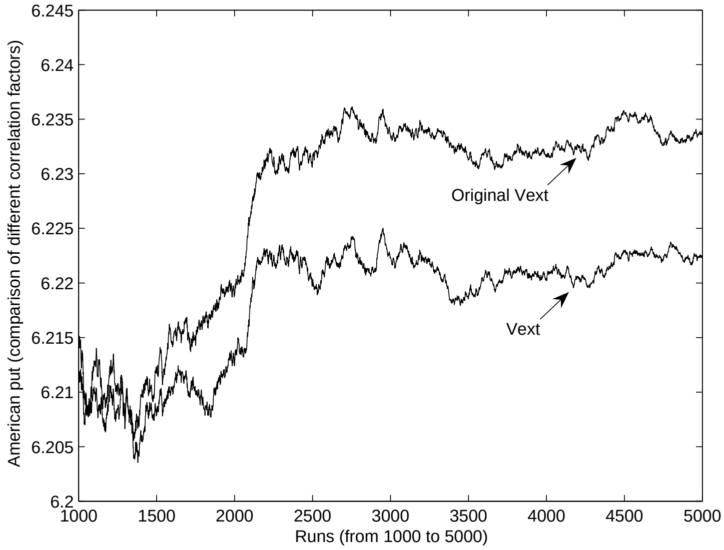


Figure 5.9: Influence of correlation parameters for an in-the-money American put V_{ext} are the processes with 10 times larger correlation than that of Original V_{ext}.

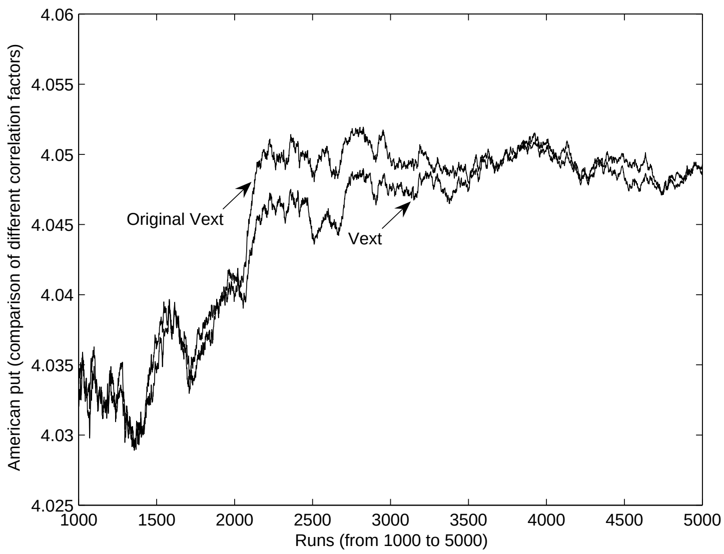


Figure 5.10: Influence of correlation parameters for an at-the-money American put V_{ext} are the processes with 10 times larger correlation than that of Original V_{ext}.

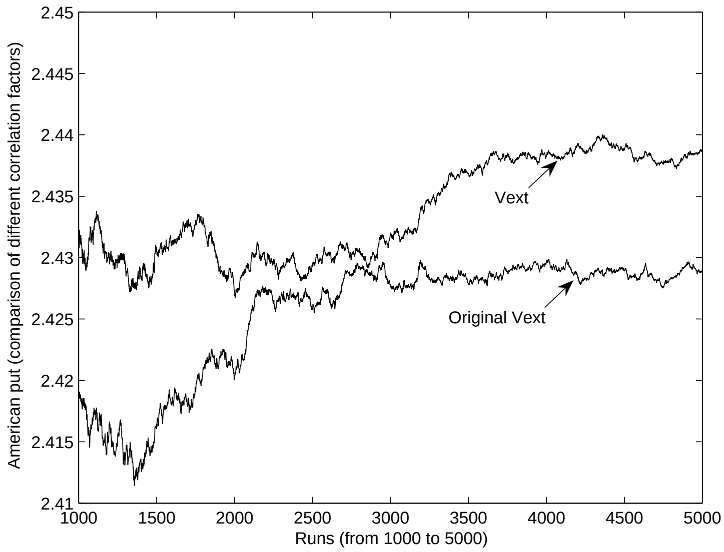


Figure 5.11: Influence of correlation parameters for an out-of-the-money American put
V_{ext} are the processes with 10 times larger correlation than that of Original V_{ext}.

Chapter 6

Discrete Barrier Currency-Options

No human investigation can be called real science if it cannot be demonstrated mathematically.

Leonardo da Vinci (1452–1519)
Treatise on Painting

Barrier options are one of the most popular first-generation exotic options, yet little theoretical research existed on them until the mid 1990s. This chapter begins by raising a realistic problem related to the currency option market. From both theoretical and hedging perspectives, barrier options are well known to be more complex than standard options. Further, it is shown that barrier options have quite different hedging properties than standard options.

One type of option heavily traded in the over-the-counter market (i.e. interbank market) is the reverse barrier option. It is a barrier option with the barrier triggered at a level when the option is in-the-money. Consequently, for a call option, the barrier would be above the strike price; for a put, below strike. If the knock-out is not triggered, the payoff is the same as for a vanilla option. Since option prices are measured by the potential profit the options carry (also mentioned in Section 5.4.3), this type of option is generally cheaper than the corresponding vanilla options.

Given the idea of barrier options, a more specific problem will be addressed in this chapter – the legal quote of the option contract delay caused by mis-hedging loss on discrete barrier currency-options. The question arises in Section 6.1, whilst Section 6.2 and Section 6.3 will focus on the European case and discretely monitored case respectively to investigate the profit and loss. Section 6.4 summarises the results of this chapter.

6.1 Introduction

Wystup and Becker (2005) addressed a mis-hedging problem due to the delay of currency fixing announcements from central banks. In most previous work in this area, the markets have been assumed to be perfect, which implies that there are neither transaction costs nor time delays in transactions. However, in reality, markets only have limited liquidity. The illiquidity affects the option prices and hedging strategies. In the present case, the hedging strategy is affected by the delay of the legal quote of the option contract.

As a simple example, suppose a client bought a European barrier currency option from the OTC market (normally, from the client's own bank). At maturity, the client has to choose whether to exercise the option or not. Of course the seller of the option will provide a quote at maturity, but in the OTC market a seller is also the "rule maker", who might move the quoted cut-off rate in favour of his/her own position. For fairness, the client prefers some independent quotes to monitor the option and the reference rate from central bank is preferable. In the present case, the European Central Bank (hereafter, ECB) was chosen. However, the ECB publishes the fixing rate with a delay about 10-20 minutes every day. This is basically because the ECB needs to gather all the exchange-rate information from all the European countries' central banks and then calculate the reference rate. Therefore it is very likely to be different from the tradable spot rate on the interbank market. This is not a problem from a buyer's perspective, as the client only needs this independent source to check the validity of the barrier option, the tradable quote is still the instantaneous spot exchange rate on the interbank market. To the seller, the client's own bank, delta-hedging becomes impossible, since the delta may become enormously large close to maturity or close to the time for monitoring the barrier. The mis-hedging problem arises on the seller's side. This only happens in the OTC market, which illustrates a drawback of this market (OTC markets are self-regulated, as mentioned in Section 3.1.1). It causes the seller to mis-hedge the position and the losses can be substantial; therefore, determining a proper price for the reverse barrier options is rather important.

6.2 European Up-and-Out Call Option

First, a European up-and-out call option will be employed as a demonstration. To address the problem, the present model is consistent with Wystup and Becker's (2005) assumption, a geometric Brownian motion is used to simulate the exchange rate process under the risk-neutral measure,

$$\frac{dx_t}{x_t} = (r - f)dt + \sigma_x dW_t, \tag{6.1}$$

where r denotes the domestic interest rate, f the foreign interest rate, σ_x the volatility and dW_t the increments of a standard Brownian motion. These parameters are assumed to be constant.

The payoff for the option is

$$V(F_T) = \max\{F_T - K, 0\}\mathbf{1}_{F_t<B}, \tag{6.2}$$

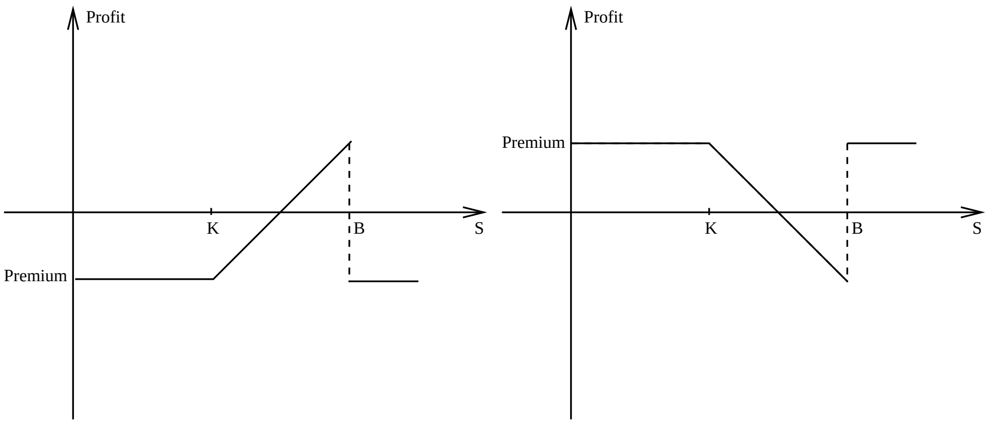
where the F_t denotes the ECB fixing rate at time t, T the maturity, K the strike price, B the knock-out barrier, and $\mathbf{1}_{(.)}$ is the indicator function defined in Section 2.2.2. The seller of the option can only trade with the spot rate, not the fixing rate; therefore the payoff for the hedging strategy is

$$V(x_T) = \max\{x_T - K, 0\}\mathbf{1}_{F_t<B}. \tag{6.3}$$

6.2.1 Hedging Error

Hedging is a strategy designed to reduce risk. It involves two positions: a position in one security and an offsetting position in another related security or securities. Normally, this counter-balancing position is adjusted when market conditions change, hence the name dynamic hedging strategy (see Benninga and Wiener, 1998a).

The seller (i.e. the writer) of the option takes the opposite position from the buyer. Figure 6.1 are shown to illustrate the difference. In the particular case shown



(a) Buyer (b) Seller

Figure 6.1: Profit and loss function for an up-and-out call option

in Figures 6.1(a) and 6.1(b), both the buyer and the seller of the reverse up-and-out call options have bounded profit or loss. Therefore, the option is relatively less risky compared to other standard options. However, this is not the case for the seller if he/she uses a hedging strategy. The 10-20 minutes delay may change the outcome of the validity of the option, which may consequently put the seller's current hedged position at risk.

There are three possible scenarios at the expiry date. Following Wystup and Becker's (2005) paper, the hedging strategy for the seller is delta hedging, and to be totally realistic, the bid/ask spread δ for the underlying asset x_t is introduced. The transaction cost is introduced into the model as it is not negligible when the seller needs to maintain his/her position covered by hedging (buying or selling the underlying to reduce the risk). Consider the three scenarios:

- $x_T \leq K$. In this case, the seller believes the option is out of the money, and therefore decides not to hedge any longer, which means $\Delta = 0$. If in 10-20 minutes (denoted by τ), the option is in the money, i.e. $K < F_T < B$. The seller has to exercise the option with this naked position. The profit and loss function (denoted as PL) is

$$PL = K - x_{T+\tau} - \delta. \tag{6.4}$$

- $K < x_T < B$. In this case, the seller believes the option is in the money, and decides to keep the covered position. Therefore, the delta $\Delta = 1$. If in 10-20 minutes, the option is out of the money, i.e. $F_T \geq B$ or $F_T \leq K$. The profit and loss function is

$$PL = x_{T+\tau} - x_T - \delta. \tag{6.5}$$

- $x_T \geq B$. This case is symmetric with the first case. The seller thinks the option is knocked out, but it turns out that it is in the money at the end of this extra 10-20 minutes. The profit and loss function is

$$PL = K - x_{T+\tau} - \delta. \tag{6.6}$$

  Note that in the first and third cases, if the fixing F_T is very volatile, it may jump over in-the-money zone, the hedge is accidentally appropriate (very rare events).

