



## Continuous Auctions and Insider Trading

Albert S. Kyle

*Econometrica*, Vol. 53, No. 6. (Nov., 1985), pp. 1315-1336.

Stable URL:

<http://links.jstor.org/sici?sici=0012-9682%28198511%2953%3A6%3C1315%3ACAAIT%3E2.0.CO%3B2-8>

*Econometrica* is currently published by The Econometric Society.

---

Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at <http://www.jstor.org/about/terms.html>. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at <http://www.jstor.org/journals/econosoc.html>.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

---

JSTOR is an independent not-for-profit organization dedicated to and preserving a digital archive of scholarly journals. For more information regarding JSTOR, please contact [support@jstor.org](mailto:support@jstor.org).

## CONTINUOUS AUCTIONS AND INSIDER TRADING

BY ALBERT S. KYLE<sup>1</sup>

A dynamic model of insider trading with sequential auctions, structured to resemble a sequential equilibrium, is used to examine the informational content of prices, the liquidity characteristics of a speculative market, and the value of private information to an insider. The model has three kinds of traders: a single risk neutral insider, random noise traders, and competitive risk neutral market makers. The insider makes positive profits by exploiting his monopoly power optimally in a dynamic context, where noise trading provides camouflage which conceals his trading from market makers. As the time interval between auctions goes to zero, a limiting model of continuous trading is obtained. In this equilibrium, prices follow Brownian motion, the depth of the market is constant over time, and all private information is incorporated into prices by the end of trading.

### 1. INTRODUCTION

HOW QUICKLY IS NEW PRIVATE INFORMATION about the underlying value of a speculative commodity incorporated into market prices? How valuable is private information to an insider? How does noise trading affect the volatility of prices? What determines the liquidity of a speculative market? The purpose of this paper is to show how answers to questions like these can be obtained as derived results by modelling rigorously the trading strategy of an insider in a dynamic model of efficient price formation.

In the particular model we investigate, one risky asset is exchanged for a riskless asset among three kinds of traders: a single insider who has unique access to a private observation of the *ex post* liquidation value of the risky asset; uninformed noise traders who trade randomly; and market makers who set prices efficiently (in the semi-strong sense) conditional on information they have about the quantities traded by others. Trading is modelled as a sequence of many auctions, structured to give the model the flavor of a sequential equilibrium as described by Kreps and Wilson [4].

At each auction trading takes place in two steps. In step one, the insider and the noise traders simultaneously choose the quantities they will trade (in effect, placing "market orders"). When making this choice, the insider's information consists of his private observation of the liquidation value of the asset, as well as past prices and past quantities traded by himself. He does not observe current or future prices, or current or future quantities traded by noise traders. The random quantity traded by noise traders is distributed independently from present or past quantities traded by the insider and independently from past quantities traded by noise traders. In step two, the market makers set a price, and trade the quantity which makes markets clear. When doing so, their information consists of observations of the current and past aggregate quantities traded by the insider and noise traders combined. We call these aggregate quantities the "order flow."

<sup>1</sup> The author thanks the Centre of Policy Studies, Monash University, and the Yale School of Organization and Management, Yale University, for their hospitality and support when this research was undertaken. The author also thanks Paul Milgrom, Peter Hartley, Phil Dybvig, and Avinash Dixit for useful comments.

Market makers do not observe the individual quantities traded by the insider or noise traders separately, nor do they have any other kind of special information. As a consequence, price fluctuations are always a consequence of order flow innovations.

The informed trader, who is risk neutral, is assumed to maximize expected profits. He acts as an intertemporal monopolist in the asset market, taking into account explicitly the effect his trading at one auction has on the price at that auction and the trading opportunities available at future auctions. The prices determined by market makers are assumed to equal the expectation of the liquidation value of the commodity, conditional on the market makers' information sets at the dates the prices are determined. Thus, market makers earn on average zero profits. Since they cannot distinguish the trading of the insider from the trading of noise traders, the noise traders in effect provide camouflage which enables the insider to make profits at their expense.

By assuming that the relevant random variables are normally distributed, the model acquires a tractable linear structure. This makes it possible to characterize explicitly a unique "sequential auction equilibrium" in which prices and quantities are simple linear functions of the observations defining the relevant information sets. In the limit as the time interval between auctions goes to zero, the discrete-time equilibrium converges to a particularly simple limit which we call a "continuous auction equilibrium." This equilibrium corresponds to what one obtains when the model is set up heuristically in continuous time.

In both the discrete model and the continuous limit, answers to the questions posed at the beginning of this paper are readily obtained. The informed trader trades in such a way that his private information is incorporated into prices gradually. In the continuous auction equilibrium where the quantity traded by noise traders follows a Brownian motion process, prices also follow Brownian motion. The constant volatility reflects the fact that information is incorporated into prices at a constant rate. Furthermore, all of the insider's private information is incorporated into prices by the end of trading in a continuous auction equilibrium. An *ex ante* doubling of the quantities traded by noise traders induces the insider and market makers to double the quantities they trade, but has no effect on prices, and thus doubles the profits of the insider.

Perhaps the most interesting properties concern the liquidity characteristics of the market in a continuous auction equilibrium. "Market liquidity" is a slippery and elusive concept, in part because it encompasses a number of transactional properties of markets. These include "tightness" (the cost of turning around a position over a short period of time), "depth" (the size of an order flow innovation required to change prices a given amount), and "resiliency" (the speed with which prices recover from a random, uninformative shock). Black [2] describes intuitively a liquid market in the following manner:

"The market for a stock is liquid if the following conditions hold:

- (1) There are always bid and asked prices for the investor who wants to buy or sell small amounts of stock immediately.
- (2) The difference between the bid and asked prices (the spread) is always small.

(3) An investor who is buying or selling a large amount of stock, in the absence of special information, can expect to do so over a long period of time at a price not very different, on average, from the current market price.

(4) An investor can buy or sell a large block of stock immediately, but at a premium or discount that depends on the size of the block. The larger the block, the larger the premium or discount.

In other words, a liquid market is a continuous market, in the sense that almost any amount of stock can be bought or sold immediately, and an efficient market, in the sense that small amounts of stock can always be bought and sold very near the current market price, and in the sense that large amounts can be bought or sold over long periods of time at prices that, on average, are very near the current market price."

Roughly speaking, Black defines a liquid market as one which is almost infinitely tight, which is not infinitely deep, and which is resilient enough so that prices eventually tend to their underlying value.

Our continuous auction equilibrium has exactly the characteristics described by Black. Furthermore, these aspects of market liquidity acquire a new prominence in our model because the insider, who does not trade as a perfect competitor, must make rational conjectures about tightness, depth, and resiliency in choosing his optimal quantity to trade. Moreover, depth and resiliency are themselves endogenous consequences of the presence of the insider and noise traders in the market. Market depth is proportional to the amount of noise trading and inversely proportional to the amount of private information (in the sense of an error variance) which has not yet been incorporated into prices. This makes our model a rigorous version of the intuitive story told by Bagehot [1]. Furthermore, our emphasis on the dynamic optimizing behavior of the insider distinguishes our model from the one of Glosten and Milgrom [3].

The plan of the rest of this paper is as follows: In Section 2, a single auction equilibrium is discussed in order to motivate the dynamic models which follow. In Section 3, a sequential auction equilibrium is defined, an existence and uniqueness result is proved, and properties of the equilibrium are derived. In Section 4, a continuous auction equilibrium is discussed heuristically, and in Section 5, it is shown that the continuous auction equilibrium is the limit of the sequential auction equilibrium as the time interval between auctions goes to zero. Section 6 makes some concluding comments.

## 2. A SINGLE AUCTION EQUILIBRIUM

In this section we motivate our equilibrium concept by discussing a simple model of one-shot trading.

*Structure and Notation.* The *ex post* liquidation value of the risky asset, denoted  $\tilde{v}$ , is normally distributed with mean  $p_0$  and variance  $\Sigma_0$ . The quantity traded by noise traders, denoted  $\tilde{u}$ , is normally distributed with mean zero and variance  $\sigma_u^2$ . The random variables  $\tilde{v}$  and  $\tilde{u}$  are independently distributed. The quantity traded by the insider is denoted  $\tilde{x}$  and the price is denoted  $\tilde{p}$ .

