Heterogeneity: measure integrating risk estimation in the case of a modeling of the observable factors

Version 1.2 (English version, 04/21/2011)

Aymric Kamega [β] **Frédéric Planchet** [*]

Université de Lyon - Université Claude Bernard Lyon 1
ISFA [γ]
WINTER & Associés [λ]

Note: This paper is the English version of a paper published in French in the Bulletin Français d'Actuariat *(Vol. 11, No. 21).*

Abstract

In the presence of a heterogeneous population, it appears that the modeling of the risk of death provides different results at the aggregated level (i.e. by considering the population in total) and the disaggregated (i.e. by segmenting the population into subpopulations), which expresses a heterogeneity bias (*cf*. for example Droesbeke and al. [1989]). This result is often explained by the "mobile-stable" phenomenon, according to which the individuals of the segment with a high mortality hazard rate leave first and thus increase the proportion of individuals of the segment with a low mortality hazard rate as time passes.

A reflection then is essential on the approach to retain to model the time to death for a heterogeneous population, taking into account in particular the problems of choice of optimal segmentation (*cf*. Planchet and Leroy [2009]) and of risk estimation (*cf*. Kamega and Planchet [2010]). Three possible approaches are quoted here: the first approach consists in modeling the behavior of each subpopulation in an independent way, the second approach consists in turning to models of survival data integrating of the observable factors of heterogeneity starting from explanatory variables, and the third approach consists in turning to models integrating of the unobservable factors of heterogeneity (frailty models). In this study one is interested in the models of the second category, in particular with the semi-parametric models of Cox [1972] and Lin and Ying [1994]. For these models, the model of Brass is used to adjust the population of reference with a table of external reference.

[β] Aymric Kamega is a PhD student in the laboratory SAF at ISFA (EA n°2429), an actuary at WINTER & Associés and member of the College of Direction at EURIA (EURo-Institut d'Actuariat, UBO). Lead author (not the contact author). Contact : akamega@winter-associes.fr

[*] Frédéric Planchet has a doctorate in management science, is research professor of laboratory SAF at ISFA (EA n°2429) and an partner actuary at WINTER & Associés. Corresponding author (not the lead author). Contact : frederic.planchet@univ-lyon1.fr

[γ] Institut de Science Financière et d'Assurances (ISFA) - 50 avenue Tony Garnier - 69366 Lyon Cedex 07 – France.

[λ] WINTER & Associés – 55 avenue René Cassin - 69009 Lyon – France.

Thanks to simulations, this study presents a measurement of the risk estimation for these two examples of models integrating heterogeneity starting from observable explanatory variables. In particular, the study makes it possible to show that the choice of such models for the measurement of heterogeneity, at the expense of the approach consisting in modeling the behavior of each subpopulation in an independent way, makes it possible to limit the level of the risk estimation.

KEYWORDS: heterogeneity, adjustment, model with external reference (Brass), multiplicative hazard model of Cox, additive hazard model of Lin and Ying, risk estimation, risk of model.

Table of Contents

1. Introduction

As recalled by Vaupel [2002], all the populations are heterogeneous: two individuals of the same age and of the same sex in a population can present two very different risks of deaths.

In practice, it appears that the modeling of the risk of death provides different results at the aggregated level (i.e. by considering the totality of population) and the disaggregated (i.e. by segmenting the population in subpopulations), reflecting a heterogeneity bias (*cf*. for example Droesbeke and al. [1989]). This result is often explained by the "mobile-stable" phenomenon, according to which the individuals of the segment with a high mortality hazard rate leave first and thus increase the proportion of individuals of the segment with a low mortality hazard rate as time passes. Thus, it is understood that when the pattern of settlement by segment remains stable over time, one can avoid modeling heterogeneity. On the other hand, as soon as the pattern of settlement evolves, like often during the evaluation of a technical provision, a taking into account of heterogeneity is essential in order to have a robust life table.

A reflection then is essential for the approach to retain the modeling of the hazard function of a heterogeneous population, taking into account in particular the problems of choice of optimal segmentation (*cf*. Planchet and Leroy [2009]) and of risk estimation (*cf*. Kamega and Planchet [2010]). Three possible approaches are quoted here.

The first approach consists in modeling the behavior of each subpopulation in an independent way. The models associated with this approach can however quickly encounter problems of insufficient data, which accentuates the problems of choice of optimal segmentation and raise the risk estimation (*cf*. Kamega and Planchet [2010]).

The second approach consists in turning to models integrating the observable factors of heterogeneity starting from explanatory variables. Here, the population is considered as a whole and we endeavor to measure the effect of the explanatory variables (which define the segments) on the observed phenomenon (which is the hazard function).

The third approach consists in turning to models integrating the unobservable factors of heterogeneity (or residual, *cf*. for example Delwarde and Denuit [2006]). For this purpose, we can be based on frailty models, which make it possible to account for heterogeneity in the risks of individual deaths. In practice, we distinguish the traditional frailty model of Vaupel, which considers the differences in level of mortality per individual and is based on the assumption of proportionality, and the combined frailty model of Barbi, which considers the differences in level per individual and the differences in slope by group of individuals and thus, it is not based on the assumption of proportionality (these models are presented in Vaupel and al. [1979] and in Barbi and al. [2003]).

In this study, we are interested in the models of the second category which integrate observable factors of heterogeneity starting from explanatory variables. In particular, this study makes it possible to appreciate the evolution of risk estimation over the passage of time from the first to the second approach.

Plan and data

The study then seeks to justify the choice of the heterogeneity model retained (section 2) and to measure the associated risk estimation (sections 3 to 6).

The presented numerical illustrations are based on the data from insurers used for the construction of the regulatory mortality tables in zone CIMA/FANAF [1] for the contracts of insurance in the event of life and death. These data more precisely cover the population of the countries of under-area UEMOA, represented here by the Côte d'Ivoire, Mali and Togo.

We consider a potential heterogeneity by country, beyond the differences related on sex and age. Also, the following tables present statistics (exposure, average age and average rate of annual mortality) of the data on the population of insured persons by country for men and women of under-area UEMOA and aged from 30 to 55 years.

Table 1 - Statistics broken down by country (UEMOA - Woman)

Man (insured population)	Population at risk	Average age	Average death rate	Average death rate (lower limit at 95%)	Average death rate (upper limit at 95%)
CI	549 656	43,9 years old	0,40%	0,38%	0,41%
ML	12 114	42,5 years old	0,22%	0,14%	0,31%
TG	133 779	43,2 years old	0,42%	0,39%	0,46%
UEMOA (CI-ML-TG)	695 549	43,8 years old	0,40%	0,38%	0,41%

Table 2 - Statistics broken down by country (UEMOA - Man)

Woman (insured population)	Population at risk	Average age	Average death rate	Average death rate (lower limit at 95%)	Average death rate (upper limit at 95%)
CI	117 199	43,2 years old	0,19%	0,17%	0,22%
ML	3 499	41,7 years old	0,11%	0,00%	0,23%
TG	22 882	42,2 years old	0,07%	0,04%	0,11%
UEMOA (CI-ML-TG)	143 580	43,0 years old	0,17%	0,15%	0,19%

These data were collected in 2009 and are presented in detail in Planchet and al. [2010]. It is retained here that they are observed over the years 2003 to 2006 and count truncations on the left (relating to the entries after 01/01/2003) and censures on the right (relating to data before 12/31/2006 for any reason other than death).

2. Choice of model

In this section, we present the steps of the process of choosing the retained model: presentation of the problems of dimension associated with the models with explanatory variables with the choice of the models of Cox and Lin and Ying.

[1] Area covering the countries of CIMA members and represented by insurance companies or reinsurance FANAF, namely Benin, Burkina Faso, the Côte d'Ivoire, Mali, Niger, Senegal and Togo (that is, the UEMOA countries, except Guinea Bissau) and Cameroon, Central African Republic, Congo Brazzaville, Gabon, Chad (that is, the CEMAC countries, excluding Equatorial Guinea).

2.1. Models with explanatory variables and problems of dimension

In statistics, when a phenomenon can be explained by several explanatory variables, one can turn to purely parametric regressions such as linear regressions. The advantage is that, in this case, one can easily find consistent estimators. The disadvantage of these models is that they are based on many assumptions on the behavior of the phenomenon observed and thus present a significant risk of not being faithful to the experience.