6.2.2 Numerical Scheme

Monte Carlo simulation is again used for the analysis. The algorithm is described as follows:

1. Divide the time period $[0, T]$ into M steps. Set $\Delta t = T/M$, and therefore $t_i = i\Delta t$, for $i = 0, 1, 2, \ldots, M$.

2. Sample N independent paths of exchange rate $x_k(t_i)$, for $k = 1, 2, \ldots, N$ using Euler discretisation.

   Set $x_k(0) = x_0$, the current value of $x(t)$. $x_k(t_{i+1})$ is determined by:

   $$x_k(t_{i+1}) = x_k(t_i) \exp\left(\left[r - f - \frac{1}{2}\sigma_x^2\right]\Delta t + \sigma_x \epsilon_i \sqrt{\Delta t}\right), \qquad (6.7)$$

   where r, f, σ_x are constant, and $\epsilon_i \sim N(0, 1)$ is a sequence of independent standard normal variables at the ith time step.

3. Recall that the fixing rate F_t is sometimes different from the spot rate; therefore, according to Wystup and Becker (2005), the following dynamics are used for the fixing rate:

   $$F_t = x_t + \phi, \quad \text{where} \quad \phi \sim N(\mu, \sigma^2). \qquad (6.8)$$

   The parameters in Table 6.1 are estimated from historical data and provided by Wystup and Becker's (2005) paper. The most liquid currency pair, Euro–U.S. dollar is chosen to analyse the extra cost due to the delay.

4. Obtain the mis-hedge quantities using the given profit and loss functions at time T.

5. Average over the mis-hedge for N sample paths.

6.2.3 Analysis of Error

The necessary parameters which are applied in the model are given below:

The mis-hedge error is shown in Figure 6.2. The profit and loss due to mis-hedging are plotted against different barriers (from 1.22 to 1.46). The losses are relatively small when the barriers are very close to the spot rate, or very far from the spot rate. Overall, the losses are relatively small for one million units of domestic currency. The largest error, about 15 units of domestic currency, occurs when the barrier is 1.31 which is at a reasonable distance from the spot rate. The cost is not substantial as the mis-hedging only occurs at maturity. The significance of this will be addressed shortly.

Table 6.1: Testing parameters

Expiry date	T	1 year
Spot rate	x_0	1.21
Domestic interest rate	r	0.0217
Foreign interest rate	f	0.0227
Exchange rate volatility	σ_x	0.104
Mean of the fixing rate	μ	$-3.125 * 10^{-6}$
Standard deviation of the fixing rate	σ	$1.264 * 10^{-4}$
Time step	M	250
Sample paths	N	$1,000,000$
Strike price	K	1.18

6.3 Discretely Monitored Up-and-out Call

The previous case in Section 6.2 assumes continuous monitoring of the barrier. Under such an assumption, Merton (1973) obtained a formula for pricing a knock-out call. However, real contracts with barrier features specify fixed times for monitoring of the barrier, typically, daily closing. Numerical examples indicate that there can be substantial price differences between discrete and continuous barrier options. Even numerical methods using standard lattice techniques or Monte Carlo simulations face significant difficulties (see Broadie, Glasserman and Kou, 1997).

The only difference with European barrier calls is that discretely monitored options have more chance to be mis-hedged for option sellers who use dynamic hedging strategy. Normally, the monitoring frequency is on a daily basis. Thus, a one year option will have 250 checking points (250 potential knock-out events, consequently 250 mis-hedge possibilities). From the results in Section 6.2.3, one may have a rough estimation of the maximum loss, say 15 units of domestic currency times 250 mis-hedge events, that is 3750 units of domestic currency. This section will show that the potential loss for a discretely monitored option is far more larger than this estimation.

Again, the payoff function for a discretely monitored up-and-out call option is

$$V(F_T, T) = \max(F_T - K, 0)\mathbf{1}_{F_t<B}, \tag{6.9}$$

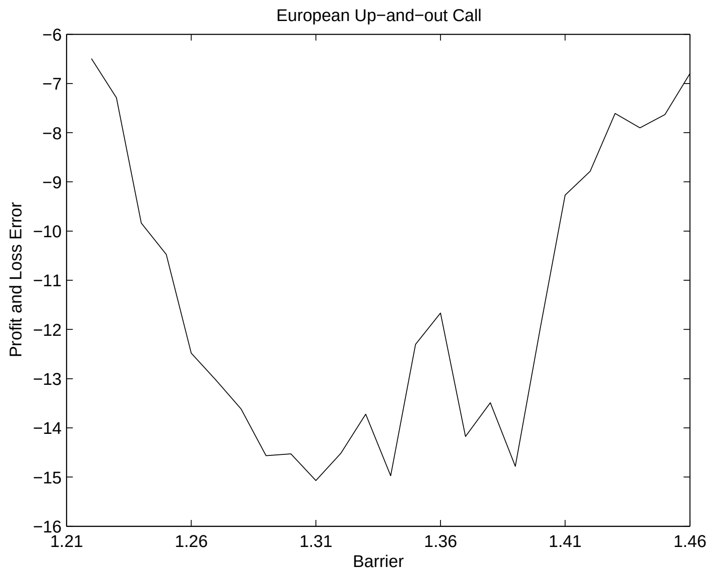


Figure 6.2: Mis-hedging error with one million units of domestic currency (U.S. dollar).

where the F_t denotes the ECB fixing rate at time t, T the maturity, K the strike price, B the knock-out barrier, and $\mathbf{1}_{(\cdot)}$ is the indicator function defined in Section 2.2.2. And the payoff for the hedging strategy is

$$V(x_T, T) = \max(x_T - K, 0)\mathbf{1}_{F_t<B}. \tag{6.10}$$

6.3.1 Hedging Error

The three possible scenarios at maturity are the same as that for the plain European call analysed in Section 6.2.1, and two additional possibilities that may cause the mis-hedge at every checking point. This two scenarios are

- $x_t < B$ and $F_t \geq B$.

  In this case, the seller holds Δ units of the underlying asset in the hedge, $\Delta(x_t)$ denotes the dynamic delta hedging quantity at time t. According to the spot rate at the checking point each day, the seller is holding the hedged position. However, 10-20 minutes later, the delayed fixing announcement shows that the option is knocked out. The underlying asset is no longer needed for the hedging. The seller has to sell the underlying asset at time $t+\tau$. The profit and loss function is,

$$PL = \Delta(x_t)\,(x_{t+\tau} - x_t - \delta)\,. \tag{6.11}$$

- $x_t \geq B$ and $F_t < B$.

  In this case, the seller thinks the option is out of the money, and decides to unwind the hedged position. Therefore, he/she sells $\Delta(x_t)$ units of underlying asset. In 10-20 minutes $F_T < B$, the seller has to build up a new hedge at time $t+\tau$, with $\Delta(x_{t+\tau})$ units. The profit and loss function is,

$$PL = \Delta(x_t)(x_t - \delta) - \Delta(x_{t+\tau})(x_{t+\tau} - \delta). \tag{6.12}$$

6.3.2 Numerical Scheme

The situation is a little more complicated than that for a European option, since the delta is no longer zero or one. Therefore, the magnitude of the dynamic delta is the key to the profit and loss computation. By offering a continuity correction to the Merton (1973) option price formula for continuous-time case, an approximation price for the discretely monitored call option proposed by Hörfelt (2003) (an extension to Broadie, Glasserman and Kou, 1997, which was introduced in Section 3.4.2) is presented:

$$V(x_t,t) = x_t e^{-f(T-t)}[G(c,d_1) - G(b,d_1)] - Ke^{-r(T-t)}[G(c,d_2) - G(b,d_2)],$$

where

$$\begin{aligned}
G(z,y) &= N(z-y) - e^{2y(c+\beta/\sqrt{M})}N(z - 2(c+\beta/\sqrt{M}) - y),\\
d_1 &= \frac{(r-f+\sigma_x^2/2)\sqrt{T-t}}{\sigma_x},\\
d_2 &= \frac{(r-f-\sigma_x^2/2)\sqrt{T-t}}{\sigma_x},\\
b &= \frac{\ln(K/x_t)}{\sigma_x\sqrt{T-t}},\\
c &= \frac{\ln(B/x_t)}{\sigma_x\sqrt{T-t}},\\
\beta &\approx 0.5826.
\end{aligned}$$

$N(\cdot)$ is defined in Section 1.4, and β is defined in Section 3.4.2. The delta at asset price x_t can be obtained either by differentiating analytically, or taking the above and differentiating numerically. The algorithm for the discretely monitored barrier options is similar as the one for European barrier option, but slightly more sophisticated:

1. Divide the time period $[0, T]$ into M steps. Set $\Delta t = T/M$, and therefore $t_i = i\Delta t$, for $i = 1, 2, \ldots, M$.

2. Sample N independent paths of exchange rate $x_k(t_i)$, for $k = 1, 2, \ldots, N$ using Euler discretisation.

   Set $x_k(0) = x_0$, the current value of $x(t)$. $x_k(t_{i+1})$ is determined by:

$$x_k(t_{i+1}) = x_k(t_i) \exp\left(\left[r - f - \frac{1}{2}\sigma_x^2\right]\Delta t + \sigma_x \epsilon_i \sqrt{\Delta t}\right), \qquad (6.13)$$

   where r, f, σ_x are constant, and $\epsilon_i \sim N(0, 1)$ is a sequence of independent standard normal variables at the ith time step.

3. Again, the fixing rate can be obtained by applying the following dynamics:

$$F_t = x_t + \phi, \quad \text{where} \quad \phi \sim N(\mu, \sigma^2) \qquad (6.14)$$

4. Obtain the mis-hedge quantities using the given profit and loss functions at time t_i, as a consequence, the contract life of the option may be shorter in some circumstances. Since additional mis-hedge opportunities exist when $t_i < T$, Equations (6.11) and (6.12) are employed. At expiry date (i.e. $t_i = T$), the equations (6.4), (6.5) and (6.6) are used to compute the profit and loss.

5. Average over the mis-hedge quantity for N sample paths.

6.3.3 Analysis of Error

The mis-hedge losses are shown in Figure 6.3. The losses due to the announcement delay are plotted against the corresponding barriers. A similar shape to that in

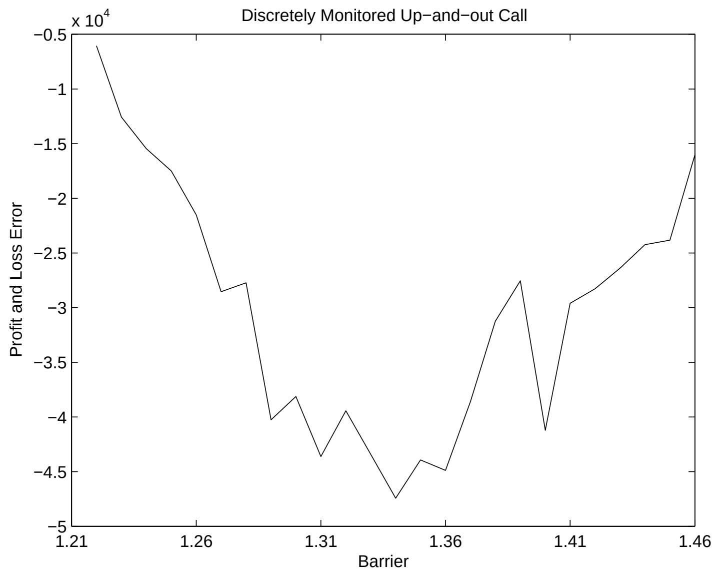


Figure 6.3: Mis-hedging error with one million units of domestic currency.

