Groupe Consultatif Actuariel Europeen
Summer School 2002
Organized by Istituto Italiano degli Attuari

Massimo De Felice Franco Moriconi

A Course on **Finance of Insurance**

vol. 2

Milano, 10-12 July 2002
Università Cattolica – Milano

Mailing Address

Massimo De Felice
Dipartimento di Scienze Attuariali e Finanziarie
Università di Roma "La Sapienza"
Via Nomentana, 41 – 00161 Roma
e-mail: massimo.defelice@uniroma1.it

Franco Moriconi
Dipartimento di Economia
Facoltà di Economia
Università di Perugia
Via A. Pascoli, 1 – 06100 Perugia
e-mail: moriconi@unipg.it

Contents

Chapter 7 – Valuation of outstanding policy portfolios

Partecipating policies with constant annul premiums

Different benefits in case of death

Portfolio valuation

Controlling the balance of assets and liabilities

Participating policies with constant annual premiums

If the annual premiums are not readjusted (i.e. $A_k \equiv A_0$), the readjustment of benefits is different from the full readjustment rule:

$$C_k = C_{k-1}\,(1+\rho_k)\,.$$

Typically, the increment ΔC_k is determined as the benefit of an additional single premium endowment over the residual life of the principal policy. The additional policy is financed by the excess return on the investment of the *savings premium* A^s_k.

The intensity of the readjustment of benefits will depend on x, n, k. *Ceteris paribus*:

- for policies with equal values of x and n, the readjustment will be increasing w.r. to k;
- for policies with equal values of n and k, the readjustment will be decreasing w.r. to x.

- *An approximating rule* (independent of x):

$$C_k = C_{k-1}\,(1+\rho_k) - C_0\,\left(1-\frac{k}{n}\right)\,\rho_k\,.$$

Different benefits in case of death

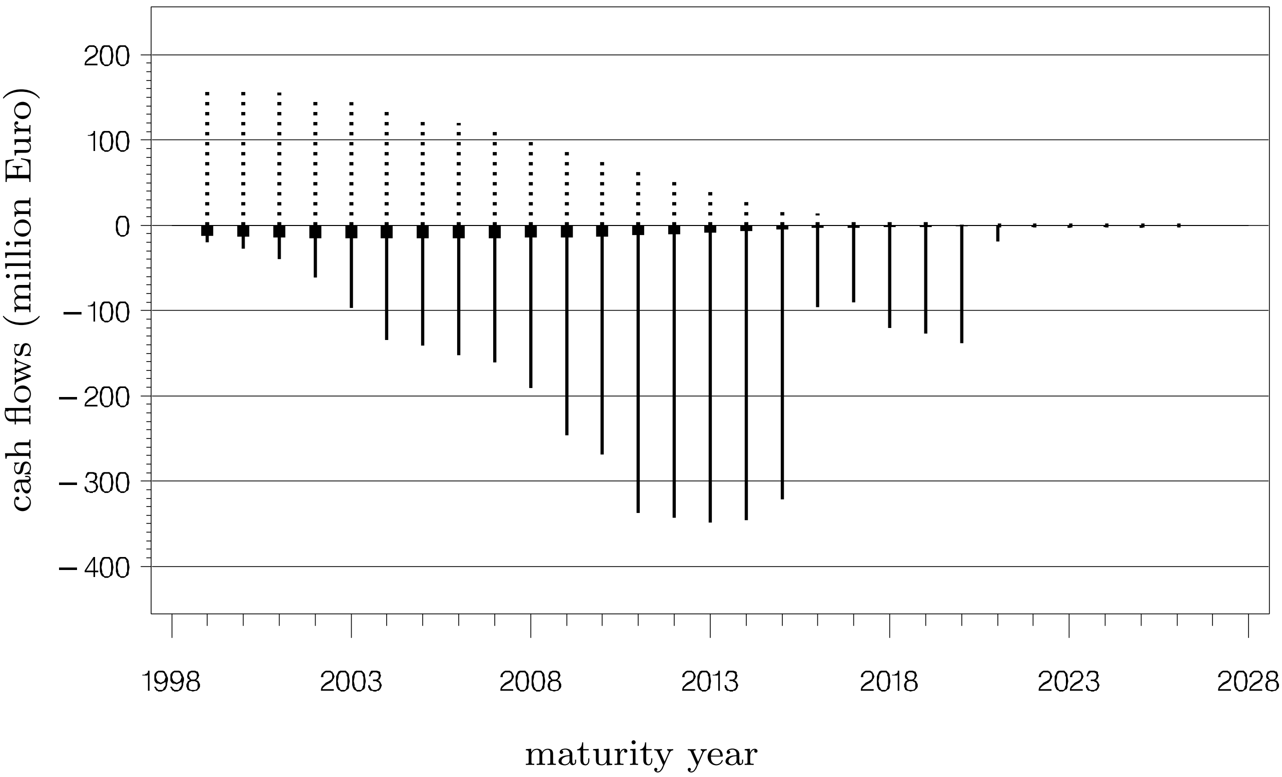
In many policies benefits payable in case of death (C_k^{D}) are different from benefits payable if the insured is alive (C_n^{L}).

