ASSESSING THE ECONOMIC VALUE OF LIFE INSURANCE CONTRACTS WITH STOCHASTIC DEFLATORS

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This appendix is attached to the article Armel and Planchet [2021]: "Assessing the economic value of life insurance contracts with stochastic deflators".

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1 Appendix 1: probability change - theoretical reminder

The objective of this section is to briefly recall the theoretical framework of probability change and deflator construction. We have relied on Duffie [2001] and El Karoui [2004] for this synthesis.

Let F be a sigma algebra and let P and Q be two probability measures F-measurable. We say that P and Q are equivalent if and only if for all $A \in F$ we have: $P(A) = 0 \Leftrightarrow Q(A) = 0$.

If P and Q are two equivalent probability measures on F then there is a unique random variable L^F on F, that is strictly positive and whose expectation under P is equal to 1, such that for any $A \in F : Q(A) = \int_A L^F dP$.

The random variable is then written: $L^F = dQ/dP$ on F and is called the Radon-Nikodym derivative of Q over P (also known as the density or likelihood of Q over P).

If $G \subseteq F$ then we have, for any random variable X F-measurable:

$$E^Q(X|G) = \frac{E^P\,(L^F.X|G)}{E^P(L^F|G)}$$

In particular: $E^Q(X) = E^P(L^F.X)$.

Furthermore, if $G \subseteq F$ is a sub sigma-algebra of F then Q and P are equivalent on G and the density of Q compared to P on G denoted L^G is written: $L^G = E^p(L^F|G) = E^P(dQ/dP|G)$.

Let $\{F_t\}_{0\le t\le T}$ be a filtration and let P and Q be two equivalent probability measures on F_T. We define the process $\{L(t)\}_{0\le t\le T}$ by:

$$L(t) = E^P(L(T)|F_t) = E^P\left(\frac{dQ}{dP}\middle|F_t\right)$$

The process of density (or likelihood) $\{L(t)\}_{0\le t\le T}$ is a martingale under P.

For all times $(t,s) \in [0,T[^2$ and $t \le s$ and for any random variable F_s-measurable X such as $E^Q(|X|) < +\infty$:

$$E^Q(X|F_t) = \frac{E^P(L(s)X|F_t)}{L(t)}$$

Thus, a process $\{X(t)\}_{0\le t\le T}$ is a $\{F_t\}_{0\le t\le T}$-martingale under Q if and only if the process $\{L(t)X(t)\}_{0\le t\le T}$ is a $\{F_t\}_{0\le t\le T}$-martingale under P.

A probability measure Q equivalent to P is an equivalent martingale measure for the pricing process X of N securities if X is a Q-martingale and if the Radon-Nikodym derivative dQ/dP has a finite variance[3]. An equivalent martingale measure is commonly referred to as a "risk neutral" measure.

Suppose there is an instantaneous short interest rate process denoted $\{r(t)\}_{0\le t\le T}$ and let $\{\delta(t)\}_{0\le t\le T}$ be a process defined by:

[3] The finite variance condition is a technical property that is not uniformly adopted in the literature.

$$\delta(t) = \exp\left(-\int_0^t r(s)ds\right)$$

Suppose after discounting by $\{\delta(t)\}_{0\le t\le T}$, that there is a martingale measure Q equivalent to P with the density process $\{L(t)\}_{0\le t\le T}$. Then a deflator $\{D(t)\}_{0\le t\le T}$ is defined by:

$$D(t) = L(t).\,\delta(t)$$

provided that $var\big(D(t)\big) < +\infty$ for all t.

Conversely, suppose that $\{D(t)\}_{0\le t\le T}$ is a deflator for the process $\{X(t)\}_{0\le t\le T}$ i.e., the process $\{D(t).\,X(t)\}_{0\le t\le T}$ is a P-martingale.

Let $\{L(t)\}_{0\le t\le T}$ be the process defined by:

$$L(t) = \exp\left(\int_0^t r(s)ds\right).\frac{D(t)}{D(0)}$$

So, provided that $var\big(L(T)\big) < +\infty$, $\{L(t)\}_{0\le t\le T}$ is the density process that defines a martingale measure equivalent to P.

Let $\{W^P(t)\}_{0\le t\le T}$ be a P-Wiener process and assume that the filtration $\{F_t\}_{0\le t\le T}$ is the completed natural filtration ($F_t = Vect(W(s), 0 \le s \le t)$).

So, for each process X (P, F_t)-martingale, there is a real number x and an adapted process h (respecting the condition of Novikov) such as:

$$X(t) = x + \int_0^t h_s dW^P(t)$$

and so:

$$dX(t) = h_t dW^P(t)$$

This theorem is known as the representation theorem. It guarantees the existence of a process h checking the above equation but does not tell us how to find or build this process h. This is what we propose in the following paragraph.

Suppose we want to change the measure from P to Q on F_T. To do this, we need a P-martingale L with $L_0 = 1$ to be used as a likelihood process. To guarantee the strict positivity of the process L and therefore the equivalence between P and Q we can choose a suitable process θ and assume that the process L is written:

$$dL(t) = L(t).\,\theta(t)dW^P(t)$$

The process L is a P-martingale and we have by Itô's lemma:

$$L(t) = \exp\left(\int_0^t \theta(t)dW^P(s) - \frac{1}{2}\int_0^t \theta(s)^2 ds\right) \quad (1)$$

Now that we have defined the likelihood process of Q, the question that naturally arises is what are the properties of $\{W^P(t)\}_{0\le t\le T}$ under the new measure Q? This problem is solved by Girsanov's theorem, which we will recall in the following.

Let W^P be a P-Wiener process and let T be a horizon of interest.

Let θ be a suitable process and define the process L by: $dL(t) = L(t)\theta(t)dW^P(t)$ and $L(0) = 1$.