Figure 6.2 is evident. The errors are relatively small when the barriers are very close to the spot rate or very far from the spot rate. However, comparing to Figure 6.2, the mis-hedging loss of the discretely monitored barrier option is increased approximately by a factor of 10^4. The errors are more than 0.6% of one unit domestic currency. The largest errors, about 4.7% of one unit domestic currency, occur when the barrier is 1.34 which is at a reasonable distance from the spot rate.

Also to demonstrate the accuracy of the Monte Carlo algorithm for this type of problem, a comparison with the analytic solution to the option price obtained by Hörfelt (2003) is shown in Figure 6.4. The parameter set chosen for the comparison is the same as in Table 6.1. Using the Monte Carlo method to approximate the discretely monitored up-and-out call, the errors are plotted averaged on one million sample paths (i.e. $N = 1,000,000$).

Figure 6.4 shows clearly that the further the barrier is from the spot rate, the larger the numerical error. When the barrier is close to the spot rate, the opportunity for the option to knock out is higher. Therefore, the option price is lower and the errors caused by Monte Carlo simulation may be insignificant. Conversely, when

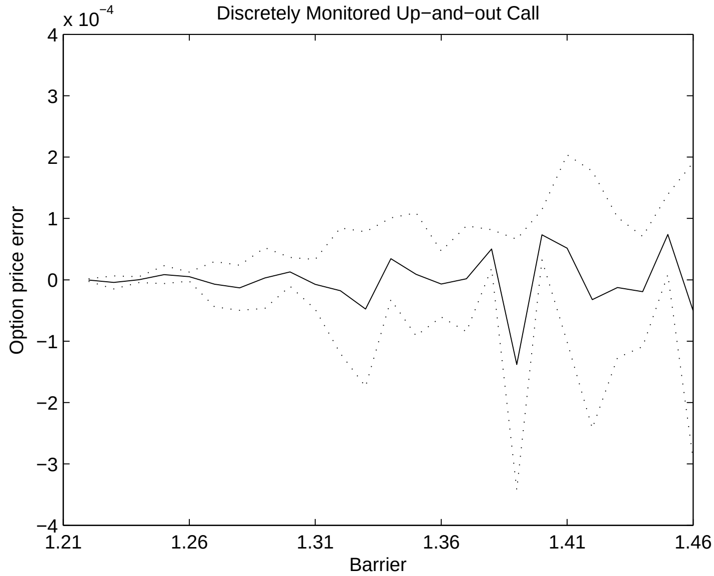


Figure 6.4: The difference between analytic solution and Monte Carlo approximation with one million units of domestic currency
The solid line is the error, the dashed lines are 95% two-sided confidence intervals.

the barrier is far from the spot rate, the option is more likely to be influenced by the discrete time monitoring, and so the option is similar to a European option. Comparing the numerical solution error to the mis-hedging errors, the numerical error is insignificant.

6.4 Summary

Barrier options are actively traded in financial markets. The feature of the discrete time for monitoring the barrier draws interest from both market professionals and academic researchers.

Wystup and Becker's (2005) paper presented a realistic problem in the currency option market. However, their result appears to be in error. In their paper, they claimed that even if the contract is with one million units of notional domestic currency (U.S. dollar) the error for a discretely monitored barrier option is a mere 14 U.S. dollars at maximum. However, since it is known that even though the mis-hedging occurs only once in the entire contract life, the seller has to re-adjust his/her hedging position by selling and buying a certain amount of the underlying asset. Therefore, he/she has to pay the transaction cost at the bid/ask spread at least once, which is two basis points of the exchange rate. This certainly costs more than 14 U.S. dollars (two basis points of one million Euros is 200 Euros, i.e. approximately 260 U.S. dollars).

Given the evidence by Easton et al. (2004), with the same parameters observed barrier option prices are greater than theoretical barrier option prices. Also the observed barrier option prices are significantly higher than the observed European option prices. These findings suggest one of the factors could be that barrier options have very high Greeks near the barrier level. The resultant instability of the Greeks may cause option sellers to require a premium, not included in standard pricing models, to compensate the hedging difficulties.

This chapter has delivered more accurate results regarding the impact of the mis-hedging, so that the importance of an improved pricing model is shown. Also, it somewhat inspires this thesis for development of a new class of barrier option, quantile Parisian-style options, which will be introduced in the next chapter.

Chapter 7

A New Class of Options: Quantile Parisian and ParAsian Options

This result is too beautiful to be false; it is more important to have beauty in one's equations than to have them fit experiment.

—— Paul Dirac (1902–1984)

The evolution of the Physicist's Picture of Nature Scientific American

7.1 Introduction

As stated in the previous chapter, the discontinuity at the barrier inherent in standard knock-out (or knock-in) options creates a number of problems for both buyers and sellers alike. Buyers might lose their entire investment due to a sudden price jump through the barrier. For sellers, hedging is difficult, since the delta of a standard barrier option is discontinuous around the barrier, and its gamma is therefore infinite (a delta function) at the barrier (see Wilmott, 2000a). More practically, the impact of the jump might tempt both buyers and sellers of such options to manipulate the market over a very short term.

The discontinuity at the barrier causes a problem. A large trading volume can drive the price of the underlying asset across the barrier. Therefore, there is a potential opportunity for sellers to manipulate the option validity. Sesit and Jereski (1995) mentioned a particular event in the foreign exchange market in 1995:

> Knock-out options can roil even the mammoth foreign-exchange markets for brief periods. David Hale, chief economist at Kemper Financial in Chicago, notes that in the past year, many Japanese exporters moved to hedge against a falling dollar with currency options. Confident at the time that the dollar would fall no further than 95 yen, the exporters chose options that would knock out at that level. Once the dollar plunged through 95 yen early last month, "they lost everything," he says. The dollar then tumbled as the Japanese companies, "which had lost their hedges, scrambled to cover" their large exposures by dumping dollars.
>
> Making matters more volatile, dealers say that pitched battles often erupt around knock-out barriers, with traders hollering across the trading floor of looming billion-dollar transactions. In three or four minutes it is all over. But in that time every trade gets sucked into the vortex.

As mentioned in Section 3.4.2, a new class of options was introduced in 1994, Parisian options. It avoids the disadvantage of standard barrier options, since Parisian options are not knocked out (or knocked in) immediately after the underlying asset price hits the barrier, but after the **consecutive** time that the price spends beyond the barrier reaches the predetermined time in the option contract (see Pechtl, 1995). Also as mentioned in Section 3.4.2, an extension to Parisian options, ParAsian options were introduced by Chesney, Jeanblanc-Picque and Yor (1997), which are not knocked out (or knocked in) unless the **total** time that the underlying asset price spends beyond the barrier reaches the predetermined time in the option contract. As a consequence, the jump in price of the underlying asset does not affect the validity of the option, and it is more difficult for both buyers and sellers to manipulate the price over a relatively long term. As a further advantage, the time-indicated features have much less extreme Greeks, in particular, the discontinuity of the delta is smoothed and the variation of the gamma is no longer so extreme.

There are several combinations of features that fully define Parisian and ParAsian options. There are eight types of options corresponding to the combinations of up/down, in/out and put/call and Parisian/ParAsian. By introducing an extra criterion, we term "quantile barrier" to be discussed next, it leads to 32 instruments in total. Since it is straightforward to vary the implementations using the Monte Carlo method, this chapter will only consider down-and-out calls for both Parisian and ParAsian options, and we focus on just four cases, namely Parisian, ParAsian, quantile Parisian, and quantile ParAsian options. The new term "quantile" will be formally defined and used in Section 7.3 to interpret the second criterion of validity for Parisian and ParAsian options.

A practical point is that the discrete monitoring effect for barrier options is very significant. Often barrier option contracts specify that the barrier is only to be monitored at the market close every day. Estimating the magnitude of the effect of this is crucial. As described in Taleb (1996), continuously monitored barrier options can tempt either the option buyer or seller to influence the underlying asset price. Discretely monitored options suffer from similar problems. Broadie, Glasserman, and Kou (1999) addressed the relation between discrete-time and continuous-time prices from three perspectives. First, nearly all closed-form solutions available for pricing barrier options are based on continuous-time modelling, but most traded options are based on discrete-time modelling (see Section 3.4.2), which implies use of a continuous formula to approximate the price of a discrete option is a practical issue. Second, if the option is based on continuous-time modelling of the underlying asset price, a discrete numerical method is often required for valuation, for example, importantly American options. Improving the quality of the numerical method involves analysing how a discrete-time, discrete-valued process approximates a continuous-time, continuous-valued process (this problem has been addressed in chapter 6). Finally, numerical methods are necessary for precise evaluation of discrete-time option prices. These are themselves based on a discretisation of time, but typically much finer time intervals than that specified in the terms of an option. Thus, numerically pricing a discrete option involves two discrete time increments – the intervals between underlying asset prices that determine the option payoff and the time step in the numerical method (i.e. the main issue in this chapter).

This chapter aims to illustrate high-dimensional path-dependent option pricing models using the Monte Carlo method, and then applies the framework to the currency option model. The pricing model algorithms are demonstrated applying the Black-Scholes (1973) framework for simplicity. Section 7.2 establishes the model for both Parisian and ParAsian options and illustrates the difference between the two options. Section 7.3 introduces a new feature, the quantile barrier, into the model. In Section 7.4, the framework of currency options with stochastic interest rates and stochastic volatilities is progressively introduced. Conclusions for this chapter are drawn in Section 7.5.

7.2 Parisian and ParAsian Options

7.2.1 Model Setup

This section focuses on the extensions of Parisian and ParAsian features. The crucial aspect of Parisian and ParAsian features is that they are path-dependent with the payoff dependent on the time that the underlying asset price spends beyond the barrier. The barrier time for a down-and-out (or down-and-in) option is the time below the barrier and for an up-and-out (or up-and-in) option it is the time above the barrier. Parisian and ParAsian options are very similar, the only distinction being the definition of the barrier time.

First, a formal definition of the new variable, τ, barrier time for a Parisian down-and-out option is introduced. The barrier time is defined as the length of time the underlying asset has been below the barrier in the current excursion, namely

$$\tau := t - \sup[u \leq t | S_u \geq B]. \tag{7.1}$$

This definition represents the difference between the current time t and the last time

S_u is below the barrier (see also Haber, Schonbucher and Wilmott, 1999; Yu, 2005). Consequently, τ is zero if S_t is above the barrier and is reset to zero if S_u moves from below to above the barrier.

The dynamics of τ for a down (down-and-in or down-and-out) barrier is given as follows:

$$d\tau_t = \begin{cases} dt & \text{if } S_t < B \\ -\tau_{t-}\delta_t & \text{if } S_t = B, \\ 0 & \text{if } S_t > B \end{cases} \tag{7.2}$$

where τ_{t-} is the left limit of τ, δ_t is the Dirac measure at t[1], and B is the barrier level which is predetermined in the option contract. This new variable, τ, can be viewed as a clock that is triggered as soon as the underlying asset price S_t hits the barrier B and starts counting, but is reset as soon as S_t returns above B. The knock-out is not activated until the clock has reached its limit, i.e. $\tau \geq \bar{T}$, where $\bar{T}$ is the occupation time, also known as the "window", which is also predetermined in the option contract. A typical sample path of the underlying asset for a down-and-out Parisian option is shown in Figure 7.1. In this case, the option is not knocked out unless $\tau_1 \geq \bar{T}$ given $\tau_1 > \tau_2$.