$\Longrightarrow$

- computation of separated streams of technical means $\overline{C}_{t,k}^{\mathrm{D}}$, $\overline{C}_{t,n}^{\mathrm{L}}$;
- computation of separated valuation factors $u^{\mathrm{D}}(t,k)$, $u^{\mathrm{L}}(t,n)$.

Technical means of premiums and benefits

valutation date 31/12/1998



Portfolio valuation

Since the calculation of the valuation factors $u(t,k)$ involves Monte Carlo procedures, the valuation of a portfolio of outstanding policies can be highly time consuming if the contracts are not properly aggregated

• For single premium policies or for policies readjusting both premiums and benefits the valuation factors only depend on $k-t$:

$$u(t,k) = u(k-t)$$

$\Longrightarrow$ a single "structure" of valuation factors is needed for each class of policies.

• In the general case, for each class of policies a different stream of valuation factors is required for different values of x, n and $n-t$.

Reserves
(million Euro)

policy	traditional (a)	stochastic (b)	diff. (a-b)	% (a-b)/a
CAP 3 +1.5	1,062	1,054	9	0.83
CAP 3 +1	739	755	-16	-2.22
CAP 3 +1	3,114	3,174	-60	-1.94
CAP 4 +0	1,450	1,453	-2	-0.17
CAP 3 +0	629	492	137	21.76
CAP 2.5+0	69	36	33	47.34
SP 3 +1	28	29	-1	-3.05
SP 4 +0	69	71	-2	-2.94
SP 3 +0	136	137	-1	-1.05
SP 2.5+0	14	14	-0	-0.04
NP 3 +0	174	161	13	7.68
PORTFOLIO	**7,485**	**7,376**	**108**	**1.45**

Legend

CAP: Constant Annual Premiums (indexed benefits)
SP: Single Premium (indexed benefits)
NP: Non Participating (constant premiums and benefits)
3+1.5: technical rate 3%, minimum guaranteed 4.5%

Components of stochastic reserves
(million Euro)

policy	benefits (a)	premiums (b)	diff. (a-b)
CAP 3 +1.5	1,306	253	1,054
CAP 3 +1	1,076	321	755
CAP 3 +1	5,789	2,615	3,174
CAP 4 +0	3,981	2,528	1,453
CAP 3 +0	2,799	2,307	492
CAP 2.5+0	435	398	36
SP 3 +1	29	0	29
SP 4 +0	71	0	71
SP 3 +0	137	0	137
SP 2.5+0	14	0	14
NP 3 +0	166	5	161
PORTFOLIO	**15,802**	**8,426**	**7,376**

policy	benefits (a+b)	survival (a)	death (b)
CAP 3 +1.5	1,306	1,242	64
CAP 3 +1	1,076	1,005	71
CAP 3 +1	5,789	5,479	310
CAP 4 +0	3,981	3,723	257
CAP 3 +0	2,799	2,587	211
CAP 2.5+0	435	397	38
SP 3 +1	29	28	1
SP 4 +0	71	69	3
SP 3 +0	137	129	8
SP 2.5+0	14	13	1
NP 3 +0	166	160	5
PORTFOLIO	**15,802**	**14,833**	**969**

