

**ACTUARIAL APPROACH TO OPTION PRICING****HANS U. GERBER**

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**Abstract**

Over sixty years ago, the Swedish actuary F. Esscher suggested that the Edgeworth approximation (a refinement of the normal approximation) yields better results, if it is applied to a modification of the original distribution of aggregate claims. In this paper, this Esscher transform is defined more generally as a change of measure for a certain class of stochastic processes that model stock prices. According to the Fundamental Theorem of Asset Pricing, security prices are calculated as expected discounted values with respect to a (or the) equivalent martingale measure. If the measure is unique, it is obtained by the method of Esscher transforms; if not, the risk-neutral Esscher measure provides a unique and transparent answer, which can be justified if there is a representative investor maximizing his expected utility. The price is unique whenever a self-financing replicating portfolio can be constructed. This is the case in the classical geometric Brownian motion model, but also in the geometric shifted Poisson process model. The latter is at the same time simpler (in view of its sample paths) and richer (the former can be retrieved as a limit). The Esscher method can be extended to pricing the derivative securities of multiple (possibly) dividend-paying stocks. We show that, in the case of a multidimensional geometric Brownian motion, the price of a European option does not depend on the interest rate, provided that the payoff is a function only of the stock prices and is homogeneous in one of them. Moreover, with the aid of Esscher transforms, a change of the numéraire can be discussed in a concise way. Finally, it is shown how certain American type options on two stocks (for example, the perpetual Margrabe option) can be priced. Applying the optional sampling theorem to certain martingales (which resemble the exponential martingale in ruin theory), we obtain several explicit results without having to deal with differential equations.

## APPROCHE ACTUARIELLE A L'EVALUATION DU PRIX D'UNE OPTION

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### Résumé

Il y a soixante ans, l'actuaire suédois F. Esscher a remarqué que l'approximation d'Edgeworth donne de meilleurs résultats après une modification préalable de la distribution du montant total des sinistres. Dans cet article, cette transformée d'Esscher est définie de façon plus générale comme étant un changement de mesure de probabilité d'une certaine classe de processus stochastiques servant à modéliser le prix d'une action. Selon un théorème fondamental, le prix d'un titre est égal à l'espérance mathématique de la valeur escomptée des paiements, espérance calculée par rapport à la (ou une) mesure de martingale équivalente. Si cette mesure est unique, elle est obtenue par la méthode de la transformée d'Esscher; sinon, la mesure d'Esscher neutre vis-à-vis du risque fournit une réponse unique et transparente, qui peut se justifier par la présence d'un investisseur représentatif maximisant son utilité espérée. Le prix est unique dès qu'il est possible de construire un portefeuille autofinancé équivalent au titre. Cette situation se rencontre dans le cas classique du modèle Brownien géométrique, mais aussi dans le cas du modèle de Poisson avec translation. Ce dernier est à la fois plus simple (notamment ses trajectoires) et plus riche, puisqu'il contient le modèle classique comme cas limite. La méthode d'Esscher peut être généralisée pour calculer le prix de produits dérivés sur plusieurs actions versant des dividendes. Il est montré que, dans le cas du mouvement Brownien géométrique multidimensionnel, le prix d'une option Européenne ne dépend pas du taux d'intérêt, à condition que le paiement soit une fonction des prix des actions qui est homogène par rapport à un des prix. De plus, à l'aide de la transformée d'Esscher, l'analyse d'un changement de numéraire peut se faire d'une façon concise. Finalement, il est montré comment on peut obtenir le prix de certaines options Américaines sur deux actions (par exemple, l'option de Margrabe perpétuelle) en appliquant le théorème d'arrêt optionnel à certaines martingales. Ceci permet d'obtenir plusieurs résultats explicites sans avoir à utiliser d'équations différentielles.

## 1. Introduction

Actuaries measure, model and manage risks. Risk associated with the investment function is a major uncertainty faced by many insurance companies. Actuaries should have knowledge of the asset side of the balance sheet of an insurance company and how it relates to the liability side. Such knowledge includes the operation of financial markets, the instruments available to the insurance companies, the options imbedded in these instruments, and the methods of pricing such options and derivative securities.

In this paper we study the pricing of financial options and contingent claims. We show that two time-honored concepts in actuarial science – the Esscher transform and the adjustment coefficient – are efficient tools for pricing many options and derivative securities if the logarithms of the prices of the primary securities are certain stochastic processes with stationary and independent increments. An Esscher transform of such a security-price process induces an equivalent probability measure on the process. The Esscher parameter or parameter vector is determined so that the discounted price of each primary security is a martingale under the new probability measure. A derivative security is valued as the expectation, with respect to this equivalent martingale measure, of the discounted payoffs.

We also study the pricing of American options on two stocks without expiration date and with payoff functions which are homogeneous with

respect to the two stock prices. An example of such options is the perpetual Margrabe option, whose payoff is the amount by which one stock outperforms the other. The method is based on the construction of two martingales with respect to the equivalent martingale measure, and applying the optional sampling theorem. The martingale construction is similar to the determination of the adjustment coefficient in collective risk theory. This approach does not involve differential equations and hence is quite different from the traditional approach in financial literature.

## 2. The Esscher Transform of a Random Variable

Let  $Y$  be a given random variable and  $h$  a real number for which the expectation

$$E[e^{hY}]$$

exists. The positive random variable

$$(2.1) \quad \frac{e^{hY}}{E[e^{hY}]}$$

can be used (as the Radon-Nikodym derivative) to define a new probability measure, which is *equivalent* to the old measure in the sense that they have the same null sets (sets of measure zero). For a measurable function  $\psi$ , the expectation of the random variable  $\psi(Y)$  with respect to the new measure is

$$(2.2) \quad E[\psi(Y); h] = \frac{E[\psi(Y)e^{hY}]}{E[e^{hY}]}$$

We call this new measure the *Esscher measure* of parameter  $h$ . The corresponding distribution is usually called the *Esscher transform* in the actuarial literature ([Es32], [Je91]). In some statistical literature, the term *exponential tilting* is used to describe this change of measure.

The method of Esscher transforms was developed to approximate the aggregate claim amount distribution around a point of interest,  $y_0$ , by applying an analytic approximation (the first few terms of the Edgeworth series) to the transformed distribution with the parameter  $h = h_0$  chosen such that the new mean is equal to  $y_0$ . Let

$$(2.3) \quad c(h) = \ln(E[e^{hY}])$$

be the cumulant-generating function. Then

$$(2.4) \quad c'(h) = \frac{E[Ye^{hY}]}{E[e^{hY}]} = E[Y; h]$$

and

$$(2.5) \quad c''(h) = \frac{E[Y^2e^{hY}]}{E[e^{hY}]} - \left( \frac{E[Ye^{hY}]}{E[e^{hY}]} \right)^2 = \text{Var}[Y; h].$$

Since  $\text{Var}[Y; h] > 0$  for a nondegenerate random variable  $Y$ , the function  $c'(h)$  is strictly increasing; thus the number  $h_0$  for which

$$y_0 = c'(h_0) = E[Y; h_0]$$

is unique. In using the Esscher transform to calculate a stop-loss premium, the parameter  $h_0$  is usually chosen such that the mean of the transformed distribution is the retention limit.

### 3. Discrete-Time Stock-Price Models

A purpose of this paper is to show that the concept of Esscher measures is an effective tool for pricing stock options and other derivative securities. We need to extend the change of measure for a single random variable to that for a stochastic process. In this section we consider the simpler case of discrete-time stochastic processes.

For  $j = 0, 1, 2, \dots$ , let  $S(j)$  denote the price of a stock at time  $j$ . Assume that there is a sequence of independent (but not necessarily identically distributed) random variables  $\{Y_k\}$  such that

$$(3.1) \quad S(j) = S(0) \exp(Y_1 + Y_2 + \dots + Y_j), \quad j = 1, 2, 3, \dots$$

Assume that the moment generating function for each  $Y_i$  exists, and write

$$(3.2) \quad M_{Y_i}(h) = E[e^{hY_i}].$$

For a sequence of real numbers  $\{h_k\}$ , define

$$(3.3) \quad Z_j = \exp\left(\sum_{k \leq j} h_k Y_k\right) / \prod_{k \leq j} M_{Y_k}(h_k).$$

Then  $\{Z_j\}$  is a positive martingale which can be used to define a change of measure for the stock-price process. For a positive integer  $m$ , let  $\psi(m)$  be a random variable that is a function  $Y_1, \dots, Y_m$ ,

$$(3.4) \quad \psi(m) = \psi(Y_1, \dots, Y_m).$$

The expected value of  $\psi(m)$ , with respect to the new measure, is

$$(3.5) \quad E[\psi(m) Z_m].$$

In (3.5) the random variable  $Z_m$  can be replaced by  $Z_j$ ,  $j \geq m$ , because of the martingale property.

We assume that the risk-free interest rate is constant through time and the stock pays no dividends. Let  $\delta$  denote the risk-free force of interest. The *risk-neutral Esscher measure* is the measure, defined by the sequence of numbers  $\{h_k^*\}$ , with respect to which

$$(3.6) \quad \{e^{-\delta j} S(j); j = 0, 1, 2, \dots\}$$

is a martingale. This leads to

$$(3.7) \quad e^\delta = M_{Y_k}(1 + h_k^*)/M_{Y_k}(h_k^*), \quad k = 1, 2, 3, \dots$$

As we pointed out at the end of the last section, the numbers  $\{h_k^*\}$  are unique.