Suppose that $E^P\big(L(T)\big) = 1$ and let Q be a probability measure verifying:

$$dQ = L(t)dP\ |F_t$$

Then Q is equivalent to P and the process W^Q defined by:

$$W^Q(t) = W^P(t) - \int_0^t \theta(s)ds$$

is Q-Wiener and we can write: $dW^Q(s) = dW^P(s) - \theta(s)ds$.

Finally, if the discounted cash flow process using the interest risk-free rate, $\exp\left(-\int_0^t r(s)ds\right)$, is a martingale under Q then the process of the deflated cash-flows by $\exp\left(-\int_0^t r(s)ds\right)L(t)$, is a martingale under P.

Under the assumptions of market completeness and the absence of arbitrage opportunities, the work of Harrisson and Kreps [1979] and Harrison and Pliska [1981] has shown that there is a unique probability measure equivalent to historical probability such that discounted prices by risk-free interest rates are, under this probability, martingales. This probability is known as the "risk-neutral probability measure".

2 Appendix 2: the price of a zero coupon bond under P

In this appendix, we present a demonstration of the closed formula of the price of a zero-coupon bond.

Let $r(t)$ be the instantaneous short interest rate at time t defined by $r(t) = x(t) + \varphi(t)$ where x is a one-factor CIR process whose stochastic differential equation under the historical probability P is written:

$$dx(t) = (k - \lambda)\left(\frac{k\theta}{k - \lambda} - x(t)\right)dt + \sigma_x\sqrt{x(t)}dW_{rate}^P(t)\ ;\ x(0) = x_0$$

And φ is a deterministic function allowing the model to reproduce the term structure of interest rates.

Suppose that the risk premium is written: $\lambda(t) = \lambda\sqrt{x(t)}/\sigma_x$.

Then the price of the zero coupon at time t and maturity T is written:

$$P(t,T) = \bar{A}'\ (t,T)e^{-B(t,T)x(t)} \qquad (2)$$

where

- $\bar{A}'(t,T) = \ (t,T) = \frac{P^M(0,T)A(0,t)exp\{-B(0,t)x_0\}}{P^M(0,t)A(0,T)exp\{-B(0,T)x_0\}}A(t,T)$;
- $P^M(0,T)$ is the market price of the risk-free zero-coupon bond observed at time 0 for maturity T ;

- $A(t,T)$ and $B(t,T)$ are defined below:

$$A\ (t,T) = \left[\frac{2h exp\left\{\frac{(k\ +\ h)(T\ -\ t)}{2}\right\}}{2h + (k+h)(exp\left\{(T-t)h\right\} - 1)}\right]^{\frac{2k\theta}{\sigma^2}}$$

$$B\ (\ t,T)\ = \frac{2(exp\left\{(T\ -\ t)h\right\} -\ 1)}{2h\ +\ (k\ +\ h)(exp\left\{(T\ -\ t)h\right\} -\ 1)}$$

$$h = \sqrt{k^2 + 2\sigma^2}$$

and

$$\exp\left(-\int_t^T \varphi(s)ds\right) = \frac{P^M(0,T)A(0,t)exp\{-B(0,t)x_0\}}{P^M(0,t)A(0,T)exp\{-B(0,T)x_0\}}$$

Demonstration

Under the risk neutral probability Q, the instantaneous short interest rate at t is defined by $r\ (t) = x(t) + \varphi(t)$ where x is a one-factor CIR process whose stochastic differential equation under the probability Q is written as:

$$dx(t) = k\big(\theta - x(t)\big)dt + \sigma_x\sqrt{x(t)}dW_{rate}^Q(t)\ ;\ x(0) = x_0$$

The price of a zero-coupon bond at t of maturity T is written under the probability Q (Brigo and Mercurio [2006]):

$$P(t,T) = \bar{A}'\ (t,T)e^{-B(t,T)x(t)}$$

Note that:

- $B(t,T)$ is the solution of the following differential equation:

$$1 - \frac{1}{2}B(t,T)^2\sigma_x^2 - k.B(t,T) + \frac{dB(t,T)}{dt} = 0$$

- $a(t,T) = \ln\big(A\ (t,T)\big)$ is the solution of the following differential equation:

$$\frac{da(t,T)}{dt} - B(t,T)k\theta = 0$$

Using Itô's lemma we can show that the differential equation of $P(t,T)$ under Q is written:

$$\frac{dP(t,T)}{P(t,T)} = \big(x(t) + \varphi(t)\big)dt\ -\ B(t,T)\sigma_x\sqrt{x(t)}dW_{rate}^Q(t)$$

We then have under the historical probability P:

$$\lambda(t) = \lambda\sqrt{x(t)}/\sigma_x$$

$$dW_{rate}^Q(t) = dW_{rate}^P(t) + \lambda(t)dt$$

and so:

$$\frac{dP(t,T)}{P(t,T)} = \varphi(t)dt + x(t)\big(1 - \lambda B(t,T)\big)dt\ -\ B(t,T)\sigma_x\sqrt{x(t)}dW_{rate}^P(t)$$

By applying Itô's lemma to $ln\big(P(t,T)\big)$ we get:

$$dln\big(P(t,T)\big) = 0.dt + \frac{dP(t,T)}{P(t,T)} - \frac{1}{2}.\frac{1}{P(t,T)^2}\left(B(t,T)P(t,T)\sigma_x\sqrt{x(t)}\right)^2 dt$$

So :

$$dln\big(P(t,T)\big) = \varphi(t)dt + x(t)\big(1-\lambda B(t,T)\big)dt - B(t,T)\sigma_x\sqrt{x(t)}dW_{rate}^P(t) - \frac{1}{2}B(t,T)^2\sigma_x^2\, x(t)dt$$

$$dln\big(P(t,T)\big) = \varphi(t)dt + x(t)\left(1-\lambda B(t,T) - \frac{1}{2}B(t,T)^2\sigma_x^2\right)dt - B(t,T)\sigma_x\sqrt{x(t)}dW_{rate}^P(t)$$