The barrier time for a ParAsian option, τ, follows the dynamics:

$$d\tau_t = \begin{cases} dt & \text{if } S_t < B \\ \bar{T} - \tau_{t-}\delta_t & \text{if } S_t = B, \\ 0 & \text{if } S_t > B \end{cases} \tag{7.3}$$

where τ_{t-} is the left limit of τ, δ_t is the Dirac measure at t, again B is the barrier. The difference with Parisian options is that τ is triggered as soon as the underlying asset price S_t hits the barrier B and starts counting, and is stopped as soon as S_t returns above B, but is not reset to zero. Again, the knock-out is not activated until

[1] Dirac measure is a probability measure that for any set A and any $x \in A$, define for any $A' \subset A$ as follows,

$$\delta_{A'} = \begin{cases} 0 & \text{if } x \notin A' \\ 1 & \text{if } x \in A'. \end{cases}$$

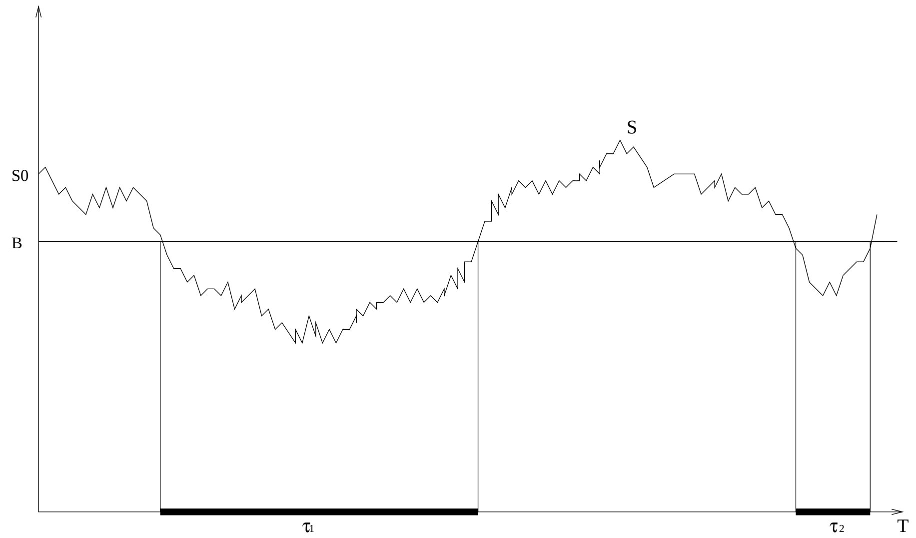


Figure 7.1: Characteristics for a down-and-out Parisian and ParAsian option

$\tau \geq \bar{T}$, where $\bar{T}$ is the window. The typical sample path of the underlying asset in Figure 7.1 can also be applied to a down-and-out ParAsian option. In this case, the option is not knocked out unless $\tau_1 + \tau_2 \geq \bar{T}$.

7.2.2 Assumptions

To illustrate the exotic feature, a simple Black-Scholes (1973) framework is applied. Assume the underlying asset is governed by a geometric Brownian motion, under a risk-neutral measure:

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t, \tag{7.4}$$

where dW_t denotes the increment of a standard Brownian motion, r is the risk-free interest rate and σ the volatility, and both r and σ are held constant.

The value function for a Parisian down-and-out call at time t, denoted as $V_{P-do}(S_t, t)$, that satisfies the equation under the risk-neutral measure $\mathbb{Q}$ is

$$V_{P-do}(S_t, t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} \max\{S_T - K, 0\} \mathbf{1}_{(\tau<\bar{T})}\right], \tag{7.5}$$

where K is the strike price, r is the risk-free interest rate, T is the expiry date, and $\mathbf{1}_{(\cdot)}$ is the indicator function (defined in Section 2.2.2) with respect to the barrier time τ. A ParAsian down-and-out option, $V_{PA-do}(S_t, t)$ has precisely the same form, but a slightly different interpretation, in particular, the barrier time τ in the formula for a ParAsian option is different from that of a Parisian option (as discussed above).

7.2.3 Numerical Scheme

The nature of Parisian and ParAsian options invites thought on numerical implementation which will now be addressed. Again, the numerical method will be described step by step and the notations will change to $S(t) = S_t(\omega)$, following Section 4.3:

1. Divide the time period $[0, T]$ into M steps. Set $\Delta t = T/M$, thus $t_i = i\Delta t$, for $i = 0, 1, 2, \ldots, M$. The window (i.e. occupation time) can be set as $\bar{T} = m\Delta t$.

2. Sample N independent paths of the underlying asset price $S_k(t_i)$, for $k = 1, 2, \ldots, N$ using Euler discretisation. $S_k(t_{i+1})$ can be determined by:
$$S_k(t_{i+1}) = S_k(t_i) \exp\left(\left[r - \frac{1}{2}\sigma^2\right] \Delta t + \sigma\sqrt{\Delta t}\epsilon_i\right), \tag{7.6}$$
where r and σ arise from Equation (7.4), and $\epsilon_i \sim N(0, 1)$ is a sequence of independent standard normal variables.

3. Use a timer "BT" as an indicator of the barrier time and "CT" for the length of the barrier time. At each time step, for both Parisian and ParAsian options
$$BT(i) = \begin{cases} 1 & \text{if } S_k(t_i) \leq B \\ 0 & \text{if } S_k(t_i) > B\ . \end{cases} \tag{7.7}$$
Note that the length of barrier times are determined differently. Namely,
$$CT(i) = \prod_{j=i-m}^{i} BT(j) \quad \text{for a Parisian option,} \tag{7.8}$$
$$CT(i) = \sum_{j=1}^{i} BT(j) \quad \text{for a ParAsian option.} \tag{7.9}$$
For the kth sample path, set $V_k(S(T)) = 0$ if $CT(i) = 1$ for the Parisian option and set $V_k(S(T)) = 0$ if $CT(i) = m + 1$ for the ParAsian option, where $V_k(S(T))$ is the payoff function at time T, namely
$$V_k(S(T)) = \max\{S(T) - K, 0\}. \tag{7.10}$$

4. Discount $V_k(S(T))$, $k = 1, 2 \ldots, N$ back to time $t = 0$ with the risk-free interest rate, namely

$$V_k(S(0)) = e^{-rT} V_k(S(T)), \quad k = 1, 2 \ldots, N. \tag{7.11}$$

5. Average over the result of $V(S(0))$,

$$\bar{V}(S(0)) = \frac{1}{N} \sum_{k=1}^{N} V_k(S(0)). \tag{7.12}$$

The Monte Carlo methods for both Parisian and ParAsian options are very straightforward. Two option prices can be computed simultaneously in the same programme.

7.2.4 Results and Analysis

The parameters used in this section are consistent with Broadie, Glasserman, and Kou (1997).

Table 7.1: Parisian and ParAsian options valuation parameters

Expiry date	T	0.2 year
Initial value of underlying asset	$S(0)$	100
Risk-free interest rate	r	0.1
Volatility	σ	0.3
Strike price	K	100
Barrier	B	85, 90, 95
Time step	M	50
Sample paths	N	1,000,000

To observe the influence of the barrier level, Figures 7.2 and 7.3 illustrate Parisian and ParAsian options prices respectively. The option price changes with respect to the different barrier levels.

The results are plotted against different window lengths (i.e. the number of time steps required to knock out). Note that the closer the barrier is to the spot price of the underlying asset (i.e. $S(0)$), the easier it is for the price to hit the barrier, and therefore the option is much more sensitive to the window. In both Figures 7.2 and 7.3, when the window length is shorter than 30 time steps (0.12 years), the option

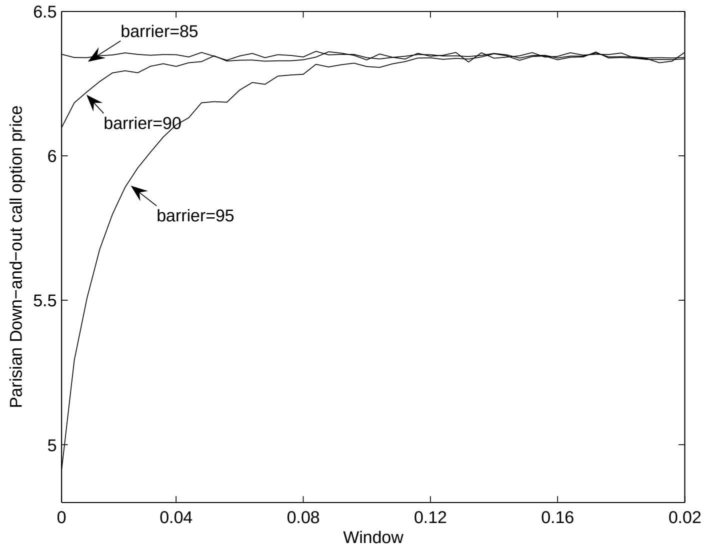


Figure 7.2: Parisian down-and-out call option value with barrier = 85, 90, 95; window length = 0.02 year

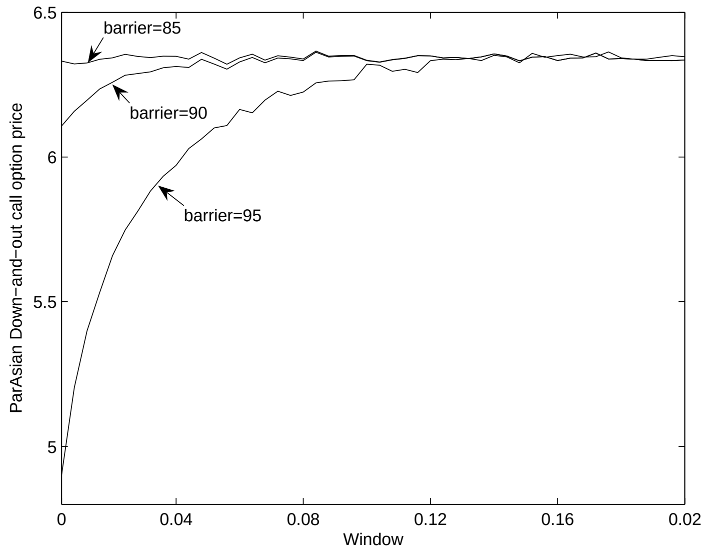


Figure 7.3: ParAsian down-and-out call option value with barrier = 85, 90, 95; window length = 0.02 year

prices increase steeply when the barrier level is 95, but when the barrier level is 85, the price only fluctuates around the value 6.33 for both Parisian and ParAsian options. This highlights the fact that the window is more influential on options with a barrier close to the spot value, which exactly confirms the importance of the window to barrier option valuation.

In order to investigate further the difference between the two types of options, Figures 7.4 and 7.5 are shown the comparison of the option values when the barrier level is set at 90 and 95 respectively, and the window length is from 0 to 0.02 years. (as mentioned above, when the barrier is set at 85, the value for both options are very close).

