Death bonds with stochastic force of mortality

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Abstract

In a financial market with stochastic interest rate following a square root process, we present a closed form solution for pricing a death bond (as a security backed by insurance contracts) when the force of mortality follows a square root stochastic process whose expected value coincides, at any time, with the force of mortality given by the so-called Gompertz Makeham density. Finally, we present how such a death bond should enter the portfolio of an agent maximizing the expected utility of both his intertemporal consumption and his final wealth when the time horizon coincides with his death time.

1 Introduction

The managers of either an insurance companies or a pension funds are concerned of both financial and actuarial risks. Nevertheless, these two kinds of risk can be partially or fully hedged with very different instruments. For instance, the inflation risk, the interest rate risk, the exchange rate risk, can all be efficiently hedged by existing financial assets (respectively, inflation indexed bonds, floating coupon bonds, forwards, futures, or options on the foreign exchange). For what concerns the actuarial risk, its hedging and diversification are more difficult because of the lack of traded assets which could be able to provide their holder with cash flows negatively correlated with the above mentioned risks.

Furthermore, since the financial assets are usually very poorly correlated with the actuarial risk sources, then any linear combination of financial asset cannot provide a suitable hedging against the actuarial risk. Accordingly, in order to be able to effectively hedge against actuarial risk, the institutional investors should issue, on the financial market, new assets correlated with death (or survival) probability of economic agents. Such issues wouldn't of course provide any hedging against the so-called basis risk, i.e. the risk that the population an actuarial-financial asset is written on diverges from the population whose demographic behaviour we are trying to hedge against. Nevertheless, these actuarial-financial assets would make many institutional investors be able to bear the so-called longevity and mortality risks. The longevity risk could be almost perfectly hedged through longevity bonds (see, for instance, Azzoppardi, 2005, and Menoncin, 2006, 2007) and, in the same spirit, there are nowadays rumors about the issue of some new assets called death bonds which should belong to the family of the Asset Backed Securities (ABS). In particular, death bonds should be backed by death insurance sold by their holder in exchange of the net present value of the final benefit.

One of the main concern about both longevity and mortality risk is that the force of mortality (given by the amount of people who die in a given period as a percentage of the whole population) is stochastic itself. In fact, once it has been estimated and foreseen on the basis of the actuarial tables, it is nevertheless affected by unforeseeable factors. In particular, the length of human life (for both men and women) has been significantly increasing during the last decades. Such an increase implies a serious risk for pension funds which will have to pay pensions for periods longer than that they had foreseen when stipulating a thirty year length pension plan.

The force of mortality (or the survival probability) can be profitably modelling by using well known results about stochastic processes (see, for instance, Dahl, 2004, Biffis, 2005, Hainaut and Devolder, 2007b).

In this paper we take into account the case of a stochastic force of mortality following a Cox et al. (1985) process and consistent with the so-called Gompertz Makeham density function (see, for instance, Milevsky 2006). In this framework, we present the price of a death insurance/death bond in a close form and we compute the role of this death bond in the portfolio of an agent maximizing the expected present utility of his intertemporal consumption and final wealth at his death time. Thus, the time horizon for the optimization problem is stochastic. One of the first paper coping with a stochastic horizon is Richard (1975). The structure of the market we present here is akin to that shown in Menoncin (2007), but we take into account a more realistic process for both the stochastic force of mortality and the interest rate (we demonstrate that, in our framework, both variables cannot take negative values).

We deal with a financial market driven by two state variables (interest rate and force of mortality) and three assets: a riskless asset, a zero-coupon bond, and a death bond. The death bond makes the financial market complete even with respect to the stochastic force of mortality.

The rest of the paper is structured as follows. Section 2 shows the model for the instantaneously riskless interest rate and a zero-coupon bond written as a derivative on the interest rate. Section 3 presents the main model assumed in the literature for the deterministic force of mortality. Section 4 presents the case of a stochastic force of mortality. Section 5 contains the main result about pricing a death insurance and a death bond both in a deterministic and stochastic framework. Section 6 shows how to compute the optimal portfolio containing a zero-coupon bond and a death bond. The technicalities about the main results are left to appendices.

2 Interest rate and bonds

The instantaneously riskless interest rate $r(t)$ is assumed to be stochastic. It follows the stochastic differential equation

$$\begin{aligned} dr(t) &= a_r(\gamma_r - r(t))\,dt + \sigma_r\sqrt{r(t)}dW_r(t), \\ r(t_0) &= r_0, \end{aligned} \tag{1}$$

with positive constant r_0 and where $dW_r(t)$ is a Brownian motion with zero mean and dt variance. For numerical simulations, we will use the following values

$$a_r = 0.1, \quad \gamma_r = 0.056, \quad \sigma_r = 0.067, \tag{2}$$

obtained from a regression on 3 month US Treasury-Bill (we will always assume that the interest rate starts at its equilibrium value, i.e. $r_0 = \gamma_r$).

For pricing purposes, we need to compute the stochastic process for $r(t)$ under a risk-neutral probability measure ($\mathbb{Q}$). After Gisanov's theorem we know that on an arbitrage free financial market there exists a market price of risk ξ_r such that

$$dW_r^{\mathbb{Q}} = \xi_r dt + dW_r.$$

The stochastic process for $r(t)$ under the new probability is given by

$$dr(t) = \left(a_r(\gamma_r - r(t)) - \sigma_r\sqrt{r(t)}\xi_r\right)dt + \sigma_r\sqrt{r(t)}dW_r^{\mathbb{Q}}.$$

A common hypothesis is that the market price of risk takes the following form

$$\xi_r = \frac{\sqrt{r(t)}}{\sigma_r}\psi,$$

where ψ is a constant. Under this hypothesis we have

$$dr(t) = a_r^{\mathbb{Q}}\left(\gamma_r^{\mathbb{Q}} - r(t)\right)dt + \sigma_r\sqrt{r(t)}dW_r^{\mathbb{Q}}, \tag{3}$$

where

$$a_r^{\mathbb{Q}} \equiv a_r + \psi, \qquad \gamma_r^{\mathbb{Q}} \equiv \frac{a_r}{a_r + \psi}\gamma_r.$$

Thanks to the particular form assumed by the market price of risk, the stochastic process under $\mathbb{Q}$ has not changed its form with respect to (1).

In an arbitrage free market the price of any asset is given by the expected value, under the risk neutral probability measure, of its future cash flows discounted by the riskless interest rate. Accordingly, the value in t of a zero-coupon paying one monetary unit in T is given by

$$B(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds}\right],$$

where $\mathbb{E}_t^{\mathbb{Q}}[\cdot]$ is the expected value operator under the probability $\mathbb{Q}$ and conditional to the information set available in t.

Some results follow.

Proposition 1 *If the riskless interest rate follows the process (1), then the price of a zero-coupon $B(t,T)$ is given by*

$$B(t,T) = e^{-a_r^{\mathbb{Q}}\gamma_r^{\mathbb{Q}}\int_t^T C_B(s,T)ds - C_B(t,T)r(t)}, \tag{4}$$

where

$$\begin{aligned} C_B(t,T) &= 2\frac{1-e^{-k(T-t)}}{k+a_r^{\mathbb{Q}}+\left(k-a_r^{\mathbb{Q}}\right)e^{-k(T-t)}}, \\ k &\equiv \sqrt{\left(a_r^{\mathbb{Q}}\right)^2+2\sigma_r^2}. \end{aligned} \tag{5}$$

Proof. See Appendix A with $X = r$ and $B(t,T) = V(t,T)|_{\chi=0}$. ■

The value of the bond in differential terms can be simply computed by applying Itô's lemma to (4):

$$\begin{aligned} \frac{dB(t,T)}{B(t,T)} &= r(t)\,dt - C_B(t,T)\,\sigma_r\sqrt{r(t)}dW_r^{\mathbb{Q}}(t) \\ &= \left(r(t) - C_B(t,T)\,\sigma_r\sqrt{r(t)}\xi_r\right)dt - C_B(t,T)\,\sigma_r\sqrt{r(t)}dW_r(t). \end{aligned} \tag{6}$$

Another way to write the value of a zero-coupon is to use the forward (instantaneous) interest rate $f(t,T)$:

$$B(t,T) = e^{-\int_t^T f(t,s)ds}.$$

An obvious no-arbitrage condition asks for the expected discounted value of $r(T)$ to equate the expected discounted value of $f(t,T)$. Accordingly, the following equality must hold:

$$\mathbb{E}_t^{\mathbb{Q}}\left[r(T)\,e^{-\int_t^T r(s)ds}\right] = \mathbb{E}_t^{\mathbb{Q}}\left[f(t,T)\,e^{-\int_t^T r(s)ds}\right],$$

but since $f(t,T)$ belongs to the information set (i.e. $\sigma-$algebra) in t, then we have

$$f(t,T) = \frac{\mathbb{E}_t^{\mathbb{Q}}\left[r(T)\,e^{-\int_t^T r(s)ds}\right]}{\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds}\right]}.$$

Proposition 2 *If the riskless interest rate $r(t)$ follows the process (1), then the forward interest rate is given by*

$$f(t,T) = \int_t^T a_r^{\mathbb{Q}}\gamma_r^{\mathbb{Q}}(s)\,e^{-\int_s^T\left(a_r^{\mathbb{Q}}+C_B(u,T)\sigma^2\right)du}ds + e^{-\int_t^T\left(a_r^{\mathbb{Q}}+C_B(u,T)\sigma_r^2\right)du}r(t), \tag{7}$$

where $C_B(t,T)$ is as in (5).