Knowing that

$$B(t,T)\sigma_x\sqrt{x(t)}dW_{rate}^P(t) = B(t,T)\left(dx(t) - (k-\lambda)\left(\frac{k\theta}{k-\lambda} - x(t)\right)dt\right)$$

Then

$$dln\big(P(t,T)\big) = \varphi(t)dt + x(t)\left(1-\lambda B(t,T) - \frac{1}{2}B(t,T)^2\sigma_x^2\right)dt - B(t,T)\left(dx(t) - (k-\lambda)\left(\frac{k\theta}{k-\lambda} - x(t)\right)dt\right)$$

Thus

$$dln\big(P(t,T)\big) = \varphi(t)dt + x(t)\left(1 - \frac{1}{2}B(t,T)^2\sigma_x^2 - k.B(t,T)\right)dt - B(t,T)dx(t) + B(t,T)k\theta dt$$

Using Itô's lemma we can write:

$$B(t,T)dx(t) = d(B(t,T)x(t)\,) - x(t).\frac{dB(t,T)}{dt}.dt$$

So

$$dln\big(P(t,T)\big) = \varphi(t)dt + x(t)\left(1 - \frac{1}{2}B(t,T)^2\sigma_x^2 - k.B(t,T)\right)dt - d\big(B(t,T)x(t)\big) + x(t).\frac{dB(t,T)}{dt}.dt + B(t,T)k\theta dt$$

And we have

$$1 - \frac{1}{2}B(t,T)^2\sigma_x^2 - k.B(t,T) + \frac{dB(t,T)}{dt} = 0$$

$$\frac{da(t,T)}{dt} - B(t,T)k\theta = 0$$
$$dln\big(P(t,T)\big) = \varphi(t)dt - \ d\big(B(t,T)x(t)\big) + B(t,T)k\theta dt$$

Knowing that $P(T,T) = 1$ and $\exp\left(-\int_t^T \varphi(s)ds\right) = \frac{P^M(0,T)A(0,t)exp\{-B(0,t)x_0\}}{P^M(0,t)A(0,T)exp\{-B(0,T)x_0\}}$ we can write $P(t,T)$ under P as:

$$P(t,T) = \overline{A}'\ (t,T)e^{-B(t,T)x(t)}$$

3 Appendix 3: the deflator form

The stochastic differential equation of the deflator under the historical probability measure is:

$$\frac{dD(t)}{D(t)} = -r(t)dt - \lambda(t)dW_{rate}^P(t)$$

The stochastic deflator is written using Itô's lemma:

$$D(T) = D(t)exp\left(-\int_t^T r(s)ds - \frac{1}{2}\int_t^T \lambda(s)^2 ds - \int_t^T \lambda(s)dW_{rate}^P(s)\right)$$

and we have:

- $\int_t^T r(s)ds = \int_t^T \varphi(s)ds + \int_t^T x(s)ds;$
- $\lambda(s) = \lambda\sqrt{x(t)}/\sigma_x$ so $\frac{1}{2}\int_t^T \lambda(s)^2 ds = \frac{\lambda^2}{2\sigma_x^2}\int_t^T x(t)ds;$
- Using the differential equation of interest rates:
$$\lambda(s)dW_{rate}^P(t) = \frac{\lambda}{\sigma_x}\sqrt{x(t)}dW_{rate}^P(t) = \frac{\lambda}{\sigma_x^2}\left(dx(t) - (k-\lambda)\left(\frac{k\theta}{k-\lambda} - x(t)\right)dt\right)$$
so:
$$\int_t^T \lambda(s)dW_{rate}^P(t) = \frac{\lambda}{\sigma_x^2}\big(x(T) - x(t)\big) - \frac{\lambda k\theta}{\sigma_x^2}(T-t) + \frac{\lambda(k-\lambda)}{\sigma_x^2}\int_t^T x(s)ds.$$

Thus:

$$D(T) = D(t)\exp\left(\frac{\lambda k\theta}{\sigma_x^2}(T-t)\right)\exp\left(-\int_t^T \varphi(s)ds\right)\exp\left(-\frac{\lambda}{\sigma_x^2}\big(x(T) - x(t)\big)\right)\exp\left(-\left(1 - \frac{\lambda^2}{2\sigma_x^2} + \frac{\lambda k}{\sigma_x^2}\right)\int_t^T x(s)ds\right) \quad (3)$$

with: $exp\left(-\int_t^T \varphi(s)ds\right) = \frac{P^M(0,T)A(0,t)exp\{-B(0,t)x_0\}}{P^M(0,t)A(0,T)exp\{-B(0,T)x_0\}}$

4 Appendix 4: the price of the risky asset under P

The risky asset process is written under the historical probability measure P:

$$\frac{dS_t}{S_t} = (r(t) + \lambda(t)^2)dt + \lambda(t)dW_{rate}^P(t)$$

The price of the risky asset is written:

$$S(T) = S(t)\exp\left(-\frac{\lambda}{\sigma_x^2}k\theta(T-t)\right).\exp\left(\int_t^T \varphi(s)ds\right).\exp\left(\frac{\lambda}{\sigma_x^2}\big(x(T) - x(t)\big)\right).\exp\left(\left(1-\frac{\lambda^2}{2\sigma_x^2}+\frac{\lambda k}{\sigma_x^2}\right)\int_t^T x(s)ds\right) \quad (4)$$

with: $exp\left(-\int_t^T \varphi(s)ds\right) = \frac{P^M(0,T)A(0,t)exp\{-B(0,t)x_0\}}{P^M(0,t)A(0,T)exp\{-B(0,T)x_0\}}$