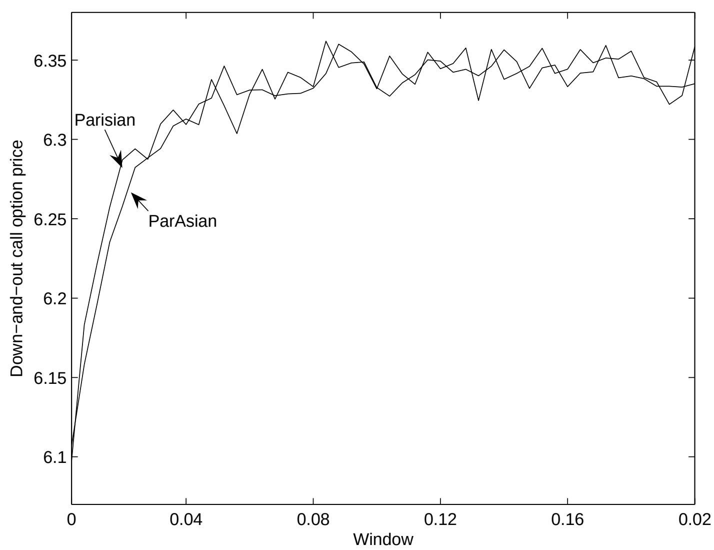


Figure 7.4: Comparison of Parisian and ParAsian down-and-out call option value with barrier $B = 90$; window length = 0.02 year

Again, both figures 7.4 and 7.5 show option values against different window lengths. At the initial point of both figures, the window length is 0, which means the options will be knocked out as soon as the underlying asset price hits the barrier. In this case, the options are numerically equivalent to standard barrier options. When the window length is equal to the expiry date T, both Parisian and ParAsian options are not knocked out until the expiry date, and therefore they are equivalent to vanilla

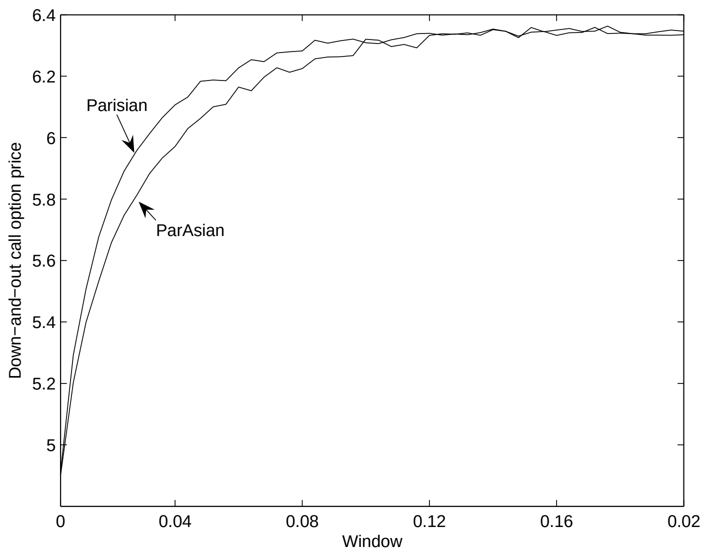


Figure 7.5: Comparison of Parisian and ParAsian down-and-out call option value with barrier $B = 95$; window length = 0.02 year

European options. As a consequence, the two curves have the same starting and end points. In general, however, Parisian options are more expensive than corresponding ParAsian option. This is because the probability for a Parisian option to knock out is lower than that of a ParAsian option. This is not especially obvious in the case where the barrier is at 90, since both options have the spot price (i.e. $S(0) = 100$) not so close to the barrier and it is not easy for either option to reach the barrier. However, the difference is very clear when the barrier is 95, since at this level the underlying asset prices can often fluctuate across the barrier.

In the next section, the new feature will be introduced to Parisian and ParAsian options.

7.3 Quantile Barriers — A New Feature

As a broadly used class of options, Parisian and ParAsian features are common in convertible bonds or for derivatives which has a relatively illiquid underlying asset. From the perspective of risk management, the Parisian and ParAsian features are used for pricing default risk (and liquidation risk) under bankruptcy procedures (see Chen and Suchanecki, 2006). In order to interpret the realistic bankruptcy procedure under the Chapter 11 provision[2], these risks are represented by either Parisian or ParAsian features. Also the feature is applicable to the valuation of bank deposit guarantees, bank deposit insurance, convertible bonds. Moreover, real option problems can be adapted into the Parisian or ParAsian framework.

It is worth pointing out that not only the occupation time (window) of the underlying asset price beyond the barrier is very important, but also the distance beyond the barrier. In risk management, the creditors will certainly not have the same tolerance when the firm asset value is one unit of currency below zero compared to one million units of currency below zero. Creditors prefer to default when the firm value is deep in debt rather than just crossing the barrier. This trigger can be regarded as a second criterion, a "quantile barrier", so the option can be knocked out when either a time or distance barrier is breached[3].

This idea, inspired by risk management, can also be used for real options — the analogy between investment decision and barrier financial derivatives further extends in this case. The feature allows the representation of a lag between an investment decision and its implementation. An investment project can be built either with a delay at a certain cost, or immediately for a higher cost (similarly see Gauthier, 2002). Overall, this is a new feature in the option markets which has the potential for a new generation of exotic options.

The term "quantile" is used in this thesis to address the integrated quantity barrier feature, since an existing class of option, "α–quantile option", has the similar characteristics. Ballotta and Kyprianou (2001) stated on page 138: "the α–quantile option's payoff at maturity is defined by the order statistics of the underlying asset

[2]The criteria to liquidate a company after the onset of financial distress vary substantially across countries and regimes. Chapter 11 of the U.S. Bankruptcy Code enables the prolonged operation of companies in financial distress but the U.K. insolvency law is characterised by the strict enforcement of creditors' contractual rights, including the liquidation rights of secured creditors. For more in-depth introduction on bankruptcy procedure, see Galai, Raviv and Wiener (2005).

[3]In fact, if the option can only be knocked out when both time and quantity barrier are breached, this type of option can be viewed as another modification, which leads another 32 instruments.

price; in particular, this order statistic or, better, the α–percentile point of the stock price for $0 < \alpha < 1$, can be thought of as the level at which the price stays below for α percent of the time during the option's contract period." The formal definition of α–quantile is also given by Higham (2004): for a given a strictly positive density function $f(x)$ and a given $0 < \alpha < 1$ we define the αth quantile of f as $z(\alpha)$, where

$$\int_{-\infty}^{z(\alpha)} f(x)dx = \alpha. \tag{7.13}$$

It might not be the most precise nomenclature for the feature that we address in this thesis, but it gives an idea of this second barrier for Parisian and ParAsian options.

As mentioned in Section 7.1, the table below shows the 16 different types of Parisian options – permutations are the same for ParAsian options; which gives all together 32 types of options

Table 7.2: Permutations of the different types of Parisian option

	Non-Quantile								Quantile							
Up	√	√	√	√					√	√	√	√				
Down					√	√	√	√					√	√	√	√
In	√	√			√	√			√	√			√	√		
Out			√	√			√	√			√	√			√	√
Call	√		√		√		√		√		√		√		√	
Put		√		√		√		√		√		√		√		√

7.3.1 Definition

The quantile barrier is defined formally as

$$\tau' = \inf \left[0 \leq t \leq T \mid \int_0^t (B - S_u) \mathbf{1}_{\{S_u \leq B\}} \, du = Q \right], \tag{7.14}$$

where S_t is the underlying asset price and B is the barrier. Here Q is a new term, the quantile barrier. This definition introduces τ' as the first time that the total quantity of S_t below the barrier B exceeds the predetermined level Q before the expiry date T.

The value function of a quantile Parisian down-and-out option, $V_{QP-do}(S_t, t)$ satisfies the following equation under the risk-neutral measure $\mathbb{Q}$:

$$V_{QP-do}(S_t, t) = \mathbb{E}^{\mathbb{Q}} \left[e^{-r(T-t)} \max\{S_T - K, 0\} \mathbf{1}_{(\rho < \bar{T})} \right], \tag{7.15}$$

where

$$\rho = \tau \wedge \tau', \tag{7.16}$$

where τ is defined in Equation (7.1), K is the strike price, r is the risk-free interest rate, T is the expiry date, and $\mathbf{1}_{(\cdot)}$ is the indicator function defined in Section 2.2.2. Again the ParAsian down-and-out option, $V_{QPA-do}(S_t, t)$ has the same form, but a slightly different interpretation, in particular, the barrier time τ in the formula for a ParAsian option is different from that of a Parisian option (as described in the previous section).

7.3.2 Numerical Scheme

The numerical algorithm is similar to that introduced in Section 7.2.3. The main difference is that an extra indicator for the quantile barrier is required at Step 3:

3. Denote "QT" as the quantile barrier indicator, for both Parisian and ParAsian options,
$$QT(i) = \sum_{j=1}^{i} \left(\Delta t \min\{S_k(t_j) - B, 0\} \right), \tag{7.17}$$
where S is the underlying asset price, B is the barrier. For the kth sample path, set $V_k(T) = 0$ if $|QT(i)| > Q$.

7.3.3 Numerical Results

In Section 7.2.4, some properties of Parisian and ParAsian options have been shown through numerical calculation. This section will focus on the new properties that the feature of quantile barrier brings to the option price.

Since it is a novel feature for options, an estimate of the range that the underlying asset price can possibly reach, is very important. To obtain this band, two extreme cases are considered, the upside price and the downside price of the underlying asset. It can be approximated by assuming that the ϵ in Equation (7.6) is 1 for the case of upside price and -1 for the downside case (this implies that the underlying asset prices cannot exceed those bounds the during the contract life). Note that the upside price is not relevant to the quantile barrier of a down-and-out option, therefore, only the downside price need be considered. Employing the parameters in Table 7.1, the possible maximum quantile barrier of the underlying asset is shown in Table 7.3 with respect to different levels of discretisation.

Table 7.3: Quantile level of a down-and-out option

Barrier	$\Delta t = 0.004$	$\Delta t = 0.0008$	$\Delta t = 0.0004$	$\Delta t = 0.00008$	$\Delta t = 0.00004$
90	5.03	9.69	11.66	15.03	15.90
95	5.94	10.65	12.64	16.02	16.89

The discretisation is one of the main issues in numerical implementation of option pricing, which is confirmed in Table 7.3. When the time step is small, the potential downside quantity over the option contract life is larger than with large time steps.

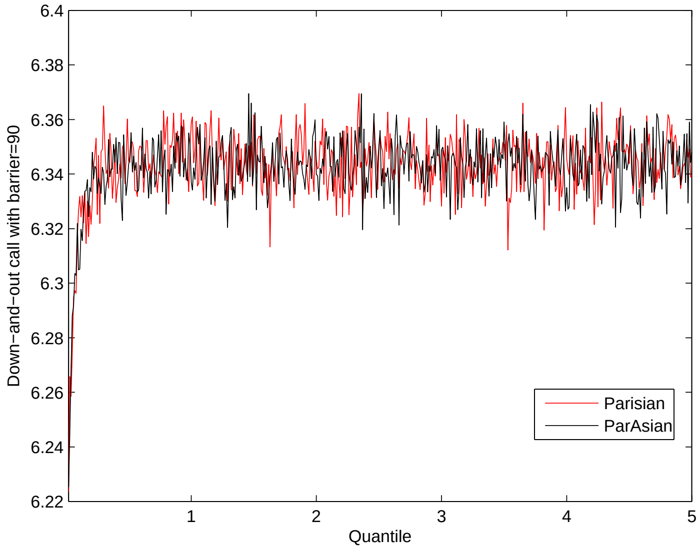
In the following section, the largest time step level is used ($\Delta t = 0.004$) without other specification, because this is equivalent to barriers monitored every working day. Based on the same parameters, some figures are shown as comparison of the quantile Parisian and quantile ParAsian down-and-out call option prices. All the figures in this subsection show that the option prices change significantly according to different quantile barrier levels. The figures are plotted with quantile barrier from 0 to 5 with increments of 0.01, and the cases of two barrier levels (90 and 95) are shown.