Suppose that each  $Y_k$  is a Bernoulli random variable, i.e., it takes on two distinct values,  $a_k$  and  $b_k$ , only. Then there is only one risk-neutral measure, given by

$$(3.8) \quad \Pr^*(Y_k = a_k) = \frac{e^\delta - e^{a_k}}{e^{b_k} - e^{a_k}}$$

and

$$(3.9) \quad \Pr^*(Y_k = b_k) = \frac{e^\delta - e^{b_k}}{e^{a_k} - e^{b_k}}$$

( $\delta$  is between  $a_k$  and  $b_k$  for each  $k$ .)

If we assume that the random variables  $\{Y_k\}$  are identically distributed in addition to being independent, then all  $h_k^*$  are the same

number. This points to an approach to extend the change of measure to certain continuous-time models, as we shall see in Section 5. On the other hand, the risk-neutral Esscher measure can also be defined for dependent random variables  $\{Y_k\}$ . In this more general situation, each  $h_k$  is a function of  $Y_1, Y_2, \dots, Y_{k-1}$  and thus a random variable itself.

#### 4. Fundamental Theorem of Asset Pricing

In this paper we assume that the market is frictionless and trading is continuous. There are no taxes, no transaction costs, and no restriction on borrowing or short sales. All securities are perfectly divisible. It is now understood that, in such a security market model, the absence of arbitrage is “essentially” equivalent to the existence of a *risk-neutral measure* or an *equivalent martingale measure*, with respect to which the *price* of a random payment is the expected discounted value. Dybvig and Ross [DR87] call this result the *Fundamental Theorem of Asset Pricing*. In general, there may be more than one equivalent martingale measure. The merit of the risk-neutral Esscher measure is that it provides a general, transparent and unambiguous solution.



That the condition of no arbitrage is intimately related to the existence of an equivalent martingale measure was first pointed out in Harrison and Kreps [HK79] and Harrison and Pliska [HP81]. Their results are rooted in the idea of risk-neutral valuation of Cox and Ross [CR76]. In a finite discrete-time model, the absence of arbitrage opportunities is equivalent to the existence of an equivalent martingale measure ([CMW90], [Sc92a]). In a more general setting the characterization is delicate, and we have to replace the term “equivalent to” by “essentially equivalent to”. It is beyond the scope of the present paper to discuss the details. Some recent papers are [Ba91], [BP91], [CH89], [De92], [DS94a], [DS94b], [Mü89], [Sc92b], [Sc94], [Scw92] and [St93].

We note that the idea of changing the probability measure to obtain a consistent positive linear pricing rule had appeared in the actuarial literature in the context of equilibrium reinsurance markets ([Bo60], [Bo90], [Bü80], [Bü84], [CM94], [Ge87], [Li86], [So91]).

## 5. Continuous-Time Stock-Price Models

In the rest of the paper we consider continuous-time stock-price models. For  $t \geq 0$ , let  $S(t)$  denote the price at time  $t$  of a non-dividend-

paying stock. We assume that there is a stochastic process  $\{X(t)\}$  with independent and stationary increments such that

$$(5.1) \quad S(t) = S(0) e^{X(t)}, \quad t \geq 0.$$

For a theoretical “justification” that stock prices should be modeled with such processes, see Samuelson [Sa65] or Parkinson [Pa77]. (Some authors call  $\{X(t)\}$  a *Lévy process*.)

We assume that the moment generating function of  $X(t)$ ,

$$M(h, t) = E[e^{hX(t)}],$$

exists and that

$$(5.2) \quad M(h, t) = M(h, 1)^t.$$

The process

$$(5.3) \quad \{e^{hX(t)} M(h, 1)^{-t}\}$$

is a positive martingale and can be used to define a change of probability measure, i.e., it can be used to define the Radon-Nikodym derivative  $\frac{dQ}{dP}$ , where  $P$  is the original probability measure and  $Q$  is the Esscher measure of parameter  $h$ . The *risk-neutral Esscher measure* is the Esscher measure of parameter  $h = h^*$  such that the process

$$(5.4) \quad \{e^{-\delta t} S(t)\}$$

is a martingale.

The condition

$$E[e^{-\delta t} S(t); h^*] = e^{-\delta 0} S(0) = S(0)$$

yields

$$\begin{aligned} e^{\delta t} &= E[e^{(1+h^*)X(t)} M(h^*, 1)^{-t}] \\ &= [M(1 + h^*, 1)/M(h^*, 1)]^t, \end{aligned}$$

or

$$(5.5) \quad e^{\delta} = M(1 + h^*, 1)/M(h^*, 1),$$

which is analogous to (3.7) with  $\{Y_k\}$  being identically distributed.

The parameter  $h^*$  is unique. There may be many other equivalent martingale measures.

Because, for  $t \geq 0$ ,

$$(5.6) \quad e^{hX(t)} M(h, 1)^{-t} = \frac{e^{hX(t)}}{E[e^{hX(t)}]} = \frac{S(t)^h}{E[S(t)^h]},$$

we have the following: Let  $g$  be a measurable function and  $h, k$  and  $t$  be real numbers,  $t \geq 0$ ; then

$$\begin{aligned} (5.7) \quad E[S(t)^k g(S(t)); h] &= E[S(t)^k g(S(t)) e^{hX(t)} M(h, 1)^{-t}] \\ &= \frac{E[S(t)^{h+k} g(S(t))]}{E[S(t)^h]}, \\ &= \frac{E[S(t)^{h+k}] E[S(t)^{h+k} g(S(t))]}{E[S(t)^h] E[S(t)^{h+k}]} \\ &= E[S(t)^k; h] E[g(S(t)); h + k]. \end{aligned}$$

This *factorization formula* simplifies many calculations, and is a main reason why the method of Esscher measures is an efficient device for valuing certain derivative securities. For example, applying (5.7) with  $k = 1$ ,  $h = h^*$  and  $g(x) = I(x > K)$  [where  $I(A)$  denotes the indicator random variable of an event  $A$ ], we obtain

$$\begin{aligned}
 (5.8) \quad E[S(\tau) I(S(\tau) > K); h^*] &= E[S(\tau); h^*] E[I(S(\tau) > K); h^* + 1] \\
 &= E[S(\tau); h^*] \Pr[S(\tau) > K; h^* + 1] \\
 &= S(0)e^{\delta\tau} \Pr[S(\tau) > K; h^* + 1].
 \end{aligned}$$

The last equality holds because (5.4) is a martingale with respect to the risk-neutral Esscher measure. Thus we have a pricing formula for a European call option on a non-dividend-paying stock,

$$\begin{aligned}
 (5.9) \quad E[e^{-\delta\tau} (S(\tau) - K)_+; h^*] \\
 &= E[e^{-\delta\tau} (S(\tau) - K) I(S(\tau) > K); h^*] \\
 &= e^{-\delta\tau} \{E[S(\tau) I(S(\tau) > K); h^*] - KE[I(S(\tau) > K); h^*]\} \\
 &= S(0)\Pr[S(\tau) > K; h^* + 1] - Ke^{-\delta\tau}\Pr[S(\tau) > K; h^*].
 \end{aligned}$$

For  $\{X(t)\}$  being a Wiener process, (5.9) is the celebrated Black-Scholes formula [BS73]; see also (9.20) below.

## 6. Representative Investor with Power Utility Function

When there is more than one equivalent martingale measure, why should the option price be the expectation, with respect to the risk-

neutral Esscher measure, of the discounted payoff? This particular choice may be justified within a utility function framework. Consider a simple economy with only a stock and a risk-free bond and their derivative securities. There is a representative investor who owns  $m$  shares of the stock and bases his decisions on a risk-averse utility function  $u(x)$ . Consider a derivative security that provides a payment of  $\pi(\tau)$  at time  $\tau$ ,  $\tau > 0$ ;  $\pi(\tau)$  is a function of the stock price process until time  $\tau$ . What is the investor's *price* for the derivative security, such that it is optimal for him not to buy or sell any fraction or multiple of it? Let  $V(0)$  denote this price. Then, mathematically, this is the condition that the function

$$(6.1) \quad \phi(\eta) = E[u(mS(\tau) + \eta[\pi(\tau) - e^{\delta\tau}V(0)])]$$

is maximal for  $\eta = 0$ . From

$$\phi'(0) = 0$$

we obtain

$$(6.2) \quad V(0) = e^{-\delta\tau} \frac{E[\pi(\tau)u'(mS(\tau))]}{E[u'(mS(\tau))]}$$

(as a necessary and sufficient condition, since  $\phi''(\eta) < 0$  if  $u''(x) < 0$ ).

In the particular case of a power utility function with parameter  $c > 0$ ,

$$(6.3) \quad u(x) = \begin{cases} \frac{x^{1-c}}{1-c} & \text{if } c \neq 1, \\ \ln x & \text{if } c = 1 \end{cases},$$

we have  $u'(x) = x^{-c}$ , and

$$\begin{aligned}
 (6.4) \quad V(0) &= e^{-\delta\tau} \frac{E[\pi(\tau)[mS(\tau)]^{-c}]}{E[[mS(\tau)]^{-c}]} \\
 &= e^{-\delta\tau} \frac{E[\pi(\tau)S(\tau)^{-c}]}{E[S(\tau)^{-c}]}
 \end{aligned}$$

Formula (6.4) must hold for all derivative securities. For  $\pi(\tau) = S(\tau)$  and therefore  $V(0) = S(0)$ , (6.4) becomes

$$\begin{aligned}
 S(0) &= e^{-\delta\tau} \frac{E[S(\tau)^{1-c}]}{E[S(\tau)^{-c}]} \\
 &= e^{-\delta\tau} S(0) \frac{M(1-c, \tau)}{M(-c, \tau)},
 \end{aligned}$$

or

$$(6.5) \quad e^{\delta\tau} = \frac{M(1-c, 1)}{M(-c, 1)}.$$

On comparing (6.5) with (5.5), we see that the value of the parameter  $c$  is  $-h^*$ . Hence  $V(0)$  is indeed the discounted expectation of the payoff  $\pi(\tau)$ , calculated with respect to the Esscher measure of parameter  $h^* = -c$ .