Proof. See Appendix A with $X = r$ and $f(t,T) = \frac{V(t,T)|_{\chi=1}}{V(t,T)|_{\chi=0}}$. ■

The forward interest rate in differential terms can be computed by simply applying Itô's lemma to (7):

$$\begin{aligned} df(t,T) &= e^{-\int_t^T \left(a_r^{\mathbb{Q}}+C_B(u,T)\sigma_r^2\right)du} C_B(t,T)\,\sigma_r^2 r(t)\,dt \\ &\quad + e^{-\int_t^T \left(a_r^{\mathbb{Q}}+C_B(u,T)\sigma_r^2\right)du} \sigma_r \sqrt{r(t)}\,dW_r^{\mathbb{Q}}, \end{aligned}$$

where it is of course true that

$$\frac{\partial C_B(t,T)}{\partial T} = e^{-\int_t^T \left(a_r^{\mathbb{Q}}+C_B(u,T)\sigma_r^2\right)du}.$$

We will show these results have a straight parallel in an actuarial framework.

3 Deterministic mortality rate

Let us call τ the stochastic death time whose density function is $\pi(t)$. In this way the probability of surviving from t_0 up to t is given by

$$({}_tp_{t_0}) = 1 - \int_{t_0}^{t} \pi(s)\,ds,$$

from which

$$\frac{d\,({}_tp_{t_0})}{dt} = -\pi(t), \tag{8}$$

and, furthermore,

$$\frac{d\,({}_tp_{t_0})}{({}_tp_{t_0})} = -\frac{\pi(t)}{1-\int_{t_0}^{t}\pi(s)\,ds}dt, \tag{9}$$

with the natural boundary condition $({}_tp_{t_0}) = 1$. In the actuarial literature, the opposite of the ratio in the r.h.s. of (9) is often called *mortality rate* (or *hazard rate*):

$$\lambda(t) \equiv \frac{\pi(t)}{1-\int_{t_0}^{t}\pi(s)\,ds}.$$

For the sake of simplicity, we will assume that t_0 (the starting time of all our computations) coincides with the age of an agent. Accordingly, also t is measured in years of age.

The (unique) solution of the ordinary differential equation (9) is

$$({}_tp_{t_0}) = e^{-\int_{t_0}^{t}\lambda(s)ds}. \tag{10}$$

One of the most common parametrizations for the mortality rate is the so called Gompertz-Makeham function:

$$\lambda(t) = \phi + \frac{1}{b}e^{\frac{t-m}{b}}, \tag{11}$$

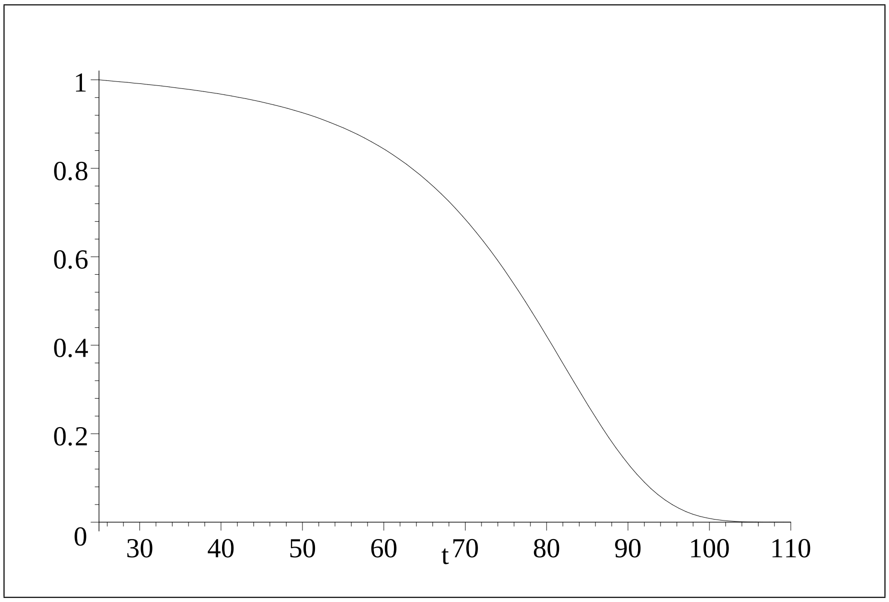


Figure 1: Survival probability from 25 till t with a Gompertz-Makeham mortality law.

where: (i) ϕ is a positive constant capturing the age independent component of mortality rate (like accidents), (ii) m (strictly positive) measures the modal value of life, and (iii) b (strictly positive) is the dispersion parameter of life. Typical value for these parameters (consistently chosen with Milvsky, 2006) are

$$\phi = 0.001, \quad m = 82.3, \quad b = 11.4. \tag{12}$$

When $\lambda(t)$ follows (11), the survival probability (10) is given by

$$({}_t p_{t_0}) = e^{-\phi(t-t_0)+e^{\frac{t_0-m}{b}}-e^{\frac{t-m}{b}}}.$$

For an agent aged of 25 (i.e. $t_0 = 25$) and with the values in (12), the survival probability till the age of t is shown in Figure 1.

4 Stochastic mortality rate

What we have presented in the previous section is not fit for describing the case of a mortality rate which may change for unforeseeable reasons. In order to fix that, we can model the mortality rate as a stochastic process itself. Thus, we assume that the mortality rate $\lambda(t)$ solves a stochastic differential equation of the following form:

$$\begin{aligned} d\lambda(t) &= \mu_\lambda(t,\lambda)\,dt + \sigma_\lambda(t,\lambda)\,dW(t), \\ \lambda(t_0) &= \lambda_0. \end{aligned}$$

In this case the survival probability cannot be written as in (10). Actually, under the information set in t, we do not know all the mortality rates from t to T. Accordingly, we can compute the survival probability from t_0 to t only under the expected value conditional to the information set in t_0:

$$({}_tp_{t_0}) = \mathbb{E}_{t_0}\left[e^{-\int_{t_0}^{t}\lambda(s)ds}\right]. \tag{13}$$

We highlight that, in this case, the expected value is computed under the historical probability measure (we haven't put any upper script on the expecte value), and not under the risk neutral probability.

If we are allowed to differentiate with respect to time (t) under the expected value[1], then we can conclude from (8) that

$$\pi(t) = -\frac{d({}_tp_{t_0})}{dt} = \mathbb{E}_{t_0}\left[\lambda(t)\,e^{-\int_{t_0}^{t}\lambda(s)ds}\right].$$

In this case the hazard rate is given by

$$l(t_0,t) \equiv -\frac{d({}_tp_{t_0})}{dt}\frac{1}{({}_tp_{t_0})} = \frac{\mathbb{E}_{t_0}\left[\lambda(t)\,e^{-\int_{t_0}^{t}\lambda(s)ds}\right]}{\mathbb{E}_{t_0}\left[e^{-\int_{t_0}^{t}\lambda(s)ds}\right]}, \tag{14}$$

which coincides with $\lambda(t)$ if and only if $\lambda(t)$ is not stochastic (of course it is true, in any case, that $l(t_0,t_0) = \lambda(t_0)$).

Such a framework has a straightforward and appealing parallel with the financial framework we have already presented above. The main difference is that the expected value on the financial market is computed under the riskless probability measure ($\mathbb{Q}$) which is different from the probability measure used for computing $({}_tp_{t_0})$.

Actuarial market	**Financial market**
Death intensity $\lambda(t)$	Interest rate $r(t)$
Suvival probability $({}_tp_{t_0})$	Zero-coupon bond price $B(t_0,t)$
Hazard rate $l(t_0,t)$	Instant. forward rate $f(t_0,t)$
Computations under $\mathbb{P}$	Computations under $\mathbb{Q}$

Now, we want to build a stochastic process for the variale $\lambda(t)$ such that its expected value is, at any instant, equal to the Gompertz-Makeham mortality rate (11). For this purpose, we use the following result.

[1] Grandell (1976) shows that the equality which follows is true if: (i) there exists a constant C such that, for any t, $\mathbb{E}_{t_0}\left[\lambda(t)^2\right] < C$, and (ii) for any $\varepsilon > 0$ and almost every time t, $\lim_{\delta\to 0}\mathbb{P}(|\lambda(t+\delta) - \lambda(t)| \geq \varepsilon) = 0$.

Proposition 3 *If the stochastic variable* $X(t)$ *solves*

$$\begin{aligned} dX(t) &= \alpha(t)\left(\frac{1}{\alpha(t)}\frac{\partial\beta(t)}{\partial t}+\beta(t)-X(t)\right)dt+\sigma(t,X)\,dW(t)\,, \\ X(t_0) &= \beta(t_0)\,, \end{aligned}$$

then

$$\mathbb{E}_{t_0}[X(t)] = \beta(t)\,.$$

Proof. Let $Y(t) = X(t)\,e^{\int_{t_0}^{t}\alpha(u)du}$, then by applying Itô's lemma we have

$$\begin{aligned} dY(t) &= e^{\int_{t_0}^{t}\alpha(u)du}dX(t)+\alpha(t)\,X(t)\,e^{\int_{t_0}^{t}\alpha(u)du}dt \\ &= e^{\int_{t_0}^{t}\alpha(u)du}\left(\frac{\partial\beta(t)}{\partial t}+\alpha(t)\,\beta(t)\right)dt+e^{\int_{t_0}^{t}\alpha(u)du}\sigma(t,X)\,dW(t) \\ &= \frac{\partial}{\partial t}\left(\beta(t)\,e^{\int_{t_0}^{t}\alpha(u)du}\right)dt+e^{\int_{t_0}^{t}\alpha(u)du}\sigma(t,X)\,dW(t)\,. \end{aligned}$$

Now, we compute the expected value under the information set in t_0:

$$\mathbb{E}_{t_0}[dY(t)] = \frac{\partial}{\partial t}\left(\beta(t)\,e^{\int_{t_0}^{t}\alpha(u)du}\right)dt,$$

and by integrating from t_0 up to t we have

$$\mathbb{E}_{t_0}[Y(t)] = Y(t_0)+\beta(t)\,e^{\int_{t_0}^{t}\alpha(u)du}-\beta(t_0)\,.$$