An alternative then consists in turning to nonparametric regressions, which are based on a limited number of assumptions for the behavior of the observed phenomenon and, thus, are less constraining. However, these models present a well-known disadvantage under the term of "curse of dimensionality" (by mathematician Richard Bellman), relating to the problem caused by an exponential increase in volume associated with adding extra dimensions to a (mathematical) space. According to this curse, the nonparametric estimators of a function of regression behave badly when the number of variables is important (for this, we can refer to Viallon [2006]).

The additive Aalen model is an example of a nonparametric model. This model presumes that the intensity of a process of Poisson $N(t)$ ($t \in [0,\tau], \tau < \infty$) of dimension n (n representing the number of individuals under risk) takes the following form (a complete description of this model is available in Martinussen and Scheike [2006] and Klein and Moeschberger [2005]):

$$\lambda(t) = Y(t) X^T(t) \beta(t),$$

where $Y(t)$ is an indicator of risk (for all $i \in [1,n]$, $Y_i(t)$ is equal to 1 if individual i is under risk at the date t, and otherwise, is equal to 0), $\beta(t)$ represents the vector of the basic coefficient and the coefficients of the p variables, and $X(t)$ represents the matrix of the basic constant term and the variables of dimension p (the first column of $X(t)$ is thus equal to the unit). If this model has great flexibility, in certain cases it could be too sophisticated, in particular when volumes of data available are limited, and can be subject to significant operational limitations, in particular because of the "curse of dimensionality".

It is thus necessary to reduce the dimension of the models. The method which is considered here is the method adopted by Lopez [2007] in his doctoral thesis on the reduction of dimension in the presence of censured data: it is the single index model.

2.2. Choice of single-index models (SIM): Cox and Lin and Ying

The single-index models (SIM) are defined by:

$$m(z) = E(Y \mid Z = z) = f\left(\delta^T z\right),$$

where Y represents the dependent variable of dimension 1, Z represents the explanatory variables of dimension p, m represents an unknown function such as $m : \mathbb{R}^p \to \mathbb{R}$, f represents a function of an unknown link such as $f : \mathbb{R} \to \mathbb{R}$ and $\delta \in \Theta \subset \mathbb{R}^p$ is an unknown parameter of a finished dimension.

If f is known, the problem becomes purely parametric; and on the other hand if δ is known, the problem becomes nonparametric but of dimension 1. In general, single-index models (SIM) are often presented as a reasonable compromise between purely parametric modeling and purely nonparametric modeling (as specified, for example, in the work of McCullagh and Nelder [1989]).

Lopez [2007] thus proposes the choice of SIM to limit the problem of dimension and shows that one can reasonably estimate these semi-parametric models in the presence of censure. To model the heterogeneity from a model taking into account explanatory variables, the semi-parametric models of type SIM thus seem to be adapted. It is now appropriate to consider retaining the SIM.

In practice, it appears that the multiplicative Cox [1972] model and the additive Lin and Ying [1994] model are typical cases of SIM, in which the assumptions do not relate on conditional expectancy but they do to the conditional instantaneous hazard rate. Indeed, the multiplicative Cox model can be written as:

$$\lambda(t \mid Z = z) = \lambda_0(t) e^{\delta^T z},$$

where λ_0 is a presumably unknown function and δ is a parameter to be estimated (thus, we easily find the representation of a SIM). In the same way, the additive Lin and Ying model can be written as:

$$\lambda(t \mid Z = z) = \lambda_0(t) + \gamma^T z,$$

where λ_0 is a presumably unknown function and γ is a parameter to be estimated (the Lin and Ying model is a typical case of the Aalen model, in which γ replaces $\gamma(t)$). Moreover, it appears that the multiplicative Cox model and the additive Lin and Ying model get together when λ_0 is constant (thus invariant in time) and the exponential term in the multiplicative model (or $e^{\delta^T z}$) is replaced by the linear expression $\left\{1 + \delta^T z\right\}$; in this case $\gamma = \lambda_0 \delta$.

Here, we limit our choice of SIM to the two models above: the Cox model and the Lin and Ying model (in practice, many alternative models could have been retained, including extensions of the multiplicative Cox model and the additive Aalen model, *cf*. Martinussen and Scheike [2006]).

The Cox model is most widely used given its sound properties which were largely studied. However, as Hill et al. [1990] point out, in this model the hazard ratio for two subpopulations of characteristics z_1 and z_2 depends only on z_1 and z_2 and not on time: $\lambda(t \mid Z = z_1) / \lambda(t \mid Z = z_2) = e^{\delta^T z_1} / e^{\delta^T z_2}$. The Cox model is thus based on the assumption of proportionality of the instantaneous death rates between different segments, which is binding.

In the Lin and Ying model, it is the absolute difference of the instantaneous risks for two subpopulations of characteristics z_1 and z_2 which depends only on z_1 and z_2 and not on time: $\lambda(t \mid Z = z_1) - \lambda(t \mid Z = z_2) = \gamma^T z_1 - \gamma^T z_2$. This assumption is also binding.

In practice, it is therefore necessary to choose between a constraint on relative differences (proportionality assumption) and a constraint on the absolute differences. Beyond any statistical test on the assumptions (*cf.* references in the following sections, in particular for the Cox model and the validation of the proportionality assumption), the choice can be guided by "expert advice", by taking into account the context of the study.

According to the statistics of WHO[2], on the general population it appears that the relative differences of mortality rates between 30 and 54 years within zone CIMA/FANAF as the age increases are more stable than the absolute differences (on this point, references are also available in Planchet et al. [2010]). In this context, a constraint on the relative differences would seem more suitable.

But, alternatively, according to the same statistics of WHO, it appears that beyond 55 years of age, the differences in mortality rates decrease as the age increases (on this point, the references are also available in Planchet and al. [2010]). In this context, a constraint on the absolute differences would seem more suitable.

In other words, according to the framework of exploitation of the data, the Cox model or the Lin and Ying model can be more or less suitable. Also, in this study, we will use these two models to measure the risk estimation within the framework of a model integrating heterogeneity starting from observable factors (in particular, the Lin and Ying model is compared to Cox model).

3. Cox model: adjustment and simulation of mortality rates

The Cox model is a traditional model in survival analysis and has been largely studied. Here, we focus on its use of the measurement of risk estimation.

The approach used here to illustrate risk estimation is by directly generating random and crude rates (as appropriate distribution), in order to deduce the impact on the estimate of parameters required to estimate adjusted rates.

[2] *Cf.* http://apps.who.int/whosis/database/life_tables/life_tables.cfm

In this context, at first this section presents the evaluation of the annual adjusted deaths rates (*cf.* 3.1), and then it presents the simulation of annual mortality rates for the measurement of risk estimation (*cf.* 3.2).

3.1. Cox: evaluation of the annual adjusted mortality rates

The Cox model, which for memory's sake can be written as following $\lambda(t \mid Z = z) = \lambda_0(t)e^{\delta^T z}$, makes it possible to measure the multiplicative effect of explanatory variables, in this case countries, on survival. For this purpose, we estimate the parameter δ by the maximum likelihood method. We consider in particular the Cox partial likelihood, which is calculated by the product of conditional probabilities observed at a given moment t_i ($i \in [1; D]$) one (or several) death knowing the composition of the population under risk at this given moment (Hill and al. [1990] present a justification of this approach).

3.1.1. Estimate of Cox (in absence and in presence of a tie)

This paragraph presents the estimate suggested by Cox, in absence and in presence of tie.

Estimate of Cox in absence of a tie

When it is supposed that one death occurs at every moment t_i, Cox [1972] indicates that the conditional probability that is the subject of a characteristic $z_{(i)}$ which dies in t_i, given that one had a group R_i of subjects at risk, is $\exp\left\{\delta^T z_{(i)}\right\} \Big/ \sum_{j \in R_i} \exp\left\{\delta^T z_{j(i)}\right\}$, where $z_{j(i)}$ represents the characteristics of the j[th] individual under risk in t_i. Partial likelihoods of Cox is calculated thus like the product of his contributions, and log likelihoods is written then as $L(\delta) = \sum_{i=1}^{D} \delta^T z_{(i)} - \sum_{i=1}^{D} \ln\left(\sum_{j \in R_i} \exp\left\{\delta^T z_{j(i)}\right\} \right)$.