Figures 7.6 and 7.7 show the influence of the quantile barrier on the option price with barriers set at 90 and 95 respectively. Here, the window is set to be 0.2 years (i.e. 50 time steps), which implies the Parisian and ParAsian are just vanilla European options, the time barrier is not one of the knock-out criteria for the options, therefore the option prices in both figures should have the same values without considering the errors result from the fluctuation caused by the Monte Carlo simulation and discretisation. Figure 7.6(a) shows a larger range of quantity barriers, from 0 to 5 with increments of 0.01, and Figure 7.6(b) plots a region of Figure 7.6(a), which is from 0 to 1 with a finer increment 0.0005. Figure 7.7(b) plots a region of Figure 7.7(a), from 0 to 1 with a finer increment 0.0005, which have the same trend as that seen in Figure 7.7(a). Again, it is very clear from Figures 7.6(b) and 7.7(b) that when the spot rate $S(0)$ is close to the barrier, the option value will be highly sensitive to the quantile barrier level. Also, for 0.2-year options the underlying asset do not vary significantly when the quantile barrier is more than one unit of currency. This implies that the possibility of the underlying asset cross below the barrier more than one unit of domestic currency is very low.

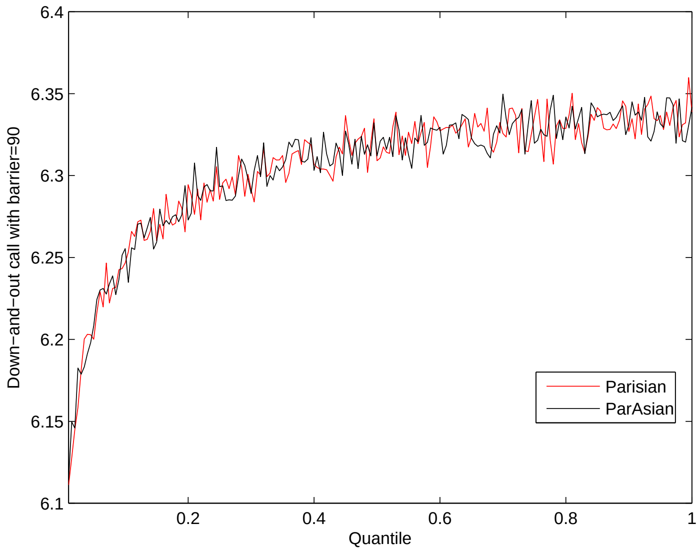
7.3.4 Quantile Parisian and Quantile ParAsian

To avoid terminology confusion, the following options we consider are the options with both time barrier and quantile barrier features. Figures 7.8 and 7.9 show the quantile Parisian and quantile ParAsian option prices with respect to different quantile barrier and different barrier levels. Option prices for the knock-outs are triggered by either the window or the integrated area excess of the barrier. In order to smooth out the fluctuation caused by the discretisation, smaller time increments are chosen. The Figures 7.8 and 7.9 are with increments of 0.0005. The window is chosen to be 0.02 year (5 days) for both of the cases, in line with an empirical paper by Easton and Gerlach (2006)[4]. In the Figures 7.8 and 7.9, when the quantile barrier is larger than 0.4, the two option values plateau. In the both cases, the option values increase dramatically when the integrated area excess of the barrier is relatively small. And again, the barrier level affects the option values too. As seen in Figure 7.8, the possibility of the underlying asset moving cross the barrier $B = 90$ is lower than that of $B = 95$, consequently the option prices differences between Parisian and ParAsian are more obvious in Figure 7.9 than in Figure is 7.8.

[4]Easton and Gerlach (2006) investigated the discretely-monitored barrier currency-option in the Australian option market.

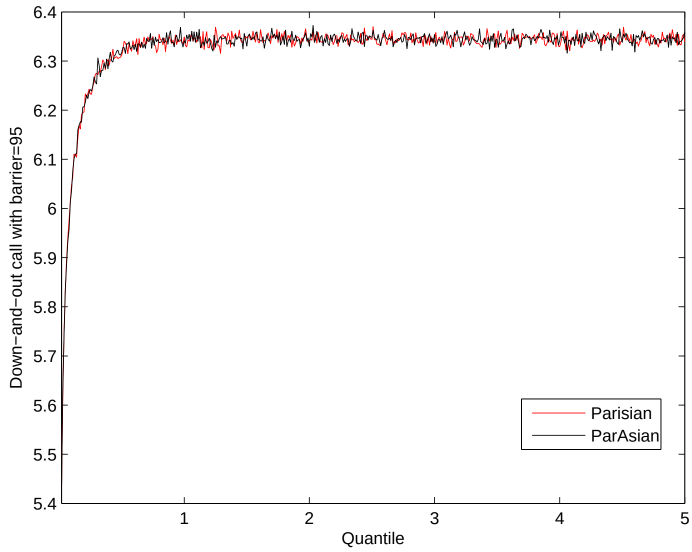


(a) Quantile barrier with increment 0.01 (from 0.01 to 5)

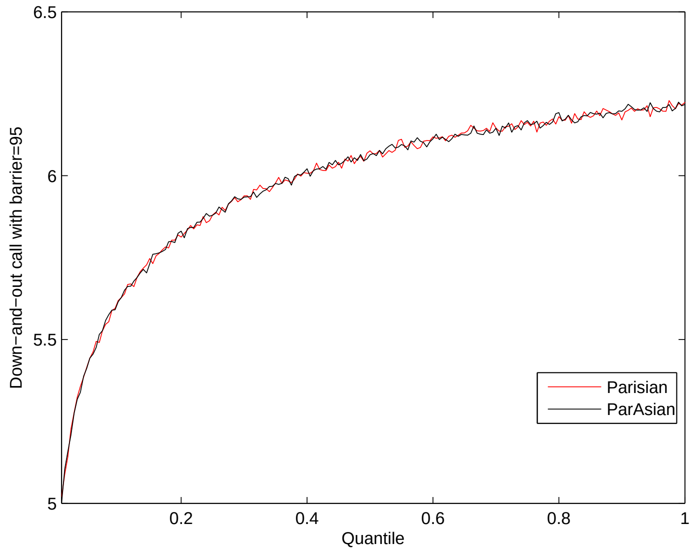


(b) Quantile barrier with increment 0.005 (zoom in from 0.005 to 1)

Figure 7.6: Comparison of Quantile European down-and-out call option value with barrier= 90



(a) Quantile barrier with increment 0.01 (from 0.01 to 5)



(b) Quantile barrier with increment 0.005 (zoom in from 0.005 to 1)

Figure 7.7: Comparison of Quantile European down-and-out call option value with barrier= 95

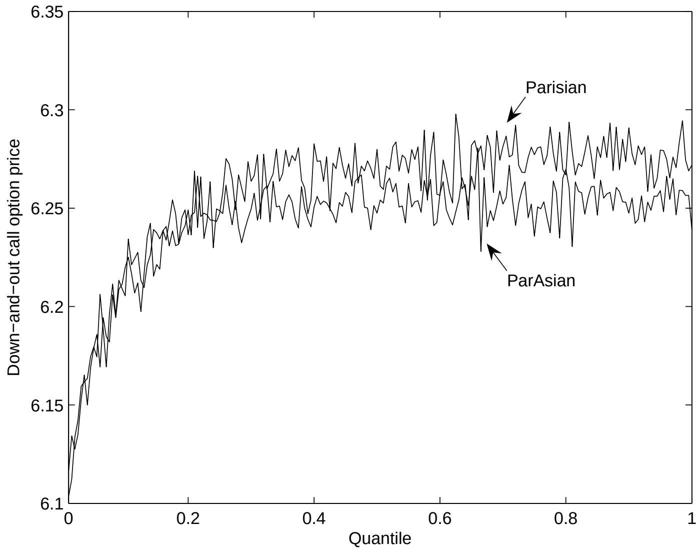


Figure 7.8: Comparison of quantile Parisian and quantile ParAsian down-and-out call option value with barrier = 90, window = 0.02 year

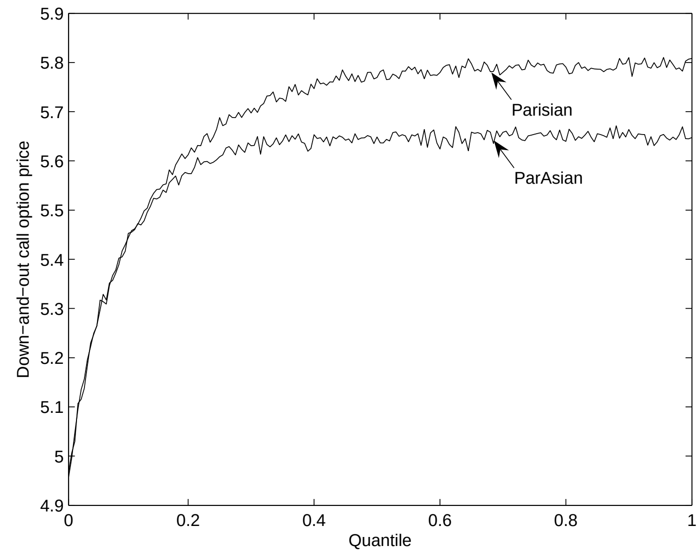


Figure 7.9: Comparison of quantile Parisian and quantile ParAsian down-and-out call option value with barrier = 95, window = 0.02 year

However, the values still fluctuate due to the discretisation of the numerical scheme and also the affect of Monte Carlo simulation.

The two figures 7.10 and 7.11 show the quantile Parisian option price differences with different window lengths. Figure 7.10 shows the values with a window length of 0.02 year (5 days) and of 0.1 year (25 days) with the barrier 90, and Figure 7.11 with barrier 95. The values are plotted against the quantile barrier level from 0 to 1. Options with a shorter window length appear to be considerably less expensive

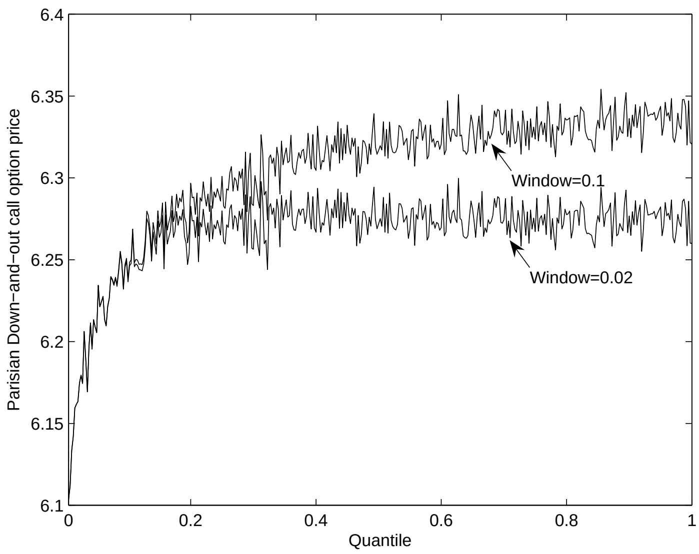


Figure 7.10: Comparison of quantile Parisian down-and-out call option value with different window length = 0.02, 0.1 years, with barrier = 90

than those with longer window lengths for both barrier levels.