After substituting for Y we finally obtain

$$\mathbb{E}_{t_0}[X(t)] = (X(t_0)-\beta(t_0))\,e^{-\int_{t_0}^{t}\alpha(u)du}+\beta(t)\,,$$

and, since $X(t_0) = \beta(t_0)$, the result of the proposition is obtained. ■

We want the expected value of $\lambda(t)$ to be always equal to the Gomopertz function (11), i.e.

$$\mathbb{E}_{t_0}[\lambda(t)] = \phi+\frac{1}{b}e^{\frac{t-m}{b}}. \tag{15}$$

By using the result of Proposition 3, we can write the process for $\lambda(t)$ as

$$\begin{aligned} d\lambda(t) &= a_\lambda(\gamma_\lambda(t)-\lambda(t))\,dt+\sigma_\lambda\sqrt{\lambda(t)}dW_\lambda(t)\,, \qquad (16) \\ \lambda(t_0) &= \phi+\frac{1}{b}e^{\frac{t_0-m}{b}}, \end{aligned}$$

where

$$\gamma_\lambda(t) \equiv \phi+\left(\frac{1}{a_\lambda b}+1\right)\frac{1}{b}e^{\frac{t-m}{b}}, \tag{17}$$

and a_λ and σ_λ are two constant (and positive) parameters that can be estimanted from the historical series on $\lambda(t)$.

In order to trace our model back to the well known results about the affine stochastic processes, we have chosen to set the diffusion term as the square root of the stochastic variable $\lambda(t)$ itself.

Some results follow.

Proposition 4 *If*

$$\sigma_\lambda^2 \leq 2a_\lambda \left(\phi + \left(\frac{1}{a_\lambda b} + 1\right)\frac{1}{b}e^{\frac{t_0-m}{b}}\right),$$

then the value of $\lambda(t)$ *in (16) never becomes negative.*

Proof. See Appendix B. ∎

Proposition 5 *If the death intensity* $\lambda(t)$ *follows the process (16), then the survival probability is given by*

$$({}_Tp_t) = e^{-a_\lambda \int_t^T C_P(s,T)\gamma_\lambda(s)ds - C_P(t,T)\lambda(t)}, \tag{18}$$

where

$$\begin{aligned} C_P(t,T) &= 2\frac{1-e^{-k(T-t)}}{k + a_\lambda + (k - a_\lambda)e^{-k(T-t)}}, \\ k &\equiv \sqrt{a_\lambda^2 + 2\sigma_\lambda^2}, \end{aligned} \tag{19}$$

and, in differential terms,

$$\frac{d({}_Tp_t)}{({}_Tp_t)} = \lambda(t)\,dt - C_P(t,T)\,\sigma_\lambda\sqrt{\lambda(t)}dW_\lambda(t).$$

Proof. See Appendix A. ∎

Proposition 6 *If the death intensity* $\lambda(t)$ *follows the process (16), then the hazard rate is given by*

$$l(t,T) = \int_t^T a_\lambda\gamma_\lambda(s)\,e^{-\int_s^T\left(a_\lambda + C_P(u,T)\sigma_\lambda^2\right)du}ds + e^{-\int_t^T\left(a_\lambda + C_P(u,T)\sigma_\lambda^2\right)du}\lambda(t), \tag{20}$$

whose differential is

$$\begin{aligned} dl(t,T) &= e^{-\int_t^T\left(a_\lambda + C_P(u,T)\sigma_\lambda^2\right)du}C_P(t,T)\,\sigma_\lambda^2\lambda(t)\,dt \\ &\quad + e^{-\int_t^T\left(a_\lambda + C_P(u,T)\sigma_\lambda^2\right)du}\sigma_\lambda\sqrt{\lambda(t)}dW_\lambda(t), \end{aligned} \tag{21}$$

where the function $C_P(t,T)$ *is as in (19).*

Proof. See Appendix A. ∎

Corollary 7 *If the death intensity* $\lambda(t)$ *follows the process (16) with* $\sigma_\lambda = 0$, *then the hazard rate is given by*

$$l(t,T) = \lambda(T).$$

Proof. When $\sigma = 0$ Equation (20) becomes

$$l(t,T)|_{\sigma=0} = \int_t^T a_\lambda \gamma_\lambda(s) e^{-a_\lambda(T-s)} ds + e^{-a_\lambda(T-t)} \lambda(t),$$

where we can substitute for $\gamma_\lambda(t)$ (Equation (17))

$$\begin{aligned} l(t,T)|_{\sigma=0} &= \int_t^T a_\lambda \left(\phi + \left(\frac{1}{a_\lambda b} + 1\right)\frac{1}{b} e^{\frac{s-m}{b}}\right) e^{-a_\lambda(T-s)} ds + e^{-a_\lambda(T-t)} \lambda(t) \\ &= \phi + \frac{1}{b} e^{\frac{T-m}{b}} - \left(\phi + \frac{1}{b} e^{\frac{t-m}{b}}\right) e^{-a_\lambda(T-t)} + e^{-a_\lambda(T-t)} \lambda(t). \end{aligned}$$

If we finally recall from (15) that, with $\sigma_\lambda = 0$,

$$\lambda(t) = \phi + \frac{1}{b} e^{\frac{t-m}{b}},$$

then the result of the corollary follows. ■

5 Death insurance and death bond

With a death insurance contract, the subscriber agrees to pay settlements (P) during his life time in order to receive, at his death time, a given amount of money (final benefit). For the sake of simplicity we will set such an amount to 1 (any other case can be trivially handled by multiplying P for any amount of money).

If the death insurance is subscribed in t_0 (i.e. when the subscriber is aged t_0), then the actuarial equilibrium for such a contract asks for the expected present value of the settlements to equate the expected present value of the final benefit (available at the deat time τ and equal to 1). If we assume that P is continuously paid, then the actuarial equilibrium asks for the following equality to hold:

$$\mathbb{E}_{t_0}^{\tau}\left[\int_{t_0}^{\tau} P(s) e^{-\int_{t_0}^{s} r_\lambda(u)du} ds\right] = \mathbb{E}_{t_0}^{\tau}\left[e^{-\int_{t_0}^{\tau} r_\lambda(u)du}\right],$$

where r_λ is a suitable discount rate taken into account by the insurance company.

When the insurance contract enters (in any way) the financial market, then the value of the contract must be computed as the value of any other asset i.e. under the riskless probability measure. Accordingly, the discount rate r_λ is replaced by the riskless interest rate r as follows

$$\mathbb{E}_{t_0}^{\mathbb{Q},\tau}\left[\int_{t_0}^{\tau} P(s) e^{-\int_{t_0}^{s} r(u)du} ds\right] = \mathbb{E}_{t_0}^{\mathbb{Q},\tau}\left[e^{-\int_{t_0}^{\tau} r(u)du}\right].$$

Now, as it is usually the case, the riskless interest rate r is assumed to be independent of the death time τ. Accordingly, the expected value computed under $\mathbb{Q}$ and τ can be separately computed:

$$\mathbb{E}_{t_0}^{\mathbb{Q},\tau}\left[\int_{t_0}^{\infty} \mathbb{I}_{s<\tau} P(s) e^{-\int_{t_0}^{s} r(u)du} ds\right] = \mathbb{E}_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^{\infty} \pi(s) e^{-\int_{t_0}^{s} r(u)du} ds\right],$$

where $\mathbb{I}_\varepsilon$ is the indicator function whose value is 1 if the event ε happens and 0 otherwise. The previous equality can be suitably written as

$$\begin{aligned} & \int_{t_0}^{\infty} \mathbb{E}_{t_0}^{\mathbb{Q}} \left[\mathbb{I}_{s<\tau}\right] \mathbb{E}_{t_0}^{\mathbb{Q}} \left[P\left(s\right) e^{-\int_{t_0}^{s} r(u)du}\right] ds \\ = & \int_{t_0}^{\infty} \mathbb{E}_{t_0}^{\mathbb{Q}} \left[\lambda\left(s\right) e^{-\int_{t_0}^{s} \lambda(u)du}\right] \mathbb{E}_{t_0}^{\mathbb{Q}} \left[e^{-\int_{t_0}^{s} r(u)du}\right] ds. \end{aligned}$$

If P is constant we have

$$P^* = \frac{\int_{t_0}^{\infty} \mathbb{E}_{t_0}^{\mathbb{Q}} \left[\lambda\left(s\right) e^{-\int_{t_0}^{s} \lambda(u)du}\right] B\left(t_0, s\right) ds}{\int_{t_0}^{\infty} \mathbb{E}_{t_0}^{\mathbb{Q}} \left[e^{-\int_{t_0}^{s} \lambda(u)du}\right] B\left(t_0, s\right) ds},$$

and by recalling (14) and (13),we finally have

$$P^* = \frac{\int_{t_0}^{\infty} l\left(t_0, t\right)^{\mathbb{Q}} \left({}_t p_{t_0}\right)^{\mathbb{Q}} B\left(t_0, s\right) ds}{\int_{t_0}^{\infty} \left({}_t p_{t_0}\right)^{\mathbb{Q}} B\left(t_0, s\right) ds}, \tag{22}$$

where we have indicated with $\left({}_t p_{t_0}\right)^{\mathbb{Q}}$ and $l\left(t_0, t\right)^{\mathbb{Q}}$ the survival probability and the hazard rate respectively, computed under the risk neutral probability.