Trading is structured in two steps as follows: In step one, the exogenous values of  $\tilde{v}$  and  $\tilde{u}$  are realized and the insider chooses the quantity he trades  $\tilde{x}$ . When

doing so, he observes  $\tilde{v}$  but not  $\tilde{u}$ . To accommodate mixed strategies, the insider's trading strategy, denoted  $X$ , assigns to outcomes of  $\tilde{v}$  probability distributions defined over quantities traded. Since, however, mixed strategies are not optimal in what follows, the more intuitive interpretation of  $X$  as a measurable function such that  $\tilde{x} = X(\tilde{v})$  is justified. In step two, the market makers determine the price  $\tilde{p}$  at which they trade the quantity necessary to clear the market. When doing so they observe  $\tilde{x} + \tilde{u}$  but not  $\tilde{x}$  or  $\tilde{u}$  (or  $\tilde{v}$ ) separately. While their pricing rule, denoted  $P$ , can be defined to accommodate randomization, an intuitive interpretation of  $P$  as a measurable real function such that  $\tilde{p} = P(\tilde{x} + \tilde{u})$  is also justified.

The profits of the informed trader, denoted  $\tilde{\pi}$ , are given by  $\tilde{\pi} = (\tilde{v} - \tilde{p})\tilde{x}$ . To emphasize the dependence of  $\tilde{\pi}$  and  $\tilde{p}$  on  $X$  and  $P$ , we write  $\tilde{\pi} = \tilde{\pi}(X, P)$ ,  $\tilde{p} = \tilde{p}(X, P)$ .

*Definition of Equilibrium.* An *equilibrium* is defined as a pair  $X, P$  such that the following two conditions hold:

(1) *Profit Maximization:* For any alternate trading strategy  $X'$  and for any  $v$ ,

$$(2.1) \quad E\{\tilde{\pi}(X, P) | \tilde{v} = v\} \geq E\{\tilde{\pi}(X', P) | \tilde{v} = v\}.$$

(2) *Market Efficiency:* The random variable  $\tilde{p}$  satisfies

$$(2.2) \quad \tilde{p}(X, P) = E\{\tilde{v} | \tilde{x} + \tilde{u}\}.$$

This model is not quite a game theoretic one because the market makers do not explicitly maximize any particular objective. We could, however, replace the market efficiency condition in step two with an explicit Bertrand auction between at least two risk neutral bidders, each of whom observes the "order flow"  $\tilde{x} + \tilde{u}$  and nothing else. The result of this explicit auction procedure would be our market efficiency condition, in which profits of market makers are driven to zero. Modelling how market makers can earn the positive frictional profits necessary to attract them into the business of market making is an interesting topic which takes us away from our main objective of studying how price formation is influenced by the optimizing behavior of an insider in a somewhat idealized setting. Kyle [5], however, discusses a model of imperfect competition among market makers, in which many insiders with different information participate.

The insider exploits his monopoly power by taking into account the effect the quantity he chooses to trade in step one is expected to have on the price established in step two. In doing so, he takes the rule market makers use to set prices in step two as given. He is not allowed to influence this rule by committing to a particular strategy in step one: The quantity he trades is required to be optimal, given his information set at the time it is chosen. This requirement seems to be reasonable given anonymous trading and the strong incentives informed traders have to cheat given any other strategy they commit to. The insider is not allowed to condition the quantity he trades on price. A model in which insiders choose demand functions ("limit orders") instead of quantities ("market orders") is considered in Kyle [6].

Fortuitously, our model has an analytically tractable equilibrium in which the rules  $X$  and  $P$  are simple linear functions, as we show in the following theorem:

**THEOREM 1:** *There exists a unique equilibrium in which  $X$  and  $P$  are linear functions. Defining constants  $\beta$  and  $\lambda$  by  $\beta = (\sigma_u^2 / \Sigma_0)^{1/2}$  and  $\lambda = 2(\sigma_u^2 / \Sigma_0)^{-1/2}$ , the equilibrium  $P$  and  $X$  are given by*

$$(2.3) \quad X(\tilde{v}) = \beta(\tilde{v} - p_0), \quad P(\tilde{x} + \tilde{u}) = p_0 + \lambda(\tilde{x} + \tilde{u}).$$

*Proof:* Suppose that for constants  $\mu, \lambda, \alpha, \beta$ , linear functions  $P$  and  $X$  are given by

$$(2.4) \quad P(y) = \mu + \lambda y, \quad X(v) = \alpha + \beta v.$$

Given the linear rule  $P$ , profits can be written

$$(2.5) \quad E\{[\tilde{v} - P(x + \tilde{u})]x | \tilde{v} = v\} = (v - \mu - \lambda x)x.$$

Profit maximization of this quadratic objective requires that  $x$  solve  $v - \mu - 2\lambda x = 0$ . We thus have  $X(v) = \alpha + \beta v$  with

$$(2.6) \quad 1/\beta = 2\lambda, \quad \alpha = -\mu\beta.$$

Note that the quadratic objective (implied by the linear pricing rule  $P$ ) rules out mixed strategies and also makes linear strategies optimal even when nonlinear strategies are allowed.

Given linear  $X$  and  $P$ , the market efficiency condition is equivalent to

$$(2.7) \quad \mu + \lambda y = E\{\tilde{v} | \alpha + \beta\tilde{v} + \tilde{u} = y\}.$$

Normality makes the regression linear and application of the projection theorem yields

$$(2.8) \quad \lambda = \frac{\beta \Sigma_0}{\beta^2 \Sigma_0 + \sigma_u^2}, \quad \mu - p_0 = -\lambda(\alpha + \beta p_0).$$

Solving (2.6) and (2.8) subject to the second order condition  $\lambda > 0$  yields the desired result. Note that we have  $\mu = p_0$ ,  $\alpha = -\beta p_0$ , and the second order condition rules out a solution with  $\beta$  and  $\lambda$  both negative. This completes the proof of the theorem.

*Properties of the Equilibrium.* The equilibrium  $X$  and  $P$  are determined by the exogenous parameters  $\Sigma_0$  and  $\sigma_u^2$ . To obtain a measure of the informativeness of prices, define  $\Sigma_1$  by  $\Sigma_1 = \text{var}\{\tilde{v} | \tilde{p}\}$ . A simple calculation shows that  $\Sigma_1 = \frac{1}{2}\Sigma_0$ ; thus, one-half of the insider's private information is incorporated into prices and the volatility of prices is unaffected by the level of noise trading  $\sigma_u^2$ .

The quantity  $1/\lambda$  measures the "depth" of the market, i.e. the order flow necessary to induce prices to rise or fall by one dollar. This measure of market liquidity is proportional to a ratio of the amount of noise trading to the amount of private information the informed trader is expected to have. In this sense, it

captures Bagehot's [1] intuition that market makers compensate themselves for bad trades due to the adverse selection of insiders by making the market less liquid. Maximized profits, given by  $v^2/(4\lambda)$ , are proportional to the depth of the market, because a proportional horizontal expansion of the supply curve induces the monopsonistic insider to trade a proportionately larger quantity without affecting prices, and this makes his profits correspondingly larger as well. Since an increase in noise trading brings forth more informed trading, it does not destabilize prices (a result which would disappear if the insider were risk averse). The expected profits of the insider (unconditional on  $\tilde{v}$ ) are given by  $E(\tilde{\pi}) = \frac{1}{2}(\Sigma_0\sigma_u^2)^{1/2}$ : The insider's profits are proportional to the standard deviations of both  $\tilde{v}$  and  $\tilde{u}$ .

As shown below, many of these properties generalize to the sequential auction model in an appropriate way.

### 3. A SEQUENTIAL AUCTION EQUILIBRIUM

In this section we generalize the model of one-shot trading by examining a model in which a number of auctions, or rounds of trading, take place sequentially. The resulting dynamic model is structured so that equilibrium prices at each auction reflect the information contained in the past and current order flow and so that the insider maximizes his expected profits, taking into account his effect on prices in both the current auction and in future auctions.

*Structure and Notation.* Trading takes place over one trading day, which begins at time  $t = 0$  and ends at time  $t = 1$ . There are  $N$  auctions, with  $t_n$  denoting the time at which the  $n$ th auction takes place. We assume

$$(3.1) \quad 0 = t_0 < t_1 < \cdots < t_N = 1,$$

so the sequence of auction dates  $\langle t_n \rangle$  partitions the interval  $[0, 1]$ .