By considering different points in time  $\tau$ , we get a consistency requirement. This is satisfied if the investor has a power utility function. We conjecture that it is violated for any other risk-averse utility function, which implies that the pricing of an option by the risk-neutral Esscher measure is a consequence of the consistency requirement. Some related papers are Rubinstein [Ru76], Bick

([Bi87], [Bi90]), Constantinides [Co89], Naik and Lee [NL90], Stapleton and Subrahmanyam [SSu90], He and Leland [HL93], Heston [He93] and Wang [Wa93].

### 7. Logarithm of Stock Price as a Shifted Poisson Process

Here we consider the so-called pure jump model. The assumption is

$$(7.1) \quad X(t) = kN(t) - ct,$$

where  $\{N(t)\}$  is a Poisson process with parameter  $\lambda$ , and  $k$  and  $c$  are constants. The price of the non-dividend-paying stock is modeled as

$$(7.2) \quad S(t) = S(0)e^{kN(t) - ct}.$$

There is only one equivalent martingale measure in this model.

Since

$$E[e^{zN(t)}] = \exp[\lambda t(e^z - 1)],$$

we have

$$(7.3) \quad \begin{aligned} M(z, t) &= E[e^{zX(t)}] \\ &= E(e^{z[kN(t) - ct]}) \\ &= \exp([\lambda(e^{zk} - 1) - zc]t). \end{aligned}$$

Because

$$\begin{aligned} E[e^{zX(t)}; h] &= \frac{M(z + h, t)}{M(h, t)} \\ &= \exp([\lambda e^{hk}(e^{zk} - 1) - zc]t), \end{aligned}$$

we see that, under the Esscher measure of parameter  $h$ , the process  $\{X(t)\}$  remains a shifted Poisson process, but with modified Poisson parameter  $\lambda e^{hk}$ . Formula (5.5) is the condition that

$$(7.4) \quad \delta = \lambda e^{h^*k}(e^k - 1) - c.$$

The equivalent martingale measure is the measure with respect to which  $\{N(t)\}$  becomes a Poisson process with parameter

$$(7.5) \quad \begin{aligned} \lambda^* &= \lambda e^{h^*k} \\ &= (\delta + c)/(e^k - 1). \end{aligned}$$

We now show that, by a replicating portfolio argument, the price of a derivative security is indeed the expectation of its discounted payoff, with the expectation taken with respect to the equivalent martingale measure. Let  $V(S(t), t)$  be the price of the derivative security at time  $t$ . We can form a self-financing portfolio of the stock and risk-free bond replicating the price  $V(S(t), t)$  through time. Let

$$(7.6) \quad \eta = \eta(S(t), t)$$

be the amount invested in the stock at time  $t$  and therefore the difference  $V(S(t), t) - \eta$  is the amount invested in the risk-free bond at time  $t$ . The amount  $\eta$  is such that the derivative security price and the portfolio value have equal instantaneous change. By considering whether there will be an instantaneous jump in the stock price or not, we have the following two conditions:



$$(7.7) \quad V(Se^k, t) - V(S, t) = \eta e^k - \eta,$$

and

$$(7.8) \quad \begin{aligned} V_t(S, t) - cSV_S(S, t) &= -c\eta + \delta[V(S, t) - \eta] \\ &= \delta V(S, t) - (\delta + c)\eta. \end{aligned}$$

Formula (7.7) yields

$$(7.9) \quad \eta = \frac{V(Se^k, t) - V(S, t)}{e^k - 1}.$$

Thus (7.8) becomes

$$(7.10) \quad V_t(S, t) - cSV_S(S, t) = \delta V(S, t) - \lambda^*[V(Se^k, t) - V(S, t)],$$

where  $\lambda^*$  is given by (7.5).

Now, let  $W(S(t), t)$  denote the value at time  $t$  of the expected discounted payoff of the derivative security; the expectation is taken with respect to the probability measure corresponding to the Poisson parameter  $\lambda^*$ . Let  $s$  be a very small positive number. By the Poisson process assumption, the probability that a jump in the stock price will occur in the time interval  $(t, t + s)$  is  $\lambda^*s + o(s)$ . Thus, conditioning on whether there are stock-price jumps in the interval  $(t, t + s)$ , we have

$$(7.11) \quad \begin{aligned} W(S, t) = & e^{-\delta s}[(1 - \lambda^*s)W(Se^{-cs}, t+s) + \lambda^*sW(Se^{k-cs}, t+s)] + o(s), \end{aligned}$$

or

$$\begin{aligned} (1 + \delta s)W(S, t) - W(Se^{-cs}, t+s) = & \lambda^*s[W(Se^{k-cs}, t+s) - W(Se^{-cs}, t+s)] + o(s). \end{aligned}$$

Dividing the last equation by  $s$  and letting  $s$  tend to 0 yields

$$(7.12) \quad \delta W(S, t) + cSW_S(S, t) - W_t(S, t) = \lambda^*[W(Se^k, t) - W(S, t)],$$

which is identical to (7.10). Consequently, the price of the derivative security,  $V(S(t), t)$ , is calculated as the expected discounted payoffs according to the provisions of the contract; the expectation is taken with respect to the measure corresponding to the Poisson process with parameter  $\lambda^*$ .

We note that, in constructing the replicating portfolio, we did not use the assumption that  $\{N(t)\}$  is a Poisson process. Thus  $N(t)$  in (7.1) and (7.2) may be assumed to come from a counting process; the equivalent martingale measure is the measure with respect to which  $\{N(t)\}$  becomes a Poisson process with parameter  $\lambda^*$  given by (7.5). A replicating portfolio can be constructed because at each point of time the stock price has only two possible movements, both with known magnitude.

It is interesting to consider the limiting case where  $k \rightarrow 0$  and  $c \rightarrow \infty$  such that the variance per unit time of  $\{X(t)\}$  in the risk-neutral measure is constant:

$$(7.13) \quad \lambda^*k^2 = \frac{\delta + c}{e^k - 1} k^2 = \sigma^2.$$

This is the classical lognormal model. In the limit (7.9) becomes

$$\eta = SV_S(S, t),$$

showing that the ratio,  $\eta(S(t), t)/S(t)$ , is given by  $V_S(S(t), t)$ , which is usually called *delta* in the option literature. Also, by means of the Taylor expansion, we have

$$\begin{aligned} & \lambda^*[V(Se^k, t) - V(S, t)] \\ &= \lambda^*\{(e^k - 1)SV_S(S, t) + [(e^k - 1)S]^2V_{SS}(S, t)/2 + O(k^3)\} \\ &= (\delta + c)SV_S(S, t) + \sigma^2S^2V_{SS}(S, t)/2 + O(k). \end{aligned}$$

Thus in the limit (7.10) becomes

$$(7.14) \quad V_t(S, t) = \delta V(S, t) - \delta SV_S(S, t) - \frac{\sigma^2}{2}S^2V_{SS}(S, t).$$

This partial differential equation was first derived by Black and Scholes [BS73] with a replicating portfolio argument.

### 8. Extension to Dividend-Paying Stocks

The results in Section 5 can be extended to the case where the stock pays dividends continuously, at a rate proportional to its price. In other words, we assume that there is a nonnegative number  $\varphi$  such that the dividend paid between time  $t$  and  $t+dt$  is

$$(8.1) \quad \varphi S(t) dt.$$

(The number  $\varphi$  may be called the dividend-yield rate.) If all dividends are reinvested in the stock, each share of the stock at time 0 grows to  $e^{\varphi t}$  shares at time  $t$ . The risk-neutral Esscher measure is the Esscher measure of parameter  $h = h^*$  such that the process

$$(8.2) \quad \{e^{-(\delta-\varphi)t}S(t)\}$$

is a martingale. Condition (5.5) now becomes

$$(8.3) \quad e^{\delta-\varphi} = M(1+h^*, 1)/M(h^*, 1).$$

Since

$$(8.4) \quad E[S(\tau); h^*] = S(0) e^{(\delta-\varphi)\tau},$$

the European call option pricing formula (5.9) is generalized as

$$(8.5) \quad E[e^{-\delta\tau} (S(\tau) - K)_+; h^*] \\ = S(0)e^{-\varphi\tau} \Pr[S(\tau) > K; h^* + 1] - Ke^{-\delta\tau} \Pr[S(\tau) > K; h^*].$$

Formula (8.5) may also be used to price currency exchange options, with  $S(\tau)$  denoting the spot exchange rate at time  $\tau$ ,  $\delta$  the domestic force of interest and  $\varphi$  the foreign force of interest. For  $\{S(t)\}$  being a geometric Brownian motion, (8.5) is known as the *Garman-Kohlhagen formula*; see also (9.20) below.