Demonstration

The risky asset process is written under the historical probability measure P:

$$\frac{dS_t}{S_t} = (r(t) + \lambda(t)^2)dt + \lambda(t)dW_{rate}^P(t)$$

So

$$\frac{dS_t}{S_t} = \left(\varphi(t) + x(t)\left(1+\frac{\lambda^2}{\sigma_x^2}\right)\right)dt + \lambda\sqrt{x(t)}/\sigma_x dW_{rate}^P(t)$$

Then

$$\sqrt{x(t)}dW_{rate}^P(t) = \frac{1}{\sigma_x}\left(dx(t) - (k-\lambda)\left(\frac{k\theta}{k-\lambda} - x(t)\right)dt\right)$$

And so we have

$$\frac{dS_t}{S_t} = \left(\varphi(t) + x(t)\left(1+\frac{\lambda^2}{\sigma_x^2}\right)\right)dt + \frac{\lambda}{\sigma_x^2}\left(dx(t) - (k-\lambda)\left(\frac{k\theta}{k-\lambda} - x(t)\right)dt\right)$$

$$\frac{dS_t}{S_t} = \left(\varphi(t) - \frac{\lambda}{\sigma_x^2}k\theta\right)dt + x(t)\left(1+\frac{\lambda^2}{\sigma_x^2}+\frac{\lambda}{\sigma_x^2}(k-\lambda)\right)dt + \frac{\lambda}{\sigma_x^2}dx(t)$$

By applying Itô's Lemme to $ln(S(t))$ we have

$$dln\big(S(t)\big) = 0.dt + \frac{dS(t)}{S(t)} - \frac{1}{2}.\frac{1}{S(t)^2}\lambda(t)^2 S(t)^2 dt$$

Thus

$$dln\big(S(t)\big) = 0.dt + \frac{dS(t)}{S(t)} - \frac{1}{2}.\frac{\lambda^2}{\sigma_x^2}x(t)dt$$

So

$$dln\big(S(t)\big) = \left(\varphi(t) - \frac{\lambda}{\sigma_x^2}k\theta\right)dt + x(t)\left(1 + \frac{\lambda^2}{\sigma_x^2} + \frac{\lambda}{\sigma_x^2}(k-\lambda)\right)dt + \frac{\lambda}{\sigma_x^2}dx(t) - \frac{1}{2}.\frac{\lambda^2}{\sigma_x^2}x(t)dt$$

Thus

$$dln\big(S(t)\big) = \left(\varphi(t) - \frac{\lambda}{\sigma_x^2}k\theta\right)dt + x(t)\left(1 + \frac{\lambda^2}{2\sigma_x^2} + \frac{\lambda}{\sigma_x^2}(k-\lambda)\right)dt + \frac{\lambda}{\sigma_x^2}dx(t)$$

Finally

$$S(T) = S(t)\exp\left(-\frac{\lambda}{\sigma_x^2}k\theta(T-t)\right).\exp\left(\int_t^T \varphi(s)ds\right).\exp\left(\frac{\lambda}{\sigma_x^2}\big(x(T)-x(t)\big)\right).\exp\left(\left(1 - \frac{\lambda^2}{2\sigma_x^2} + \frac{\lambda k}{\sigma_x^2}\right)\int_t^T x(s)ds\right)$$

with: $exp\left(-\int_t^T \varphi(s)ds\right) = \frac{P^M(0,T)A(0,t)exp\{-B(0,t)x_0\}}{P^M(0,t)A(0,T)exp\{-B(0,T)x_0\}}$

5 Appendix 5: Expected returns on risky assets

Note s_t the logarithmic return of the risky asset at t on a one-year horizon. By definition:

$$s_{t+1} = ln\left(\frac{S(t+1)}{S(t)}\right)$$

The mathematical expectation of the random variable s_{t+1} under the historical probability P is written:

$$E^p(s_{t+1}) = R^M(t,t+1) - \left(\ln\left(\frac{A(0,t)}{A(0,t+1)}\right) + x_0\big(B(0,t+1) - B(0,t)\big)\right) + \left(1 + \frac{\lambda^2}{2\sigma_x^2}\right)\left(e^{-(k-\lambda)t}\frac{\big(k\theta - x_0(k-\lambda)\big)}{(k-\lambda)^2}\big(e^{-(k-\lambda)} - 1\big) + \frac{k\theta}{k-\lambda}\right) \tag{5}$$

With:

- $R^M(t,t+1)$ the observed market risk-free interest rate between t and $t+1$;
- $A(t,T)$ and $B(t,T)$ are deterministic functions defined in section 2;
- x_0 is the initial value of the process x_t.

The expectation of excess return, denoted e_t, over market risk-free interest rate is therefore written:

$$E^p(e_{t+1}) = -\left(\ln\left(\frac{A(0,t)}{A(0,t+1)}\right) + x_0\big(B(0,t+1) - B(0,t)\big)\right) + \left(1 + \frac{\lambda^2}{2\sigma_x^2}\right)\left(e^{-(k-\lambda)t}\frac{\big(k\theta - x_0(k-\lambda)\big)}{(k-\lambda)^2}\big(e^{-(k-\lambda)} - 1\big) + \frac{k\theta}{k-\lambda}\right) \quad (6)$$

Demonstration

We have under P:

$$S(T) = S(t)\exp\left(-\frac{\lambda}{\sigma_x^2}k\theta(T-t)\right).\exp\left(\int_t^T \varphi(s)ds\right).\exp\left(\frac{\lambda}{\sigma_x^2}\big(x(T) - x(t)\big)\right).\exp\left(\left(1 - \frac{\lambda^2}{2\sigma_x^2} + \frac{\lambda k}{\sigma_x^2}\right)\int_t^T x(s)ds\right)$$

And

$$dx(t) = k_2\big(\theta_2 - x(t)\big)dt + \sigma_x\sqrt{x(t)}dW_{rate}^P(t);\ x(0) = x_0$$

With:

- $k_2 = k - \lambda$;
- $\theta_2 = \frac{k\theta}{k-\lambda}$.