Once the value of the premium is obtained, the value of the death insurance, at any time t, is given by the difference between the expected present value of the final benefit and the expected present value of the premia still due:

$$D\left(t\right) = \mathbb{E}_t^{\mathbb{Q},\tau} \left[e^{-\int_t^{\tau} r(u)du}\right] - P^* \mathbb{E}_t^{\mathbb{Q},\tau} \left[\int_t^{\tau} e^{-\int_t^{s} r(u)du} ds\right],$$

which can be simplified as we have done above by obtaining

$$D\left(t\right) = \int_t^{\infty} \left(l\left(t, s\right)^{\mathbb{Q}} - P^*\right) \left({}_s p_t\right)^{\mathbb{Q}} B\left(t, s\right) ds. \tag{23}$$

As it is evident from this last equation, the value of the premium for a death insurance subscribed in t_0 can also be computed from (23) by imposing the condition

$$D\left(t_0\right) = 0.$$

From Equation (23) it is evident that the death insurance cannot be distinguished from an infinitely living bond whose coupons are given by $\left(l\left(t, s\right)^{\mathbb{Q}} - P^*\right) \left({}_s p_t\right)^{\mathbb{Q}}$.

The differential form of (23) is

$$\begin{aligned} \frac{dD\left(t\right)}{D\left(t\right)} = & \left(r\left(t\right) + \lambda\left(t\right) + \frac{P^* - \lambda\left(t\right)}{D\left(t\right)}\right) dt \\ & + \frac{D_r\left(t\right)}{D\left(t\right)} \sigma_r \sqrt{r} dW_r^{\mathbb{Q}} + \frac{D_\lambda\left(t\right)}{D\left(t\right)} \sigma_\lambda \sqrt{\lambda} dW_\lambda^{\mathbb{Q}}, \end{aligned} \tag{24}$$

where D_λ and D_r are the partial derivatives of D with respect to λ and r respectively and, in particular,

$$\begin{aligned}\frac{\partial D(t)}{\partial \lambda(t)} &= \int_t^\infty \frac{\partial l(t,s)^{\mathbb{Q}}}{\partial \lambda(t)} ({}_s p_t)^{\mathbb{Q}} B(t,s)\, ds \\ &\quad + \int_t^\infty \left(l(t,s)^{\mathbb{Q}} - P^* \right) \frac{\partial ({}_s p_t)^{\mathbb{Q}}}{\partial \lambda(t)} B(t,s)\, ds \\ &= \int_t^\infty e^{-\int_t^s \left(a_\lambda + C_P(u,s)\sigma_\lambda^2\right) du} ({}_s p_t)^{\mathbb{Q}} B(t,s)\, ds \\ &\quad - \int_t^\infty C_P(t,s) \left(l(t,s)^{\mathbb{Q}} - P^* \right) ({}_s p_t)^{\mathbb{Q}} B(t,s)\, ds,\end{aligned}$$

$$\begin{aligned}\frac{\partial D(t)}{\partial r(t)} &= \int_t^\infty \left(l(t,s)^{\mathbb{Q}} - P^* \right) ({}_s p_t)^{\mathbb{Q}} \frac{\partial B(t,s)}{\partial r(t)} ds \\ &= -\int_t^\infty C_B(t,s) \left(l(t,s)^{\mathbb{Q}} - P^* \right) ({}_s p_t)^{\mathbb{Q}} B(t,s)\, ds.\end{aligned}$$

In this case, the (opposite of the) semielasticity of $D(t)$ with respect to $r(t)$ coincides with the duration of the death insurance.

5.1 The deterministic case

When the interest rate is constant and the death intensity $\lambda(t)$ is deterministic (i.e. $\sigma = 0$), the premium P^* and the value of a death insurance contract $D(t)$ can be obtained in an easy closed form by using an incomplete Gamma function.

Proposition 8 *If the interest rate is constant (i.e. $r(t) = r$) and the mortality rate $\lambda(t)$ is deterministic (i.e. $\sigma_\lambda = 0$ in Equation (16)), then the premium of a death insurance is*

$$P^* = \phi + \frac{1}{b} \frac{\Gamma\left(-\left(\phi + r - \frac{1}{b}\right) b, e^{\frac{t_0 - m}{b}}\right)}{\Gamma\left(-(\phi + r) b, e^{\frac{t_0 - m}{b}}\right)},$$

and the value of the insurance contract at any time t is

$$\begin{aligned}D(t) &= e^{(\phi+r)(t-m) + e^{\frac{t-m}{b}}} \Gamma\left(-(\phi+r)b, e^{\frac{t-m}{b}}\right) \\ &\quad \times \left(\frac{\Gamma\left(-\left(\phi + r - \frac{1}{b}\right) b, e^{\frac{t-m}{b}}\right)}{\Gamma\left(-(\phi+r)b, e^{\frac{t-m}{b}}\right)} - \frac{\Gamma\left(-\left(\phi + r - \frac{1}{b}\right) b, e^{\frac{t_0-m}{b}}\right)}{\Gamma\left(-(\phi+r)b, e^{\frac{t_0-m}{b}}\right)} \right),\end{aligned} \tag{25}$$

where Γ is the incomplete Gamma function as defined in (31).

Proof. See Appendix C. ■

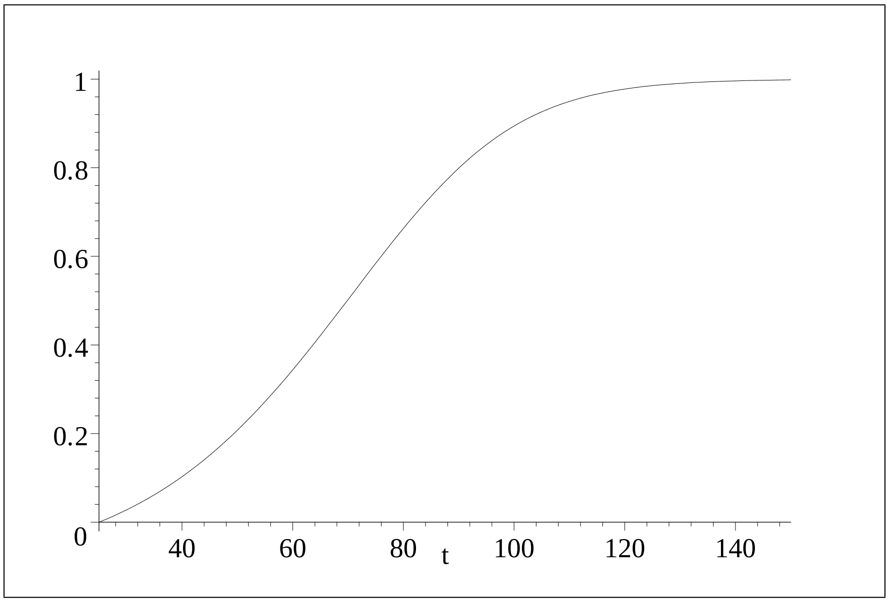


Figure 2: The value of a death insurance for an agent aged of 25 as a function of his age

With the values of the parameters given in (12) and $r = 0.05$, the value of the death insurance at any time t for an agen who entered the contract at the oage of $t_0 = 25$ is shown in Figure 2. With the same data the premium is $P^* = 0.0066$.

In the deterministic case, if we are at time t, the dividends of (23) at time s are

$$
\begin{aligned}
& (\lambda(s) - P^*)\, e^{-\phi(s-t)+e^{\frac{t-m}{b}}-e^{\frac{s-m}{b}}} \\
= \quad & \frac{1}{b}\left(e^{\frac{s-m}{b}} - \frac{\Gamma\left(-\left(\phi+r-\frac{1}{b}\right)b, e^{\frac{t-m}{b}}\right)}{\Gamma\left(-\left(\phi+r\right)b, e^{\frac{t-m}{b}}\right)}\right) e^{-\phi(s-t)+e^{\frac{t-m}{b}}-e^{\frac{s-m}{b}}}.
\end{aligned}
$$

If we assume the values in (12) and $r = 0.05$, then for an agent aged of $t = 65$, the coupons of the bond (23) for any time s (from 65 up to infinity) are represented in Figure 3.

5.2 Death bond

A death bond is like any other *Asset Backed Security* (ABS). The process for changing a death insurance into a death bond is made by 4 steps (as in Figure 5). Let us see such steps in details.

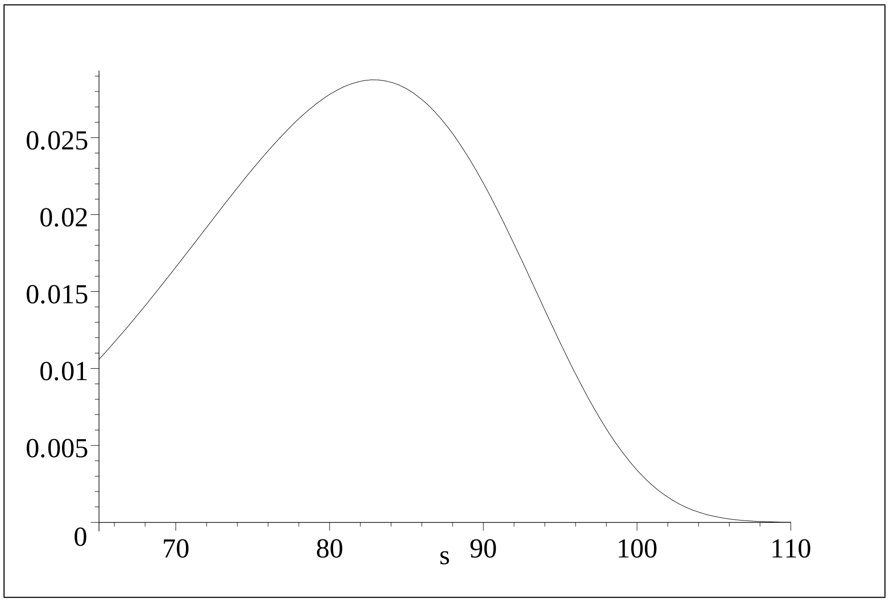


Figure 3: Devidends of a death bond when both interest and mortality rates are deterministic, for an agent aged of 65.