Estimate of Cox in presence of a tie

When it is supposed that several deaths occur at every moment t_i, Cox [1972] provides a new specification of a multiplicative model in a discrete case:

$$\frac{\lambda(t \mid Z = z)}{1 - \lambda(t \mid Z = z)} = \frac{\lambda_0(t)}{1 - \lambda_0(t)} e^{\delta^T z}.$$

Indeed, if Cox supposes that in continuous time (when only one death occurs at every moment t_i) its model is written as following $\lambda(t \mid Z = z) = \lambda_0(t)e^{\delta^T z}$, in discrete time (when several deaths occur at every moment t_i) he supposes that the instantaneous hazard rates are not sufficiently close to 0 to consider $1-\lambda(t \mid Z=z)=1$ and $1-\lambda_0(t)=1$.

With this new specification, the contribution to the probability of deaths d_i in time t_i, knowing that one had the group R_i of subjects at risk, is written:

$$\exp\left\{\delta^T s_{(i)}\right\} \Big/ \sum_{j\in(R_i;\, d_i)} \exp\left\{\delta^T s_{j(i)}\right\},$$

where $s_{(i)}$ represents the sum of $z_{(i)}$ for all individuals who died in t_i, and the notation of the denominator means that the sum is taken over all individuals d_i is quite distinctive from the one taken on R_i. Log likelihoods is then written as:

$$L(\delta)=\sum_{i=1}^{D}\delta^T s_{(i)}-\sum_{i=1}^{D}\ln\left(\sum_{j\in(R_i;\, d_i)}\exp\left\{\delta^T s_{j(i)}\right\}\right).$$

3.1.2. Estimate of Breslow (in presence of tie)

One of specificities of the death risk is that the frequency of supervening of the risk is weak and that the exposure to the risk is relatively high. Also, in this case, the number of possible combinations of the sum of the denominator $\sum_{j\in(R_i;\, d_i)}\exp\left\{\delta^T s_{j(i)}\right\}$ in the Cox estimator in the presence of tie is particularly important and limits the implementation of the estimate (in particular when the estimate lies within the scope of simulations, as is the case here).

In this context, one can turn to the approximations of Breslow and Elfron (a comparison of the approaches of Cox, Breslow and Elfron starting from simple quantified examples is presented in Klein and Moeschberger [2005]). Here, one retains the Breslow simplification, which is the most practiced, according to which the contribution to the probability of d_i deaths in t_i time, knowing that one had the group R_i of subjects at risk, is written as $\exp\left\{\delta^T s_{(i)}\right\} \Big/ \left[\sum_{j\in R_i}\exp\left\{\delta^T z_{j(i)}\right\}\right]^{d_i}$ with the notations defined above. Log-likelihoods is then written as:

$$L(\delta)=\sum_{i=1}^{D}\delta^T s_{(i)}-\sum_{i=1}^{D}d_i\times\ln\left(\sum_{j\in R_i}\exp\left\{\delta^T z_{j(i)}\right\}\right).$$

3.1.3. Test statistic

Within the framework of this study, we limit ourselves to the illustrations of the test of total significance and the test of parameter significance. It is supposed here that the tests are carried out starting from the statistics of the likelihoods ratio which follows a distribution of Chi-2 under the null hypothesis H_0 (these statistics count among the most used in this context, with the Wald's statistic, *cf.* for example Klein and Moeschberger [2005] for an illustrated presentation).

In the case of the test of total significance, we test the hypothesis of simultaneous nullity of the whole of the parameters, and we thus consider $H_0 : \delta = 0$, or (p being the dimension of Z representing explanatory variables of the model):

$$\chi_L^2(p) = 2\left[L\left(\hat{\delta}\right) - L(0)\right].$$

In the case of a significance test of the parameters, we test the null hypothesis of each parameter δ_j (with $j = 1,..., p$ and $\delta = \left(\delta_1,...,\delta_p\right)$), and we thus consider $H_0 : \delta_j = 0$, or:

$$\chi_{L_j}^2(1) = 2\left[L\left(\hat{\delta}\right) - L\left(\tilde{\delta} \setminus \tilde{\delta}_j, \delta_j = 0\right)\right],$$

where the expression "$\tilde{\delta} \setminus \tilde{\delta}_j, \delta_j = 0$" represents the estimate of the parameters δ_g (with $g \in \left(1,..., p\right) \setminus j$) while fixing $\delta_j = 0$.

Only these two tests of significance of the model and the parameters are carried out here within the framework of our study. It is noted, however, that in practice, the use of the Cox model turns to many complementary statistical tests. The literature is abundant on this subject (a detailed review of the literature on the principal existing statistical tests on the Cox model is presented in the works of Hill and al. [1990], Therneau and Grambsch [2000] and Martinussen and Scheike [2006]). While limiting ourselves to the most traditional tests and relating only to the hypothesis of proportional hazard rates (in practice, the tests also relate to the link function of the model, the form of the variables of the model, the proprieties of the residuals, etc.), we enumerate three approaches. The first (graphic approach of Kay) consists in considering a stratified model and illustrating the evolution of the differences of the logarithms of the hazard's cumulated functions of the stratum as age increases: if they are about constant, the hypothesis of risk proportional is deemed to be adequate for the selected stratification. The second (approach of Therneau and Grambsch [2000]) consists of considering an extension of the Cox model by considering a parameter dependent on time: if the parameter depends significantly on time, the hypothesis of proportionality is not suitable. Third (approach of Lin and al. [1993]) consists in testing the hypothesis of proportionality starting from the cumulated residuals of the model.

3.1.4. Evaluation of the adjusted annual mortality rates

Once the parameters are considered, we deduce the adjusted annual mortality rates for each subpopulation.

In practice, we initially determine the adjusted mortality rates for the subpopulation of the Côte d'Ivoire (basic subpopulation in our applications) starting from the Brass model with external reference. For this purpose, the crude rates are estimated according to the Hoem approach and the reference rates are those of regulatory French life tables TH/TF00-02 (for deaths). These adjusted rates are noted as $q_{x,CI}\left(\hat{\theta}\right)$, where $\hat{\theta}=\left(\hat{a},\hat{b}\right)$ is the parameter of the Brass model (*cf.* Kamega and Planchet [2010]).

Next, we deduce the adjusted mortality rates of Mali and Togo, starting from the parameters of the Cox model by the following relations (by retaining the hypothesis that the rates of instantaneous hazard are constant between two entire ages): $q_{x,ML}\left(\hat{\theta};\hat{\delta}_{ML}\right)=1-\left(1-q_{x,CI}\left(\hat{\theta}\right)\right)^{\exp\left(\hat{\delta}_{ML}\right)}$ and $q_{x,TG}\left(\hat{\theta};\hat{\delta}_{TG}\right)=1-\left(1-q_{x,CI}\left(\hat{\theta}\right)\right)^{\exp\left(\hat{\delta}_{TG}\right)}$, where $\hat{\delta}=\left(\hat{\delta}_{TG};\hat{\delta}_{ML}\right)$ is the estimated parameter of the Cox model.

3.2. Cox: evaluation of the simulated annual mortality rates

We place ourselves in the case where the crude rates are estimated according to the Hoem approach for each subpopulation. These crude rates are written as $\hat{q}_{x,h}$ for a country h.

The simulation technique selected consists in considering a Monte Carlo method to simulate the distribution of a normal law (starting from the simulation of a standard normal distribution, which itself is deduced from a simulation of uniform distribution between 0 and 1). Thus, for each country h, we generate, as the first step, k simulations ($k\in\left[1,K\right]$) of the crude mortality rates for all ages x ($x\in\left[x_m,x_M\right]$), according to $Q_{x,h}\sim N\left(\hat{q}_{x,h};\sqrt{\frac{\hat{q}_{x,h}\left(1-\hat{q}_{x,h}\right)}{R_{x,h}}}\right)$.

In second step, we deduce for each simulation k and all ages x the value close to the quantity of deaths by country by $\hat{d}^k_{x,h}\approx \mathrm{A}\left(\hat{q}^k_{x,h}\times R_{x,h}\right)$, where $\mathrm{A}(.)$ is rounded to the nearest whole, $\hat{q}^k_{x,h}$ represents a realization k of $Q_{x,h}$, and $R_{x,h}$ represents the group of subjects at risk of age x for country h.