We can write

$$s_{t+1} = -\frac{\lambda}{\sigma_x^2}k\theta + \int_t^{t+1}\varphi(s)ds + \left(\frac{\lambda}{\sigma_x^2}\big(x(t+1) - x(t)\big)\right) + \left(1 - \frac{\lambda^2}{2\sigma_x^2} + \frac{\lambda k}{\sigma_x^2}\right)\int_t^{t+1} x(s)ds$$

And therefore

$$E^p(s_t) = \Lambda_0 + \Lambda_1 + \Lambda_2 + \Lambda_3$$

With

- $\Lambda_0 = -\frac{\lambda}{\sigma_x^2}k\theta$
- $\Lambda_1 = E^P\left(\int_t^{t+1}\varphi(s)ds\right)$
- $\Lambda_2 = E^p\left(\frac{\lambda}{\sigma_x^2}\big(x(t+1) - x(t)\big)\right)$
- $\Lambda_3 = E^p\left(\left(1 - \frac{\lambda^2}{2\sigma_x^2} + \frac{\lambda k}{\sigma_x^2}\right)\int_t^{t+1}x(s)ds\right)$

We have then:

$$exp\left(-\int_t^{t+1}\varphi(s)ds\right) = \frac{\mathrm{P^M(0,t+1)}}{\mathrm{P^x(0,t+1)}}.\frac{\mathrm{P^x(0,t)}}{\mathrm{P^M(0,t)}} = \frac{\exp\big(-R^M(t,t+1)\big)}{\exp\big(-U^x(t,t+1)\big)}$$

Where:

$$U^x(t,t+1) = \ln\big(P^x(0,t)\big) - \ln\big(P^x(0,t+1)\big)$$
$$= \ln\left(\frac{A(0,t)}{A(0,t+1)}\right) + x_0\big(B(0,t+1) - B(0,t)\big)$$

And so:

$$\Lambda_1 = E^P\left(\int_t^{t+1} \varphi(s)ds\right) = R^M(t,t+1) - U^x(t,t+1)$$

Also, the expectation of x is written under P:

$$E^p\{x(t)|F_s\} = x(s)e^{-k_2(t-s)} + \theta_2\big(1 - e^{-k_2(t-s)}\big)$$

So

$$E^p\big(x(t)\big) = x(0)e^{-k_2 t} + \theta_2(1 - e^{-k_2 t})$$

and

$$E^p\big(x(t)\big) = x_0 e^{-k_2 t} + \theta_2(1 - e^{-k_2 t})$$

And so

$$\Lambda_2 = E^p\left(\frac{\lambda}{\sigma_x^2}\big(x(t+1) - x(t)\big)\right) = \frac{\lambda}{\sigma_x^2} e^{-k_2 t}(\theta_2 - x_0)(1 - e^{-k_2})$$

In addition

$$E^p\left(\int_t^{t+1} x(s)ds\right) = \int_t^{t+1} E^p\big(x(s)\big)ds = \int_t^{t+1}\big(x(0)e^{-k_2 s} + \theta_2(1 - e^{-k_2 s})\big)ds$$

Thus

$$E^p\left(\int_t^{t+1} x(s)ds\right) = e^{-k_2 t}\frac{(\theta_2 - x_0)}{k_2}(e^{-k_2} - 1) + \theta_2$$

So :

$$\Lambda_3 = E^p\left(\left(1 - \frac{\lambda^2}{2\sigma_x^2} + \frac{\lambda k}{\sigma_x^2}\right)\int_t^{t+1} x(s)ds\right)$$

$$\Lambda_3 = \left(1 - \frac{\lambda^2}{2\sigma_x^2} + \frac{\lambda k}{\sigma_x^2}\right)\left(e^{-k_2 t}\frac{(\theta_2 - x_0)}{k_2}(e^{-k_2} - 1) + \theta_2\right)$$

$$\Lambda_3 = \left(1 + \frac{\lambda^2}{2\sigma_x^2} + \frac{\lambda k_2}{\sigma_x^2}\right)\left(e^{-k_2 t}\frac{(\theta_2 - x_0)}{k_2}(e^{-k_2} - 1) + \theta_2\right)$$

$$\Lambda_3 = \left(1+\frac{\lambda^2}{2\sigma_x^2}\right)\left(e^{-k_2 t}\frac{(\theta_2 - x_0)}{k_2}(e^{-k_2}-1)+\theta_2\right) + \frac{\lambda k_2}{\sigma_x^2}\left(e^{-k_2 t}\frac{(\theta_2 - x_0)}{k_2}(e^{-k_2}-1)+\theta_2\right)$$

$$\Lambda_3 = \left(1+\frac{\lambda^2}{2\sigma_x^2}\right)\left(e^{-k_2 t}\frac{(\theta_2 - x_0)}{k_2}(e^{-k_2}-1)+\theta_2\right) + \frac{\lambda k_2}{\sigma_x^2}\left(e^{-k_2 t}\frac{(\theta_2 - x_0)}{k_2}(e^{-k_2}-1)\right) + \frac{\lambda k_2 \theta_2}{\sigma_x^2}$$

$$\Lambda_3 = \left(1+\frac{\lambda^2}{2\sigma_x^2}\right)\left(e^{-k_2 t}\frac{(\theta_2 - x_0)}{k_2}(e^{-k_2}-1)+\theta_2\right) + e^{-k_2 t}\frac{\lambda(\theta_2 - x_0)}{\sigma_x^2}(e^{-k_2}-1) + \frac{\lambda k\theta}{\sigma_x^2}$$

$$\Lambda_3 = \left(1+\frac{\lambda^2}{2\sigma_x^2}\right)\left(e^{-k_2 t}\frac{(\theta_2 - x_0)}{k_2}(e^{-k_2}-1)+\theta_2\right) - \Lambda_2 - \Lambda_0$$