Let  $\tilde{u}(t)$  denote a Brownian motion partition process with instantaneous variance  $\sigma_u^2$ , and define  $\tilde{u}_n$  and  $\Delta\tilde{u}_n$  by  $\tilde{u}_n = \tilde{u}(t_n)$  and  $\Delta\tilde{u}_n = \tilde{u}_n - \tilde{u}_{n-1}$ . We assume that the quantity traded by noise traders at the  $n$ th auction is  $\Delta\tilde{u}_n$ . The Brownian motion assumption implies that  $\Delta\tilde{u}_n$  is normally distributed with zero mean and variance  $\sigma_u^2 \Delta t_n$ , where  $\Delta t_n = t_n - t_{n-1}$ , and that the quantity traded at one auction is independent of the quantity traded at other auctions. The liquidation value of the asset,  $\tilde{v}$ , is still assumed to be normally distributed with mean  $p_0$  and variance  $\Sigma_0$ . The random variable  $\tilde{v}$  is distributed independently of the entire process  $\tilde{u}(t)$ .

The  $N$  auctions take place sequentially. Let  $\tilde{x}_n$  denote the aggregate position of the insider after the  $n$ th auction, so that  $\Delta\tilde{x}_n$  (defined by  $\Delta\tilde{x}_n = \tilde{x}_n - \tilde{x}_{n-1}$ ) denotes the quantity traded by the insider at the  $n$ th auction. Let  $\tilde{p}_n$  denote the market clearing price at the  $n$ th auction. At each auction, trade is structured in two steps as before, with information sets modified to take into account relevant information from past auctions. Since mixed trading strategies and random pricing rules are not optimal in what follows, we are justified in interpreting the trading rules and pricing rules as functions of the relevant observations. According to this interpretation, when the insider chooses the quantity to trade at step one of

an auction, he not only observes the liquidation value of the asset,  $\tilde{v}$ , but also past prices as well. Accordingly, for some measurable function  $X_n$ , his position after the  $n$ th auction is given by

$$(3.2) \quad \tilde{x}_n = X_n(\tilde{p}_1, \dots, \tilde{p}_{n-1}, \tilde{v}) \quad (n = 1, \dots, N),$$

from which the actual quantity traded is easily determined using information in the information set. When market makers set a market clearing price at step two of the  $n$ th auction, they not only observe the current value of the order flow,  $\Delta\tilde{x}_n + \Delta\tilde{u}_n$ , but they observe past values of the order flow as well. Accordingly, for some measurable function  $P_n$ , the price  $\tilde{p}_n$  is determined by

$$(3.3) \quad \tilde{p}_n = P_n(\tilde{x}_1 + \tilde{u}_1, \dots, \tilde{x}_n + \tilde{u}_n) \quad (n = 1, \dots, N).$$

Note that in the absence of mixed strategies, the insider can infer from his information set the quantities he has traded at past auctions, and the market makers can infer from their information set the prices they have set in the past. Note also that the insider can infer the quantities traded by noise traders in the past if the functions  $P_n$  are monotonic in their last arguments.

Now define the vectors of functions  $X$  and  $P$  by

$$(3.4) \quad X = \langle X_1, \dots, X_N \rangle, \quad P = \langle P_1, \dots, P_N \rangle.$$

We refer to  $X$  as the informed trader's "trading strategy" and to  $P$  as the market makers' "pricing rule."

For  $n = 1, \dots, N$ , let  $\tilde{\pi}_n$  denote the profits of the insider on positions acquired at auctions  $n, \dots, N$ . Clearly,  $\tilde{\pi}_n$  is given by

$$(3.5) \quad \tilde{\pi}_n = \sum_{k=n}^N (\tilde{v} - \tilde{p}_k) \tilde{x}_k \quad (n = 1, \dots, N).$$

To emphasize the dependence of  $\tilde{p}_n$ ,  $\tilde{x}_n$ , and  $\tilde{\pi}_n$  on  $P$  and  $X$ , we sometimes write

$$(3.6) \quad \tilde{p}_n = \tilde{p}_n(X, P), \quad \tilde{x}_n = \tilde{x}_n(X, P), \quad \tilde{\pi}_n = \tilde{\pi}_n(X, P).$$

*Equilibrium.* A *sequential auction equilibrium* is defined as a pair  $X, P$  such that the following two conditions hold:

1. *Profit Maximization:* For all  $n = 1, \dots, N$  and for all  $X' = \langle X'_1, \dots, X'_N \rangle$  such that  $X'_1 = X_1, \dots, X'_{n-1} = X_{n-1}$ , we have

$$(3.7) \quad E\{\tilde{\pi}_n(X, P) | \tilde{p}_1, \dots, \tilde{p}_{n-1}, \tilde{v}\} \geq E\{\tilde{\pi}_n(X', P) | \tilde{p}_1, \dots, \tilde{p}_{n-1}, \tilde{v}\}.$$

2. *Market Efficiency:* For all  $n = 1, \dots, N$ , we have

$$(3.8) \quad \tilde{p}_n = E\{\tilde{v} | \tilde{x}_1 + \tilde{u}_1, \dots, \tilde{x}_n + \tilde{u}_n\}.$$

A *linear equilibrium* is defined as a (sequential auction) equilibrium in which the component functions of  $X$  and  $P$  are linear, and a *recursive linear equilibrium* is defined as a linear equilibrium in which there exist constants  $\lambda_1, \dots, \lambda_N$  such that for  $n = 1, \dots, N$ ,

$$(3.9) \quad \tilde{p}_n = \tilde{p}_{n-1} + \lambda_n(\Delta\tilde{x}_n + \Delta\tilde{u}_n).$$



The market efficiency condition implies that trading prices follow a martingale whose pattern of volatility over time reflects the rate at which information is incorporated into prices. In a linear equilibrium, price increments are normally and independently distributed with zero means; thus, the distribution function for the pricing process is characterized by a sequence of variance parameters measuring the volatility of price fluctuations from auction to auction.

The profit maximization condition gives our equilibrium the flavor of a sequential equilibrium (as discussed by Kreps and Wilson [4]). The quantity  $\tilde{x}_n$  chosen at the  $n$ th auction maximizes expected profits over the remaining rounds of trading given the information available to the insider when he chooses it. There is no commitment to strategies. This means that the insider cannot influence the pricing rule by committing to a trading rule before prices are established. Conversely, while the market makers impute a trading strategy to the insider, they do not observe it; they only observe the order flow. Note, however, that the profit maximization condition implies that for all trading strategies  $X'$ ,

$$(3.10) \quad E\{\tilde{\pi}_n(X, P)\} \geq E\{\tilde{\pi}_n(X', P)\}.$$

*Characterization of Equilibrium.* In the rest of this section, we prove existence of a unique linear equilibrium, show that it is a recursive linear equilibrium, and characterize it as the solution to a difference equation system subject to boundary conditions. We suspect, but have not been able to prove, that equilibria with nonlinear  $X_n$  and  $P_n$  do not exist.