The following steps consist in estimating K -the achievements (as the fluctuations of sampling) of the adjusted rates (one speaks then about simulated rates), and for this purpose, one distinguishes the case of the Côte d'Ivoire from the case of Mali and Togo.

In the case of the Côte d'Ivoire, the adjusted rates are obtained starting from the Brass model. For each simulation k, one thus has $q_{x,CI}\left(\hat{\theta}^k\right)=\frac{\exp\left(\hat{a}^k z_x+\hat{b}^k\right)}{1+\exp\left(\hat{a}^k z_x+\hat{b}^k\right)}$, where for memory's sake $z_x=\ln\left(q_x^{ref}/\left(1-q_x^{ref}\right)\right)$ and $\hat{\theta}^k=\left(\hat{a}^k,\hat{b}^k\right)$ is estimated by the least squares method for each simulation k.

In the case of Mali and Togo, the adjusted rates are obtained starting from the Cox model. For each simulation k, we estimate the parameters $\delta^k=\left(\delta^k_{ML};\delta^k_{TG}\right)$ of the aforesaid model with the method of Breslow; in this case, log likelihood is written:

$$L\left(\delta^k\right)=\sum_{i=1}^{D}\left(\delta^k\right)^T s^k_{(i)}-\sum_{i=1}^{D}\hat{d}_i^k\times\ln\left(\sum_{j\in R_i}\exp\left\{\left(\delta^k\right)^T z_{j(i)}\right\}\right),$$

where $\hat{d}_i^k=\sum_h \hat{d}^k_{i,h}$ (with $h=CI,ML,TG$ in our study). For each simulation k, we then deduce from the parameters $\delta^k=\left(\delta^k_{ML};\delta^k_{TG}\right)$ the adjusted rates of Mali and Togo by $q_{x,ML}\left(\hat{\theta}^k;\hat{\delta}^k_{ML}\right)=1-\left(1-q^k_{x,CI}\left(\hat{\theta}^k\right)\right)^{\exp\left(\hat{\delta}^k_{ML}\right)}$ and $q_{x,TG}\left(\hat{\theta}^k;\hat{\delta}^k_{TG}\right)=1-\left(1-q^k_{x,CI}\left(\hat{\theta}^k\right)\right)^{\exp\left(\hat{\delta}^k_{TG}\right)}$.

4. Lin and Ying model: adjustment and simulation of mortality rates

According to the studies presented for the Cox model, we concentrate here on the use of the Lin and Ying model within the framework of the measurement of risk estimation.

Thus, we show on one hand the adjusted rates (*cf.* 4.1), and on the other hand the simulated rates (*cf.* 4.2).

4.1. Lin and Ying: evaluation of the rates of adjusted annual deaths

The Lin and Ying model, which for memory can be written $\lambda(t \mid Z=z)=\lambda_0(t)+\gamma^T z$, is a typical case of the Aalen additive model. It makes it possible to measure the additional risk due to the effect of explanatory variables of the model, which are the countries in this case, in absolute terms (a reminder: the Cox multiplicative model measures the excess of risk in relative terms). For this purpose, we estimate the parameter γ using an explicit formula.

4.1.1. Estimate of Klein and Moeschberger

Lin and Ying [1994] and Klein and Moeschberger [2005] show that starting with the decomposition martingale of the Poisson process, the estimate of the coefficients of the model is:

$$\hat{\gamma}=\mathrm{A}^{-1}\mathrm{B},$$

where $\mathrm{A}=\sum_{i=1}^{D}\sum_{j\in R_i}\left(z_{j(i)}-\bar{z}_{j(i)}\right)^T\left(z_{j(i)}-\bar{z}_{j(i)}\right)$, $\mathrm{B}=\sum_{i=1}^{D}\sum_{h}d_{i,h}\left(z_{(i),h}-\bar{z}_{(i)}\right)$

and $\bar{z}_{(i)}=\frac{1}{R_i}\sum_{j\in R_i}z_{j(i)}$ (with the same notations as those of section 3 relating to the Cox model).

4.1.2. Test statistic

Like the work done for the Cox model, within the framework of this study, we limit ourselves to illustrations of the test of total significance and the test of parameter significance.

According to the same authors, the total significance of the model can be appreciated starting from the Wald statistics which follows a distribution of Chi-2 to p degrees of freedom (p being the dimension of Z representing the explanatory variables of the model) under the assumption $H_0:\gamma=0$, or:

$$\chi_W^2=\hat{\gamma}^T\,\widehat{\mathrm{V}}^{-1}\hat{\gamma},$$

where $\widehat{\mathrm{V}}=\mathrm{A}^{-1}\mathrm{C}\mathrm{A}^{-1}$, with $\mathrm{C}=\sum_{i=1}^{D}\sum_{h}d_{i,h}\left(z_{(i),h}-\bar{z}_{(i)}\right)^T\left(z_{(i),h}-\bar{z}_{(i)}\right)$.

In the case of the test of parameter significance, we test the null hypothesis for each parameter γ_j (with $j=1,\ldots,p$ and $\gamma=\left(\gamma_1,\ldots,\gamma_p\right)$), and we thus consider $H_0:\gamma_j=0$, or:

$$\chi_{W_j}^2=\hat{\gamma}_j^2\Big/\widehat{\mathrm{V}}_{jj}.$$

4.1.3. Evaluation of the adjusted mortality rates

Once the parameters are considered, we deduce the adjusted mortality rates for each subpopulation.

In practice, at first we determine the adjusted mortality rates for the subpopulation of the Côte d'Ivoire (which concerns the basic subpopulation) starting from the approach adopted for work on the Cox model (*cf.* paragraph 3.1.4). These adjusted rates are noted $q_{x,CI}\left(\hat{\theta}\right)$, where $\hat{\theta}=\left(\hat{a},\hat{b}\right)$ are the parameters of the Brass model (*cf.* Kamega et Planchet [2010]).

Secondly, we deduce the adjusted mortality rates of Mali and Togo, starting from the parameters of the Lin and Ying model, by the following relationship (under the assumption that the instantaneous rates of hazard are constant between two entire ages): $q_{x,ML}\left(\hat{\theta};\hat{\gamma}_{ML}\right)=1-\left(1-q_{x,CI}\left(\hat{\theta}\right)\right)\exp\left(-\hat{\gamma}_{ML}\right)$ and $q_{x,TG}\left(\hat{\theta};\hat{\gamma}_{TG}\right)=1-\left(1-q_{x,CI}\left(\hat{\theta}\right)\right)\exp\left(-\hat{\gamma}_{TG}\right)$, where $\hat{\gamma}=\left(\hat{\gamma}_{TG};\hat{\gamma}_{ML}\right)$ are the estimated parameters of the Lin and Ying model.

4.2. Lin and Ying: evaluation of the simulated annual mortality rates

The first steps in the simulation of the mortality rates for the Lin and Ying model are identical to those are carried out for simulations of the Cox model (*cf.* sub-section 3.2).

For each simulation k ($k\in\left[1,K\right]$) of the crude death rate, and for all ages x ($x\in\left[x_m,x_M\right]$), we thus have a simulation of the number of deaths by country by $\hat{d}_{x,h}^{k}$ and of the mortality rate simulated for the Côte d'Ivoire $q_{x,CI}\left(\hat{\theta}^k\right)$.

One can then determine, for each simulation k, the mortality rates of Mali and Togo. For this purpose, we are based on the Lin and Ying model and we estimate for simulation k the parameters $\gamma^k=\left(\gamma_{ML}^k;\gamma_{TG}^k\right)$ of the aforesaid model, using the relation:

$$\hat{\gamma}=\mathrm{A}^{-1}\mathrm{B}^k,$$

where $\mathrm{B}^k=\sum_{i=1}^{D}\sum_{h}\hat{d}_{i,h}^{k}\left(z_{(i),h}-\bar{z}(i)\right)$. For each simulation k, we then deduce the mortality rates simulated for Mali and Togo by

$$q_{x,ML}\left(\hat{\theta}^{k};\hat{\gamma}^{k}_{ML}\right)=1-\left(1-q^{k}_{x,CI}\left(\hat{\theta}^{k}\right)\right)\exp\left(-\hat{\gamma}^{k}_{ML}\right)$$ and

$$q_{x,TG}\left(\hat{\theta}^{k};\hat{\gamma}^{k}_{TG}\right)=1-\left(1-q^{k}_{x,CI}\left(\hat{\theta}^{k}\right)\right)\exp\left(-\hat{\gamma}^{k}_{TG}\right).$$

5. Comparison of the adjustments of the Cox model and Lin and Ying model and backtesting on heterogeneity

This section presents the results of the adjustments of the models integrating heterogeneity starting from observable factors and presents a backtesting of the capacity of these models to take into account heterogeneity, compared with an approach selecting an independent model for each subpopulation.