Figures 7.12 and 7.13 show the quantile ParAsian option price differences with different window lengths. Figure 7.12 shows the values with a window length of 0.02 year (5 days) and of 0.1 year (25 days) with barrier 90, and Figure 7.13 with barrier 95. Again, the values are plotted for different values of the quantile barrier level from 0 to 1. Again, those with shorter window length appear to be considerably less expensive than those with longer window lengths. The reason for this is quite straightforward, it is because with shorter window length the options are easier to be knocked out than that of longer window length.

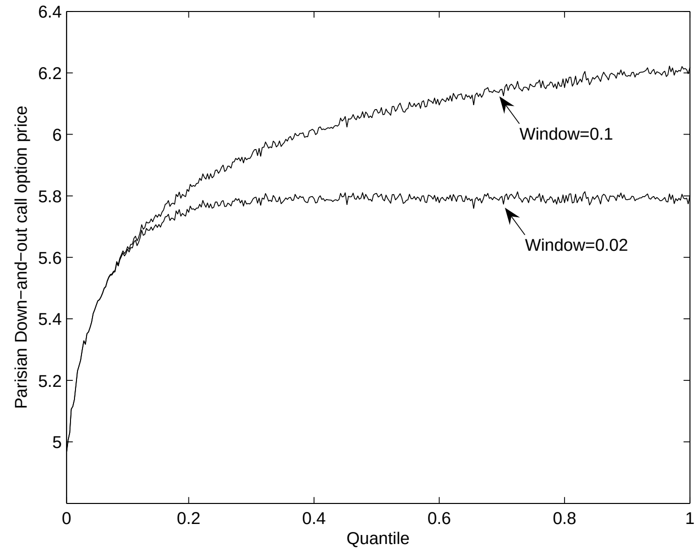


Figure 7.11: Comparison of quantile Parisian down-and-out call option value with different window length = 0.02, 0.1 years, with barrier = 95

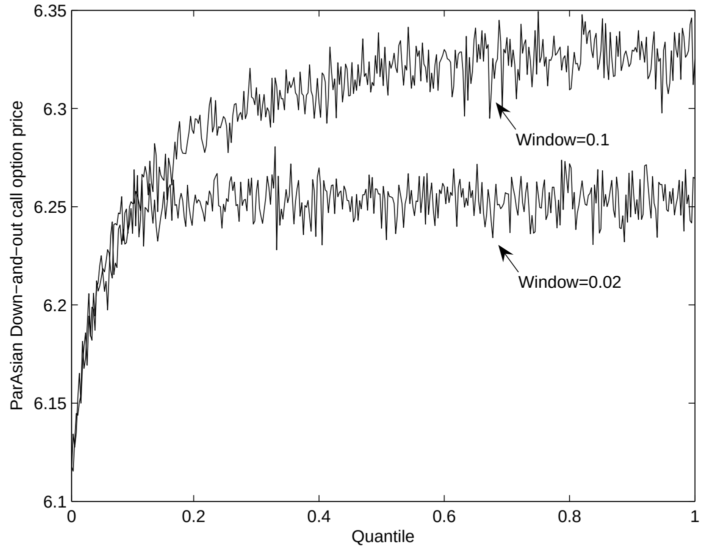


Figure 7.12: Comparison of quantile ParAsian down-and-out call option value with barrier = 90, window = 0.02, 0.1 years

7.4 Application to Currency Options

To be consistent with the overall theme of this thesis, this section extends the basic quantile Parisian and quantile ParAsian feature to the currency option pricing problem. The model with stochastic interest rates and stochastic volatilities is considered, using the same assumptions as used in Section 5.4.1. The parameters employed are those from Chapter 5, whenever applicable.

Before applying the new features, a benchmark must be obtained. A plain down-and-out European option value is 8.003, provided by a 10-million simulation, we now progressively add new features into the model. Figures 7.14 and 7.15 show the down-and-out currency options with a barrier set at 10 percent lower than the spot price of the underlying asset and 5 percent lower than the spot price respectively. To avoid fluctuating result from Monte Carlo random number generator, the same set of random numbers are used for both quantile Parisian and ParAsian options. Two sets of results are shown for window length 0.06 years. In Figures 7.14(a) and 7.15(a) are with quantile barrier from 0 to 1 with time increment 0.01, whereas Figures 7.14(b) and 7.15(b) are with quantile barrier from 0 to 10 with a slightly coarse time increment 0.1. ParAsian options return lower values than corresponding Parisian options. However the time-discretisation of numerical implementation makes the value curve fluctuate, even with the same set of random numbers for the sample paths. From Figures 7.14 and 7.15, a conclusion can be drawn, namely the quantile Parisian is always more expensive than the corresponding ParAsian option and less expensive than the corresponding European option (the exceptions shown in the results are due to sampling error). Again, one needs to bear in mind that the discretisation of the numerical technique has a substantial impact on this class of options. As mentioned in Section 3.4.2, all the numerical methods are affected by it.

Table 7.4: Quantile Parisian and ParAsian currency-option valuation parameters

Barrier	B	$0.90x(0)$ and $0.95x(0)$
window	$\bar{T}$	0.06 year
Expiry date	T	1 year
Time step	M	250
Sample paths	N	100,000
Initial value of exchange rate	$x(0)$	0.0079101
Strike price	K	0.0079101
Initial value of domestic interest rate	$r(0)$	0.0585
Initial value of foreign interest rate	$f(0)$	0.00704
Mean-reversion rate of domestic interest	κ_r	0.3334
Mean-reversion rate of foreign interest	κ_f	0.1279
Long term growth rate of domestic interest	θ_r	0.0585
Long term growth rate of foreign interest	θ_f	0.00704
Mean-reversion rate of exchange rate volatility	κ_1	6.17
Long term growth rate of exchange rate volatility	θ_1	0.0097
Volatility of the exchange rate volatility	σ_1	0.21
Correlation between $x(t)$ and $\upsilon_1(t)$	ρ_{x1}	-0.13
Initial value of exchange rate volatility	$\upsilon_1(0)$	0.1236
Initial value of domestic volatility	$\upsilon_2(0)$	0.0161
Initial value of foreign volatility	$\upsilon_3(0)$	0.0571
Mean-reversion rate of domestic volatility	κ_2	2.5
Mean-reversion rate of foreign volatility	κ_3	2.0
Long term growth rate of domestic volatility	θ_2	0.01
Long term growth rate of foreign volatility	θ_3	0.02
Volatility of domestic interest rate volatility	σ_2	0.1
Volatility of foreign interest rate volatility	σ_3	0.1
Correlation between $x(t)$ and $\upsilon_2(t)$	ρ_{x2}	-0.008
Correlation between $x(t)$ and $\upsilon_3(t)$	ρ_{x3}	0.007
Correlation between $r(t)$ and $\upsilon_1(t)$	ρ_{1r}	-0.008
Correlation between $\upsilon_1(t)$ and $\upsilon_2(t)$	ρ_{12}	-0.006
Correlation between $f(t)$ and $\upsilon_1(t)$	ρ_{1f}	0.008
Correlation between $\upsilon_1(t)$ and $\upsilon_3(t)$	ρ_{13}	0.005
Correlation between $r(t)$ and $\upsilon_2(t)$	ρ_{r2}	0.02
Correlation between $r(t)$ and $\upsilon_3(t)$	ρ_{r3}	0.003
Correlation between $f(t)$ and $\upsilon_2(t)$	ρ_{2f}	0.008
Correlation between $\upsilon_2(t)$ and $\upsilon_3(t)$	ρ_{23}	0.002
Correlation between $f(t)$ and $\upsilon_3(t)$	ρ_{f3}	0.01

7.5 Summary

This chapter explores a new class of options, quantile Parisian and quantile ParAsian options. This class of options offers a large range of flexibility to deal with more realistic credit risk products. In credit derivatives literature it is highly recommended that Parisian and ParAsian options are used for pricing defaultable bonds in structural models.

To capture the characteristics of defaultable bonds (also applicable to real options — the analogy between investment decision), a new feature has been introduced, which allows the bond to default more easily because of the tolerance of creditors, which also allows the representation of a lag between an investment decision and its implementation. An investment project can be built either with a delay at a certain cost, or immediately for a higher cost. Overall, this new feature has the potential for a new generation of exotic options.

Finally, the application of these ideas to currency options has been illustrated, and is quite easy to apply the new feature to the currency option framework. Two important cases are considered, as shown in Section 7.4, and these can be extended to any other combinations in Table 7.2. For currency option applications, the numerical implementation has been shown to have a noticeable impact on option prices. Given the limited accuracy of Monte Carlo simulations, there is much scope for further investigation into option valuations of this type.

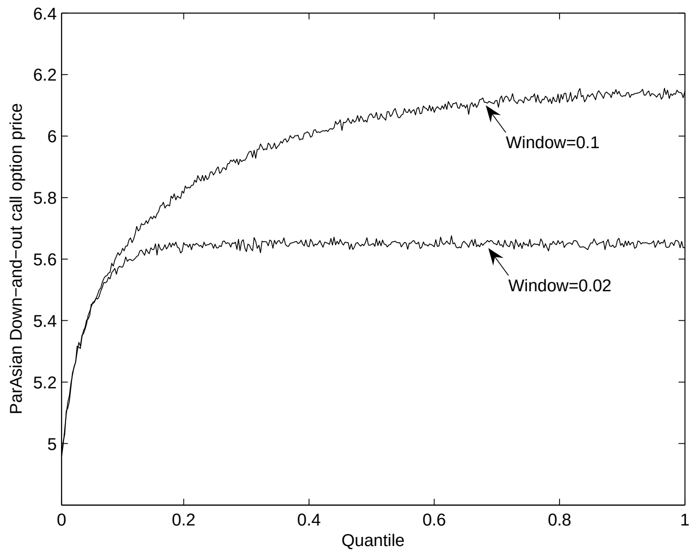
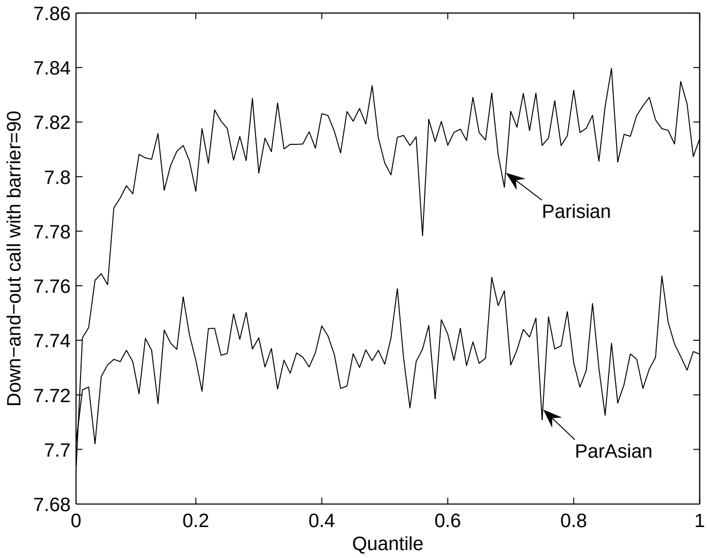
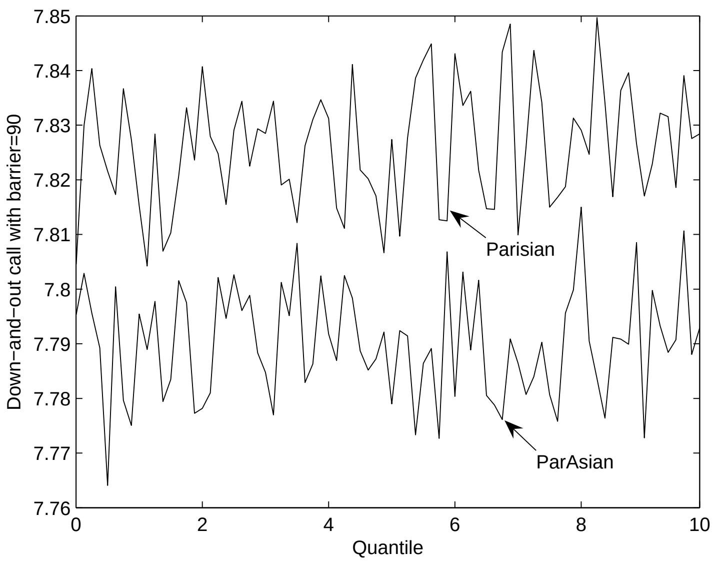