**THEOREM 2:** *There exists a unique linear equilibrium and this equilibrium is a recursive linear equilibrium. In this equilibrium there are constants  $\beta_n, \lambda_n, \alpha_n, \delta_n,$  and  $\Sigma_n$  such that for*

$$(3.11) \quad \Delta \tilde{x}_n = \beta_n (\tilde{v} - \tilde{p}_{n-1}) \Delta t_n,$$

$$(3.12) \quad \Delta \tilde{p}_n = \lambda_n (\Delta \tilde{x}_n + \Delta \tilde{u}_n),$$

$$(3.13) \quad \Sigma_n = \text{var}(\tilde{v} | \Delta \tilde{x}_1 + \Delta \tilde{u}_1, \dots, \Delta \tilde{x}_n + \Delta \tilde{u}_n),$$

$$(3.14) \quad E\{\tilde{\pi}_n | p_1, \dots, p_{n-1}, v\} = \alpha_{n-1} (v - p_{n-1})^2 + \delta_{n-1} \quad (n = 1, \dots, N).$$

Given  $\Sigma_0$ , the constants  $\beta_n, \lambda_n, \alpha_n, \delta_n, \Sigma_n$  are the unique solution to the difference equation system

$$(3.15) \quad \alpha_{n-1} = \frac{1}{4\lambda_n(1 - \alpha_n\lambda_n)},$$

$$(3.16) \quad \delta_{n-1} = \delta_n + \alpha_n \lambda_n^2 \sigma_u^2 \Delta t_n,$$

$$(3.17) \quad \beta_n \Delta t_n = \frac{1 - 2\alpha_n\lambda_n}{2\lambda_n(1 - \alpha_n\lambda_n)},$$

$$(3.18) \quad \lambda_n = \beta_n \Sigma_n / \sigma_u^2,$$

$$(3.19) \quad \Sigma_n = (1 - \beta_n \lambda_n \Delta t_n) \Sigma_{n-1} \quad (n = 1, \dots, N),$$

subject to  $\alpha_N = \delta_N = 0$  and the second order condition

$$(3.20) \quad \lambda_n(1 - \alpha_n \lambda_n) > 0.$$

REMARK: The parameters  $\beta_n (n = 1, \dots, N)$ , which characterize the insider's trading strategy  $X_n$ , measure the intensity with which the insider trades on the basis of his private observation, and the parameters  $\lambda_n (n = 1, \dots, N)$ , which characterize the recursive pricing rule, measure the depth of the market (with small  $\lambda_n$  corresponding to a deep market). The parameters  $\Sigma_n (n = 1, \dots, N)$ , which give the error variance of prices after the  $n$ th auction, measure how much of the insider's private information is not yet incorporated into prices (as estimated by market makers). Note that  $\Sigma_0$  is just the variance of the initial prior price  $p_0$ . The parameters  $\alpha_{n-1}$  and  $\delta_{n-1}$  define a quadratic profit function which gives the value of trading opportunities at auctions  $n, \dots, N$ .

*Outline of Proof:* The proof of the theorem is divided into three steps. In the first step, which is the most important one, a backward induction argument is used to obtain the insider's trading strategy and expected trading profits as a function of the pricing rule. Since the pricing rule is characterized by the market depth parameters  $\lambda_n$ , the insider's problem is intuitively one of deciding how intensely to trade on the basis of his private information, given the pattern of market depth expected at current and future auctions. If market depth at future auctions is greater than market depth at the current auction, the insider has an incentive to "save" his private information by trading small quantities now and large quantities later. Conversely, if market depth declines in future auctions, the insider has an incentive to trade intensely at the current auction, where profits are greater.

Intuitively, the second order condition (3.20) rules out a situation in which the insider can make unbounded profits by first destabilizing prices with unprofitable trades made at the  $n$ th auction, then recouping the losses and much more with profitable trades made at future auctions. When  $\lambda_n$  is large, it does not cost much to destabilize prices at the  $n$ th auction (because trading small quantities is sufficient), but when  $\alpha_n$  is large, the value of future trading opportunities to the insider from moving the price far away from its liquidation value is large. The second order condition accordingly rules out unbounded destabilization schemes by placing an upper bound on  $\lambda_n$  which decreases in  $\alpha_n$ .

The backward induction argument in step one of the proof simultaneously shows that the insider's profit function is quadratic and that the linear equilibrium is recursive. In addition, it shows explicitly how the parameter  $\alpha_n$ , which measures the value of private information at future auctions  $n + 1, \dots, N$  as a function of market depth at those auctions, combines with the current market depth parameter  $\lambda_n$  to generate via backward induction values of  $\beta_n$  and  $\alpha_{n-1}$ .

In step two of the proof, the market efficiency condition is used to derive  $\lambda_n$  and  $\Sigma_n$  from  $\beta_n$  and  $\Sigma_{n-1}$ . The idea here is that, given the level of noise trading ( $\sigma_u^2 \Delta t_n$ ), the depth of the market at a particular auction ( $\lambda_n$ ) depends negatively upon how much private information the insider has ( $\Sigma_{n-1}$ ) and how intensely

the insider trades upon the basis of his private information ( $\beta_n$ ), and this also determines how much of the insider's remaining private information is revealed at the particular auction and how much still remains private ( $\Sigma_n$ ). This step of the proof makes precise Bagehot's idea that market makers respond to insider trading by reducing the liquidity of the market.

In step three of the proof, it is shown that the relationships derived in the first two steps generate a difference equation system which characterizes the unique linear equilibrium.

PROOF: We now give the details of the three steps of the proof.

*Step 1.* To prove by backward induction that the informed trader's expected profits are of the quadratic form specified in (3.14), we begin with the boundary condition  $\alpha_N = \delta_N = 0$ , which states that no profits on new positions are made after trade is completed. Now make the inductive hypothesis that for constants  $\alpha_n$  and  $\delta_n$ , we have

$$(3.21) \quad E\{\tilde{\pi}_{n+1}(X, P) | p_1, \dots, p_n, v\} = \alpha_n (v - p_n)^2 + \delta_n.$$

Since  $\tilde{\pi}_n$  is given recursively by  $\tilde{\pi}_n = (\tilde{v} - \tilde{p}_n) \Delta \tilde{x}_n + \tilde{\pi}_{n+1}$ , we obtain

$$(3.22) \quad E\{\tilde{\pi}_n | p_1, \dots, p_{n-1}, v\} = \max_{\Delta x} E\{(\tilde{v} - \tilde{p}_n) \Delta x + \alpha_n (\tilde{v} - \tilde{p}_n)^2 + \delta_n | p_1, \dots, p_{n-1}, v\}.$$

In a linear equilibrium,  $p_n$  is given by

$$(3.23) \quad p_n = p_{n-1} + \lambda_n (\Delta x_n + \Delta u_n) + h,$$

where  $h$  is some linear function of  $\Delta x_1 + \Delta u_1, \dots, \Delta x_{n-1} + \Delta u_{n-1}$ . Plugging (3.23) into (3.22) and evaluating the conditional expectation yields

$$(3.24) \quad E\{\tilde{\pi}_n | p_1, \dots, p_{n-1}, v\} = \max_{\Delta x} \{(v - p_{n-1} - \lambda_n \Delta x - h) \Delta x + \alpha_n (v - p_{n-1} - \lambda_n \Delta x - h)^2 + \alpha_n \lambda_n^2 \sigma_u^2 \Delta t_n + \delta_n\}.$$

Since maximized profits are quadratic in  $\Delta x$ , the maximizing  $\Delta x$  (which we write  $\Delta x_n$ ) is easily shown to be given by

$$(3.25) \quad \Delta x_n = \beta_n (v - p_{n-1} - h) \Delta t_n,$$

where  $\beta_n \Delta t_n$  is given by (3.17). The second order condition is (3.20).

We now claim that  $\tilde{h} = 0$ . To prove this, observe that the market efficiency condition implies

$$(3.26) \quad E\{\Delta \tilde{p}_n | \Delta \tilde{x}_1 + \Delta \tilde{u}_1, \dots, \Delta \tilde{x}_{n-1} + \Delta \tilde{u}_{n-1}\} = 0.$$

An explicit calculation shows, however, that

$$(3.27) \quad E\{\Delta \tilde{p}_n | \Delta \tilde{x}_1 + \Delta \tilde{u}_1, \dots, \Delta \tilde{x}_{n-1} + \Delta \tilde{u}_{n-1}\} = \frac{\tilde{h}}{2(1 - \alpha_n \lambda_n)}.$$

Clearly, then, we must have  $\tilde{h} = 0$  (with probability one). From this it follows from (3.23) and (3.25) that  $\Delta\tilde{p}_n$  and  $\Delta\tilde{x}_n$  have the recursive form given by (3.11) and (3.12), as stated in the theorem. Furthermore, it is also easy to show from (3.24) that  $\alpha_{n-1}$  and  $\delta_{n-1}$  are given by (3.15) and (3.16).