We can extend the model to more than one dividend-paying stock. For each  $j$ ,  $j = 1, 2, \dots, n$ , let  $S_j(t)$  denote the price of stock  $j$  at time  $t$ ,  $t \geq 0$ , and we assume that there exists a nonnegative constant  $\varphi_j$  such that stock  $j$  pays dividends of amount

$$\varphi_j S_j(t) dt$$

between time  $t$  and  $t+dt$ . Write

$$(8.6) \quad X_j(t) = \ln[S_j(t)/S_j(0)], \quad j = 1, 2, \dots, n,$$

and

$$(8.7) \quad \mathbf{X}(t) = (X_1(t), X_2(t), \dots, X_n(t))'$$

Let  $R^n$  denote the linear space of column vectors with  $n$  real entries, and

$$(8.8) \quad M(\mathbf{z}, t) = E[e^{\mathbf{z}'\mathbf{X}(t)}], \quad \mathbf{z} \in R^n,$$

be the moment generating function of  $\mathbf{X}(t)$ . We assume that  $\{\mathbf{X}(t)\}_{t \geq 0}$  is a stochastic process with independent and stationary increments and that

$$(8.9) \quad M(\mathbf{z}, t) = [M(\mathbf{z}, 1)]^t, \quad t \geq 0.$$

Let  $\mathbf{h} = (h_1, h_2, \dots, h_n)' \in R^n$  for which  $M(\mathbf{h}, 1)$  exists. The positive martingale

$$(8.10) \quad \{e^{\mathbf{h}'\mathbf{X}(t)} M(\mathbf{h}, 1)^{-t}\}_{t \geq 0}$$

can be used to define a new measure, the Esscher measure of parameter vector  $\mathbf{h}$ . The risk-neutral Esscher measure is the Esscher measure of parameter vector  $\mathbf{h} = \mathbf{h}^*$  such that, for each  $j, j = 1, 2, \dots, n$

$$(8.11) \quad \{e^{-(\delta - \varphi_j)t} S_j(t)\}$$

is a martingale. Condition (8.3) is generalized as  $n$  simultaneous conditions :

$$(8.12) \quad e^{\delta - \varphi_j} = M(\mathbf{1}_j + \mathbf{h}^*, 1)/M(\mathbf{h}^*, 1), \quad j = 1, \dots, n.$$

Here

$$(8.13) \quad \mathbf{1}_j = (0, \dots, 0, 1, 0, \dots, 0)',$$

where the 1 in the column vector  $\mathbf{1}_j$  is in the  $j$ -th position.

For  $\mathbf{k} = (k_1, \dots, k_n)'$ , write

$$(8.14) \quad S(t)^{\mathbf{k}} = S_1(t)^{k_1} \dots S_n(t)^{k_n}.$$

Then

$$\begin{aligned} E[S(t)^{\mathbf{k}} g(S(t)); \mathbf{h}] &= \frac{E[S(t)^{\mathbf{k}} g(S(t)) e^{\mathbf{h}'\mathbf{X}(t)}]}{E[e^{\mathbf{h}'\mathbf{X}(t)}]} \\ &= \frac{E[S(t)^{\mathbf{k}} g(S(t)) S(t)^{\mathbf{h}}]}{E[S(t)^{\mathbf{h}}]} \\ &= \frac{E[S(t)^{\mathbf{k}+\mathbf{h}}] E[g(S(t)) S(t)^{\mathbf{k}+\mathbf{h}}]}{E[S(t)^{\mathbf{h}}] E[S(t)^{\mathbf{k}+\mathbf{h}}]} \\ (8.15) \quad &= E[S(t)^{\mathbf{k}}; \mathbf{h}] E[g(S(t)); \mathbf{k} + \mathbf{h}], \end{aligned}$$

which generalizes the factorization formula (5.7). An immediate consequence of formula (8.15) and that (8.11) is a martingale under the risk-neutral Esscher measure is

$$\begin{aligned} (8.16) \quad E[e^{-\delta t} S_j(t) g(S(t)); \mathbf{h}^*] &= E[e^{-\delta t} S_j(t); \mathbf{h}^*] E[g(S(t)); \mathbf{h}^* + \mathbf{1}_j] \\ &= S_j(0) e^{-\phi_j t} E[g(S(t)); \mathbf{h}^* + \mathbf{1}_j]. \end{aligned}$$

The *Margrabe option* [Ma78] is the option to exchange one stock for another at the end of a stated period, say time  $\tau$ ,  $\tau > 0$ . The payoff of this European option is

$$(8.17) \quad [S_1(\tau) - S_2(\tau)]_+.$$

Its value at time 0, calculated with respect to the risk-neutral Esscher measure, is

$$(8.18) \quad E(e^{-\delta\tau}[S_1(\tau) - S_2(\tau)]_+; \mathbf{h}^*).$$

Since

$$(s_1 - s_2)_+ = s_1 I(s_1 > s_2) - s_2 I(s_1 > s_2),$$

it follows from (8.16) that

$$\begin{aligned} (8.19) \quad & E(e^{-\delta\tau}[S_1(\tau) - S_2(\tau)]_+; \mathbf{h}^*) \\ &= S_1(0)e^{-\varphi_1\tau} E(I[S_1(\tau) > S_2(\tau)]; \mathbf{h}^* + \mathbf{1}_1) \\ &\quad - S_2(0)e^{-\varphi_2\tau} E(I[S_1(\tau) > S_2(\tau)]; \mathbf{h}^* + \mathbf{1}_2) \\ &= S_1(0)e^{-\varphi_1\tau} \Pr[S_1(\tau) > S_2(\tau); \mathbf{h}^* + \mathbf{1}_1] \\ &\quad - S_2(0)e^{-\varphi_2\tau} \Pr[S_1(\tau) > S_2(\tau); \mathbf{h}^* + \mathbf{1}_2]. \end{aligned}$$

A special case of (8.19) is (8.5).

### 9. Change of Numéraire, Homogeneous Payoff Function and Wiener Process

Consider a European option or derivative security with exercise date  $\tau$  and payoff

$$(9.1) \quad \Pi(S_1(\tau), \dots, S_n(\tau)).$$

For example, the Margrabe option has the payoff function

$$(9.2) \quad \Pi(s_1, s_2) = (s_1 - s_2)_+.$$

Let  $E_t[\cdot]$  denote the expectation conditional on all information up to time  $t$ . For  $0 \leq t \leq \tau$ , let  $V(t)$  denote the price of the security at time  $t$ , calculated with respect to the risk-neutral Esscher measure,

$$\begin{aligned} V(t) &= E_t[e^{-\delta(\tau-t)} \Pi(S_1(\tau), \dots, S_n(\tau)); \mathbf{h}^*] \\ &= E_t[e^{-\delta(\tau-t)} S_j(\tau) \Pi(S_1(\tau), \dots, S_n(\tau))/S_j(\tau); \mathbf{h}^*] \\ &= E_t[e^{-\delta(\tau-t)} S_j(\tau); \mathbf{h}^*] E_t[\Pi(S_1(\tau), \dots, S_n(\tau))/S_j(\tau); \mathbf{h}^* + \mathbf{1}_j] \\ &= e^{-\varphi_j(\tau-t)} S_j(t) E_t[\Pi(S_1(\tau), \dots, S_n(\tau))/S_j(\tau); \mathbf{h}^* + \mathbf{1}_j]. \end{aligned}$$

Thus

$$(9.3) \quad \frac{V(t)}{e^{\varphi_j t} S_j(t)} = E_t\left[\frac{1}{e^{\varphi_j \tau} S_j(\tau)} \Pi(S_1(\tau), \dots, S_n(\tau)); \mathbf{h}^* + \mathbf{1}_j\right],$$

from which it follows that, with respect to the Esscher measure of parameter vector  $\mathbf{h}^* + \mathbf{1}_j$ , the process

$$(9.4) \quad \left\{ \frac{V(t)}{e^{\varphi_j t} S_j(t)}; 0 \leq t \leq \tau \right\}$$

is a martingale. In particular, with respect to this measure, the processes

$$(9.5) \quad \left\{ \frac{e^{\delta t}}{e^{\varphi_j t} S_j(t)} \right\}$$

and

$$(9.6) \quad \left\{ \frac{e^{\varphi_k t} S_k(t)}{e^{\varphi_j t} S_j(t)} \right\}$$

are martingales. To explain the denominator  $e^{\varphi_j t} S_j(t)$ , we consider stock  $j$  as a standard of value or a *numéraire*. In other words, there is a



mutual fund consisting of stock  $j$  only and all dividends are reinvested; all other securities are measured in terms of the value of this mutual fund. See also [GER94].

Now, we assume that the payoff function  $\Pi$  is homogeneous of degree one with respect to the  $j$ -th variable,

$$(9.7) \quad \Pi(s_1, \dots, s_n) = s_j \Pi(s_1/s_j, \dots, s_{j-1}/s_j, 1, s_{j+1}/s_j, \dots, s_n/s_j),$$

which is a condition satisfied by (9.2) with both  $j = 1$  and  $j = 2$ . Then (9.3) becomes

$$(9.8) \quad \frac{V(t)}{e^{\phi_j t} S_j(t)} = E_t[\Pi(\frac{S_1(\tau)}{e^{\phi_j \tau} S_j(\tau)}, \dots, \frac{S_n(\tau)}{e^{\phi_j \tau} S_j(\tau)}); \mathbf{h}^* + \mathbf{1}_j].$$

The right-hand side is a conditional expectation, with respect to the Esscher measure of parameter vector  $\mathbf{h}^* + \mathbf{1}_j$ , of a function of the  $(n-1)$ -dimensional random vector

$$(9.9) \quad (X_1(\tau) - X_j(\tau), \dots, X_{j-1}(\tau) - X_j(\tau), X_{j+1}(\tau) - X_j(\tau), \dots, X_n(\tau) - X_j(\tau))'.$$