Basis Risk

Stochastic duration

policy	premiums	benefits	survival	death
CAP 3 +1.5	2.15	2.57	2.59	2.23
CAP 3 +1	2.69	2.91	2.94	2.45
CAP 3 +1	3.24	3.33	3.35	2.87
CAP 4 +0	3.55	3.91	3.95	3.33
CAP 3 +0	3.70	4.14	4.19	3.55
CAP 2.5+0	3.87	4.34	4.41	3.74
SP 3 +1	.	1.94	1.94	2.03
SP 4 +0	.	2.07	2.07	1.87
SP 3 +0	.	2.18	2.20	1.95
SP 2.5+0	.	2.23	2.25	1.95
NP 3 +0	1.51	2.94	2.93	3.14
PORTFOLIO	**3.42**	**3.49**	**3.52**	**3.07**

Delta

policy	premiums	benefits	survival	death
CAP 3 +1.5	0.00	0.00	0.00	0.00
CAP 3 +1	0.00	0.07	0.07	0.06
CAP 3 +1	0.00	0.05	0.05	0.05
CAP 4 +0	0.00	0.03	0.03	0.03
CAP 3 +0	0.00	0.02	0.02	0.02
CAP 2.5+0	0.00	0.02	0.02	0.01
SP 3 +1	.	0.10	0.10	0.11
SP 4 +0	.	0.11	0.11	0.11
SP 3 +0	.	0.13	0.13	0.13
SP 2.5+0	.	0.14	0.14	0.14
NP 3 +0	0.00	0.00	0.00	0.00
PORTFOLIO	**0.00**	**0.04**	**0.04**	**0.03**

Embedded options

(million Euro)

Put decomposition of benefits

policy	benefits (a+b)	base (a)	put (b)	% b/(a+b)
CAP 3 +1.5	1,264	1,190	73	5.80
CAP 3 +1	1,073	975	97	9.09
CAP 3 +1	5,776	5,152	624	10.81
CAP 4 +0	3,978	3,526	453	11.37
CAP 3 +0	2,798	2,558	240	8.58
CAP 2.5+0	435	402	33	7.62
SP 3 +1	29	26	2	7.89
SP 4 +0	71	67	5	6.67
SP 3 +0	137	129	8	5.86
SP 2.5+0	14	13	1	5.32
NP 3 +0	0	0	0	.
PORTFOLIO	**15,574**	**14,038**	**1,537**	**9.87**

Call decomposition of benefits

policy	benefits (a+b)	guaranteed (a)	call (b)	% b/(a+b)
CAP 3 +1.5	1,264	1,207	57	4.47
CAP 3 +1	1,073	941	132	12.30
CAP 3 +1	5,776	4,817	959	16.61

Value of Business In Force
(million Euro)

VBIF without minimum guarantees	(a)	2,841
Value of minimum guarantees	(b)	982
VBIF	**(a-b)**	**1,858**
Investment gain		108
Mortality gain		157
Surrender gain		269
Value of loadings		1,324

The value of minimum guarantees (b) is computed on third order basis.

Value of Business In Force by policy type
(million Euro)

	Inv. gain	Mort. gain	Sur. gain	Load.	**VBIF**	Value of m.g.	VBIF without min. guar.
CAP 3 +1.5	9	5	15	65	**94**	66	161
CAP 3 +1	-16	10	14	82	**89**	82	171
CAP 3 +1	-60	43	109	517	**608**	437	1,046
CAP 4 +0	-2	49	56	345	**447**	254	702
CAP 3 +0	137	44	55	267	**504**	111	615
CAP 2.5+0	33	8	6	46	**93**	14	107
SP 3 +1	-1	-0	1	0	**-0**	2	2
SP 4 +0	-2	-0	2	0	**-0**	5	4
SP 3 +0	-1	-0	8	0	**6**	8	13
SP 2.5+0	-0	-0	1	0	**1**	1	2
NP 3 +0	13	-0	2	1	**16**	2	18
PORTFOLIO	**108**	**157**	**269**	**1,324**	**1,858**	**982**	**2,841**

The value of minimum guarantees is computed on third order basis.