*Step 2.* Consider the values of  $\lambda_n$  and  $\Sigma_n$  consistent with the market efficiency condition. Because  $\tilde{v} - \tilde{p}_{n-1}$  is independent from  $\Delta\tilde{x}_1 + \Delta\tilde{u}_1, \dots, \Delta\tilde{x}_{n-1} + \Delta\tilde{u}_{n-1}$ , we obtain from the market efficiency condition

$$(3.28) \quad \tilde{p}_n - \tilde{p}_{n-1} = E\{\tilde{v} - \tilde{p}_{n-1} | \Delta\tilde{x}_n + \Delta\tilde{u}_n\}.$$

Using (3.11), a simple application of the projection theorem for normally distributed random variables confirms that  $\Delta\tilde{p}_n$  is indeed of the form specified in (3.12), as was shown necessary in step one, and yields the following explicit expressions for  $\lambda_n$  and  $\Sigma_n$ :

$$(3.29) \quad \lambda_n = \frac{\beta_n \Sigma_{n-1}}{\beta_n^2 \Sigma_{n-1} \Delta t_n + \sigma_u^2},$$

$$\Sigma_n = \frac{\sigma_u^2 \Sigma_{n-1}}{\beta_n^2 \Sigma_{n-1} \Delta t_n + \sigma_u^2}.$$

These are equivalent to (3.18) and (3.19).

*Step 3.* In steps one and two above, it is shown that given  $\Sigma_0$ , equations (3.11)-(3.20) and the boundary condition  $\alpha_N = \delta_N = 0$  are necessary for a linear equilibrium. Clearly, they are also sufficient. We now show that given  $\Sigma_0$ , the difference equation system (3.15)-(3.19) has a unique solution satisfying the boundary condition  $\alpha_N = \delta_N = 0$  and the second order condition (3.20); it thus characterizes the unique linear equilibrium.

We first claim that given nonnegative values of  $\alpha_n$  and  $\Sigma_n$ , there is a unique way to iterate the system backwards such that the second order condition is satisfied. To prove this, combine equations (3.18) with (3.17) and simplify, obtaining

$$(3.30) \quad (1 - \lambda_n^2 \sigma_u^2 \Delta t_n / \Sigma_n)(1 - \alpha_n \lambda_n) = \frac{1}{2}.$$

Given nonnegative  $\alpha_n$  and  $\Sigma_n$ , this is a cubic equation in  $\lambda_n$ , which has three real roots. While neither the largest nor the smallest root satisfies the second order condition, the middle root does satisfy the second order condition. Thus,  $\lambda_n$  is uniquely defined by (3.30). Furthermore, given  $\lambda_n$ , it is a simple matter to obtain  $\beta_n$ ,  $\alpha_{n-1}$ ,  $\delta_{n-1}$ , and  $\Sigma_{n-1}$  from (3.15)-(3.19), and we have therefore iterated the system backwards one step.

Given the boundary condition  $\alpha_N = \delta_N = 0$ , the backward iteration procedure defines a family of solutions to the difference equation system parametrized by the terminal value  $\Sigma_N$  used to start off the backward iteration. We now claim that only one terminal value exists such that the correct initial value  $\Sigma_0$  is obtained at the last step of the backward iteration. To prove this, observe that if  $\alpha_{n-1}$ ,  $\delta_{n-1}$ ,  $\Sigma_{n-1}$ ,  $\beta_n$ ,  $\lambda_n$  ( $n = 1, \dots, N$ ) solve the difference equation system given arbitrary  $\Sigma_N$  and  $\alpha_N = \delta_N = 0$ , then for any positive constant  $\zeta$ , it is also true that

$\zeta\alpha_{n-1}, \zeta\delta_{n-1}, \zeta^2\Sigma_{n-1}, \zeta\beta_n, \zeta^{-1}\lambda_n$  ( $n = 1, \dots, N$ ) solve the difference equation system when the terminal value is  $\zeta^2\Sigma_N$ . Since this implies that  $\Sigma_N$  is proportional to  $\Sigma_0$  in any solution, there is a unique solution for any initial value  $\Sigma_0$ . This completes the proof of the theorem.

*Properties of the Equilibrium.* It is apparent from inspecting the difference equation system that many of the properties of the single auction equilibrium generalize to the sequential auction equilibrium. From (3.19), it is clear that the parameter  $\Sigma_n$ , which measures the informativeness of prices, declines monotonically, reflecting the fact that information is gradually incorporated into prices. While we have  $\Sigma_N > 0$  (so not all information is incorporated into prices by the end of trading), we show below that  $\Sigma_N$  may be very small. It is clear from inspection of the difference equation system that if  $\sigma_u$  doubles, then  $\lambda_n$  halves;  $\alpha_n, \delta_n$ , and  $\beta_n$  double; and  $\Sigma_n$  is unaffected. Thus, increasing the amount of noise trading increases market depth proportionately, increases proportionately the profits of the insider by encouraging him to trade more, and leaves the informativeness of prices unchanged. It is clear from the proof that if the amount of prior inside information, as measured by  $\Sigma_0^{1/2}$ , increases, then market depth decreases proportionately but expected (ex ante) profits increase proportionately. Thus, profits are proportional to  $(\Sigma_0\sigma_u^2)^{1/2}$ , as in the single auction model.

The rest of this paper investigates the properties of the equilibrium in more detail by examining what happens to the equilibrium when the interval between auctions becomes very small. In particular, we are interested in learning more about the dynamic behavior of  $\Sigma_n$  and  $\lambda_n$ .

## 5. A CONTINUOUS AUCTION EQUILIBRIUM

In this section we discuss unrigorously a model in which trading takes place continuously rather than at discrete intervals. Our main results are that the depth of the market is constant over time and the volatility of prices is constant. In Section 6, we show rigorously that the sequential auction equilibrium converges to the continuous auction equilibrium discussed here when auctions are held frequently.

Proceeding intuitively, let us define a *continuous auction equilibrium* exactly analogously to the sequential auction equilibrium discussed above. We take it for granted that a unique linear equilibrium with a structure analogous to the recursive equilibrium Theorem 2 exists. Write the analogues of (3.5), (3.11), and (3.12) as

$$(4.1) \quad d\pi(t) = [v - p(t) - dp(t)] dx(t) = [v - p(t)] dx(t),$$

$$(4.2) \quad dx(t) = \beta(t)[v - p(t)] dt,$$

$$(4.3) \quad dp(t) = \lambda(t)[dx(t) + du(t)].$$

Note that in this notation, we suppress tildes over random variables. Here we assume that the trading strategy  $dx$  and the pricing rule  $dp$  are characterized by

the functions  $\beta(\cdot)$  and  $\lambda(\cdot)$ , respectively; we will not worry about trading strategies which have a more complicated structure. Readers unwilling to *assume* that a linear equilibrium has the simple structure of (4.1)–(4.3) are invited to interpret the following discussion as referring to a modified equilibrium concept in which  $\pi$ ,  $x$ , and  $p$  are *constrained* to have the simple linear form which we give them here.

Because the  $dp dx$  term in (4.1) is of order  $dt^{3/2}$ , whether trades are priced at the beginning or the end of the instant in which they occur has an inconsequential effect on the profits of the insider, given that he buys and sells smoothly. If trades were priced at the beginning of the instant in which they occurred, however, the insider would not want to trade smoothly: Instead, he would want to trade unbounded quantities at the prices quoted by market makers in advance, since his quantity traded would have no effect on the immediate execution price. Thus, we make the assumption that trades are priced at the end of the instant in which they occur, which is consistent with the sequential auction equilibrium discussed above. The assumption as to whether pricing occurs at the beginning or the end of the instant in which trades occur does affect the profits of the market makers and the noise traders because we have  $du dp = -\lambda\sigma_u^2 dt$ , i.e.,  $du dp$  is of order  $dt$  and not of higher order like the corresponding term for the insider. “End-of-instant pricing” makes the market efficiency condition a zero profit condition for market makers by requiring that noise traders bear the losses incurred by virtue of the fact that they drive prices against themselves as they trade.

In a continuous auction equilibrium, the profit maximization condition states that given a pricing rule, the trading rule  $dx$  maximizes expected profits given by

$$(4.4) \quad E\{\pi(t) | \langle p(s) \rangle_{s \in [0,t]}, v\} = E \left\{ \int_{s=t}^1 d\pi(s) \mid \langle p(s) \rangle_{s \in [0,t]}, v \right\}.$$

The market efficiency condition states that the pricing rule satisfies

$$(4.5) \quad E\{v | \langle dx + du \rangle_{s \in [0,t]}\} = p(t).$$

Given arbitrary  $\beta(\cdot)$  and  $\lambda(\cdot)$ , define  $\Sigma^*(t)$  and  $\Sigma(t)$  by

$$(4.6) \quad \Sigma^*(t) = E\{[v - p(t)]^2\},$$

$$(4.7) \quad \Sigma(t) = \text{var} \{ \tilde{v} | \langle dx + du \rangle_{s \in [0,t]} \}.$$

Clearly, the market efficiency condition implies  $\Sigma^*(t) = \Sigma(t)$ .