Consider the special case that  $\{\mathbf{X}(t)\}$  is an  $n$ -dimensional Wiener process, with  $\underline{\mu} = (\mu_1, \mu_2, \dots, \mu_n)'$  and  $\mathbf{V} = (\sigma_{ij})$  denoting the mean vector and the covariance matrix of  $\mathbf{X}(1)$ , respectively. It is assumed that  $\mathbf{V}$  is nonsingular. Because

$$(9.10) \quad M(\mathbf{z}, t) = \exp[t(\mathbf{z}'\underline{\mu} + 1/2\mathbf{z}'\mathbf{V}\mathbf{z})], \quad \mathbf{z} \in \mathbb{R}^n,$$

we have, for  $\mathbf{h} \in \mathbb{R}^n$ ,

$$(9.11) \quad E[e^{z'X(t)}; \mathbf{h}] = M(\mathbf{z} + \mathbf{h}, t)/M(\mathbf{h}, t) \\ = \exp\{t\{\mathbf{z}'(\underline{\mu} + \mathbf{V}\mathbf{h}) + 1/2\mathbf{z}'\mathbf{V}\mathbf{z}\}\}, \mathbf{z} \in \mathbb{R}^n,$$

showing that, under the Esscher measure of parameter vector  $\mathbf{h}$ ,  $\{\mathbf{X}(t)\}$  remains an  $n$ -dimensional Wiener process with modified mean vector per unit time

$$\underline{\mu} + \mathbf{V}\mathbf{h}$$

and unchanged covariance matrix per unit time  $\mathbf{V}$ . It follows from (8.12) that, for  $k = 1, 2, \dots, n$ ,

$$(9.12) \quad \delta - \varphi_k = \mathbf{1}_k'(\underline{\mu} + \mathbf{V}\mathbf{h}^*) + 1/2\mathbf{1}_k'\mathbf{V}\mathbf{1}_k.$$

Thus

$$(9.13) \quad \underline{\mu}^* = E[\mathbf{X}(1); \mathbf{h}^*]$$

$$(9.14) \quad = \underline{\mu} + \mathbf{V}\mathbf{h}^*$$

$$(9.15) \quad = \delta\mathbf{1} - (\varphi_1 + 1/2\sigma_{11}, \varphi_2 + 1/2\sigma_{22}, \dots, \varphi_n + 1/2\sigma_{nn})',$$

where

$$(9.16) \quad \mathbf{1} = (1, 1, 1, \dots, 1)'$$

Also,

$$(9.17) \quad E[\mathbf{X}(1); \mathbf{h}^* + \mathbf{1}_k] = \underline{\mu} + \mathbf{V}(\mathbf{h}^* + \mathbf{1}_k) = \underline{\mu}^* + \mathbf{V}\mathbf{1}_k \\ = \delta\mathbf{1} - (\varphi_1 - \sigma_{1k} + 1/2\sigma_{11}, \varphi_2 - \sigma_{2k} + 1/2\sigma_{22}, \dots, \varphi_n - \sigma_{nk} + 1/2\sigma_{nn})'.$$

For  $\{\mathbf{X}(t)\}$  being an  $n$ -dimensional Wiener process, (9.9) is a normal random vector under the Esscher measure of parameter vector  $\mathbf{h}^* + \mathbf{1}_j$ , and it follows from (9.17) that its mean does not involve the

force of interest  $\delta$ , and of course its  $(n - 1)$ -dimensional covariance matrix, which is the same for all  $\mathbf{h}$ , does not depend on  $\delta$ . Thus  $V(t)$ , the price of a derivative security with a payoff function which is homogeneous with respect to one of its arguments, does not depend on  $\delta$ .

For example, consider the European Margrabe option. Here  $n =$

2. Let

$$(9.18) \quad \begin{aligned} v^2 &= \text{Var}[X_1(1) - X_2(1)] \\ &= \sigma_{11} - 2\sigma_{12} + \sigma_{22}, \end{aligned}$$

$$(9.19) \quad \zeta(\tau) = \frac{1}{v\sqrt{\tau}} \ln\left(\frac{e^{-\varphi_1\tau}S_1(0)}{e^{-\varphi_2\tau}S_2(0)}\right),$$

and  $\Phi$  denote the standardized normal distribution function. Then

(8.19) becomes

$$(9.20) \quad \begin{aligned} E(e^{-\delta\tau}[S_1(\tau) - S_2(\tau)]_+; \mathbf{h}^*) \\ = e^{-\varphi_1\tau}S_1(0)\Phi(\zeta(\tau) + 1/2v\sqrt{\tau}) - e^{-\varphi_2\tau}S_2(0)\Phi(\zeta(\tau) - 1/2v\sqrt{\tau}), \end{aligned}$$

which does not depend on  $\delta$ . For non-dividend-paying stocks ( $\varphi_1 = \varphi_2 = 0$ ), formula (9.20) has been given by Margrabe [Ma78]. Fischer [Fi78] has also derived (9.20) with  $\varphi_1 = 0$  as a European call option formula; for him,  $S_2(\tau)$  is the stochastic exercise price at time  $\tau$ .

**Remarks.** In the model of  $n$  stocks, the risk-neutral Esscher measure is the Esscher measure corresponding to the  $n$ -dimensional vector  $\mathbf{h}^*$

such that, for  $j = 1, 2, \dots, n$ , (8.11) is a martingale. Let us now consider modeling only a subset of the  $n$  stocks, say stock 1 to stock  $k$ ,  $k < n$ . Then the risk-neutral Esscher measure is the Esscher measure corresponding to the  $k$ -dimensional vector  $\mathbf{h}^*$  such that, for  $j = 1, 2, \dots, k$ , (8.11) is a martingale. To avoid confusion, we write the second  $\mathbf{h}^*$  as  $\mathbf{h}_1^*$ . One may wonder how  $\mathbf{h}^*$  and  $\mathbf{h}_1^*$  are related. An explicit answer can be given when  $\{\mathbf{X}(t)\}$  is a Wiener process. Let  $\mathbf{P}$  denote the projection matrix from  $\mathbb{R}^n$  onto its first  $k$  coordinates,

$$(9.21) \quad \mathbf{P} = (\mathbf{I} \ \mathbf{O}),$$

where  $\mathbf{I}$  is the  $k$ -by- $k$  identity matrix and  $\mathbf{O}$  is the  $k$ -by- $(n-k)$  zero matrix. (The dimension of  $\mathbf{P}$  is  $k$  by  $n$ .) Then the  $k$ -by- $k$  matrix  $\mathbf{PVP}'$  is the covariance matrix of the random vector

$$(X_1(1), X_2(1), \dots, X_k(1))'.$$

It now follows from (9.14) that

$$(9.22) \quad \mathbf{PVh}^* = \mathbf{PVP}'\mathbf{h}_1^*,$$

or

$$(9.23) \quad \mathbf{h}_1^* = (\mathbf{PVP}')^{-1}\mathbf{PVh}^*.$$

There is another way to express the relationship between  $\mathbf{h}^*$  and  $\mathbf{h}_1^*$ . Let  $\mathbf{V}_1$  denote the covariance matrix  $\mathbf{PVP}'$ . Similarly, let  $\mathbf{V}_2$  denote the covariance matrix of the  $(n-k)$ -dimensional random vector

$$(X_{k+1}(1), X_{k+2}(1), \dots, X_n(1))'.$$

Consider the model consisting only of stock  $k+1$  to stock  $n$ ; let  $\mathbf{h}_2^*$  denote the  $(n-k)$ -dimensional vector determining the risk-neutral measure in the model. Then

$$(9.24) \quad \mathbf{V}\mathbf{h}^* = \begin{pmatrix} \mathbf{V}_1 & \mathbf{O} \\ \mathbf{O} & \mathbf{V}_2 \end{pmatrix} \begin{pmatrix} \mathbf{h}_1^* \\ \mathbf{h}_2^* \end{pmatrix}.$$

### 10. Probability of Ruin

The idea of replacing the original probability measure by an Esscher measure with an appropriately chosen parameter has an elegant application in classical actuarial risk theory. Let  $\{U(t)\}$  be the surplus process,

$$(10.1) \quad U(t) = u + X(t),$$

where  $u \geq 0$  is the initial surplus, and  $X(t)$  the aggregate gains (premiums minus claims) up to time  $t$ . We suppose that the process  $\{X(t)\}$  has independent and stationary increments, satisfies (5.2), and has a positive drift,

$$(10.2) \quad E[X(1)] > 0.$$

Let

$$(10.3) \quad T = \inf\{t \mid U(t) < 0\}$$

be the time of ruin. The probability of ruin before time  $m$ ,  $m > 0$ , is

$$(10.4) \quad \psi(u, m) = \Pr(T < m) = E[I(T < m)].$$

Let  $a \wedge b$  denote of the minimum of  $a$  and  $b$ . By a change of measure,

$$(10.5) \quad \begin{aligned} \psi(u, m) &= E[I(T < m) e^{-hX(T \wedge m)} M(h, 1)^{T \wedge m}; h] \\ &= E[I(T < m) e^{-hX(T)} M(h, 1)^T; h], \end{aligned}$$

which can be simplified if  $h$  is chosen as the nontrivial solution of the equation

$$(10.6) \quad M(h, 1) = 1.$$

For simplicity we write

$$M(h) = M(h, 1).$$

It follows from

$$M''(h) = E[X(1)^2 e^{hX(1)}] > 0$$

that  $M(h)$  is a convex function. Thus equation (10.6) has at most one other solution besides  $h = 0$ . Because

$$M'(0) = E[X(1)] > 0,$$

the nontrivial solution for (10.6) is a negative  $h$ . Following the usual notation in risk theory, we write this solution of (10.6) as  $-R$ . ( $R$  is called the *adjustment coefficient*.) With  $h = -R$ , (10.5) becomes

$$(10.7) \quad \psi(u, m) = E[I(T < m) e^{RX(T)}; -R].$$

The probability of ruin over an infinite horizon is

$$(10.8) \quad \begin{aligned} \psi(u) &= \psi(u, \infty) \\ &= E[I(T < \infty) e^{RX(T)}; -R]. \end{aligned}$$

Now,

$$\begin{aligned} E[X(1); -R] &= E[X(1) e^{-RX(1)}] \\ &= M'(-R) \\ &< 0, \end{aligned}$$

because  $M$  is a convex function. An aggregate gains process with a negative drift means that ruin is certain. Thus, under the Esscher measure of parameter  $-R$ ,

$$I(T < \infty) = 1$$

almost surely, and (10.8) simplifies as

$$\begin{aligned} \psi(u) &= E[e^{RX(T)}; -R] \\ (10.9) \quad &= E[e^{RU(T)}; -R]e^{-Ru}. \end{aligned}$$

This approach to the ruin problem can be found in Chapter XII of Asmussen's book [As87] and he has attributed the idea to von Bahr [vB74] and Siegmund [Si75]. Formula (10.9) should be compared with (12.3.4) on page 352 of *Actuarial Mathematics* [BGHJN86],

$$\psi(u) = \frac{e^{-Ru}}{E[e^{-RU(T)} \mid T < \infty]},$$

where the conditional expectation in the denominator is taken with respect to the original probability measure.

