SHORT CONTRIBUTION

A TEACHER'S REMARK ON EXACT CREDIBILITY

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In the classical Bayesian approach to credibility the claims are conditionally independent and identically distributed random variables, with common density $f(x, \vartheta)$. The unknown parameter ϑ is a realization of a random variable Θ having initial (prior) density $u(\vartheta)$. Let

$$\mu(\vartheta) = \int x f(x, \vartheta)\, dx.$$

The initial pure premium is

$$E[X_1] = E[\mu(\Theta)].$$

The premium for X_{t+1}, given $X_1, \ldots, X_t$, is the conditional expectation

$$E[\mu(\Theta) \mid X_1, \ldots, X_t] = \int \mu(\vartheta)\, u(\vartheta \mid X_1, \ldots, X_t)\, d\vartheta.$$

A central question is for which pairs $f(x, \vartheta)$ and $u(\vartheta)$ this expression is linear, i.e. of the form

$$Z \cdot \bar{X} + (1 - Z) \cdot E[\mu(\Theta)]$$

where $\bar{X} = (X_1 + \ldots + X_t)/t$ is the observed average. This is indeed the case for about half a dozen famous examples. JEWELL (1974) has found an elegant and general approach to unify these examples, see also GOOVAERTS and HOOGSTAD (1987, chapter 2). The classical examples can be retrieved as special cases; however a preliminary reparameterization has to be performed on a case by case basis. The purpose of this note is to propose an alternative (but of course strongly related) formulation of the general model, from which the classical examples can be retrieved in a *straightforward* way.

The common density of the claims is supposed to be of the form

$$f(x, \vartheta) = \frac{a(x) \cdot b(\vartheta)^x}{c(\vartheta)}, \qquad x \in A.$$

Here A is the set of possible values of the claims (discrete or continuous), and $c(\vartheta)$ is the normalizing constant:

$$c(\vartheta) = \int_A a(x) \cdot b(\vartheta)^x\, dx.$$

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Then

$$\mu(\vartheta) = \frac{b(\vartheta)}{b'(\vartheta)} \cdot \frac{c'(\vartheta)}{c(\vartheta)} .$$

As a prior density we choose

$$u(\vartheta) = \frac{c(\vartheta)^{-n_0} \cdot b(\vartheta)^{x_0} \cdot b'(\vartheta)}{d(n_0, x_0)} ,$$

where ϑ varies in some interval,

$$d(n_0, x_0) = \int c(\vartheta)^{-n_0} \cdot b(\vartheta)^{x_0} \cdot b'(\vartheta)\, d\vartheta$$

is the normalizing constant and n_0 and x_0 are two parameters. Then it is easy to see that the posterior density

$$u(\vartheta \mid X_1, \ldots, X_t)$$

is a member of the same family, with updated parameter values:

$$n_t = n_0 + t, \qquad x_t = x_0 + X_1 + \ldots + X_t .$$

Hence, if we have an expression for $E[\mu(\Theta)]$, it suffices to replace n_0 by n_t and x_0 by x_t to obtain $E[\mu(\Theta) \mid X_1, \ldots, X_t]$.

By definition,

$$E[\mu(\Theta)] = \int \mu(\vartheta) \cdot u(\vartheta)\, d\vartheta = \frac{1}{d(n_0, x_0)} \int c(\vartheta)^{-n_0 - 1} \cdot c'(\vartheta) \cdot b(\vartheta)^{x_0 + 1}\, d\vartheta$$

Now we perform a partial integration and assume that the function

$$c(\vartheta)^{-n_0} \cdot b(\vartheta)^{x_0 + 1}$$

vanishes at the integration limits.

Then we obtain

$$E[\mu(\Theta)] = \frac{x_0 + 1}{n_0 \cdot d(n_0, x_0)} \int c(\vartheta)^{-n_0} \cdot b(\vartheta)^{x_0} \cdot b'(\vartheta)\, d\vartheta = \frac{x_0 + 1}{n_0} .$$

Hence the premium for X_{t+1} is

$$\frac{x_t + 1}{n_t} = \frac{x_0 + t \cdot \bar{X} + 1}{n_0 + t} = Z \cdot \bar{X} + (1 - Z) \cdot E[\mu(\Theta)] \text{ with } Z = \frac{t}{n_0 + t} .$$

The classical examples can be retrieved directly as follows:

a. POISSON-GAMMA

$$A = \{0, 1, 2, ...\}, \; 0<\vartheta<\infty$$

$$a(x) = \frac{1}{x!}, \; b(\vartheta)=\vartheta, \; c(\vartheta)=e^{\vartheta}$$

$$u(\vartheta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \vartheta^{\alpha-1} e^{-\beta\vartheta} \text{ with } \alpha = x_0+1, \; \beta=n_0$$

b. GEOMETRIC-BETA

$$A = \{0, 1, 2, ...\}, \; 0<\vartheta<1$$

$$a(x)=1, \; b(\vartheta)=1-\vartheta, \; c(\vartheta) = \frac{1}{\vartheta}$$

$$u(\vartheta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \vartheta^{\alpha-1}(1-\vartheta)^{\beta-1} \text{ with } \alpha = n_0+1, \; \beta = x_0+1$$

c. EXPONENTIAL-GAMMA

$$A = (0, \infty), \; 0<\vartheta<\infty$$

$$a(x)=1, \; b(\vartheta)=e^{-\vartheta}, \; c(\vartheta) = \frac{1}{\vartheta}$$

$$u(\vartheta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \vartheta^{\alpha-1} e^{-\beta\vartheta} \text{ with } \alpha = n_0+1, \; \beta = x_0+1$$

d. NORMAL-NORMAL

$$A = (-\infty, \infty), \; -\infty<\vartheta<\infty$$

$$a(x) = \exp\left(-\frac{x^2}{2\sigma^2}\right), \; b(\vartheta) = \exp\left(\frac{\vartheta}{\sigma^2}\right), \; c(\vartheta) = \sqrt{2\pi}\,\sigma\exp\left(\frac{\vartheta^2}{2\sigma^2}\right)$$

$$u(\vartheta) = \frac{1}{\sqrt{2\pi}\,v} e^{-\left(\frac{(\vartheta-\mu)^2}{2v^2}\right)} \text{ with } \mu = \frac{x_0+1}{n_0}, \; v^2 = \frac{\sigma^2}{n_0}$$

e. BERNOULLI-BETA

$$A = \{0, 1\}, \ 0<\vartheta<1$$

$$a(x) = 1, \ b(\vartheta) = \frac{\vartheta}{1-\vartheta}, \ c(\vartheta) = \frac{1}{1-\vartheta}$$

$$u(\vartheta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \vartheta^{\alpha-1}(1-\vartheta)^{\beta-1} \text{ with } \alpha = x_0+1, \ \beta = n_0 - x_0 - 1$$

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REFERENCES

GOOVAERTS, M.J. and HOOGSTAD, W.J. (1987) *Credibility Theory*. Surveys of Actuarial Studies, No. 4, Nationale-Nederlanden.

JEWELL, W.S. (1974) Credible means are exact for exponential families. *ASTIN Bulletin*, **8**, 77-90.

JEWELL, W.S. (1975) Regularity conditions for exact credibility. *ASTIN Bulletin*, **8**, 336-341.