Assuming that the equilibrium has the linear form given in (3.2) and (3.3), we obtain the following theorem:

**THEOREM 3:** *In the recursive continuous auction equilibrium, the function  $\lambda(t)$  is a constant given by*

$$(4.8) \quad \lambda(t) = (\Sigma_0 / \sigma_u^2)^{1/2},$$

and the functions  $\Sigma(t)$ ,  $\beta(t)$ ,  $\alpha(t)$ , and  $\delta(t)$  are given by

$$(4.9) \quad \Sigma(t) = (1-t)\Sigma_0,$$

$$(4.10) \quad \beta(t) = \sigma_u^2 \lambda(t) / \Sigma(t) = \sigma_u \Sigma_0^{-1/2} / (1-t),$$

$$(4.11) \quad \alpha(t) = \frac{1}{2}(\sigma_u^2 / \Sigma_0)^{1/2}, \quad t \in (0, 1),$$

$$(4.12) \quad \delta(t) = \frac{1}{2}(\sigma_u^2 \Sigma_0)^{1/2}(1-t).$$

PROOF: Let us first examine the optimal trading rule given an arbitrary pricing rule characterized by some function  $\lambda(t)$ . Since the trading rule is assumed to be linear, we need only maximize ex ante profits, given by

$$(4.13) \quad E\{\pi(0)\} = E\left\{\int_{t=0}^1 d\pi(t)\right\}.$$

From (4.1)–(4.3), we obtain

$$(4.14) \quad E\{d\pi(t)\} = \beta(t)\Sigma^*(t) dt.$$

We also have

$$(4.15) \quad \Sigma^*(t+dt) = E\{(v-p-dp)^2\} = (1-\lambda\beta dt)^2 \Sigma^*(t) + \lambda^2 \sigma_u^2 dt,$$

from which we obtain

$$(4.16) \quad d\Sigma^*/dt = -2\lambda\beta\Sigma^*(t) + \lambda^2 \sigma_u^2.$$

Plugging (4.14) into (4.13), we obtain

$$(4.17) \quad E\{\pi(0)\} = \int_{t=0}^1 \beta(t)\Sigma^*(t) dt,$$

where  $\Sigma^*(t)$  evolves according to the differential equation (4.16). Now (4.16) is equivalent to

$$(4.18) \quad \beta(t)\Sigma^*(t) = \frac{1}{2}[\lambda(t)\sigma_u^2 - \Sigma^{*'}(t)/\lambda(t)],$$

and plugging this into (4.17) yields

$$(4.19) \quad E\{\pi(0)\} = \frac{1}{2} \int_{t=0}^1 \lambda(t)\sigma_u^2 dt + \frac{1}{2} \int_{t=0}^1 \lambda^{-1}(t) d(-\Sigma^*(t)).$$

In this equation, the control  $\beta(t)$  has been eliminated from the optimization problem and only the state  $\Sigma^*(t)$  remains. While  $\Sigma^*(t)$  is defined for all functions of bounded variation, it is clear that all such functions attainable (even in a limiting sense) with controls  $\beta(t)$  must satisfy  $\Sigma^*(t) \geq 0$ . Thus, the insider's problem is equivalent to choosing an (otherwise attainable)  $\Sigma^*(t)$  which satisfies this nonnegativity constraint.

We now show using (4.19) that only a constant function  $\lambda(t)$  is consistent with equilibrium. First, observe that we must have  $\Sigma(1) = 0$  if the right-hand-side of (4.19) is to be maximized; in other words, the price  $p(t)$  must be driven by the

insider to its underlying value  $v$  by the end of trading. Next, observe that if  $\lambda(t)$  ever decreases (i.e., if market depth ever increases), then unbounded profits can be generated by letting  $\Sigma(t)$  increase a large amount before the increase in  $\lambda(t)$  and then letting it decrease the same amount after the decrease. Since unbounded profits are inconsistent with equilibrium, we conclude that  $\lambda(t)$  must be monotonically nondecreasing in any equilibrium.

Intuitively, the requirement that  $\lambda(t)$  is nondecreasing eliminates profitable destabilization schemes and thus generalizes the second order condition (3.20) of the discrete model. With continuous trading, an insider who destabilizes prices by acquiring a large position in many small parcels over a short time period acts much like a perfectly discriminating monopsonist who moves up along a given supply curve. Since the supply curve is linear, the average price paid is approximately the mean of the highest and lowest prices paid on the small parcels. If the supply curve subsequently flattens (i.e., market depth increases), the insider can liquidate his position at a more favorable average price and thus generate unbounded profits by acquiring a large enough position in the first place.

It is also clear from inspecting (4.19) that in order to maximize profits, we must have  $\Sigma^*(t) = 0$  at a point where  $\lambda(t)$  is minimized. If  $\lambda(t)$  were ever to increase, we would therefore have  $\lambda(t^*) = 0$  for some  $t^*$  satisfying  $t^* < 0$ . From the market efficiency condition, this would imply that all information would be incorporated into prices before the end of trading and thus that prices would cease to fluctuate; but the only way for this to happen is to have  $\lambda(t) = 0$  for  $t^* < t < 1$  and this is inconsistent with  $\lambda(t)$  never decreasing. We conclude that  $\lambda(t)$  never increases either. We have thus proved that  $\lambda(t)$  must be a constant in equilibrium.

From (4.19), it is clear that if  $\lambda(t)$  is constant, any function  $\Sigma^*(t)$  satisfying  $\Sigma^*(1) = 0$  satisfies the profit maximization condition as long as it is attainable with some function  $\beta(t)$ . To calculate the values of  $\lambda$ ,  $\beta(t)$ , and  $\Sigma(t)$  consistent with equilibrium, we therefore turn to the market efficiency condition. Observe that (if  $\beta(t)$  is finite) the instantaneous variance of  $dp$  is  $\lambda^2 \sigma_u^2 dt$ , i.e., the volatility of prices is completely dominated by noise trading. In order to have  $\Sigma(1) = 0$  and market efficiency, the integral of volatilities must add up to the prior variance  $\Sigma_0$ . This gives us  $\lambda^2 \sigma_u^2 = \Sigma_0$ , from which (4.10) and (4.11) are immediate implications. To determine the values of  $\beta(t)$  consistent with market efficiency, observe that  $\beta(t)$  must be such that  $\lambda$  is the correct regression coefficient in the equation

$$(4.18) \quad E\{v - p(t) | \beta(t)[v - p(t)] dt + du(t)\} = \lambda[\beta(v - p(t)) dt + du(t)].$$

The appropriate Kalman filtering formula for the regression coefficient is  $\lambda = \beta(t)/\Sigma(t)$ , from which (4.10) is an immediate consequence. We leave (4.11) and (4.12) for the reader to derive. This completes the proof of the theorem.

*Properties of the Continuous Auction Equilibrium.* The fact that  $\Sigma'(t)$  is a constant (or, equivalently, that  $\lambda(t)$  is a constant) in a continuous auction equilibrium implies that trading prices have constant volatility over time and therefore that information is gradually incorporated into prices at a constant rate. From the



fact that  $\Sigma(1) = 0$ , we infer that all of the insider's private information is incorporated into prices by the end of trading, i.e.  $p(t)$  converges to  $v$  (in mean square) at  $t \rightarrow 1$ . Because of normality and the martingale properties inherent in the market efficiency condition, the price actually follows a Brownian motion process with instantaneous variance  $\Sigma_0$ . Of course, the insider knows that the price path will eventually converge to the liquidation value  $\tilde{v}$ , but to market makers, who do not observe  $\tilde{v}$  explicitly, price fluctuations appear to have no drift.