### 11. Perpetual American Options on Two Stocks

The actuarial concept of the adjustment coefficient turns out to be the right tool for pricing certain *American* options without expiration date. In this section we consider two stocks with positive dividend-yield rates. For  $\mathbf{k} = (k_1, k_2)' \in \mathbb{R}^2$ , we write

$$\mathbf{S}(t)^{\mathbf{k}} = S_1(t)^{k_1} S_2(t)^{k_2}$$

[the same notation as (8.14)]. The condition on  $\mathbf{k}$  so that the process

$$(11.1) \quad \{e^{-\delta t} \mathbf{S}(t)^{\mathbf{k}}\}_{t \geq 0}$$

becomes a martingale under the risk-neutral Esscher measure is:

$$(11.2) \quad e^{-\delta t} \mathbb{E}[e^{\mathbf{k}'\mathbf{X}(t)}; \mathbf{h}^*] = 1.$$

Actually, we are only interested in  $\mathbf{k}$  of the form

$$(11.3) \quad (\theta, 1 - \theta)'$$

With the definition

$$(11.4) \quad f(\theta) = e^{-\delta} \mathbb{E}(\exp[\theta X_1(1) + (1 - \theta)X_2(1)]; \mathbf{h}^*),$$

condition (11.2) becomes

$$(11.5) \quad f(\theta) = 1,$$

which is analogous to (10.6). Because

$$f(0) = e^{-\delta} \mathbb{E}[e^{X_2(1)}] = e^{-\varphi_2} < 1,$$

$$f(1) = e^{-\delta} \mathbb{E}[e^{X_1(1)}] = e^{-\varphi_1} < 1$$

and

$$f''(\theta) = e^{-\delta} \mathbb{E}[(X_1(1) - X_2(1))^2 \exp[\theta X_1(1) + (1 - \theta)X_2(1)]; \mathbf{h}^*) > 0,$$



we gather that equation (11.5) has, under fairly mild regularity conditions, exactly two solutions,  $\theta_0 < 0$  and  $\theta_1 > 1$ . [Equation (10.6) also has two solutions, 0 and  $-R$ .] Thus, for  $i = 0, 1$ , the process

$$(11.6) \quad \left\{ e^{-\delta t} S_2(t) \left( \frac{S_1(t)}{S_2(t)} \right)^{\theta_i}; t \geq 0 \right\}$$

is a martingale with respect to the risk-neutral Esscher measure.

For the rest of this paper we assume that  $\{X(t)\}$  is a two-dimensional Wiener process, so that there is only one equivalent martingale measure, and  $\{S_1(t)\}$  and  $\{S_2(t)\}$  have continuous sample paths. With  $\underline{\mu}^*$  given by (9.13) ( $n = 2$ ), we have

$$(11.7) \quad E[e^{\mathbf{k}'X(t)}; \mathbf{h}^*] = \exp(\mathbf{k}'\underline{\mu}^* + 1/2\mathbf{k}'\mathbf{V}\mathbf{k}).$$

Hence (11.4) becomes

$$f(\theta) = \exp[-1/2v^2\theta(1-\theta) - \phi_1\theta - \phi_2(1-\theta)],$$

where  $v^2$  is defined by (9.18). Equation (11.5) is now equivalent to the quadratic equation:

$$(11.8) \quad 1/2v^2\theta(1-\theta) + \phi_1\theta + \phi_2(1-\theta) = 0.$$

Again it is clear that one root is less than zero and the other greater than one. The roots are

$$(11.9) \quad \theta_0 = \omega - \Delta$$

and

$$(11.10) \quad \theta_1 = \omega + \Delta,$$

where

$$(11.11) \quad \omega = \frac{1}{2} + \frac{\varphi_1}{\sqrt{2}} - \frac{\varphi_2}{\sqrt{2}}$$

and

$$(11.12) \quad \Delta = \sqrt{\omega^2 + \frac{2\varphi_2}{\sqrt{2}}}$$

$$= \sqrt{\frac{1}{4} + \frac{\varphi_1}{\sqrt{2}} + \frac{\varphi_2}{\sqrt{2}} + \left(\frac{\varphi_1 - \varphi_2}{\sqrt{2}}\right)^2}.$$

Note that  $\Delta$  is symmetric with respect to the parameters of the two stocks, but  $\omega$  is not. The roots  $\theta_0$  and  $\theta_1$  do not depend on the force of interest  $\delta$ . Also,  $\theta_0 + \theta_1 = 1$  if and only if  $\varphi_1 = \varphi_2$ .

We are interested in the pricing of a perpetual American option whose payoff is  $\Pi(S_1(t), S_2(t))$  if it is exercised at time  $t$ . Its price is the supremum, taken over all stopping times  $T$ , of

$$E[e^{-\delta T} \Pi(S_1(T), S_2(T)); \mathbf{h}^*].$$

We assume that the payoff function  $\Pi(s_1, s_2)$  is nonnegative and homogeneous of degree 1. Thus

$$\Pi(s_1, s_2) = s_2 \pi\left(\frac{s_1}{s_2}\right),$$

where

$$(11.13) \quad \pi(x) = \Pi(x, 1).$$

Examples are

$$\Pi(s_1, s_2) = (s_1 - s_2)_+,$$

which is the payoff function for the Margrabe option,

$$\Pi(s_1, s_2) = \text{Max}(s_1, s_2),$$

the payoff function for the maximum option, and

$$\Pi(s_1, s_2) = |s_1 - s_2|,$$

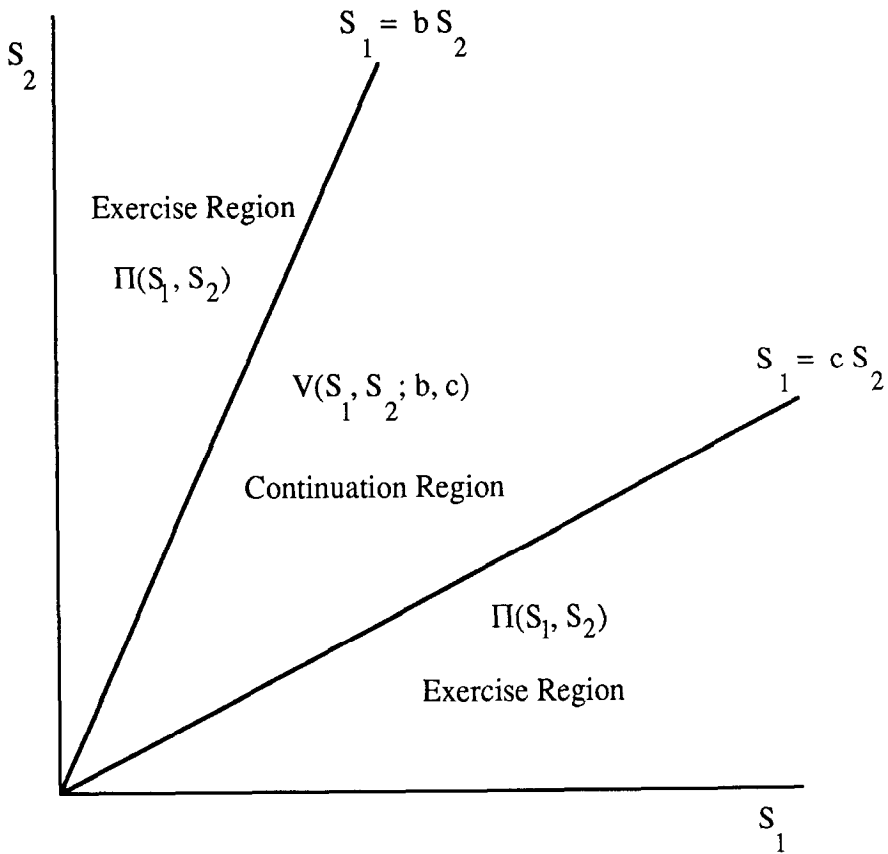
the payoff function for the symmetric Margrabe option.