5.1. Comparison of the adjustments of Cox and Lin and Ying

This subsection presents the results of model fitting for the Cox model and the Lin and Ying model respectively. A comparison of these results is also presented.

5.1.1. Results of the adjustments of Cox

We consider here population UEMOA, represented here by the Côte d'Ivoire, Mali and Togo (*cf.* Table 1 and Table 2 abovementioned).

The results of the estimate $\hat{\delta}=\left(\hat{\delta}_{TG};\hat{\delta}_{ML}\right)$ starting from the two approaches taken in consideration in case of the presence of a tie (Cox or Breslow approach), are presented in the following table (for women only). These estimates are made from deaths and exposures by age.

Table 3 – Cox model: comparison adjustment (UEMOA - Woman)

Statistic	Cox (with tie)	Breslow (with tie)
Minimum(*)	1747	2146
Iterations	8	8
χ_L^2 (*p-value* model)	$18,02\left(p=0,01.10^{-2}\right)$	$18,07\left(p=0,01.10^{-2}\right)$
$\hat{\delta}_{ML}$ (initial value)	$-0,4638\,(0)$	$-0,4709\,(0)$
$\exp\left(\hat{\delta}_{ML}\right)$	$0,6289$	$0,6244$
$\chi_{L_{ML}}^2$ (*p-value* parameter)	$1,03\left(p=31,01.10^{-2}\right)$	$1,07\left(p=30,10.10^{-2}\right)$
$\hat{\delta}_{TG}$ (initial value)	$-0,9199\,(0)$	$-0,9198\,(0)$
$\exp\left(\hat{\delta}_{TG}\right)$	$0,3986$	$0,3986$
$\chi_{L_{TG}}^2$ (*p-value* parameter)	$17,49\left(p=2,89.10^{-5}\right)$	$17,51\left(p=2,86.10^{-5}\right)$

(*) because in practice we minimize the opposite of the log-likelihood.

It appears that the estimates of Cox and Breslow lead to comparable results in this example.

By adopting from now on the approach of Breslow only, the table below presents the results of the estimates for the female and male population.

Table 4 – Cox model : Breslow adjustment (UEMOA – W. et M.)

Statistic	Woman	Man
χ_L^2 (*p-value* model)	$18,07\left(p=0,01.10^{-2}\right)$	$12,79\left(p=0,17.10^{-2}\right)$
$\exp\left(\hat{\delta}_{ML}\right)$	$0,6244$	$0,6084$
$\chi_{L_{ML}}^2$ (*p-value* parameter)	$1,07\left(p=30,10.10^{-2}\right)$	$7,89\left(p=0,50.10^{-2}\right)$
$\exp\left(\hat{\delta}_{TG}\right)$	$0,3986$	$1,1035$
$\chi_{L_{TG}}^2$ (*p-value* parameter)	$17,51\left(p=2,86.10^{-5}\right)$	$4,38\left(p=3,64.10^{-2}\right)$

The results of the parameters are coherent with the descriptive statistics presented in Table 1 and Table 2. For the significance test of the models, it arises that the value of the empirical test is higher than that of the table to a threshold of 5% at 2 degrees of freedom (equal to 5.99, for memory), whether it is for the female or the male population. Thus, the models are significant with the threshold of 5%. On the level of the variables, however, it appears that the parameter for Mali for women is not significant with a threshold of 5% (for this individual parameter, the value of the empirical test is lower than that of the table to a threshold of 5% at 1 degree of freedom, equal to 3.84, for memory). The parameters of the other variables are on the other hand significant with the threshold of 5%.

5.1.2. Results of the adjustments of Lin and Ying

Like the illustration of the Cox model, we consider here the female and male population of population UEMOA, represented here by the Côte d'Ivoire, Mali and Togo (*cf.* Table 1 and Table 2).

The results of the estimate of $\hat{\gamma}=\left(\hat{\gamma}_{TG};\hat{\gamma}_{ML}\right)$ are presented in the following table.

Table 5 – Lin and Ying model: adjustment (UEMOA – W. & M.)

Statistic	Woman	Man
χ^2_W (*p-value* model)	$26,55\left(p=1,71.10^{-6}\right)$	$16,01\left(p=0,03.10^{-2}\right)$
$\hat{\gamma}_{ML}$	$-0,0693$ %	$-0,1429$ %
$\chi^2_{W_{ML}}$ (*p-value* parameter)	$1,43\left(p=23,25.10^{-2}\right)$	$10,68\left(p=0,11.10^{-2}\right)$
$\hat{\gamma}_{TG}$	$-0,1129$ %	$0,0399$ %
$\chi^2_{W_{TG}}$ (*p-value* parameter)	$26,14\left(p=3,17.10^{-7}\right)$	$4,07\left(p=4,37.10^{-2}\right)$

The results of the parameters are coherent with the descriptive statistics presented in Table 1 (for women) and in Table 2 (for men) on one hand, and with the coefficients of the Cox model presented in Table 4 (for women and men) on the other hand. In statistical terms of tests, like the results of the Cox model, it appears that the models for women and men are significant with a threshold of 5%, even if the parameter of Mali for women is not significant.

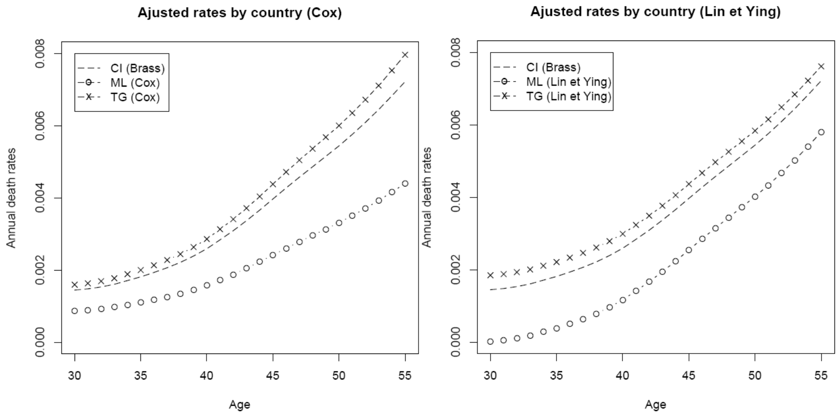
Moreover, concerning the comparison between the multiplicative model (Cox) and the additive model (Lin and Ying) on an operational component, we find the principal conclusions of the work of Cao [2005]. Thus, it appears that these two multiplicative and additive models are exploitable for the truncated and/or censured data, can account for observable factors, and present consistent *p-values* for the coefficients of variables in the model.

5.1.3. Illustration and comparison of the results of the adjustments of Cox and Lin and Ying

The two graphs below (Figure 1) present a comparison between the results of the Cox model and those of the Lin and Ying model, for the male population only. The first graph shows the adjusted mortality rates of the Côte d'Ivoire, resulting from the Brass model, like those of Mali and Togo, derived from the parameters $\hat{\delta}=\left(\hat{\delta}_{TG};\hat{\delta}_{ML}\right)$ of the Cox model. On the same principle, the second graph shows the adjusted mortality rates of the Côte d'Ivoire resulting from the Brass model, like those of Mali and Togo, derived from the parameters $\hat{\gamma}=\left(\hat{\gamma}_{TG};\hat{\gamma}_{ML}\right)$ of the Lin and Ying model (the curves of the two graphs relating to the Côte d'Ivoire are thus identical).

In statistical terms, within the framework of the adjustment of the rates from the Brass model for the Côte d'Ivoire, it can be seen that the model is significant at a threshold of 5% (Fisher test) and that the adjusted R2 is equal to 83.8%. One can also point out that the retained parameters for the evaluation of the adjusted rates from Lin and Ying for Togo and Mali are significant at a threshold of 5%.