Note that since the volatility of prices is determined by noise traders and not by the insider, there is a sense in which the "trading volume" of the insider is small. Despite his small trading volume, however, the insider ultimately determines what price is established at the end of trading. He does this because his trades, unlike the trades of noise traders, are positively correlated from period to period.

The expected (ex ante) profits of the insider, which equal the expected losses of noise traders, can be shown to equal  $\Sigma_0^{1/2}\sigma_u$ . This is exactly double the profits the insider expects in the single auction equilibrium.

*Market Liquidity.* It is interesting to compare the liquidity properties of the continuous auction equilibrium with the corresponding properties of a sequential auction equilibrium. It was pointed out above that "market liquidity" refers to several different elements of transactions costs, including "tightness," "depth," and "resiliency."

"Tightness" refers to the cost of turning over a position in a short period of time. In the continuous auction equilibrium, the market is infinitely tight, in the sense that it is costless to turn over a position very quickly. This occurs because a trader acts like a perfectly discriminating monopsonist, who moves along a given "expected residual supply curve." In a sequential auction equilibrium, however, the monopolist is not able to trade at every price along the supply curve because auctions are not held closely enough together. As a result, the market is not infinitely tight, and the cost of turning over a position is an increasing function of how quickly it must be done.

"Depth" refers to the ability of the market to absorb quantities without having a large effect on price. It is measured by the reciprocal of the liquidity parameter  $\lambda(t)$ . In the continuous auction equilibrium, the depth of the market is constant. In the proof of Theorem 3, we showed that this is the case because neither increasing nor decreasing depth is consistent with behavior by the informed trader which is "stable" enough to sustain an equilibrium. If depth ever increases, the insider wants to destabilize prices (before the increase in depth) to generate unbounded profits. If depth ever decreases, the insider wants to incorporate all of his private information into the price immediately. This constancy of market depth also explains why the volatility of prices is constant in a continuous auction equilibrium. In a sequential auction equilibrium, depth is not constant over time.

Market "resiliency" refers to the speed with which prices tend to converge towards the underlying liquidation value of the commodity. Resiliency also

measures the rate at which prices bounce back from an uninformative shock. In both the continuous and sequential auction equilibria, the resiliency of prices is determined by the trading of the insider. Noise trading causes the price to wander aimlessly, with no tendency to return to an underlying value. Note that in the continuous auction equilibrium, the fact that  $\beta(t)$  increase in  $t$  means that the resiliency of the market increases in  $t$ . Since  $\beta(t) \rightarrow \infty$  as  $t \rightarrow 1$ , the market becomes infinitely resilient near the end of trading.

It is clear from this description of the liquidity characteristics of the market that a continuous auction equilibrium has essentially the same features Black uses to characterize a liquid market. What makes our model different from what is already in the literature is that these characteristics of market liquidity are results derived within an appropriate model of maximizing behavior.

#### 5. A CONVERGENCE RESULT

How are the properties of the discrete model of sequential trading related to the properties of the continuous auction equilibrium when the interval between auctions is small? In this section, we answer this question by proving a convergence result. We show that as the interval between auctions in the discrete model becomes uniformly small, the unique sequential auction equilibrium characterized in Theorem 2 converges to the continuous auction equilibrium of Theorem 3.

Define the *mesh* of a partition of the interval  $[0, 1]$  into auction dates, which we denote  $|\Delta t|$ , as the maximum interval between auctions. Let  $\lambda_n, \beta_n, \Sigma_n, \alpha_n, \delta_n$  be defined as continuous functions  $\lambda(t), \beta(t)$ , etc., by the conventions  $\lambda(t) = \lambda_{n-1}$  for all  $t \in [t_{n-1}, t_n)$ , etc. We have the following theorem:

**THEOREM 4:** *Holding  $\Sigma_0$  and  $\sigma_u^2$  constant, consider a sequence of sequential auction equilibria (with different partitions defining auction dates) such that  $|\Delta t| \rightarrow 0$ . Then the values of  $\lambda(t), \beta(t), \Sigma(t), \alpha(t), \delta(t)$  characterized in Theorem 2 converge to the corresponding values in the continuous auction equilibrium obtained in Theorem 3. For  $\Sigma(t)$  and  $\delta(t)$ , the convergence is uniform for all  $t \in [0, 1]$ . For  $\lambda(t), \beta(t)$ , and  $\alpha(t)$  the convergence is uniform in all closed intervals which do not contain  $t = 1$ .*

**PROOF:** To prove this theorem, we would like to show that the difference equation system in Theorem 2 converges to a differential equation system. This approach does not lead to a simple proof, however, because the difference equation system is so badly behaved around  $t = 1$  that standard convergence theorems cannot be applied: Note, for example the discontinuity in  $\alpha(t)$  when  $t = 1$ , apparent from Theorem 3 and the boundary condition  $\alpha_N = 0$ . As an alternative, we show that the difference equation system in Theorem 2 can be tackled by first obtaining a difference equation in one variable (denoted  $\phi$  below), then characterizing explicitly the behavior of this difference equation, and finally using the limiting solution to this difference equation to determine the limiting behavior of the original difference equation system.

We can write equations (3.17) and (3.30), respectively as

$$(5.1) \quad \beta_n \lambda_n \Delta t_n = z_1 / (1 + z_1), \quad z_1 = 1 - 2\alpha_n \lambda_n,$$

$$(5.2) \quad \beta_n \lambda_n \Delta t_n = z_2 / (1 + z_2), \quad z_2 = \beta_n^2 \Sigma_{n-1} \Delta t_n / \sigma_u^2.$$

Equating  $z_1$  and  $z_2$  yields

$$(5.3) \quad 1 - 2\alpha_n \lambda_n = \beta_n^2 \Sigma_{n-1} \Delta t_n / \sigma_u^2.$$

Equations (3.17) and (3.15) are equivalent to

$$(5.4) \quad \beta_n \Delta t_n = 2\alpha_{n-1} (1 - 2\alpha_n \lambda_n),$$

$$(5.5) \quad (\alpha_n - \alpha_{n-1}) / \alpha_{n-1} = -(1 - 2\alpha_n \lambda_n)^2.$$

Now define  $\phi_n$  by

$$(5.6) \quad \phi_n = 4\alpha_n^2 \Sigma_n / \sigma_u^2 \quad (n = 0, \dots, N).$$

Combining (5.3) and (5.4) to eliminate  $\beta_n$  yields

$$(5.7) \quad 1 - 2\alpha_n \lambda_n = \Delta t_n / \phi_{n-1}.$$

Combining (5.7) with (5.3) and substituting into (3.31) yields

$$(5.8) \quad \Sigma_n / \Sigma_{n-1} = (1 + \Delta t_n / \phi_{n-1})^{-1},$$

and combining (5.5) with (5.7) yields

$$(5.9) \quad \alpha_n / \alpha_{n-1} = 1 - \Delta t_n^2 / \phi_{n-1}^2.$$

These two difference equations define  $\Sigma_n$  and  $\alpha_n$  in terms of  $\phi_{n-1}$ . Multiplying (5.8) and (5.9) together yields the following difference equation for  $\phi_n$  in terms of itself:

$$(5.10) \quad \phi_n / \phi_{n-1} = (1 - \Delta t_n^2 / \phi_{n-1}^2) [1 - \Delta t_n / (\phi_{n-1} + \Delta t_n)].$$

This can be simplified to

$$(5.11) \quad \phi_n - \phi_{n-1} = -\Delta t_n - \Delta t_n^2 / \phi_{n-1} + \Delta t_n^3 / \phi_{n-1}^2.$$

The boundary condition is  $\phi_N = 0$ . To iterate this difference equation for  $\phi_n$  backwards, a cubic equation must be solved at each step. We leave it to the reader to show that of the three solutions to this cubic equation, only one solution makes economic sense, and this solution satisfies

$$(5.12) \quad -5/4 < (\phi_n - \phi_{n-1}) / \Delta t_n < -1,$$

$$(5.13) \quad (\phi_n - \phi_{n-1}) / \Delta t_n \rightarrow -1 \quad \text{as} \quad \phi_n / \Delta t_n \rightarrow \infty.$$

From (5.12) and (5.13) it is clear that for the continuous version of  $\phi$ , denoted  $\phi(t)$ , we have

$$(5.14) \quad \phi(t) \rightarrow 1 - t,$$

and the convergence is uniform for all  $t \in [0, 1]$ . We have thus calculated the limiting behavior of  $\phi(t)$ .