Because of the homogeneity assumption it suffices to consider stopping strategies where the decision to exercise the option or not at any time  $t$  depends only on the ratio of  $S_1(t)$  to  $S_2(t)$ . Then, under some fairly general conditions, we can restrict ourselves to stopping times of the form

$$(11.14) \quad T_{b,c} = \inf \left\{ t \mid \frac{S_1(t)}{S_2(t)} = b \text{ or } \frac{S_1(t)}{S_2(t)} = c \right\},$$

$0 < b < S_1(0)/S_2(0) < c$ . For simplicity we write  $S_1 = S_1(0)$  and  $S_2 = S_2(0)$ . The value of the option-exercise strategy is

$$(11.15) \quad V(S_1, S_2; b, c) = E[e^{-\delta T_{b,c}} \Pi(S_1(T_{b,c}), S_2(T_{b,c})); \mathbf{h}^*].$$



With the definitions

(11.16)

$$\beta(S_1, S_2; b, c) = E(e^{-\delta T_{b,c}} S_2(T_{b,c}) I[S_1(T_{b,c}) = b S_2(T_{b,c})]; \mathbf{h}^*)$$

and

(11.17)

$$\gamma(S_1, S_2; b, c) = E(e^{-\delta T_{b,c}} S_2(T_{b,c}) I[S_1(T_{b,c}) = c S_2(T_{b,c})]; \mathbf{h}^*),$$

equation (11.15) becomes

$$(11.18) \quad V(S_1, S_2; b, c) = \pi(b)\beta(S_1, S_2; b, c) + \pi(c)\gamma(S_1, S_2; b, c).$$

To determine the expectations  $\beta = \beta(S_1, S_2; b, c)$  and  $\gamma = \gamma(S_1, S_2; b, c)$ , we stop the two martingales (11.6) at time  $T_{b,c}$  and apply the *optional sampling theorem*. This leads to the equations

$$S_2(S_1/S_2)^{\theta_0} = \beta b^{\theta_0} + \gamma c^{\theta_0}$$

and

$$S_2(S_1/S_2)^{\theta_1} = \beta b^{\theta_1} + \gamma c^{\theta_1}.$$

Their solution is

$$(11.19) \quad \begin{pmatrix} \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} b^{\theta_0} & c^{\theta_0} \\ b^{\theta_1} & c^{\theta_1} \end{pmatrix}^{-1} \begin{pmatrix} S_2(S_1/S_2)^{\theta_0} \\ S_2(S_1/S_2)^{\theta_1} \end{pmatrix}.$$

The optimal option-exercise ratios  $b = \tilde{b}$  and  $c = \tilde{c}$  are obtained from the first order conditions

$$(11.20) \quad V_b(S_1, S_2; \tilde{b}, \tilde{c}) = 0$$

and

$$(11.21) \quad V_c(S_1, S_2; \tilde{b}, \tilde{c}) = 0,$$

where the subscripts denote partial differentiation. Here we assume that  $\tilde{b} > 0$  and  $\tilde{c} < \infty$ . We shall see that  $\tilde{b}$  and  $\tilde{c}$  depend on neither  $S_1$  nor  $S_2$ . Since matrix notation facilitates the further discussion of (11.20) and (11.21), we define the vector-valued functions:

$$(11.22) \quad \mathbf{f}(S_1, S_2; b, c) = (\beta(S_1, S_2; b, c), \gamma(S_1, S_2; b, c))',$$

$$(11.23) \quad \mathbf{g}(x) = (x^{\theta_0}, x^{\theta_1})'$$

and

$$(11.24) \quad \mathbf{h}(x_1, x_2) = x_2 \mathbf{g}\left(\frac{x_1}{x_2}\right).$$

With these definitions, we can rewrite (11.19) as

$$(11.25) \quad \mathbf{f}(S_1, S_2; b, c) = (\mathbf{g}(b) \ \mathbf{g}(c))^{-1} \mathbf{h}(S_1, S_2),$$

and (11.18) as

$$(11.26) \quad V(S_1, S_2; b, c) = (\pi(b) \ \pi(c)) \mathbf{f}(S_1, S_2; b, c).$$

Hence

$$(11.27) \quad V(S_1, S_2; b, c) = (\pi(b) \ \pi(c)) (\mathbf{g}(b) \ \mathbf{g}(c))^{-1} \mathbf{h}(S_1, S_2).$$

To find the partial derivatives of  $V$  with respect to  $b$  and  $c$ , we need the partial derivatives of the inverse of the matrix  $(\mathbf{g}(b) \ \mathbf{g}(c))$ . Let  $\mathbf{A}$  be an invertible matrix with elements that are functions of a parameter. If we differentiate the identity

$$\mathbf{A}^{-1} \mathbf{A} = \mathbf{I}$$

with respect to the parameter, we get

$$(\mathbf{A}^{-1})' \mathbf{A} + \mathbf{A}^{-1} \mathbf{A}' = \mathbf{O},$$

or

$$(\mathbf{A}^{-1})' = -\mathbf{A}^{-1} \mathbf{A}' \mathbf{A}^{-1}.$$

(Note that  $'$  denotes differentiation, while  $'$  denotes matrix transposition.) Thus, differentiating (11.25) with respect to  $b$  yields

$$(11.28) \quad \begin{aligned} \mathbf{f}_b(S_1, S_2; b, c) &= -(\mathbf{g}(b) \ \mathbf{g}(c))^{-1} (\mathbf{g}'(b) \ \mathbf{0}) (\mathbf{g}(b) \ \mathbf{g}(c))^{-1} \mathbf{h}(S_1, S_2) \\ &= -(\mathbf{g}(b) \ \mathbf{g}(c))^{-1} (\mathbf{g}'(b) \ \mathbf{0}) \mathbf{f}(S_1, S_2; b, c) \\ &= -\beta(S_1, S_2; b, c) (\mathbf{g}(b) \ \mathbf{g}(c))^{-1} \mathbf{g}'(b). \end{aligned}$$

It now follows from (11.26) that

$$\begin{aligned}
 &V_b(S_1, S_2; b, c) \\
 &= (\pi'(b) \ 0)\mathbf{f} + (\pi(b) \ \pi(c))\mathbf{f}_b \\
 (11.29) \quad &= \beta(S_1, S_2; b, c)[\pi'(b) - (\pi(b) \ \pi(c))(\mathbf{g}(b) \ \mathbf{g}(c))^{-1}\mathbf{g}'(b)].
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 (11.30) \quad V_c(S_1, S_2; b, c) = \\
 \gamma(S_1, S_2; b, c)[\pi'(c) - (\pi(b) \ \pi(c))(\mathbf{g}(b) \ \mathbf{g}(c))^{-1}\mathbf{g}'(c)].
 \end{aligned}$$

Because  $\beta > 0$  and  $\gamma > 0$ , the first order conditions (11.20) and (11.21) are equivalent to the matrix equation

$$(11.31) \quad (\pi(\tilde{b}) \ \pi(\tilde{c}))(\mathbf{g}(\tilde{b}) \ \mathbf{g}(\tilde{c}))^{-1}(\mathbf{g}'(\tilde{b}) \ \mathbf{g}'(\tilde{c})) = (\pi'(\tilde{b}) \ \pi'(\tilde{c})).$$

With (11.31) we can determine the optimal option-exercise ratios  $\tilde{b}$  and  $\tilde{c}$ , which depend on neither  $S_1$  nor  $S_2$ . The price of the perpetual option is

$$(11.32) \quad \left\{ \begin{array}{ll} V(S_1, S_2; \tilde{b}, \tilde{c}) & \tilde{b} \leq S_1/S_2 \leq \tilde{c} \\ \Pi(S_1, S_2) & \text{otherwise} \end{array} \right. ,$$

where  $V$  is given by (11.27).

### 12. High Contact or Smooth Pasting Condition

The first order conditions (11.20) and (11.21) are closely related to the *high contact condition* in the finance literature [Sa65] and the

*smooth pasting condition* in the optimal stopping literature ([Sh78], [SS93]). Also see Dixit [Di93]. In the present context, it means that the gradients of the option-price function  $V(\cdot, \cdot; \tilde{b}, \tilde{c})$  and the payoff function  $\Pi(\cdot, \cdot)$  coincide on the optimal option-exercise boundaries  $S_1 = \tilde{b}S_2$  and  $S_1 = \tilde{c}S_2$ , i.e., for  $S_1 > 0, S_2 > 0$ ,

$$(12.1) \quad V_{S_1}(\tilde{b}S_2, S_2; \tilde{b}, \tilde{c}) = \Pi_{S_1}(\tilde{b}S_2, S_2),$$

$$(12.2) \quad V_{S_2}(\tilde{b}S_2, S_2; \tilde{b}, \tilde{c}) = \Pi_{S_2}(\tilde{b}S_2, S_2),$$

$$(12.3) \quad V_{S_1}(\tilde{c}S_2, S_2; \tilde{b}, \tilde{c}) = \Pi_{S_1}(\tilde{c}S_2, S_2)$$

and

$$(12.4) \quad V_{S_2}(\tilde{c}S_2, S_2; \tilde{b}, \tilde{c}) = \Pi_{S_2}(\tilde{c}S_2, S_2).$$

To see this, let  $\mathbf{v}$  denote the row vector

$$(12.5) \quad (\pi(\tilde{b}) \ \pi(\tilde{c}))(\mathbf{g}(\tilde{b}) \ \mathbf{g}(\tilde{c}))^{-1},$$

which depends on neither  $S_1$  nor  $S_2$ . Then

$$(12.6) \quad V(S_1, S_2; \tilde{b}, \tilde{c}) = \mathbf{v}\mathbf{h}(S_1, S_2),$$

and (11.31) becomes

$$(12.7) \quad \mathbf{v}(\mathbf{g}'(\tilde{b}) \ \mathbf{g}'(\tilde{c})) = (\pi'(\tilde{b}) \ \pi'(\tilde{c})).$$

Because

$$\Pi(S_1, S_2) = S_2\pi(S_1/S_2)$$

and

$$\mathbf{h}(S_1, S_2) = S_2\mathbf{g}(S_1/S_2),$$

we have



$$\Pi_{S_1}(S_1, S_2) = \pi'(S_1/S_2)$$

and

$$\mathbf{h}_{S_1}(S_1, S_2) = \mathbf{g}'(S_1/S_2).$$

Now (12.1) and (12.3) follow from (12.6) and (12.7).