Controlling the balance of assets and liabilities

The corresponding asset portfolio is evaluated using the same pricing model used for the policy portfolio

Same pricing model, same valuation date, same calibration
$\longrightarrow$ the values A_t and V_t (and their sensitivities) can be coherently compared.

ALM analysis

Policy portfolio
(million Euro)

	Price	Duration	Delta
Premiums	8,426	3.42	0.00
Benefits	15,802	3.49	0.04
Reserve/Gap	**7,376**	**0.07**	**0.04**

Investment portfolio
(million Euro)

	Price	%	Duration	Delta
Bond	6,176	71.7	1.41	0.00
Stock	2,441	28.3	0.00	0.28
Total	**8,617**	**100.0**	**1.41**	**0.28**

VaR of the investment portfolio
(99%, 10 days)

	Price	Amm	VaR	%
Bond	6,176	95.06 bp	61	0.99
Stock	2,441	-8.30 %	203	8.30
Total	**8,617**	**.**	**264**	**3.06**

ALM analysis

Asset-liability portfolio

		Price	Duration	Delta
Investments	(a)	8,617	1.41	0.28
Premiums	(b)	8,426	3.42	0.00
Asset	(a+b)	17,043	2.41	0.14
Liabilities	(c)	15,802	3.49	0.04
A/L Portfolio	(a+b-c)	**1,241**	**-1.07**	**0.11**

VaR of the asset-liability portfolio

(99%, 10 days)

	Price	Interest VaR (pb)	%	Stock VaR (%)	%
Investments	8,617	-50	-0.6	203	2.4
Premiums	8,426	-121	-1.4	0	0.0
Asset	17,043	-170	-1.0	203	1.2
Liabilities	15,802	227	1.4	-49	-0.3
A/L Portfolio	**1,241**	**56**	**4.5**	**153**	**12.3**

The interest rate VaR of the A/L portfolio corresponds to Amm=-76.35 bp.
For an interest rate movement of +95.06 bp the VaR is negative.

Netting the VaR of the investments

	Amm (Inv.)	VaR (Inv.)	%	Amm (A/L)	VaR (A/L)	%
Bond	95.06 pb	61	0.99	-76.35 pb	56	0.91
Stock	-8.30 %	203	8.30	-8.30 %	153	6.27
Total		**264**	**3.06**		**209**	**2.43**

Chapter 8 – Alternative valuation methods

VBIF: the annual profits approach

Equivalence with the stochastic reserve approach

Actuarial expectation of future annual profits

- Mortality gain
- Investment gain
- Valuation of the annual profits

Alternative valuation methods

- Risk-neutral probabilities
- Risk-adjusted discounting
- RAD under scenario

VBIF: the annual profits approach

• The standard approach for determining VBIF is based on an investment argument.

We refer for simplicity to a single premium endowment.

During the life of the policy the company must maintain a capital at the level of the reserve process:

$$\widetilde{R}_k := \mathbf{1}_{\{T_x > k-1\}}\, R_k \,,$$

However, at time $k-1$ the reserve $\widetilde{R}_{k-1}$ can be invested in the reference fund, providing an annual rate of return I_k.
The amount $\widetilde{R}_{k-1}\,(1+I_k)$ realized at time k, net of the new reserve level $\widetilde{R}_k$ and of the liability $\widetilde{Y}_k$ represents the *technical gain* in year k.
→ The VBIF at time 0 can be obtained as the present value of the sequence of the annual gains.