Figure 1 - Adjusted mortality rates (Cox and Lin and Ying, UEMOA - M)



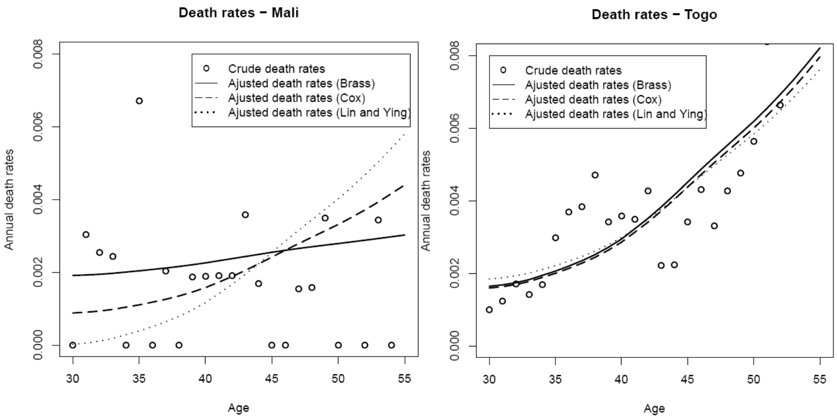
To continue the comments relating to the choice between these two models (*cf*. sub-section 2.2), it appears that for the Cox multiplicative model of heterogeneity, the absolute differences between the countries increase significantly as age increases, contrary to the Lin and Ying model of heterogeneity in which the absolute differences are constant across all ages.

5.2. Backtesting on heterogeneity (comparison of the models of heterogeneity)

The aim of the study is to compare the risk estimation of two approaches for the treatment of heterogeneity. This comparison is presented in the following section (section 6). We consider a population made up of three subpopulations: the Côte d'Ivoire (basic subpopulation), Mali and Togo. In the first approach, heterogeneity is taken into account for the independent models of each subpopulation (in practice, the mortality rates of each population are adjusted according to a Brass model). In the second approach, heterogeneity is taken into account from the models integrating the observable factors (here, the mortality rates are adjusted from the Brass model for the Côte d'Ivoire and from the Cox model or Lin and Ying model for Mali and Togo).

The results of adjustmented mortality rates according to these two approaches (which count three models in total) are presented in the graph below for Mali and Togo (the graph relating to the Côte d'Ivoire is not presented because it is identical for the two approaches).

Figure 2 – Presentation of mortality rate of Mali and Togo (M)



In Mali's case, it appears that some crude death rates equal zero (*cf*. graph above) limiting the use of the logit of crude death rates, which are still essential to the modeling of Brass. Thus, the following convention was retained for the adjustment (for the Brass model only): we consider that the zero death rates are equal to the lower level of crude death rates greater than zero for the age range retained for the adjustment. Alternative approaches could have been to not retain the ages where one observes the zero crude death rates or to substitute the crude death rates equal to zero by average observed rates, but these solutions significantly raise the mortality rates retained for the adjustment.

On these bases, the adjusted mortality rates for Mali diverge and depend on the model. On a statistical level, the Brass model for Mali (first approach) is not significant at a threshold of 5% (Fisher's test), and the adjusted R2 is equal to 1.4%. In addition, as mentioned before, the parameters of Mali for the Cox model and Lin and Ying model (models of the second approach) are significant at a threshold of 5% (*cf*. Table 4 and Table 5). Thus, we retain that the models integrating heterogeneity allow us, contrary to the Brass model, to avoid retaining conventions where crude death rates equal zero while also getting satisfactory results in terms of statistical tests.

For Togo, the adjusted mortality rates are similar regardless of the model considered. We note that the Brass model for Togo is significant at a threshold of 5% (Fisher's test) and that adjusted R2 is equal to 69.2%. To reiterate, the parameters of Togo for the Cox model and Lin and Ying model are significant at a threshold of 5% (*cf*. Table 4 and Table 5), in spite of the weak variations within the Côte d'Ivoire rates (*cf*. Figure 1).

To appreciate the relevance of the mortality rates adjusted in these three models, the following table presents the comparisons between the achievements and predictions of the deaths (established from the adjusted mortality rates and exposures to risk by age).

Table 6 – Comparison achievement/prediction of deaths (UEMOA – M.)

Global model UEMOA - H (Brass global)			
Country	Observed deaths	Predicted deaths	Différence
Côte d'Ivoire	2 188	2 203	0,7%
Mali	27	44	63,8%
Togo	565	511	-9,6%

Integrating heterogeneity models (without and with obs. fact.)			
Country (model)	Observed deaths	Predicted deaths	Différence
Côte d'Ivoire (Brass)	2 188	2 144	-2,0%
Mali (Brass(*))	27	29	8,4%
Mali (Cox)		26	-3,2%
Mali (Lin et Ying)		26	-4,9%
Togo (Brass)	565	565	-0,1%
Togo (Cox)		548	-3,0%
Togo (Lin et Ying)		550	-2,7%

(*) agreement with the treatment of crude death rate equal to zero.

The first sub-table presents the differences when the adjusted mortality rates are given in total without taking into account heterogeneity between subpopulations. The second sub-table presents the differences when the adjusted mortality rates are given from the models integrating heterogeneity, either from the independent models for each subpopulation (as with the Brass model), or from the models integrating heterogeneity from observable factors (as with the Cox model and Lin and Ying model, the factors here being countries). A comparison of the two sub-tables confirms the need to take into account the heterogeneity of the population.

In the sub-table integrating heterogeneity, it appears that except for the adjusted rates of Mali with the Brass model, the differences between the observed deaths and the theoretical deaths are all less than 5%, which confirms the need to account for heterogeneity.

In detail, for this second sub-table, we note initially that the number of theoretical deaths for the Côte d'Ivoire underestimate the number of observed deaths by 2%. This difference reflects the estimation bias introduced by the logit function of the Brass model. Indeed, the concave character of the logit function between 0 and ½ led, by using Jensen's inequality, to the undervaluation of the mortality rates (*cf.* Planchet and Thérond [2006] for a more through description of this phenomenon).

The estimates of the Brass model for the Côte d'Ivoire are used to estimate the Cox model and the Lin and Ying model for Mali and Togo. Also, the observed differences for Mali and Togo using these models include the observed differences for the Côte d'Ivoire given by the Brass model.

Finally, in spite of the retained convention for the treatment of Mali with the Brass model (substitution of the zero crude death rates by the observed lower non-zero crude death rates), we note in this second sub-table that the estimated theoretical deaths of Mali with the Brass model gross up the observed deaths. As comparison, with convention consisting in retaining only the ages where the crude death rates are not zero, the predicted number of deaths for Mali rises to 38 (or a difference of 39.7% against theoretical deaths, compared to 8.4% with the selected convention). Thus, the convention selected seems more suited although it presents important limits (the differences obtained for the Brass model applied to Mali are not satisfactory in comparison to differences observed for other countries and other models).

In the end, these analyses lead to the observation that the choice of a model integrating heterogeneity from observable factors (for example the Cox model or the Lin and Ying model), makes it possible to preserve a good statistical appreciation of the risk of death for the heterogeneous subpopulations within a population. Moreover, these models make it possible to account for the annual mortality rates equal to zero for the given ages (frequent phenomenon in the small populations), and thus allows the avoidance of retaining the assumptions or convention (contrary to the Brass model).

6. Comparison of the risks estimation between the independent models for each subpopulation and those integrating heterogeneity from observable factors

This section presents the systematic risk related to the sampling fluctuations (that is, the risk estimation) in the different models selected for each of the two approaches: the Brass model (the first approach), and the Cox model and Lin and Ying model (the second approach). For the three studied models, the measuring instruments are presented through the adjusted mortality rates and through life tables.

In general, the approaches presented here are in line with those detailed in Kamega and Planchet [2010] on measuring risk estimation under the Brass model.

In addition, in all the presented numerical applications, we let $K = 1\,000$ simulations of the crude rates and we consider that for each simulation k, the sample results of the standard normal distribution $N(0;1)$ from which we deduce the suitable outcomes for each subpopulation, are exactly the same as the ones in the three models used (for a given subpopulation). It also notes that crude rates are in practice generated under the constraint $q_{x,h}^{k} > 0$ (when this condition is not met for a simulation k, the outcome is not counted and we resample for country h).