Similarly, (12.2) and (12.4) can be obtained from (12.6), (12.7),

$$\Pi_{S_2}(S_1, S_2) = \pi(S_1/S_2) - (S_1/S_2)\pi'(S_1/S_2),$$

$$\mathbf{h}_{S_2}(S_1, S_2) = \mathbf{g}(S_1/S_2) - (S_1/S_2)\mathbf{g}'(S_1/S_2)$$

and

$$\mathbf{v}(\mathbf{g}(\tilde{\mathbf{b}}) \ \mathbf{g}(\tilde{\mathbf{c}})) = (\pi(\tilde{\mathbf{b}}) \ \pi(\tilde{\mathbf{c}})).$$

**Remark.** We note that the common gradient, along the line  $S_1 = \tilde{\mathbf{b}}S_2$ , is the constant vector

$$(\pi'(\tilde{\mathbf{b}}), \ \pi(\tilde{\mathbf{b}}) - \tilde{\mathbf{b}}\pi'(\tilde{\mathbf{b}}))',$$

and, on the line  $S_1 = \tilde{\mathbf{c}}S_2$ , is the constant vector

$$(\pi'(\tilde{\mathbf{c}}), \ \pi(\tilde{\mathbf{c}}) - \tilde{\mathbf{c}}\pi'(\tilde{\mathbf{c}}))'.$$

### 13. Perpetual Margrabe Option

An interesting limiting case of (11.32) is the pricing formula for the perpetual American Margrabe option. Here,  $\tilde{\mathbf{b}} = 0$  and

$$(13.1) \quad \tilde{\mathbf{c}} = \frac{\theta_1}{\theta_1 - 1}.$$

The current price of the option is

$$(13.2) \quad W(S_1, S_2) = \begin{cases} V(S_1, S_2; 0, \tilde{c}) & \text{if } S_1/S_2 < \tilde{c} \\ S_1 - S_2 & \text{if } S_1/S_2 \geq \tilde{c} \end{cases}$$

$$= \begin{cases} \left( \frac{S_1}{\theta_1} \right)^{\theta_1} / \left( \frac{S_2}{\theta_1 - 1} \right)^{\theta_1 - 1} & \text{if } \frac{S_1}{\theta_1} < \frac{S_2}{\theta_1 - 1} \\ S_1 - S_2 & \text{if } \frac{S_1}{\theta_1} \geq \frac{S_2}{\theta_1 - 1} \end{cases}$$

We remark that special cases of the Margrabe option are the call and put options. Also,  $\{X(t)\}$  need not be a Wiener process for the pricing formula (13.2) to be valid: let  $\{X(t)\}$  be a process with stationary and independent increments, satisfying (8.9); if  $\{X_1(t) - X_2(t)\}$  is a skip-free (jump-free) upward process, then (13.2) is a pricing formula for the perpetual American Margrabe option.

#### 14. Forward and Futures Price of the Perpetual Margrabe Option

We conclude this paper by deriving the forward price and futures price of the perpetual Margrabe option. Because of the constant interest rate assumption, these two prices are the same. The current price of the perpetual Margrabe option is given by (13.2),

$$(14.1) \quad W(S_1, S_2) = \Theta S_1^{\theta_1} S_2^{1 - \theta_1} I(S_1/S_2 \leq \tilde{c}) + (S_1 - S_2) I(\tilde{c} < S_1/S_2),$$

where  $S_1$  and  $S_2$  are the current stock prices and

$$(14.2) \quad \Theta = \frac{(\theta_1 - 1)^{\theta_1 - 1}}{\theta_1^{\theta_1}}.$$

The m-year forward/futures price of the perpetual option is

$$(14.3) \quad \begin{aligned} & E[W(S_1(m), S_2(m)); \mathbf{h}^*] \\ &= \Theta E[S_1(m)^{\theta_1} S_2(m)^{1 - \theta_1} I(S_1(m)/S_2(m) \leq \tilde{c}); \mathbf{h}^*] \\ &+ E[(S_1(m) - S_2(m)) I(\tilde{c} < S_1(m)/S_2(m)); \mathbf{h}^*]. \end{aligned}$$

Applying the factorization formula (8.15) and that :

$\{e^{-\delta t} S_1(t)^{\theta_1} S_2(t)^{1 - \theta_1}\}$  is a martingale under the risk-neutral measure, we have

$$(14.4) \quad \begin{aligned} & E[S_1(m)^{\theta_1} S_2(m)^{1 - \theta_1} I(S_1(m)/S_2(m) \leq \tilde{c}); \mathbf{h}^*] \\ &= E[S_1(m)^{\theta_1} S_2(m)^{1 - \theta_1}; \mathbf{h}^*] E[I(S_1(m)/S_2(m) \leq \tilde{c}); \mathbf{h}^* + (\theta_1, 1 - \theta_1)'] \\ &= e^{\delta m} S_1^{\theta_1} S_2^{1 - \theta_1} \Pr[S_1(m)/S_2(m) \leq \tilde{c}; \mathbf{h}^* + (\theta_1, 1 - \theta_1)']. \end{aligned}$$

Similarly,

$$(14.5) \quad \begin{aligned} & E[(S_1(m) - S_2(m)) I(\tilde{c} < S_1(m)/S_2(m)); \mathbf{h}^*] \\ &= E[S_1(m); \mathbf{h}^*] E[I(\tilde{c} < S_1(m)/S_2(m)); \mathbf{h}^* + (1, 0)'] \\ &\quad - E[S_2(m); \mathbf{h}^*] E[I(\tilde{c} < S_1(m)/S_2(m)); \mathbf{h}^* + (0, 1)'] \\ &= e^{(\delta - \varphi_1)m} S_1 \Pr[\tilde{c} < S_1(m)/S_2(m); \mathbf{h}^* + (1, 0)'] \\ &\quad - e^{(\delta - \varphi_2)m} S_2 \Pr[\tilde{c} < S_1(m)/S_2(m); \mathbf{h}^* + (0, 1)']. \end{aligned}$$

Let us illustrate (14.3), (14.4) and (14.5) with a perpetual put option on a non-dividend-paying stock with a constant exercise price K. Thus we consider

$$S_1(t) \equiv K, \sigma_1 = 0, \varphi_1 = \delta,$$

$$S_2(t) \equiv S(t), \sigma_2 = \sigma, \varphi_2 = 0.$$

[Recall that  $S(t) = S(0)e^{X(t)}$ , where  $\{X(t)\}$  is a Wiener process with variance per unit time  $\sigma^2$ .] Equation (11.8) simplifies as

$$(14.6) \quad \frac{1}{2}\sigma^2\theta(1 - \theta) + \delta\theta = 0,$$

yielding

$$(14.7) \quad \theta_1 = 1 + \frac{2\delta}{\sigma^2}.$$

By (13.1)

$$(14.8) \quad \tilde{c} = \frac{\sigma^2 + 2\delta}{2\delta}.$$

With the definition

$$(14.9) \quad \kappa = \ln\left(\frac{K}{S(0)\tilde{c}}\right),$$

equation (14.5) becomes

$$(14.10) \quad \begin{aligned} & E[(K - S(m)) I(\tilde{c} < K/S(m)); h^*] \\ &= K\Pr[X(m) < \kappa; h^*] - e^{\delta m}S(0)\Pr[X(m) < \kappa; h^* + 1] \\ &= K\Phi\left(\frac{\kappa - (\delta - \frac{\sigma^2}{2})m}{\sigma\sqrt{m}}\right) - e^{\delta m}S(0)\Phi\left(\frac{\kappa - (\delta + \frac{\sigma^2}{2})m}{\sigma\sqrt{m}}\right), \end{aligned}$$

where  $\Phi$  is the standardized normal distribution.

The probability term in (14.4) is

$$(14.11) \quad \begin{aligned} \Pr[X(m) \geq \kappa; h^* + 1 - \theta_1] &= \Pr[X(m) \geq \kappa; h^* - \frac{2\delta}{\sigma^2}] \\ &= 1 - \Phi\left(\frac{\kappa - (-\delta - \frac{\sigma^2}{2})m}{\sigma\sqrt{m}}\right) \end{aligned}$$

$$= \Phi\left(\frac{-\kappa - (\delta + \frac{\sigma^2}{2})m}{\sigma\sqrt{m}}\right).$$

Also,

$$(14.12) \quad \Theta K^{\theta_1} S(0)^{1-\theta_1} = \frac{K}{\theta_1} \left(\frac{K}{S(0)\tilde{c}}\right)^{\theta_1-1} \\ = \frac{\sigma^2 K}{\sigma^2 + 2\delta} \exp(2\delta\kappa/\sigma^2).$$

It follows from (14.3), (14.4), (14.11), (14.12) and (14.10) that the m-year futures/ forward price of the perpetual Margrabe option is:

$$(14.13) \quad \frac{\sigma^2 K}{\sigma^2 + 2\delta} \exp\left[\delta\left(m + \frac{2\kappa}{\sigma^2}\right)\right] \Phi\left(\frac{-\kappa - (\delta + \frac{\sigma^2}{2})m}{\sigma\sqrt{m}}\right) \\ + K \Phi\left(\frac{\kappa - (\delta - \frac{\sigma^2}{2})m}{\sigma\sqrt{m}}\right) - S(0)e^{\delta m} \Phi\left(\frac{\kappa - (\delta + \frac{\sigma^2}{2})m}{\sigma\sqrt{m}}\right).$$

Another derivation of (14.13) can be found in Gerber and Shiu [GS93].

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