1. The so-called seller is the subscriber of the death insurance. When the agent becomes older (typically 70) and he does not have any further need for the insurance on his life and we would like to cash our his policty.

2. The seller hires a *life settlement* broker who will find a buyer for his policy. The buyer pays the net present value of the policy (we have called $D(t)$ in the previous section) and receives the insurance policy. Thus, it will be the buyer to continue paying the settlements to the insurance company. The buyer will also receive the final benefit from the insurance company when the seller dies. The up-front payout to the seller varies widely, from 20% of the death benefit to 40%. These percentages coincide with the values of $D(t)$ we have shown in Fig. 2. The seller pays to the broker a commission ranged from 5% to 6%.

3. Another character in this game is the so-called *life settlement provider*. Through him, a hedge fund or an investment bank buys a pool of death insurances from insurance company (or insurance companies). Now the hedge fund will receive the premia from the buyer and will pay the final benefit.

4. In the last step, after a sufficient number of policies has been collected, these policies can back the emission of a death bond. Accordingly, the policies play the same role as the assets in an ABS or the mortgages in

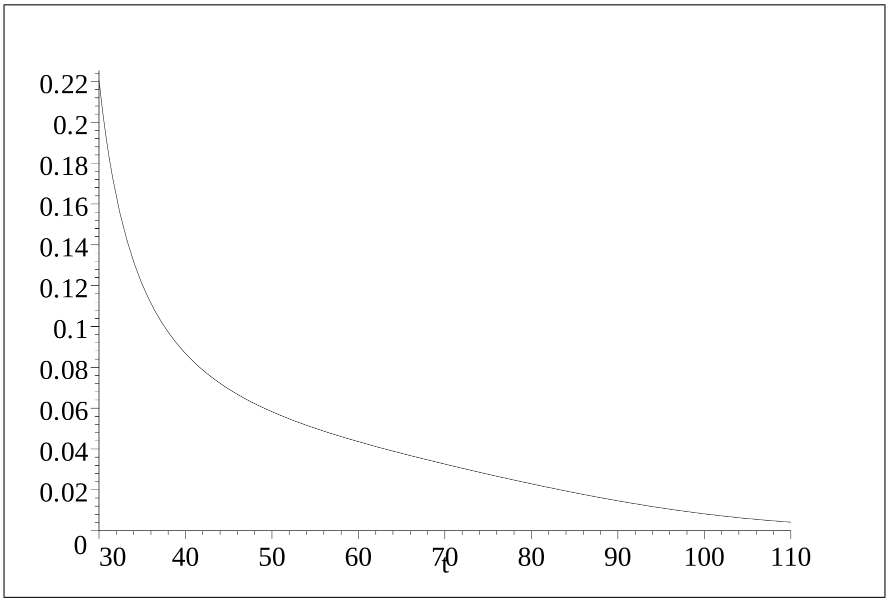


Figure 4: Return on a death bond ($\frac{dD}{D}$) with $t_0 = 25$ and $r = 0.05$ for t going from 30 to 110.

a mortgage backed security. We say that the new death bond is a *pass through* asset if the premia received by the hedge fund are directly paid to the investors.

From Equation (24) we can see that the return on a death bond is given by $r(t) + \lambda(t) + \frac{P^* - \lambda(t)}{D(t)}$. Now, let us take into account the fully deterministic case with the values of the parameters given in (12), $r = 0.05$, and $t_0 = 25$. The, the premium is given by $P^* = 0.0066$ and the return on the death bond for time t going from 30 to 110 is plotted in Figure 4.

It is evident that the bond return decreases while time goes on. In fact, the best case for the buyer of the death bond is when the seller immediately dies after receiving the first premium P^*.

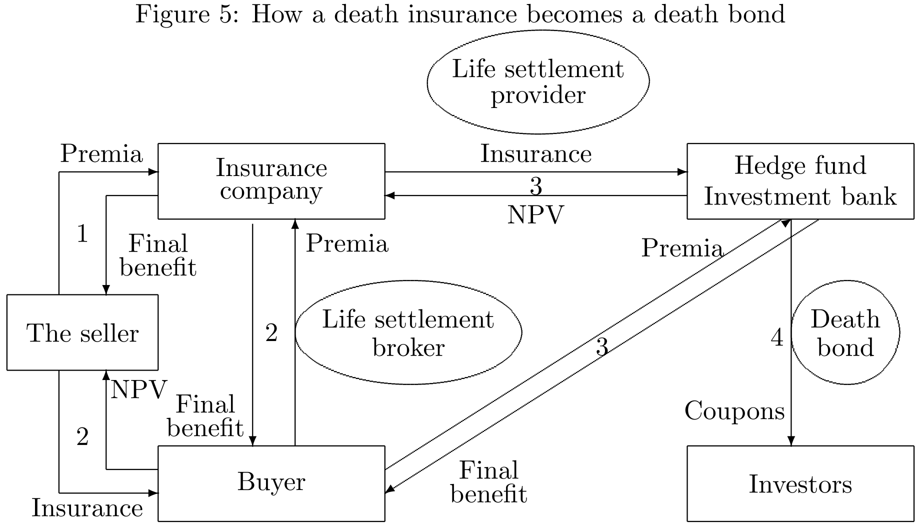
5.3 Correlation with interest rate

Some financial advisors suggest to invest in death bonds because the are uncorrelated with interest rates. Accordingly, they allow to diversify a bond portfolio.

From Equations (1) and (6) we can immediately check that bonds and interest rate are perfectly correlated. In fact, their covariance is

$$\mathbb{C}_t \left[\frac{dB(t,T)}{B(t,T)}, dr(t) \right] = -\sigma_r^2 r(t) C_B(t,T) dt,$$

Figure 5: How a death insurance becomes a death bond



and, accordingly, the correlation is

$$\rho_{B,r} = \frac{\mathbb{C}_t\left[\frac{dB(t,T)}{B(t,T)}, dr\left(t\right)\right]}{\sqrt{\mathbb{V}_t\left[\frac{dB(t,T)}{B(t,T)}\right]\mathbb{V}_t\left[dr\left(t\right)\right]}} = \frac{-\sigma_r^2 r\left(t\right) C_B\left(t,T\right)}{\sqrt{\sigma_r^4 r\left(t\right)^2 C_B\left(t,T\right)^2}} = -1.$$

Instead, if we take into account Equations (6) and (24), then the correlation index is

$$\rho_{B,D} = \frac{\mathbb{C}_t\left[\frac{dB(t,T)}{B(t,T)}, \frac{dD(t)}{D(t)}\right]}{\sqrt{\mathbb{V}_t\left[\frac{dB(t,T)}{B(t,T)}\right]\mathbb{V}_t\left[\frac{dD(t)}{D(t)}\right]}} = -\frac{1}{\sqrt{1+\frac{\left(\frac{D_\lambda(t)}{D(t)}\right)^2\sigma_\lambda^2\lambda(t)}{\left(\frac{D_r(t)}{D(t)}\right)^2\sigma_r^2 r(t)}}},$$

which is, of course, negative and not zero. The higher the mortality rate volatility (σ_λ), the lower the correlation between death and ordinary bonds. Accordingly, the death bonds are suitable for diversify a bond portfolio only with a suitably high level of mortality uncertainty (the correlation $\rho_{B,D}$ tends towards zero when σ_λ tends towards infnity).

Would it be possible to create an asset which is fully uncorrelated with the interest rate?

Let us call $\eta\left(t,r,\lambda\right)$ the coupon at time t of such an asset. Then, if this title exipires in T and it is an amortizing bond,[2] then its value is given by

$$V\left(t,r,\lambda\right) = \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \eta\left(s,r,\lambda\right) e^{-\int_t^s r(u)du} ds\right].$$

[2] In this case the coupons also contain the repayment of the nominal value.

Now, we can have

$$\frac{\partial V(t,r,\lambda)}{\partial r} = 0$$

if and only if the coupon η depends on r. For instance, η could be given by the compounded value of a function depending only on time and λ:

$$\eta(s,r,\lambda) = \hat{\eta}(s,\lambda)\, e^{\int_t^s r(u)du}.$$

In this case, in fact, we would have

$$V(t,r,\lambda) = \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \hat{\eta}(s,\lambda)\, ds\right].$$

Nevertheless, in the case of the death bond, the premia do not depend on interest rate. This is exactly the reason why the value of the bond does depend on interest rate.