Finally:

$$\left(1+\frac{\lambda^2}{2\sigma_x^2}\right)\left(e^{-k_2 t}\frac{(\theta_2 - x_0)}{k_2}(e^{-k_2}-1)+\theta_2\right) = \left(1+\frac{\lambda^2}{2\sigma_x^2}\right)\left(e^{-(k-\lambda)t}\frac{\left(\frac{k\theta}{k-\lambda}-x_0\right)}{k-\lambda}\left(e^{-(k-\lambda)}-1\right)+\frac{k\theta}{k-\lambda}\right)$$

$$= \left(1+\frac{\lambda^2}{2\sigma_x^2}\right)\left(e^{-(k-\lambda)t}\frac{\left(k\theta - x_0(k-\lambda)\right)}{(k-\lambda)^2}\left(e^{-(k-\lambda)}-1\right)+\frac{k\theta}{k-\lambda}\right)$$

In conclusion:

$$E^p(s_{t+1}) = \Lambda_0 + \Lambda_1 + \Lambda_2 + \Lambda_3$$

so:

$$E^p(s_{t+1}) = R^M(t,t+1) - \left(\ln\left(\frac{A(0,t)}{A(0,t+1)}\right) + x_0\big(B(0,t+1) - B(0,t)\big)\right) + \left(1+\frac{\lambda^2}{2\sigma_x^2}\right)\left(e^{-(k-\lambda)t}\frac{\left(k\theta - x_0(k-\lambda)\right)}{(k-\lambda)^2}\left(e^{-(k-\lambda)}-1\right)+\frac{k\theta}{k-\lambda}\right)$$

Therefore:

$$E^p(e_{t+1}) = -\left(\ln\left(\frac{A(0,t)}{A(0,t+1)}\right) + x_0\big(B(0,t+1) - B(0,t)\big)\right) + \left(1+\frac{\lambda^2}{2\sigma_x^2}\right)\left(e^{-(k-\lambda)t}\frac{\left(k\theta - x_0(k-\lambda)\right)}{(k-\lambda)^2}\left(e^{-(k-\lambda)}-1\right)+\frac{k\theta}{k-\lambda}\right)$$

6 Appendix 6: excess return on risky assets

In the long term, in a steady state ($t >> 0$), the excess return depends only on the risk factor λ and the parameters of the CIR model (k, θ and σ_x) and is written:

$$E^p(e_\infty) = \frac{k\theta}{\sigma_x^2}(k - h) + \frac{k\theta}{k-\lambda}\left(1 + \frac{\lambda^2}{2\sigma_x^2}\right) \quad (7)$$

Demonstration

The expectation of excess return is written:

$$E^p(e_{t+1}) = -\left(\ln\left(\frac{A(0,t)}{A(0,t+1)}\right) + x_0\big(B(0,t+1) - B(0,t)\big)\right) + \left(1 + \frac{\lambda^2}{2\sigma_x^2}\right)\left(e^{-(k-\lambda)t}\frac{\big(k\theta - x_0(k-\lambda)\big)}{(k-\lambda)^2}\big(e^{-(k-\lambda)} - 1\big) + \frac{k\theta}{k-\lambda}\right)$$

$$E^p(e_{t+1}) = F_1(t+1) + F_2(t+1) + F_3(t+1)$$

where:

- $F_1(t+1) = -\ln\left(\frac{A(0,t)}{A(0,t+1)}\right)$ and so :

$$F_1(t+1) = -\frac{2k\theta}{\sigma_x^2} ln\left[\frac{exp\left\{\frac{(k+h)t}{2}\right\}}{exp\left\{\frac{(k+h)(t+1)}{2}\right\}} \times \frac{2h + (k+h)(exp\{(t+1)h\} - 1)}{2h + (k+h)(exp\{th\} - 1)}\right]$$

$$F_1(t+1) = -\frac{2k\theta}{\sigma_x^2} ln\left[exp\left\{\frac{-(k+h)}{2}\right\} \times \frac{2h + (k+h)(exp\{(t+1)h\} - 1)}{2h + (k+h)(exp\{th\} - 1)}\right]$$

- $F_2(t+1) = -x_0\big(B(0,t+1) - B(0,t)\big)$ and so :

$$F_2(t+1) = -x_0\left(\frac{2(exp\{(t+1)h\} - 1)}{2h + (k+h)(exp\{(t+1)h\} - 1)} - \frac{2(exp\{th\} - 1)}{2h + (k+h)(exp\{th\} - 1)}\right)$$

- $F_3(t+1) = \left(1 + \frac{\lambda^2}{2\sigma_x^2}\right)\left(e^{-(k-\lambda)t}\frac{(k\theta - x_0(k-\lambda))}{(k-\lambda)^2}\big(e^{-(k-\lambda)} - 1\big) + \frac{k\theta}{k-\lambda}\right)$

In a steady state, $t >> 0$ we can observe that:

$$\lim_{t\to+\infty} F_1(t+1) = -\frac{2k\theta}{\sigma_x^2} ln\left[exp\left\{\frac{-(k+h)}{2}\right\} \times exp(h)\right]$$

$$\lim_{t\to+\infty} F_1(t+1) = \frac{k\theta}{\sigma_x^2}(k - h)$$

$$\lim_{t\to+\infty} F_2(t+1) = 0$$

$$\lim_{t\to+\infty} F_3(t+1) = \frac{k\theta}{k-\lambda}\left(1 + \frac{\lambda^2}{2\sigma_x^2}\right)$$