Figure 7.13: Comparison of quantile ParAsian down-and-out call option value with barrier = 95, window = 0.02, 0.1 years

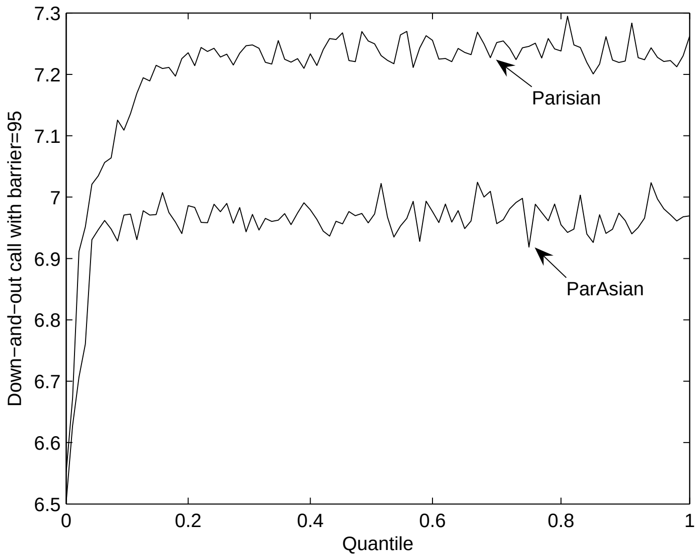


(a) Quantile barrier with increment 0.01 (from 0 to 1)

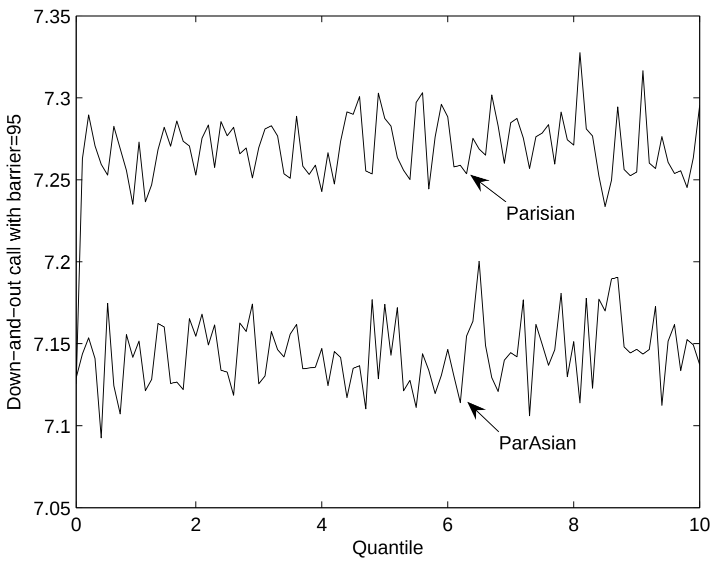


(b) Quantile barrier with increment 0.1 (zoom out from 0 to 10)

Figure 7.14: Comparison of down-and-out quantile call with barrier $B = 0.9x(0)$, window $\bar{T} = 0.06$ year



(a) Quantile barrier with increment 0.01 (from 0 to 1)



(b) Quantile barrier with increment 0.1 (zoom out from 0 to 10)

Figure 7.15: Comparison of down-and-out quantile call with barrier $B = 0.95x(0)$, window $\bar{T} = 0.06$ year

Chapter 8

Conclusions

The whole of science is nothing more than a refinement of everyday thinking.

—— Albert Einstein (1879-1955)

Throughout this thesis, pricing of high-dimensional options is addressed using Monte Carlo simulation approach which is the only well established approach to date for these mathematically challenging problems.

8.1 Summaries

The research presented in this thesis addresses the development of four important types of currency option models: American options, discretely-monitored barrier options, quantile Parisians and quantile ParAsian options. By setting the underlying asset, exchange rate process, into a totally stochastic environment, the model becomes complex but more realistic. The Monte Carlo method, modified for speed and handling early exercise has allowed modelling with stochastic interest rates and volatilities with correlation.

The goal of Chapter 5 had been to develop a more realistic but practical model for American currency-options. First, the new method has been applied to the Amin and Bodurtha (1995) framework as a benchmark. In order to develop a new model in the totally stochastic environment, an extended model employs the CIR model, which provides flexibility for further extension to a more sophisticated framework, including stochastic interest rates and stochastic volatilities. One of the most useful findings in this chapter is that the correlations between the various stochastic factors do not significantly impact the valuation. This has been addressed in the final part of Chapter 5. It has allowed further development of an easily implementable model covering the fullest range of parameters yet available including American-style early exercise. This chapter has developed not only a practical model for currency-option evaluation, but also a promising multi-dimensional option pricing technique which offers better accuracy than the Longstaff and Schwartz (2001). This has been proved using Amin and Bodurtha (1995) framework settings as well as the parameters shown in Chapter 5). Furthermore, the numerical technique has the potential to be applied in many other areas, such as credit spread option pricing, quanto, basket options, or sophisticated high-dimensional term structure derivatives.

Chapter 6 used a realistic example to address the mis-hedge problem of plain barrier options. For the case of discretely monitored barrier options, the options are checked only once a day, and the delay of the announcement for the reference rate will put the option seller at risk. By referring to Wystup and Becker (2005), corrected results are obtained. A huge potential loss can happen to the seller (for the case in Chapter 6, the loss is up to 5%). Also the issue of discontinuity shows the importance of the birth of a new class of option which is addressed in Chapter 7.

In Chapter 7, quantile Parisian-style options, a new class of options offers a very large range of flexibility to deal with more realistic credit risk products and also provides more sensible features for investment decision in real options. Provided the soft trigger feature (the option is not knocked out/in at the moment the underlying asset reaches the barrier, but takes time to make the knock out/in) of standard barrier options, Parisian and ParAsian options are highly recommended for pricing defaultable bonds in structural models. To capture the characteristics of defaultable bonds, also applicable to real options, a new feature has been introduced that allows the bond to default more easily because of the tolerance of creditors (they prefer the company to default when it is deep in debt instead of just reaching a barrier). It also allows the representation of a lag between an investment decision and its implementation (an investment project can be built either with a delay at a certain cost, or immediately for a higher cost). Overall, this new feature has potential for a new generation of exotic options. Finally, the application to currency options has been illustrated, and it is quite easy to apply the new feature to the currency option framework. Two important cases are considered, and clear characteristics can be observed.

8.2 Future Research

Future research regarding to this thesis can be addressed in the following three aspects.

In practice, it is important not only to evaluate the option price accurately and efficiently, but also to evaluate the hedging parameters. Calculation of the Greeks using Monte Carlo methods would be an interesting area to explore. As mentioned in Chapter 4, extreme Greeks result from the discontinuity of numerical methods, in particular delta and gamma. Overcoming this disadvantage of Monte Carlo methods in this respect would be useful in the future work.

In Chapter 7, for quantile Parisian and ParAsian options — down-and-out call options are considered. The options are knocked out when either the time barrier or the quantile barrier are breached. Other modifications shown in Table 7.2 can also be considered. Furthermore, the options introduced in Chapter 7 are of European type. Early exercise feature may be added in, giving the option more flexibility and therefore attracting a wider market of buyers. As the Parisian-style options suffer the same problems as that of barrier options. For discretely-monitored Parisian and ParAsian options, the hedging difficulty is one of the priority issues in practice. The potential hedging errors can be substantial, consequently affecting option prices, which is worthy of attention in the future.

The modelling and numerical methods research carried out in this thesis is underpinned by data drawn from the empirical work of others, that work itself based on simpler theoretical models. The reliability of such an analysis is a condition of the currency option modelling, especially in a highly stochastic environment. However, it would be satisfying to see later empirical work in other research groups employ modelling of the type developed here.

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Appendix A

Matlab "randn" Test

For Monte Carlo methods, "well behaved" random numbers are crucial. Therefore, a high quality of random number generator is essential for the programs. Kahaner, Moler and Nash (1989) defined five criteria to judge the generator:

- Quality: pass all the statistical tests and have a very long period.
- Efficiency: quick and less storage consuming.
- Repeatability: minimal change in the starting condition required.
- Portability: work universally.
- Simplicity: easy to implement.

The simulation in this thesis was implemented in Matlab 7.1.0.183(R14) programming environment. In this appendix we prove that the built in function "`randn`" in Matlab is good enough for the programs in this thesis. We generate 50 million normal random variables and test the mean and the variance of those variables. The tests are repeated 10 times and shown individually. The random number generator "`randn`" is proven to provide sufficient normal distribution behaviour. According to Matlab software official documentation[1], the period of the generator is around $1.37 * 10^{449}$, whereas the longest period required by the simulations in this thesis is around $1.92 * 10^{11}$. For more detailed test, see Higham (2004).

[1] The full document can be found at: `http://www.mathworks.com/moler/random.pdf`.

Table A.1: Random number generator testing

Mean	Variance
0.0000	1.0003
-0.0001	1
0.0002	1
0.0000	1.0002
-0.0004	1.0004
0.0001	0.9998
0.0002	0.9999
-0.0002	0.9998
0.0001	0.9999
0.0000	0.9999

Appendix B

Exchange Rate Process

Under the risk-neutral measure, the exchange rate process is initially assumed to have a general form as:

$$\frac{dx_t}{x_t} = \mu_x dt + \sigma_x dW_x, \tag{B.1}$$

where μ_x is the drift of the exchange rate, a function with respect to two short rates of interest r_d and r_f, σ_x is the volatility of the exchange rate, and dW_x is the increments of a standard Brownian motion. Moreover, we assume

$$\frac{dB_d}{B_d} = r_d dt, \tag{B.2}$$

$$\frac{dB_f}{B_f} = r_f dt, \tag{B.3}$$

$$\frac{B_f^*}{B_f} = x_t, \tag{B.4}$$

where

B_d = the domestic zero-coupon bond,

B_f = the foreign zero-coupon bond,

B_f^* = the foreign zero-coupon bond in domestic currency.

Following Björk (2004), the model is based in the domestic economy, therefore B_d is chosen to be the numeraire. Using Itô's lemma, we have

$$dB_f^* = B_f^*(\mu + r_f)dt + B_f^* \sigma_x dW_x. \tag{B.5}$$

Equation (B.5) is a risk-free process in the domestic economy. As B_f is risk-free in the foreign economy:

$$dB_f^* = B_f^* r_d dt + B_f^* \sigma_x dW_x. \tag{B.6}$$

Since it is assumed that there is no arbitrage, the same product should have the same price no matter which economy it is issued from. Equations (B.5) and (B.6) are identical if $\mu_x = r_d - r_f$, the exchange rate process dx_t is then given by

$$\frac{dx_t}{x_t} = (r_d - r_f)dt + \sigma_x dW_x, \tag{B.7}$$

corresponding to Equation 5.1.