6 An asset allocation problem

Now we wan to investigate the role of a death bond (23) in an asset allocation problem for an agent maximizing the expected utility of his intertemporal consumption. We also assume that the agent takes some utility from the wealth he still has at the time of his death (in τ). The problem can thus be written as

$$\max_{c,w} \mathbb{E}_{t_0}^{\tau}\left[\int_{t_0}^{\tau} e^{-\rho(t-t_0)} U_1(c(t))\, dt + e^{-\rho(\tau-t_0)} U_2(R(\tau))\right].$$

Here U_1 is the intertemporal utility of consumption, U_2 is the so-called bequest utility function, and ρ is the subjective discount rate. The control variables are: consumption (c) and portfolio weights (in the vector w). The state variable R is the consumer-investor's wealth. With passages similar to those already made for computing the value of the insurance contract, we can rewrite the optimization problem as

$$\begin{aligned} &\max_{c,w} \mathbb{E}_{t_0}\left[\int_{t_0}^{\infty} e^{-\int_{t_0}^{t} \lambda(s)ds - \rho(t-t_0)} U_1(c(t))\, dt\right] \\ &+\mathbb{E}_{t_0}\left[\int_{t_0}^{\infty} \lambda(t)\, e^{-\int_{t_0}^{t} \lambda(s)ds - \rho(t-t_0)} U_2(R(t))\, dt\right], \end{aligned}$$

or

$$\max_{c,w} \mathbb{E}_{t_0}\left[\int_{t_0}^{\infty} e^{-\int_{t_0}^{t} \lambda(s)ds - \rho(t-t_0)} \left(U_1(c(t)) + \lambda(t)\, U_2(R(t))\right) dt\right]. \tag{26}$$

For the sake of simplicity and in order to have a quasi-explicit solution for both the optimal consumption and the optimal asset allocation, we assume U_1 to be equal to U_2. In particular, both the utility functions are assumed to belong to the Constant Relative Risk Aversion family:

$$U_1(x) = U_2(x) = \frac{x^{1-\delta}}{1-\delta}. \tag{27}$$

6.1 The financial market and investor's wealth

On the financial market we assume to have three assets:

1. a riskless asset whose price $G(t)$ solve the (deterministic) differential equation

$$\begin{aligned} dG(t) &= G(t)\,r(t)\,dt, \\ G(t_0) &= 1, \end{aligned}$$

2. a zero coupon bond whose price solves

$$\frac{dB(t,T)}{B(t,T)} = \left(r(t) - C_B(t,T)\,\sigma_r\sqrt{r(t)}\xi_r\right)dt - C_B(t,T)\,\sigma_r\sqrt{r(t)}dW_r(t)\,,$$

3. a death bond whose price solves

$$\begin{aligned} \frac{dD(t)}{D(t)} &= \left(r(t) + \lambda(t) + \frac{P^* - \lambda(t)}{D(t)} + \frac{D_r(t)}{D(t)}\sigma_r\sqrt{r(t)}\xi_r + \frac{D_\lambda(t)}{D(t)}\sigma_\lambda\sqrt{\lambda(t)}\xi_\lambda\right)dt \\ &\quad + \frac{D_r(t)}{D(t)}\sigma_r\sqrt{r(t)}dW_r + \frac{D_\lambda(t)}{D(t)}\sigma_\lambda\sqrt{\lambda(t)}dW_\lambda. \end{aligned}$$

Here, we have assumed that both bonds have the same time to maturity. This is the reason why we have $C_B(t,T)$ in both differential equations for $D(t)$ and $B(t,T)$.

Since the semielasticity $\frac{D_r(t)}{D(t)}$ is negative, then the value of the market price of risk for the interest rate must be negative too.

If we call $w = \begin{bmatrix} w_B \\ w_D \end{bmatrix}$ the vector of the bond weights and w_G the amount of riskless asset held in portfolio, then the investor's wealth R is given by[3]

$$R = w_G G + w_B B + w_D D.$$

The differential of R is given by (we apply Itô's differential rule with stochastic w_G, w_B, and w_D):

$$\begin{aligned} dR &= \underbrace{w_G dG + w_B dB + w_D dD}_{dR_1} \\ &\quad + \underbrace{G dw_G + dw_B(B + dB) + dw_D(D + dD)}_{dR_2}. \end{aligned}$$

If the portfolio is self-financed, then dR_2 must equate the dividends one obtains from the assets in the portfolio, diminished by the consumption c. In our case, we must then have

$$dR_2 = D(t)\left(\frac{\lambda(t) - P^*}{D(t)} - \lambda(t)\right)dt - cdt,$$

[3] In what follows we forget about the functional dependences.

since the death bond is the only asset paying dividends. This means that the wealth differential, in matrix form, can be written as[4]

$$\begin{aligned} dR &= \left(Rr + \begin{bmatrix} w_B B \\ w_D D \end{bmatrix}' \begin{bmatrix} -C_B(t,T)\,\sigma_r \sqrt{r(t)}\xi_r \\ \frac{D_r(t)}{D(t)}\sigma_r \sqrt{r(t)}\xi_r + \frac{D_\lambda(t)}{D(t)}\sigma_\lambda \sqrt{\lambda(t)}\xi_\lambda \end{bmatrix} - c \right) dt \\ &\quad + \begin{bmatrix} w_B B \\ w_D D \end{bmatrix}' \begin{bmatrix} -C_B(t,T)\,\sigma_r \sqrt{r(t)} & 0 \\ \frac{D_r(t)}{D(t)}\sigma_r \sqrt{r(t)} & \frac{D_\lambda(t)}{D(t)}\sigma_\lambda \sqrt{\lambda(t)} \end{bmatrix} \begin{bmatrix} dW_r \\ dW_\lambda \end{bmatrix}, \end{aligned}$$

where the prime denotes transposition.

6.2 The optimal consumption and portfolio

In the section we present some results about the optimal portfolio. The results are presented in decreasing order of generality. Then, we start by showing the most general result.

Proposition 9 *The optimal consumption and portfolio solving Problem (26) are given by*

$$\begin{aligned} c^* &= J_R^{-\frac{1}{\delta}} e^{-\frac{1}{\delta}\int_{t_0}^{s}\lambda(s)ds - \frac{1}{\delta}\rho(s-t_0)}, \\ w_D^* D(t) &= \frac{D(t)}{D_\lambda(t)} \left(-\frac{J_R}{J_{RR}} \frac{\xi_\lambda}{\sigma_\lambda \sqrt{\lambda(t)}} - \frac{J_{\lambda R}}{J_{RR}} \right), \\ w_B^* B(t) &= -\frac{1}{C_B(t,T)} \left(-\frac{J_R}{J_{RR}} \frac{\xi_r}{\sigma_r \sqrt{r(t)}} - \frac{J_{rR}}{J_{RR}} \right) + \frac{1}{C_B(t,T)} \frac{D_r(t)}{D(t)} w_D^* D(t), \end{aligned}$$

where the function $J(t,r,\lambda,R)$ solves the partial differential equation (32) and the subscripts on J indicate partical derivatives.

Proof. See Appendix D. ■

Proposition 10 *The optimal consumption and portfolio solving Problem (26) with (27), are given by*

$$\begin{aligned} \frac{1}{R}c^* &= \frac{e^{-\frac{1}{\delta}\int_{t_0}^{s}\lambda(s)ds - \frac{1}{\delta}\rho(s-t_0)}}{F(t,z)}, \\ \frac{w_D^* D(t)}{R} &= \frac{D(t)}{D_\lambda(t)} \left(\frac{1}{\delta} \frac{\xi_\lambda}{\sigma_\lambda \sqrt{\lambda(t)}} + \frac{1}{F(t,r,\lambda)} \frac{\partial F(t,r,\lambda)}{\partial \lambda} \right), \\ \frac{w_B^* B(t)}{R} &= -\frac{1}{C_B(t,T)} \left(\frac{1}{\delta} \frac{\xi_r}{\sigma_r \sqrt{r(t)}} + \frac{1}{F(t,r,\lambda)} \frac{\partial F(t,r,\lambda)}{\partial r} \right) \\ &\quad + \frac{1}{C_B(t,T)} \frac{D_r(t)}{D(t)} \frac{w_D^* D}{R}, \end{aligned}$$

[4] We substitute w_G by

$$w_G = \frac{R - w_B B - w_D D}{G}.$$

where the function $F(t,r,\lambda)$ solves the partial differential equation (33).

Proof. See Appendix D. ■

Proposition 11 *The optimal consumption and portfolio solving Problem (26) with $U_1(x) = U_2(x) = \ln x$, are given by*

$$\begin{aligned}
\frac{1}{R}c^* &= H(t,\lambda)^{-1}, \\
\frac{w_D^* D(t)}{R} &= \frac{D(t)}{D_\lambda(t)}\left(\frac{\xi_\lambda}{\sigma_\lambda\sqrt{\lambda(t)}} + \frac{1}{H(t,\lambda)}\frac{\partial H(t,\lambda)}{\partial\lambda}\right), \\
\frac{w_B^* B(t)}{R} &= -\frac{1}{C_B(t,T)}\frac{\xi_r}{\sigma_r\sqrt{r(t)}} + \frac{1}{C_B(t,T)}\frac{D_r(t)}{D(t)}\frac{w_D^* D}{R},
\end{aligned}$$

where

$$H(t,\lambda) = \mathbb{E}_t\left[\int_t^\infty (1+\lambda(s))\, e^{-\int_t^s \lambda(u)du} e^{-\rho(s-t)} ds\right].$$

Proof. See Appendix D. ■

A Computation of $\mathbb{E}_t\left[(1-\chi+\chi X(T))\, e^{-\int_t^T X(s)ds}\right]$

If the stochastic variable $X(t)$ follows the process

$$\begin{aligned}
dX(t) &= a(\gamma(t) - X(t))\,dt + \sigma\sqrt{X(t)}dW(t), \\
X(t_0) &= X_0,
\end{aligned}$$

then the expected value

$$V(t,T) = \mathbb{E}_t\left[(1-\chi+\chi X(T))\, e^{-\int_t^T X(s)ds}\right],$$

must solve the partial differential equation

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial X}a(\gamma(t) - X) + \frac{1}{2}\frac{\partial^2 V}{\partial X^2}\sigma^2 X = XV,$$

with the boundary condition

$$V(T,T) = 1 - \chi + \chi X(T),$$

where the parameter χ can take either value 1 or value 0. If $\chi = 0$, then the function V coincides with the probability $({}_T p_t)$ if $X = \lambda$ and with the value of a zero-coupon if $X = r$. Instead, if $\chi = 1$, then the function V coincides with the numerator of $l(t,T)$ in Equation (14).