To calculate the limiting behavior of  $\Sigma(t)$ , observe that we can use (5.8) to write

$$(5.15) \quad (\Sigma_n - \Sigma_{n-1})/\Sigma_{n-1} = -\Delta t_n/(1-t_n) + o(|\Delta t|),$$

and standard convergence results for converting difference equations into differential equations allow us to conclude that the solution to (5.15) converges to the solution of the difference equation

$$(5.16) \quad \Sigma'/\Sigma = -1/(1-t)$$

uniformly for all  $t$  bounded away from  $t = 1$ . Furthermore, the solution to (5.16) is

$$(5.17) \quad \Sigma(t) = (1-t)\Sigma_0.$$

To calculate the limiting behavior of  $\alpha(t)$ , it is clear from the definition of  $\phi$  in (5.6) and from the limit results for  $\phi(t)$  and  $\Sigma(t)$  in (5.14) and (5.17) that for all  $t$  bounded away from  $t = 1$ ,  $\alpha(t)$  converges uniformly to the constant  $\frac{1}{2}\sigma_u/\Sigma_0^{1/2}$ . Convergence results for  $\beta$ ,  $\lambda$ , and  $\delta$  are simple exercises which we leave to the reader. This completes the proof of the theorem.

*Specific Examples.* In the process of proving the convergence result in Theorem 4, we also demonstrate a simple procedure for calculating solutions to the difference equation system which characterizes the sequential auction equilibrium. That procedure is to calculate  $\gamma_n$  from (5.10) by backward iteration from  $\gamma_N = 0$ , next to calculate  $\Sigma_n$  from (5.8) by forward iteration given  $\Sigma_0$ , then to use (5.6) to calculate  $\alpha_n$ , and finally to calculate other parameters from  $\alpha_n$  and  $\Sigma_n$ . In Figure 1, values of the liquidity parameter  $\lambda_n$  and error variance  $\Sigma_n$  are plotted for the particular cases  $N = 4$ ,  $N = 20$ , and  $N = 100$ . In these cases, exogenous parameters are normalized by setting  $\Sigma_0 = \sigma_u^2 = 1$ , and auctions are assumed to occur at equally spaced intervals. For the purpose of comparison, results for the continuous case ( $\Sigma_n = 1-t$ ,  $\lambda_n = 1$ ) are also given. Figure 1 illustrates clearly convergence of parameters to the continuous model as  $N$  becomes large.

## 6. CONCLUSION

We have investigated a model of speculative trading in which an insider maximizes profits by exploiting strategically his monopoly power in a dynamic context. The model is important for a number of reasons.

It illustrates that modeling price innovations as functions of quantities traded is not inconsistent with modelling price innovations as the consequence of new information. Simultaneously, it illustrates that the strategic exercise of monopoly power by an insider is in no way inconsistent with prices being set efficiently in the semi-strong sense. The model shows how a discrete model of sequential trading (structured to resemble a sequential equilibrium) converges in the limit as trading takes place very frequently to a simple model of continuous trading.

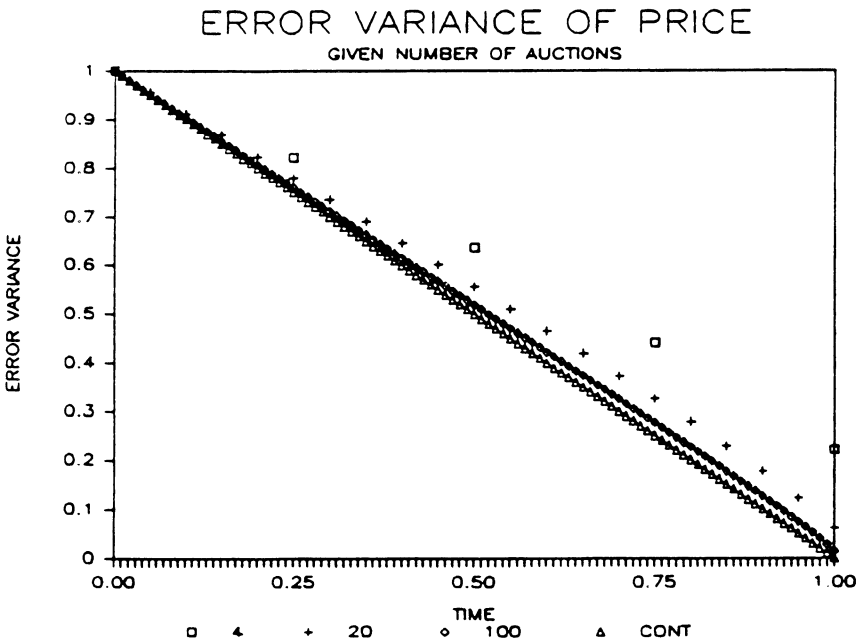
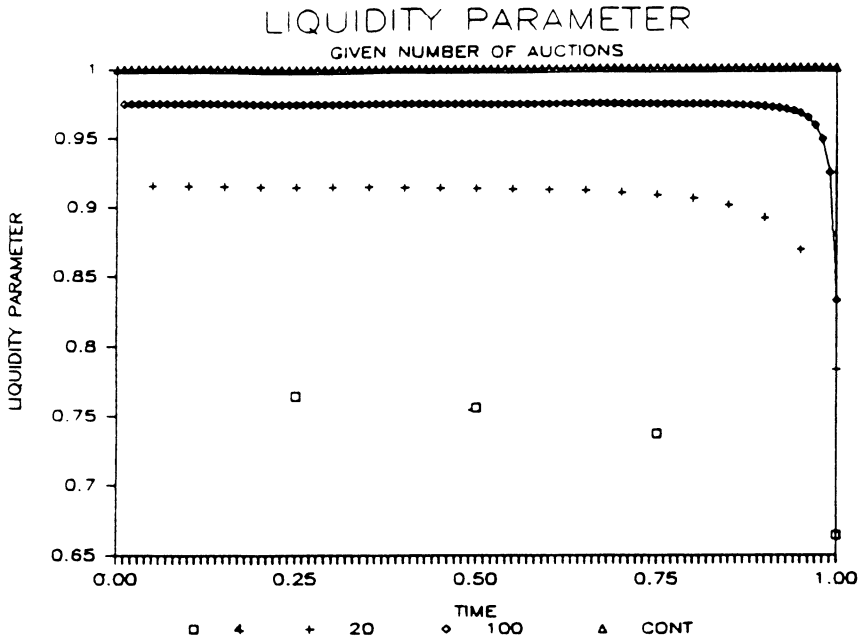


FIGURE 1

In doing so, it illustrates that constant volatility of prices need not require that the information upon which trades are based be produced in a smooth manner. Finally, the model demonstrates how the liquidity characteristics of an "efficient," "frictionless" market can be derived from underlying information asymmetries in a dynamic trading environment which captures some relevant features of trading in organized exchanges.

*Princeton University*

*Manuscript received January 1984; revision received December, 1984.*

#### REFERENCES

- [1] BAGEHOT, WALTER: "The Only Game in Town," *Financial Analysts Journal*, 27(1971), 12-14, 22.
- [2] BLACK, FISCHER: "Towards a Fully Automated Exchange, Part I," *Financial Analysts Journal*, 27(1971), 29-34.
- [3] GLOSTEN, LAWRENCE R., AND PAUL R. MILGROM: "Bid, Ask and Transaction Prices in a Specialist Market with Heterogeneously Informed Traders," J. L. Kellogg Graduate School of Management Working Paper No. 570, 1984.
- [4] KREPS, DAVID M., AND ROBERT WILSON: "Sequential Equilibria," *Econometrica*, 50(1982), 863-894.
- [5] KYLE, ALBERT S.: "Market Structure, Information, Futures Markets, and Price Formation," in *International Agricultural Trade: Advanced Readings in Price Formation, Market Structure, and Price Instability*, ed. by Gary G. Storey, Andrew Schmitz, and Alexander H. Sarris. Boulder and London: Westview Press, 1984, 45-64.
- [6] ———: "Equilibrium in a Speculative Market with Strategic Informed Trading," unpublished manuscript, 1984.