• At time 0 the annual profits emerging from the policy can be represented by the cash flow stream:

$$\widetilde{\mathbf{G}} = \{\widetilde{G}_k \,, \quad k = 1, 2, \ldots, n\} \,,$$

where:

$$\widetilde{G}_k = \widetilde{R}_{k-1}\,(1+I_k) - \widetilde{R}_k - \widetilde{Y}_k \,.$$

Then the VBIF at time 0 is given by:

$$E_0 = V(0; \widetilde{\mathbf{G}}) = \sum_{k=1}^{n} V\big(0; \widetilde{G}_k\big) \,.$$

Equivalence with the stochastic reserve approach

Under the no arbitrage assumption in perfect market the annual profits approach is equivalent to the stochastic reserve approach. The previous expression can be explicitely written as:

$$E_0 = \sum_{k=1}^{n} V\big(0;\, \widetilde{R}_{k-1}\,(1+I_k)\big) - \sum_{k=1}^{n-1} V\big(0;\, \widetilde{R}_k\big) - \sum_{k=1}^{n} V\big(0;\, \widetilde{Y}_k\big)\,. \quad (^*)$$

By the "reinvestment security theorem":

$$V\big(t;\, \widetilde{R}_{k-1}\,(1+I_k)\big) = V\big(t;\, \widetilde{R}_{k-1}\big)\,.$$

Thus the first sum in (*) can be expressed as:

$$\sum_{k=1}^{n} V\big(0;\, \widetilde{R}_{k-1}\,(1+I_k)\big) = \sum_{k=1}^{n} V\big(0;\, \widetilde{R}_{k-1}\big) = R_0 + \sum_{k=1}^{n-1} V\big(0;\, \widetilde{R}_k\big)\,;$$

thus expression (*) reduces to:

$$E_0 = R_0 - \sum_{k=1}^{n} V_0 = R_0 - V_0\,.$$

■

Actuarial expectation of future annual profits

Taking the expectation of the actuarial random variables, the expected future gains $\widetilde{\mathbf{G}}$ are defined by:

$$\widehat{G}_k = \begin{cases} {}_{k-1}p_x \left[R_{k-1}(1+I_k) - (1-q_{x+k-1})R_k - q_{x+k-1}C_k\right], & k < n\,, \\ {}_{n-1}p_x \left[R_{n-1}(1+I_n) - C_n\right], & k = n\,. \end{cases}$$

Subtracting the quantity:

$$R_{k-1}\,(1+\rho_k)(1+i) - (1-q'_{x+k-1})\,R_k - q'_{x+k-1}\,C_k\,,$$

which is equal to zero by the equilibrium constraint, we get the "Homans formula":

$$\widehat{G}_k = \begin{cases} {}_{k-1}p_x \Big[\,R_{k-1}\,(I_k - m_k) \\ \qquad\qquad + (C_k - R_k)\,(q'_{x+k-1} - q_{x+k-1})\Big], & k < n\,, \\ {}_{n-1}p_x\,R_{n-1}\,(I_n - m_n)\,, & k = n\,, \end{cases}$$

where $m_k := (1+\rho_k)(1+i) - 1$, that is:

$$m_k = \max\{\beta\,I_k,\, i\}\,.$$

- Under first order basis, i.e. if:

$$I_k \equiv i \quad \text{and} \quad q_{x+k-1} \equiv q'_{x+k-1}\,,$$

then all the expected profits are zero:

$$\widehat{G}_k = 0\,, \quad k = 1, 2, \ldots, n\,.$$

• *Mortality gain*

If $I_k \equiv i$, then the annual gains are:

$$\widehat{G}_k^D = \begin{cases} {}_{k-1}p_x\,(C_k - R_k)\,(q'_{x+k-1} - q_{x+k-1})\,, & k < n\,, \\ 0\,, & k = n\,, \end{cases}$$

which can be referred to as *mortality gains*.

Typically, the $\mathbf{P}'$ measure is "conservative" with respect to the $\mathbf{P}$ measure; that is, for any k: $q_{x+k-1} \le q'_{x+k-1}$. Therefore the mortality gain $\widehat{G}_k^D$ is not negative.