Now we use the guess function

$$V(t,X) = (E(t) + F(t)\,X)\, e^{-A(t)-C(t)X},$$

where the function A, C, E, and F must be computed in order to solve the previous differential equation. The boundary condition translates into the following conditions:

$$\begin{aligned} E(T) &= 1-\chi, \\ F(T) &= \chi, \\ A(T) &= 0, \\ C(T) &= 0. \end{aligned}$$

Once the partial derivatives of V are substituted into the differential equation we obtain[5]

$$\begin{aligned} 0 \;=\; & \left(\frac{\partial E}{\partial t}+\frac{\partial F}{\partial t}X\right)+(E+FX)\left(-\frac{\partial A}{\partial t}-\frac{\partial C}{\partial t}X\right) \\ & +(F-(E+FX)C)\,a\,(\gamma(t)-X) \\ & +\frac{1}{2}\left(-2CF+(E+FX)C^2\right)\sigma^2X-X(E+FX), \end{aligned}$$

which is an ordinary differential equation in A, C, E, and F. Since this equation must hold for any value of X then we can split it into three ordinary differential equations as follows

$$\begin{cases} 0=\frac{\partial E}{\partial t}+Fa\gamma(t)-E\left(A_t+Ca\gamma(t)\right), \\ 0=\frac{\partial F}{\partial t}-F\left(A_t+Ca\gamma(t)\right)-Fa-CF\sigma^2, \\ 0=-\frac{\partial C}{\partial t}+aC+\frac{1}{2}C^2\sigma^2-1. \end{cases} \tag{28}$$

We immediately see that the value of function $C(t)$ can be computed from the third equation. With the suitable boundary condition the only solution of the differential equation for $C(t)$ is given by

$$\begin{aligned} C(t,T) &= 2\frac{1-e^{-k(T-t)}}{k+a+(k-a)\,e^{-k(T-t)}}, \\ k &\equiv \sqrt{a^2+2\sigma^2}. \end{aligned}$$

The values of all the other functions can be written as functions of $C(t,T)$. Now, if we wanted to compute just the survival probability, then we would have $E=1$ and $F=0$ with the function A accordingly solving

$$0=\frac{\partial A}{\partial t}+Ca\gamma(t),$$

with the boundary condition $A(T)=0$. The only solution of this equation is

$$A(t)=a\int_t^T C(s)\,\gamma(s)\,ds.$$

[5] For the sake of simplicity, we have omitted the functional dipendences (except for the function $\gamma(t)$).

Given this value for $A(t)$, the two first equations of system (28) become

$$\begin{aligned} 0 &= \frac{\partial E(t)}{\partial t} + F(t)\, a\gamma(t), \\ 0 &= \frac{\partial F(t)}{\partial t} - F(t)\left(a + C(t)\sigma^2\right). \end{aligned}$$

We now compute the value of F from the second equation by obtaining

$$F(t) = \chi e^{-\int_t^T \left(a + C(s)\sigma^2\right) ds},$$

and the value of E can then be computed from the first equation

$$E(t) = 1 - \chi + a \int_t^T F(s)\, \gamma(s)\, ds.$$

Finally, we can write

$$\begin{aligned} V(t,T) &= \left(1 - \chi + \chi \int_t^T a\gamma(s)\, e^{-\int_s^T \left(a + C(u)\sigma^2\right) du} ds + \chi e^{-\int_t^T \left(a + C(u)\sigma^2\right) du} X(t)\right) \\ &\quad \times e^{-a\int_t^T C(u)\gamma(u) du - C(t) X(t)}. \end{aligned}$$

The two values we are interested into are given by

$$V(t,T)\big|_{\chi=0} = e^{-a\int_t^T C(u)\gamma(u) ds - C(t)X(t)},$$

and

$$\frac{V(t,T)\big|_{\chi=1}}{V(t,T)\big|_{\chi=0}} = \int_t^T a\gamma(s)\, e^{-\int_s^T \left(a + C(u)\sigma^2\right) du} ds + e^{-\int_t^T \left(a + C(u)\sigma^2\right) du} X(t).$$

B Proof of Proposition 4

We start by citing a well known result about the Cox, Ingersoll, and Ross process:

$$dX(t) = a(\gamma - X(t))\, dt + \sigma\sqrt{X(t)} dW(t),$$

with a, γ, and σ positive constants: the value of $X(t)$ never becomes negative if $\sigma^2 \leq 2a\gamma$. In order to prove the proposition, we use the following result.

Proposition 12 *Let us assume we have two continuous, adapted processes $X_i(t), i = 1, 2$, such that*

$$X_i(t) = X_i(t_0) + \int_{t_0}^t \mu_i(s, X_i(s))\, ds + \int_{t_0}^t \sigma(s, X_i(s))\, dW(s),$$

and $\forall t \in [t_0, \infty[, x \in \mathbb{R}, y \in \mathbb{R}$: (i) the coefficients $\mu_i(t,x)$ and $\sigma(t,x)$ are continuous, real-valued functions, (ii) $|\sigma(t,x) - \sigma(t,y)| \leq h(|x-y|)$ where $h:$

$[0,\infty[\times[0,\infty[$ *is a strictly increasing function with* $h(0)=0$ *and* $\int_{(0,\varepsilon)} h^{-2}(u)\,du = \infty$, $\forall\varepsilon>0$, *(iii)* $X_1(t_0)\le X_2(t_0)$ *a.s., (iv)* $\mu_1(t,x)\le\mu_2(t,x)$, *(v) there exists a positive constant* K *such that either* $\mu_1(t,x)$ *or* $\mu_2(t,x)$ *satisfies* $|\mu_i(t,x)-\mu_i(t,y)|\le K|x-y|$. *Then*

$$\mathbb{P}\{X_1(t)\le X_2(t),\forall t\in[t_0,\infty[\}=1.$$

Proof. See Karatzas and Shreve (1991), Proposition 2.18 p. 293. ■

Here, we take the following processes

$$\begin{aligned}\lambda_1(t) &= \left(\phi+\frac{1}{b}e^{\frac{t_0-m}{b}}\right)+\int_{t_0}^{t}a\left(\gamma(t_0)-\lambda_1(s)\right)ds+\int_{t_0}^{t}\sigma\sqrt{\lambda_1(s)}dW(s),\\ \lambda_2(t) &= \left(\phi+\frac{1}{b}e^{\frac{t_0-m}{b}}\right)+\int_{t_0}^{t}a\left(\gamma(s)-\lambda_2(s)\right)ds+\int_{t_0}^{t}\sigma\sqrt{\lambda_2(s)}dW(s),\end{aligned}$$

where $\gamma(t)$ is defined in (17). It is evident that both the drift and the diffusion terms respect all the conditions in Proposition 12.

Since we have set $\lambda_1(t_0)=\lambda_2(t_0)$ and we know that $\lambda_1(t)$ never becomes negative if

$$\sigma^2\le 2a\gamma(t_0), \tag{29}$$

then we also know that $\lambda_2(t)$ never becomes negative if its drift is greater than $\lambda_1(t)$'s:

$$a(\gamma(t)-\lambda(t))\ge a(\gamma(t_0)-\lambda(t)),$$

for any real λ and for any $t\in[t_0,\infty[$. Such inequality holds if and only if

$$\gamma(t)\ge\gamma(t_0).$$

Nevertheless, since $\gamma(t)$ is strictly increasing in t, then this inequality always holds. This means that $\lambda_2(t)$ never becomes negative if just (29) holds.

C Proof of Proposition 8

Under the conditions of Proposition 8, the value

$$\int_{t_0}^{\infty} l(t_0,t)\,({}_tp_{t_0})^{\mathbb{Q}}\,B(t_0,t)\,dt,$$

can be written as

$$\begin{aligned}&\int_{t_0}^{\infty}\left(\phi+\frac{1}{b}e^{\frac{t-m}{b}}\right)e^{-\phi(t-t_0)+e^{\frac{t_0-m}{b}}-e^{\frac{t-m}{b}}}e^{-r(t-t_0)}dt\\ =\;&\phi\int_{t_0}^{\infty}e^{-(\phi+r)(t-t_0)+e^{\frac{t_0-m}{b}}-e^{\frac{t-m}{b}}}dt\\ &+\frac{1}{b}e^{\frac{t_0-m}{b}}\int_{t_0}^{\infty}e^{-\left(\phi+r-\frac{1}{b}\right)(t-t_0)+e^{\frac{t_0-m}{b}}-e^{\frac{t-m}{b}}}dt.\end{aligned}$$

This means that we have to compute the following integral:

$$\int_{t_0}^{\infty} e^{-h(t-t_0)+e^{\frac{t_0-m}{b}}-e^{\frac{t-m}{b}}}dt,$$

where h can take different values according to the case we are taking into account. The first step is to change the variable:

$$z=e^{\frac{t-m}{b}} \Rightarrow dz=\frac{1}{b}e^{\frac{t-m}{b}}dt,$$

from which we have

$$\begin{aligned} m+b\ln z &= t, \\ e^{-\frac{t-m}{b}}bdz &= dt. \end{aligned}$$

The integral becomes

$$\begin{aligned} & \int_{t_0}^{\infty} e^{-h(t-t_0)+e^{\frac{t_0-m}{b}}-e^{\frac{t-m}{b}}}dt \qquad (30) \\ = \; & be^{h(t_0-m)+e^{\frac{t_0-m}{b}}} \int_{e^{\frac{t_0-m}{b}}}^{\infty} z^{-hb-1}e^{-z}dz = be^{h(t_0-m)+e^{\frac{t_0-m}{b}}}\Gamma\left(-hb, e^{\frac{t_0-m}{b}}\right), \end{aligned}$$

where Γ is the incomplete Gamma funcion defined as

$$\Gamma(a,x)=\int_x^{\infty} z^{a-1}e^{-z}dz. \qquad (31)$$

The premium defined as in (22) can then be writte as

$$\begin{aligned} P^* &= \phi+\frac{\frac{1}{b}e^{\frac{t_0-m}{b}}\int_{t_0}^{\infty} e^{-\left(\phi+r-\frac{1}{b}\right)(t-t_0)+e^{\frac{t_0-m}{b}}-e^{\frac{t-m}{b}}}dt}{\int_{t_0}^{\infty} e^{-(\phi+r)(t-t_0)+e^{\frac{t_0-m}{b}}-e^{\frac{t-m}{b}}}dt} \\ &= \phi+\frac{1}{b}\frac{\Gamma\left(-\left(\phi+r-\frac{1}{b}\right)b, e^{\frac{t_0-m}{b}}\right)}{\Gamma\left(-(\phi+r)b, e^{\frac{t_0-m}{b}}\right)}. \end{aligned}$$