So, in the long term, in a steady state, excess return is written:

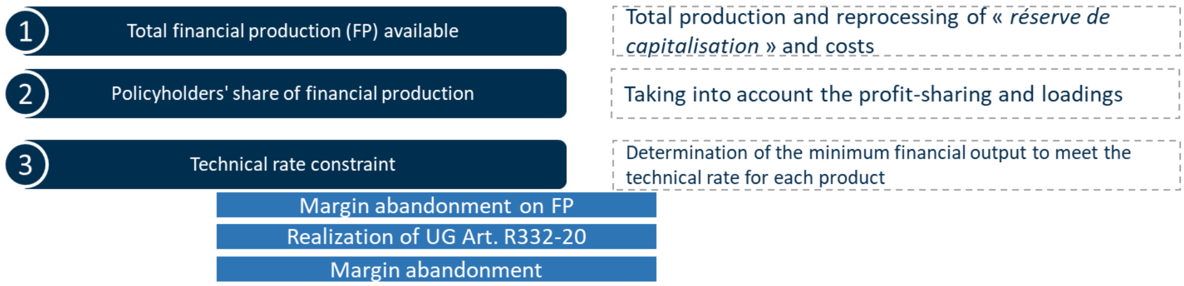
$$E^{p}(e_{\infty}) = \frac{k\theta}{\sigma_x^2}(k - h) + \frac{k\theta}{k-\lambda}\left(1 + \frac{\lambda^2}{2\sigma_x^2}\right)$$

7 Appendix 7: revaluation algorithm, a review of market practices

An examination of the revaluation algorithms of certain major players in the French euro savings market has enabled to draw up a standard diagram of the revaluation process. This diagramm is presented below[4]. It presents the steps for optimizing profitability (referred to as "margin" in the following sections) under the constraints implemented in the models and reflects the contract revaluation processes implemented in practice by insurers.

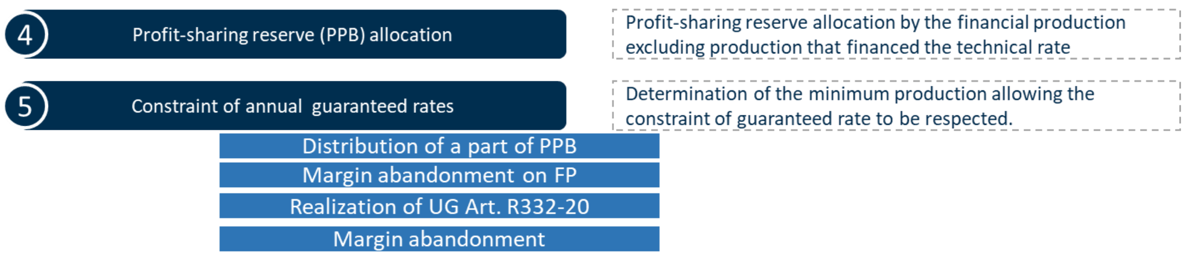
The first three steps aim to reassert the value of the contracts at the technical rate as shown in the next figure. It should be pointed out that the sub-step of margin abandonment on FP (financial products in the accounting sense) can occur for some companies after the UGL (Unrealized Gains or Losses) realization.

Figure 1: Technical Rate Service



If the financial production is enough to serve the technical rates, the profit sharing reserve (designated in the following by PPB for "*Provision pour Participation aux Bénéfices*") is provided with the balance. The increased PPB is then used to serve the minimum guaranteed rates (some companies may give up their margin on FP after the UGL realization step).

Figure 2: Guaranteed Minimum Rate Service



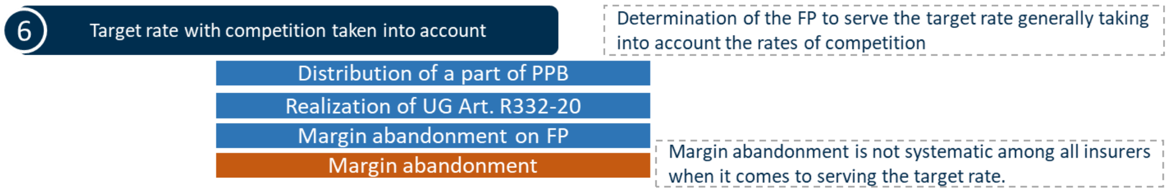
If the PPB is enough to pay the minimum guaranteed rates, we can then look at what is called the "target revaluation rate"[5]. When the wealth (financial production and PPB) is

[4] Simbel uses the same algorithm to assess the revaluation rate.

[5] See below for a definition of the target revaluation rate and a presentation of market practices.

significant, some insurers realize unrealized losses to adjust the distributed wealth downwards and remove depreciated assets from the portfolio. If the wealth is significantly lower, we can observe a loss of margin on financial production or a realization of unrealised gains before considering a loss of margin on the result.

Figure 3: Service of the target rate



The last step is to verify distribution constraints of mandatory minimum profits (including PPB that has been provisioned for more than 8 years).

Figure 4: Target rate correction to satisfy the minimum profit-sharing constraint

In practice, there is little room for life insurance companies to revalue contracts using technical rates or minimum guaranteed rates. We observe little differences in financial products' generating models:

- On step 1 - financial production: systematic realisation of X% of UGL (systematic turnover on equities and real estate), reallocation of assets, etc.
- On steps 3 and 5: some insurers realise UGL before any margin abandonment on financial products (FP).