• *Investment gain*

If $q_{x+k-1} \equiv q'_{x+k-1}$, then $\widehat{G}_k$ can be interpreted as the actuarial expectation of the *investment gain* in year k; it is given by:

$$\widehat{G}_k^I = {}_{k-1}p'_x\,R_{k-1}\,(I_k - m_k)\,. \quad k = 1, 2, \ldots, n\,.$$

⊙ Using the language of the technical means, that is defining:

$$\overline{K}_{k-1} = R^*_{k-1}\ {}_{k-1}p'_x\ (1+i)^{-(k-1)}\,,$$

(where R^*_{k-1} is the technical reserve at time k–1 of the corresponding non participating policy) the investment gain can be rewritten as:

$$\widehat{G}_k^I = \overline{K}_{k-1} \prod_{j=1}^{k-1} (1+m_j)\ (I_k - m_k)\,.$$

Since $m_j = \max\{\beta\, I_j,\, i\}$, this equation makes apparent the dependence of $\widehat{G}_k^I$ on all the sample path $\{I_1, I_2, \ldots, I_k\}$ of the fund returns $\longrightarrow$ the minimum return guarantee is an *annual guarantee*.

• To better characterize the minimum guarantee embedded in the policy, let us consider the investment gain of an analogous policy without minimum guarantee; this is the *base payoff*, defined as:

$$\widehat{B}_k = \overline{K}_{k-1} \prod_{j=1}^{k-1} (1 + \beta\, I_j)\ (1 - \beta)\, I_k\,.$$

Of course: $\widehat{B}_k \geq \widehat{G}^I_k$.

The *guarantee payoff*, or the *put payoff*, is the difference:

$$\widehat{P}_k = \widehat{B}_k - \widehat{G}^I_k \geq 0\,.$$

• For $k = 1$ we have:

$$\widehat{G}^I_1 = R^*_0\,(I_1 - m_1) = R^*_0 \Big[(1 - \beta)\, I_1 - \max\{i - \beta\, I_1,\, 0\}\Big]\,.$$

That is

$$\widehat{G}^I_1 = \widehat{B}_1 - \widehat{P}_1\,,$$

where:

$$\widehat{B}_1 = R^*_0\,(1 - \beta)\, I_1\,,$$

$$\widehat{P}_1 = R^*_0\, \max\{i - \beta\, I_1,\, 0\}\,.$$

• *Valuation of the annual profits*

The VBIF at time 0 can be obtained as the value of the future annual profits $\widetilde{\mathbf{G}}$; under our assumptions:

$$V(0; \widetilde{G}_k) = V(0; \widehat{G}_k) \,;$$

hence:

$$V(0; \widetilde{\mathbf{G}}) = \sum_{k=1}^{n} V(0; \widehat{G}_k) \,.$$

We are mainly interested in the investment component of the annual profits; we have:

$$V(0; \widetilde{G}_k^I) = V(0; \widehat{G}_k^I) \,,$$

which can be written as:

$$V(0; \widetilde{G}_k^I) = V(0; \widehat{B}_k) - V(0; \widehat{P}_k) \,.$$

Under the fair valuation approach, the sum of this values over the life of the policy must be equal to the investment component given by the stochastic reserve approach.

Defining:

$$\Psi_k := (I_k - m_k) \prod_{j=1}^{k-1} (1 + m_j) \,,$$

the value of the investment gain can be expressed as:

$$\widehat{G}_k^I = \overline{K}_{k-1} \, \Psi_k \,,$$

where:

- the technical mean $\overline{K}_{k-1}$ is determined by actuarial assumptions on the probability measure $\mathbf{P}^{(1)}$;
- the factors Ψ_k are determined by capital market uncertainty.

Alternative valuation methods

- *Risk-neutral probabilities* (RNP)

The risk-neutral probability (RNP) approach is natural when the valuation problem is set up in the framework of contingent claims pricing. Under the arbitrage principle in a perfect market:

$$V\big(0;\ \Psi_k\big) = \mathbf{E}_0^Q\left[\Psi_k\ \chi(0,k)\right]\ , \tag{RNP}$$

where:

$\mathbf{E}_0^Q$ is the expectation operator taken with respect to the risk-neutral probability $\mathbf{Q}$, conditional on the information at time 0;

$\chi(0,k)$ is a stochastic discount factor on the time interval $[0,k]$.

The discount factor $\chi(0,k)$ and the risk-neutral probability $\mathbf{Q}$ must be specified under an appropriate stochastic model.