The value of the insurance contract is now given by

$$\begin{aligned} D(t) &= \int_t^{\infty} \lambda(s)\left({}_sp_t\right)^{\mathbb{Q}} e^{-r(s-t)}ds - P^*\int_t^{\infty}\left({}_sp_t\right)^{\mathbb{Q}} e^{-r(s-t)}ds \\ &= \int_t^{\infty}\left(\lambda(s)-P^*\right)e^{-\phi(s-t)+e^{\frac{t-m}{b}}-e^{\frac{s-m}{b}}}e^{-r(s-t)}ds \\ &= \phi\int_t^{\infty} e^{-(\phi+r)(s-t)+e^{\frac{t-m}{b}}-e^{\frac{s-m}{b}}}ds \\ &\quad +\frac{1}{b}e^{\frac{t-m}{b}}\int_t^{\infty} e^{-\left(\phi+r-\frac{1}{b}\right)(s-t)+e^{\frac{t-m}{b}}-e^{\frac{s-m}{b}}}ds \\ &\quad -P^*\int_t^{\infty} e^{-(\phi+r)(s-t)+e^{\frac{t-m}{b}}-e^{\frac{s-m}{b}}}ds. \end{aligned}$$

By using the result (30) we can write

$$\begin{aligned} D(t) &= (\phi - P^*)\, b e^{(\phi+r)(t-m)+e^{\frac{t-m}{b}}} \Gamma\left(-(\phi+r)\, b, e^{\frac{t-m}{b}}\right) \\ &\quad + \frac{1}{b} e^{\frac{t-m}{b}} b e^{\left(\phi+r-\frac{1}{b}\right)(t-m)+e^{\frac{t-m}{b}}} \Gamma\left(-\left(\phi+r-\frac{1}{b}\right) b, e^{\frac{t-m}{b}}\right), \end{aligned}$$

and, after substituting for the value of P^*, the result of the proposition follows.

D Optimal portfolio

The problem can be written as

$$\max_{c,w} \mathbb{E}_{t_0} \left[\int_{t_0}^{\infty} f(t_0, s) \frac{c(s)^{1-\delta}}{1-\delta} ds + \int_{t_0}^{\infty} \lambda(s) f(t_0, s) \frac{R(s)^{1-\delta}}{1-\delta} ds \right],$$

where

$$f(t_0, s) \equiv e^{-\int_{t_0}^{s} \lambda(s) ds - \rho(s-t_0)},$$

the state variable R follows

$$dR = (Rr + w'\Sigma'\xi - c)\, dt + w'\Sigma' dW,$$

where

$$\Sigma' = \begin{bmatrix} -C_B(t,T)\, \sigma_r \sqrt{r(t)} & 0 \\ \frac{D_r(t)}{D(t)} \sigma_r \sqrt{r(t)} & \frac{D_\lambda(t)}{D(t)} \sigma_\lambda \sqrt{\lambda(t)} \end{bmatrix},$$

and $\xi = \begin{bmatrix} \xi_r & \xi_\lambda \end{bmatrix}'$, and, finally, all the other state variables $z = \begin{bmatrix} r & \lambda \end{bmatrix}'$ follow

$$dz = \mu_z dt + \Omega' dW,$$

with

$$\mu_z = \begin{bmatrix} a_r(\gamma_r - r) \\ a_\lambda(\gamma_\lambda - r) \end{bmatrix}, \quad \Omega' = \begin{bmatrix} \sigma_r \sqrt{r(t)} & 0 \\ 0 & \sigma_\lambda \sqrt{\lambda(t)} \end{bmatrix}.$$

The Hamiltonian of the problem is

$$\begin{aligned} \mathcal{H} &= f(t_0, t) \frac{c^{1-\delta}}{1-\delta} + \lambda(t) f(t_0, t) \frac{R^{1-\delta}}{1-\delta} + J_R (Rr + w'\Sigma'\xi - c) \\ &\quad + \frac{1}{2} J_{RR} w'\Sigma'\Sigma w + J_z \mu_z + \frac{1}{2} tr(\Omega'\Omega J_{zz}) + w'\Sigma'\Omega J_{zR}, \end{aligned}$$

where $J(R, z, t)$ is the value function solving the optimization problem. The first order on consumption is

$$\frac{\partial \mathcal{H}}{\partial c} = f(t_0, t)\, c^{-\delta} - J_R,$$

and the first order on portfolio is

$$\frac{\partial \mathcal{H}}{\partial w} = J_R \Sigma' \xi + J_{RR} \Sigma' \Sigma w + \Sigma' \Omega J_{zR}.$$

The optimal consumption and portfolio in terms of the value function J are

$$c^* = \left(\frac{J_R}{f(t_0, t)} \right)^{-\frac{1}{\delta}},$$

$$w^* = -\frac{J_R}{J_{RR}} \Sigma^{-1} \xi - \frac{1}{J_{RR}} \Sigma^{-1} \Omega J_{zR}.$$

Once the values of Σ, ξ, and Ω are substituted, we obtain the result in Proposition 9.

When these optimal values are substituted into the Hamiltonian, we have the so-called Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned} 0 \quad = \quad & J_t + \frac{\delta}{1-\delta} f(t_0, t)^{\frac{1}{\delta}} J_R^{1-\frac{1}{\delta}} + \lambda(t) f(t_0, t) \frac{R^{1-\delta}}{1-\delta} \\ & + J_R R r - \frac{1}{2} \frac{J_R^2}{J_{RR}} \xi' \xi - \frac{J_R}{J_{RR}} J_{zR}' \Omega' \xi \\ & - \frac{1}{2} \frac{1}{J_{RR}} J_{zR}' \Omega' \Omega J_{zR} + J_z \mu_z + \frac{1}{2} tr\left(\Omega' \Omega J_{zz} \right), \end{aligned} \tag{32}$$

whose boundary condition is

$$\lim_{t \to \infty} J(t, z, R) = 0.$$

Now, we try the following guess value function

$$J(t, z, R) = F(t, z)^{\delta} \frac{R^{1-\delta}}{1-\delta},$$

where $F(t, z)$ is a function whose value must be found in order to satisfy the HJB equation. After substituting for $J(t, z, R)$ into the HJB, we have

$$\begin{aligned} 0 \quad = \quad & F_t + \left(\mu_z' + \frac{1-\delta}{\delta} \xi' \Omega \right) F_z + \frac{1}{2} tr\left(\Omega' \Omega F_{zz} \right) + \frac{1-\delta}{\delta} \left(r + \frac{1}{2} \frac{1}{\delta} \xi' \xi \right) F \\ & + f(t_0, t)^{\frac{1}{\delta}} + \lambda(t) f(t_0, t) \frac{1}{\delta} F^{1-\delta}, \end{aligned} \tag{33}$$

with boundary condition

$$\lim_{t \to \infty} F(t, z) = 0.$$

The optimal consumption and portfolio are then given by

$$\frac{1}{R} c^* = \frac{f(t_0, t)^{\frac{1}{\delta}}}{F(t, z)},$$

$$\frac{1}{R}w^* = \frac{1}{\delta}\Sigma^{-1}\xi + \frac{1}{F(t,z)}\Sigma^{-1}\Omega\frac{\partial F(t,z)}{\partial z}.$$

If we substitute for the values of Σ, ξ, and Ω, we obtain what presented in Proposition 10.

Here, we can obtain a closed form solution for this differential equation only if $\delta = 1$, i.e. if the utility function is logaritm. In this case, in fact, we have

$$0 = F_t + \mu_z' F_z + \frac{1}{2} tr\left(\Omega'\Omega F_{zz}\right) + f(t_0, t) + \lambda(t_0, s) f(t_0, t),$$

whose solution is

$$\begin{aligned} F(t,z) &= \mathbb{E}_t\left[\int_t^\infty (1+\lambda(s)) e^{-\int_{t_0}^s \lambda(u)du} e^{-\rho(s-t_0)} ds\right] \\ &= e^{-\int_{t_0}^t \lambda(u)du} e^{-\rho(t-t_0)} \mathbb{E}_t\left[\int_t^\infty (1+\lambda(s)) e^{-\int_t^s \lambda(u)du} e^{-\rho(s-t)} ds\right]. \end{aligned}$$

Accordingly, the optimal consumption and portfolio are

$$\begin{aligned} \frac{c^*}{R} &= \frac{e^{-\int_{t_0}^t \lambda(u)du} e^{-\rho(t-t_0)}}{e^{-\int_{t_0}^t \lambda(u)du} e^{-\rho(t-t_0)} \mathbb{E}_t\left[\int_t^\infty (1+\lambda(s)) e^{-\int_t^s \lambda(u)du} e^{-\rho(s-t)} ds\right]} \\ &= \frac{1}{\mathbb{E}_t\left[\int_t^\infty (1+\lambda(s)) e^{-\int_t^s \lambda(u)du} e^{-\rho(s-t)} ds\right]}. \end{aligned}$$

$$\frac{1}{R}w^* = \Sigma^{-1}\xi + \frac{1}{F}\Sigma^{-1}\Omega F_z.$$

Now, if we substitute for the suitable matrices Σ and Ω we finally have the result of Proposition (11).[6]

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