Finally, in this section, only the numerical illustrations relating to Togo will be presented: the illustrations of the Côte d'Ivoire are not presented because its modeling is identical in the two approaches, and those of Mali are not presented either because they need arbitrary conventions which could create a bias in the exploitation and the analysis of the results. In addition, the aim of the study is to present measurements of risk estimation according to the choice of the selected model taking into account heterogeneity. Therefore, analysis of the risk estimation of the various models while limiting ourselves to the Togolese subpopulation thus improves the clarity of the illustrations.

6.1. Measure risk estimation on the adjusted rates

We present here the measuring instruments of the risk estimation on the adjusted rates, as well as an illustration and interpretation of the results for the three studied models.

6.1.1. Measuring instruments of risk estimation on the rates

We have the mortality rates adjusted according to the Brass model, the Cox model or the Lin and Ying model. To simplify the notations, we write $q_{x,h}\left(\hat{\theta}\right)$ as the adjusted mortality rates for a country h (for Mali and Togo, the adjusted mortality rates should normally be noted as $q_{x,h}\left(\hat{\theta};\hat{\delta}_h\right)$ and $q_{x,h}\left(\hat{\theta};\hat{\gamma}_h\right)$, respectively, for the Cox model and Lin and Ying model). Given k simulations of the crude rates for each country h, we also have the mortality rates simulated according to the Brass model, the Cox model or the Lin and Ying model. To simplify the notations, we denote $q_{x,h}\left(\hat{\theta}^k\right)$ as the mortality rates for simulation k and country h (for Mali and Togo, the simulated mortality rates should normally be noted as $q_{x,h}\left(\hat{\theta}^k;\hat{\delta}_h^k\right)$ and $q_{x,h}\left(\hat{\theta}^k;\hat{\gamma}_h^k\right)$, respectively, for the estimates of the Cox model and the Lin and Ying model).

The risk estimation of the mortality rates for a country h can then be measured using the coefficient $c\left(\psi_{x,h}\right)=\frac{\psi_{x,h}}{q_{x,h}\left(\hat{\theta}\right)}$, where $\psi_{x,h}=\sqrt{E\left[\left(q_{x,h}\left(\hat{\theta}^k\right)-q_{x,h}\left(\hat{\theta}\right)\right)^2\right]}$. This coefficient constitutes a measure of dispersion of the mortality rates simulated around the rate of adjusted death (expressed as a percentage and under risk estimation).

6.1.2. Comparison of risk estimation on the rates

The comparison of risk estimation on adjusted rates is carried out by means of the average of the coefficient $c\left(\psi_{x,h}\right)$, for all ages $x\in\left[x_m,x_M\right]$.

Table 7 – Risk estimation on the mortality rates (average) (Togo - M)

Population	Brass model (approach 1)	Cox model (approach 2)	Lin et Ying model (approach 2)
Togo $c\left(\psi_{TG}\right)$	9,89 %	6,19 %	6,78 %

It appears that the use of the Brass model (independent model for each subpopulation) led to a risk estimation higher than that obtained by the Cox model (multiplicative model integrating heterogeneity using observable factors) or with the Lin and Ying model (additive model integrating heterogeneity using observable factors). For the Lin and Ying model, the comparisons must however be treated with prudence, taking into account the assumption that the absolute differences of the rates of instantaneous deaths are constant.

Note that in addition, the risk estimation is the more important as the population presents a weak exposure. Thus, for example, the risk estimation with the Brass model for the Côte d'Ivoire is equal to 4.73 %. In the same way, the risk estimation for Mali with the Cox model is equal to 22.27 %.

The comparisons presented above are limited to the simulated mortality rates; the comparisons of the mortality tables built starting from the simulated mortality rates are presented in the sub-section below.

6.2. Measure of risk estimation in the tables

In addition to the presentation and the comparison of the mortality rates, it is necessary to compare the associated mortality tables.

6.2.1. Measuring instrument of risk estimation in the tables

For the measurement of risk estimation in the mortality tables, it is advisable to use a specific function for each table which associates a positive number to it. The life expectancy is, from this point of view, a natural functional within the framework of insurance, the amount of the liability is another. Thus, these two functions are used in the work below.

Initially, the risk estimation in the tables is measured from the residual life expectancy at 30 years, between 30 and 55 years. For this purpose, we use the distribution of residual life expectancies established from simulated mortality rates.

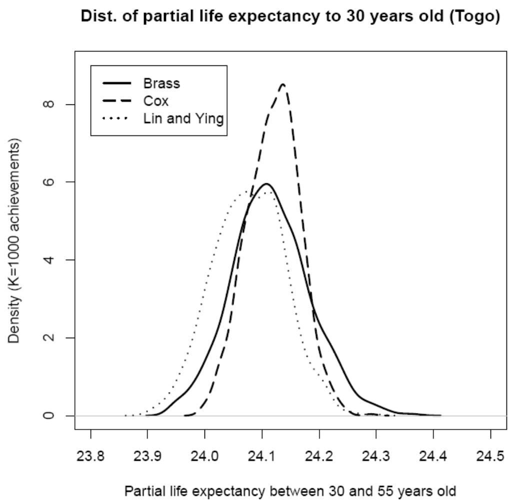
Next, the risk estimation in the tables is measured from the liabilities. We consider more precisely the deterministic provisions relative to temporary obligations to the deaths (the term of the obligations is d years, with $d \geq 1$) and evaluated from the simulated mortality rates (in this case, the mortality rates are the only sources of risk). By assuming a death to be in middle of year, we deduce that the amount of liabilities of a death for k is $L_{0,h}^{k} = \sum_{t=0}^{d-1} F_{x,h}^{k}(t) \times (1+r_{t+1})^{-t-1/2}$, where $F_{x,h}^{k}(t)$ represents the probable cash flows of the services to pay in time t for an individual of age x of country h (for k simulated mortality rates) and r_t represents the discount rate of cash flows at time t. The impact of the risk estimation on the liabilities can then be measured by the coefficient $c(\Upsilon_h) = \frac{\Upsilon_h}{L_{0,h}}$, where $L_{0,h}$ corresponds to the liability calculated from the adjusted mortality rates $q_{x,h}(\hat{\theta})$ and $\Upsilon_h = \sqrt{E\left[\left(L_{0,h}^{k} - L_{0,h}\right)^2\right]}$. This coefficient makes it possible to have a measurement of dispersion expressed as a percentage (as the risk estimation) around the liability calculated from the adjusted mortality rates.

Finally, we denote $\overline{L}_{0,h} = \frac{1}{K}\sum_{j=1}^{K} L_{0,h}^{j}$ as the average of the deterministic liabilities calculated with simulated mortality rates.

6.2.2. Comparison of the risk estimation in the tables

Initially, we present the risk estimation in the tables through the partial life expectancies. In this context, the following graph presents, for the subpopulation of Togo, the estimates of the functions of density (by the kernel estimator) of the residual life expectancies (between 30 and 55 years) established from the simulated mortality rates.

Figure 3 - Distribution of the partial life expectancies of Togo (M)



Following the observations and conclusions above, it appears in this graph that the distribution of the partial life expectancy resulting from the Cox model presents a tail lower than that resulting from the Brass model. In other words, the mortality tables resulting from the simulated mortality rates from the multiplicative model integrating heterogeneity using the observable factors have a lower volatility than the tables resulting from the simulated mortality rates from the Brass model used independently for each subpopulation. In the case of the additive Lin and Ying model on the other hand, it appears that the mortality tables have a volatility comparable with that resulting from the Brass model.

Moreover, it is a question of quantifying this additional risk estimation in the tables in an insurance context. For this purpose, we introduce the concept of liability, calculated with the discount rate curve of the *Institut des Actuaires* on

12/31/2009[3]. Moreover, we consider an insured person of age $x = 31$, an ensured capital equal to $C = 1$ and a term of contract equal to $d = 20$.

For this reason, we present in the table below the synthetic results of the estimates and simulations of deterministic liabilities, respectively, with the adjusted mortality rates and the simulated mortality rates.