Once the sources of market uncertainty are specified in the model, χ and $\mathbf{Q}$ are the same for all the securities which depend on these risk factors
$\Longrightarrow$ if the model is calibrated in order to match the observed price of traded securities, it can be applied to non-traded securities, providing coherent pricing.

Remark. The valuation of the options embedded in life insurance policies with the RNP method can be considered a problem in *Real Option Analysis.*
[Copeland, Antikarov, 2001] ■

The case of a deterministic interest rate is not realistic in life insurance applications; however it is often considered in order to simplify the exposition.

If the force of interest r (the spot rate) is constant over time, expression (RNP) reduces to:

$$V\big(0;\,\Psi_k\big) = e^{-r\,k}\,\mathbf{E}_0^Q\big[\,\Psi_k\,\big]\,.$$

In the celebrated Black and Scholes model Ψ_k can be expressed as a function of an underlying price process $\{S_t\}$, which is specified as a geometric brownian motion.

If $\{S_t\}$ has drift parameter μ and volatility parameter σ, the arbitrage argument demands that $\mathbf{Q}$ is lognormal with parameters r and σ, instead of μ and σ.

The istantaneous expected return μ of the underlying does not enter in the determination of price, since the model prescribes that taking the average under the modified (r,σ)-distribution provides the appropriate adjustment for the risk aversion.

• *Risk-adjusted discounting* (RAD)

The standard approach to calculating VBIF consists in taking the natural expectation of the random payoff Ψ_k and then discounting it at an appropriate risk-adjusted force of interest r_a; that is:

$$V\big(0;\, \Psi_k\big) = e^{-r_a\, k}\, \mathbf{E}_0\big[\, \Psi_k \,\big]\,. \tag{RAD}$$

The RAD method is widely used in capital budgeting applications, where it is also referred to as the *Net Present Value* method.

The risk premium $r_a - r$ is usually determined by the observation of past returns on assets of similar insurance firms, using popular models as the Capital Asset Pricing Model or the Dividend Discount Model.

Pros:

· RAD method is easier to communicate to practitioners

· is the most intuitive in a single-period setting

Cons:

· RAD method becomes very complicated when the problem is inherently intertemporal

⊙ high degree of subjectivity is involved in the practical assessment of both the expected payoff and the risk-adjusted rate; this problem is even more important when option-like payoffs are considered

$\rightarrow$ it can be argued that just this difficulty gave an impetus to the development of the option pricing theory and of the RNP method.

⊙ *RAD under scenario*

Scenario methods are typically used in practical applications of the RAD approach, in order to derive an estimate of the natural expectation $\mathbf{E}_0\left[\,\Psi_k\,\right]$.

Since the technical mean Ψ_k is a function of the realized return I_k of the reference fund, a "best estimate" I_k^* of this random variable is taken and the "expectation" of Ψ_k is derived correspondingly.

For illustration purposes, let us assume:

$$I_k^* = \mathbf{E}_0\left[\,I_k\,\right].$$

A problem obviously arises since if Ψ_k is a non linear function of I_k the property:

$$\mathbf{E}_0\left[\,\Psi_k(I_k)\,\right] = \Psi_k\left(I_k^*\right)$$

in general does not hold.

The embedded options are far-out-of-the-money at the policy issuance and in typical market conditions they remain out-of-the-money during the life of the policy; i.e. normally the assumed scenario is such that, for each future year k:

$$I_k^* > i/\beta\,,$$

which corresponds to:

$$m_k = \beta\, I_k^*\,;$$

hence:

$$\overline{K}_{k-1}\,\mathbf{E}_0\left[\,\Psi_k\,\right] = \overline{K}_{k-1}\prod_{j=1}^{k-1}\left(1+\beta\, I_j^*\right)\,(1-\beta)\, I_k^* = \mathbf{E}_0\left[\,\widehat{B}_k\,\right].$$

$\longrightarrow$ the embedded options are not captured under the scenario method.