For step 6, heterogeneous approaches are observed on the market for the definition of the target revaluation rate. We usually distinguish between logics involving "a rate expected by the policyholder" and one or more references in the rate construction:

- Interest rates possibly restated from the loadings rate (e.g. *TME*, 10-year *swap* or zero-coupon rate, weighted average of 1-year and 10-year *swap* rates, *Livret* A, 10-year *swap* rate plus volatility adjustment, etc.);
- Financial performance of an index (e.g., adjusting the CAC40 performance over 3 years);
- Internal benchmark (e.g., revaluation rate served to policyholders in year N or N-1);
- Competitive rate such as the rate published by the ACPR (ACPR [2018]) or the market average revaluation rate.

Further examples of references are provided by the French Institute of Actuaries ([2016], p. 42).

The majority of the approaches used in practice use one or two indicators, including, very often, an interest rate indicator. This logic is justified in particular by the close relationship between OAT rate and revaluations observed in the past (see Borel-Mathurin and al. [2018]).

For instance, some insurers assume that the rate expected by policyholders is a weighted average of a "memory effect" and a rate served by the supposed competition equal to the 10-year OAT rate:

$$Tx\_expected(t) = max\big(TMG, a \times tx\_Servi(t-1) + (1-a) \times OAT(t, 10ans)\big)$$

The final target rate corresponds to the expected rate minus a subjective $Spread$ that materializes product characteristics representing a brake on lapses, such as a rate guarantee or a particularly advantageous taxation.

$$Tx\_target(t) = max\big(TMG, Tx\_expected(t) - Spread(t)\big)$$

The final revaluation rate may be different (upward or downward) from the target rate defined in Step 6.

The difference between the revaluation rate and the rate expected by policyholders is used by practitioners as a dynamic lapse determinant variable. The following section summarizes market practices.

8 Appendix 8: policyholders' dynamic behaviour

National guidelines (ACPR [2013]) specify that "In addition to the *structural surrenders that the insurer may observe in a "normal" economic context on euro savings life insurance contracts, the insurer must take into account cyclical surrenders. These occur in particular in a highly competitive context when the policyholder arbitrates their insurance contract in favour of other financial supports (insurance, banking or real estate products)*". ACPR recommends using experience or market tables to model structural surrenders.

Policyholders' dynamic behaviour is modelled by dynamic lapse. It is therefore assumed in market models that policyholders modulate their lapses upwards or downwards according to the financial arbitrage opportunities that occur.

According to ACPR [2013], dynamic lapses are commonly modelled by a function exclusively depending on the difference between the paid revaluation rate and a rate dependent on the economic environment, often referred to as the policyholder's expected revaluation rate (see section 7). The dynamic lapse rate should be added to the structural surrender rate.

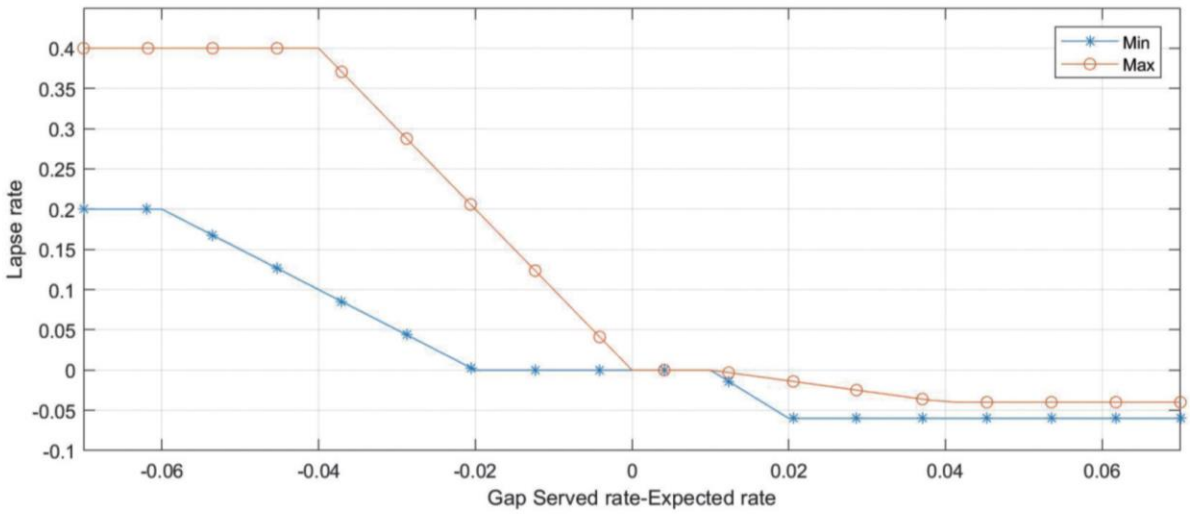
If the served rate (TS) is lower than the expected rate (TA) by the policyholder, the latter will tend to withdraw more than indicated by the structural lapse curve.

Conversely, if policyholders are offered a higher rate than they expected, they will withdraw less in the following year than in the past.

The ACPR (ACPR [2013]) proposes to maintain in the models the dynamic lapses as a function of the gap ($TS - TA$) inside a tunnel presented in *Figure 5*.

The majority of organizations use the proposed legislation of ACPR (ACPR [2013]) to model dynamic lapse. This kind of model consists in assuming that the dynamic lapse is piecewise affine function of value (TS-TA).

Figure 5: Min-max tunnel proposed by ACPR for dynamic lapse modelling



Therefore the dynamic lapse model implemented by the market explicitly assumes that the lapse decision results from a reasoning based on historical data (the served revaluation rates and the rates of the competition to date) and not on the policyholder's rational expectations.

The modeling discrepancies that can be seen between insurers concern the setting of the piecewise affine reaction function (expected rate, thresholds, etc.), but not the basic framework. On the academic level, the few existing references on the subject concern the rationalization of the parameters of the piecewise affine function or the study of explanatory variables for lapses (e.g. Suru [2011] and Rakah [2015]). There are also some works proposing modeling using logistic regressions (see. Sakho [2018]).

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