Table 8 - Risk estimation in the liabilities (20 years) (Togo - M)

Statistic	Brass		Cox		Lin et Ying	
	Liab. and adjusted rate	Liab. and simula-ted rate	Liab. and adjusted rate	Liab. and simula-ted rate	Liab. and adjusted rate	Liab. and simula-ted rate
Mean (L_0 ou $\bar{L}_0$)	$4{,}18.10^{-2}$	$4{,}03.10^{-2}$	$4{,}06.10^{-2}$	$4{,}01.10^{-2}$	$4{,}22.10^{-2}$	$4{,}17.10^{-2}$
Quantile at 0,5 %	NA	$3{,}18.10^{-2}$	NA	$3{,}52.10^{-2}$	NA	$3{,}54.10^{-2}$
Quantile at 5 %	NA	$3{,}52.10^{-2}$	NA	$3{,}70.10^{-2}$	NA	$3{,}74.10^{-2}$
Quantile at 95 %	NA	$4{,}50.10^{-2}$	NA	$4{,}34.10^{-2}$	NA	$4{,}61.10^{-2}$
Quantile at 99,5 %	NA	$4{,}74.10^{-2}$	NA	$4{,}50.10^{-2}$	NA	$4{,}83.10^{-2}$
Coefficient $c(\Upsilon)$	NA	$7{,}91\ \%$	NA	$5{,}15\ \%$	NA	$6{,}41\ \%$

It arises that with the male data of Togo, accounting for systematic risk decreases the calculated liability by 3.6 % when the Brass model is retained, where as the impact is weaker with the Cox model and Lin and Ying model (it falls to 1.2 % for these two models). Concerning quantiles, it arises that the differences of the averages are more important for the Brass model than for the Cox model and Lin and Ying model. Lastly, it appears that the coefficient $c(\Upsilon)$ relating to modeling starting from the Cox model or Lin and Ying model undervalues the one obtained starting from the Brass model (respective falls of 20% and 35%).

Thus, we retain that the choice of a model integrating heterogeneity makes it possible to significantly reduce the risk estimation associated with the life table based on an insured population. In our example, it appears however that

[3] Curve available on the website of the *Institut des Actuaires* : http://www.institutdesactuaires.com/gene/link.php?doc_link=../docs/2010003190313_IA20091231.xls.

the weight of the risk estimation in the evaluation of a liability (measured for each model by the difference between the liability calculated from the adjusted mortality rates and that calculated from the simulated mortality rates) is comparable to the weight of the risk of model (measured by the difference between the liability calculated from the adjusted mortality rates for the three models suggested). Part of the risk estimation is transformed into model risk.

7. Synthesis and conclusion

In a context of heterogeneity, this study aims to appreciate the evolution of the risk estimation during the passage of an evaluation of mortality rates of independent models for each subpopulation (first approach), to a model directly integrating heterogeneity of observable factors (second approach).

If it is acquired in this study that the choice of the model for the first approach is the Brass model (in continuity of the works completed in Kamega and Planchet [2010]), it is appropriate to inquire on the choice(s) of model(s) for the second approach. The analysis carried out in the section 2 led to the choices of the multiplicative Cox model and the additive Lin and Ying model.

Consequently, the adjustment of the mortality rates for each subpopulation was presented and implemented for these two models of the second approach. A comparison of calibration results and statistical tests associated with these two models illustrate their consistency. However, the Cox model and Lin and Ying model present differences which can justify the choice of one of the two models according to our needs. Thus, whereas the Cox model considers that the relative differences of the hazard rates are stable (the absolute differences thus increase significantly with age), the Lin and Ying model considers that the absolute differences are stable (the relative differences thus decrease significantly with age). As a result, if these two models can lead to comparable results on the ages retained for the adjustment, they can lead to mortality rates considerably different outside this age range (moreover, the model of Lin and Ying can present important operational limits for the younger ages).

After having verified in sub-section 5.2 that these two models take into account the heterogeneous character of the population, a measurement of risk estimation was carried out. For this purpose, several estimates of the mortality rates resulting from the Brass model (first approach) and models of Cox and Lin and Ying (second approach) were simulated on crude mortality rates (generated from the distribution of the initial crude rates in order to reflect the sampling fluctuations).

The obtained results illustrate a reduction in the impact of risk estimation on the liabilities of more than 50 % with the models of the second approach (Cox model and Lin and Ying model), compared to the impact observed with the model of the first approach (Brass model). Moreover, with the models of the second approach, the measurement of the risk estimation on the liabilities decreases by 20% to 35%. Nevertheless, in our example we note in parallel that the weight of the risk estimation on the provisions (which reflects the dispersion of the estimates of liability due to the crude rate sampling fluctuations for a given model) is comparable with the weight of the risk of a model (which reflects the dispersion of the liability estimates due to the model choice). In search of a model which makes it possible to reduce risk estimation, special attention must also be given to the consequences in terms of model risk.

Finally, the choice of a model integrating heterogeneity from observable factors (second approach) has several advantages, among which we emphasize:

- the reduction of the impact of the risk estimation on the estimated liability (drops by more than 50% compared to the impact observed with the Brass model used independently for each subpopulation);
- the capacity to model the mortality rates when the data are noticeably limited (in particular when there are the data with crude rates equals zero, knowing that such data are not exploitable with the Brass model, except when retaining the arbitrary conventions).

The choice of models of the second approach, however, presents some disadvantages, among which we note:

- the assumption that relative differences are constant (model of Cox) or absolute (model of Lin and Ying) of the hazard rates;
- the general potential impact in terms of model risk.

References

Barbi E., Caselli G., Vallin J. [2003] « Hétérogénéité des générations et âge extrême de la vie », *Institut Nationale d'Études Démographiques / Population, Vol. 58, No. 2006-1*.

Cao H. [2005] *A comparison Between the Additive and Multiplicative Risk Models*, Mémoire d'actuariat, Université Laval.

Cox D. R. [1972] « Regression Models and Life-Tables », *Journal of the Royal Society. Series B (Methodological), Vol. 34, No. 2*.

Delwarde A., Denuit M. [2006] « Construction de tables de mortalité périodiques et prospectives », *Economica*.

Droesbeke J.-J., Fichet B., Tassi P. [1989] (éditeurs) « Analyse statistique des durées de vie », *Economica*.

Hill C., Com-Nougué C., Kramar A., Moreau T., O'Quigley J., Senoussi R., Chastang C. [1990] « Analyse statistique des données de survie », *Inserm, Médecine – Sciences, Flammarion*.

Kamega A., Planchet F. [2010], « Mesure du risque d'estimation associé à une table d'expérience », *Cahiers de recherche de l'ISFA, WP2136*.

Klein J. P., Moeschberger M. L. [2005] « Survival Analysis – Techniques for Censored and Truncated Data », *Springer, 2nd edition*.

Lin D. Y., Wei L.J., Ying Z. [1993] « Checking the Cox model with cumulative sums of martingale-based residuals », *Biometrika, n. 80*.

Lin D. Y., Ying Z. [1994] « Semiparametric analysis of the additive risk model », *Biometrika, n. 81*.

Lopez O. [2007] *Réduction de dimension en présence de données censurées*, Thèse de doctorat, Université de Rennes 1.

Martinussen T., Scheike T. H. [2006] « Dynamic Regression Models for Survival Data », *Springer*.

McCullagh P., Nelder J. A. [1989] « Generalized Linear Models », *Monographs on Statistics and Applied Probability 37, Chapman & Hall/CRC, 2nd edition*.

Planchet F., Kamega A., Ziguélé M. [2010], « Confection des tables de mortalité réglementaires pour la zone CIMA », *CIMA/WINTER & Associés (confidentiel).*

Planchet F., Leroy G. [2009] « Quel niveau de segmentation pertinent ? », *La Tribune de l'Assurance, n. 142*.

Planchet F., Thérond P. [2006] « Modèles de Durée - Applications actuarielles », *Economica*.

Therneau T. M., Grambsch P. M. [2000] « Modeling Survival Data – Extending the Cox Model », *Springer*.

Vaupel J. W. [2002] « Life Expectancy at Current Rates vs. Current Conditions: A Reflexion Stimulated by Bongaarts and Feeney's "How Long Do We Live?" », *Demographic Research, Vol. 7, Art. 8*.

Vaupel J. W., Manton K. G. Stallard E. [1979] « The impact of heterogeneity in individual frailty on the dynamics of mortality », *Demography, Vol. 16, No. 3*.

Viallon V. [2006] *Processus empiriques, estimation non paramétrique et données censurées*, Thèse de doctorat, Université Paris 6.