VBIF as of DEC 31, 2001 - RAD method

(Traditional policies)

Financial assumptions

Return of the segregated fund:	5.00% p.a.
Risk adjusted discount rate:	7.50% p.a.

Actuarial assumptions

2nd order mortality tables:	SIM92 "30% discounted"
Redemption tables:	"A.G. 85-87"
Renewal rates for recurring:	...

VBIF – RAD method
(cost of solvency capital not included)

(thousand of Euro)

policy type	**technical rate**	**technical reserve**	**investment gain** (a)	**mortality gain** (b)	**surrender gain** (c)	**total gain** (d=a+b+c)	**Zillmer value** (e)	**total+ Zillmer** (f=d+e)	**overall loadings** (g)	**gross VBIF** (h=d+g)
CAP_1	4%	1,620,000	100,911	-10,467	16,945	107,389	57,020	164,408	72,453	179,843
CAP_2	2.5-3%	93,948	19,757	-25	-595	19,138	6,368	25,506	20,274	39,412
CAP_2	4%	502,756	45,060	4,432	4,254	53,746	15,373	69,118	49,565	103,310
IAP_1	4%	939,297	59,994	-3,597	2,727	59,124	43,190	102,314	48,200	107,324
IAP_2	2,5-3%	35,427	8,908	-21	-316	8,571	4,183	12,754	9,644	18,215
IAP_2	4%	148,743	17,783	1,820	2,343	21,948	5,706	27,654	18,408	40,355
SP	2,5-3%	97,315	3,531	0	-307	3,224	0	3,224	0	3,224
SP	4%	44,319	1,599	0	-120	1,480	0	1,480	0	1,480
RSP	2,5-3%	322,632	70,196	839	-15,153	55,881	0	55,881	32,335	88,216
RSP	4%	79,811	6,808	0	-891	5,917	0	5,917	1,778	7,694
TI	4%	6,405	438	4,730	69	5,237	595	5,832	1,651	6,888
group	2,5-3%	284,550	12,691	-443	-5,050	7,198	0	7,198	0	7,198
group	4%	824,794	36,977	-6,383	-784	29,810	652	30,462	2,880	32,690
portfolio		**5,000,000**	**384,656**	**-9,115**	**3,122**	**378,663**	**133,087**	**511,749**	**257,189**	**635,852**

legend

- CAP: Constant Annual Premiums (indexed benefits)
- IAP: Indexed Annual Premiums (indexed premiums and benefits)
- SP: Single Premium (indexed benefits)
- RSP: Recurring Single Premium (indexed benefits)
- TI: Term Insurance

- Zillmer value (e) is the value of future loadings for acquisition costs
- gross VBIF (h) is the VBIF gross of expenses
- total+Zillmer (f) is the VBIF net of expenses (assuming collection and administration costs are paid by the corresponding loadings)

RAD vs RNP method: investment component of VBIF

(first order valuation)

(thousand of Euro)

			RAD		RNP	
policy type	**technical rate**	**technical reserve**	**total inv. gain**	**total inv. gain** (a)	**base value** (b)	**put value** (b-a)
CAP_1	4%	1,620,000	100,911	47,516	103,416	55,900
CAP_2	2.5-3%	93,948	19,757	21,289	26,914	5,626
CAP_2	4%	502,756	45,060	18,629	54,828	36,199
IAP_1	4%	939,297	59,994	7,955	61,629	53,674
IAP_2	2,5-3%	35,427	8,908	9,464	12,540	3,076
IAP_2	4%	148,743	17,783	7,332	21,347	14,016
SP	2,5-3%	97,315	3,531	1,518	3,597	2,079
SP	4%	44,319	1,599	-363	1,350	1,713
RSP	2,5-3%	322,632	70,196	82,257	103,913	21,657
RSP	4%	79,811	6,808	941	6,871	5,930
TI	4%	6,405	438	422	422	0
group	2,5-3%	284,550	12,691	8,081	16,820	8,739
group	4%	824,794	36,977	6,985	82,261	75,275
portfolio		**5,000,000**	**384,656**	**212,025**	**495,907**	**283,882**

legend

- CAP: Constant Annual Premiums (indexed benefits)
- IAP: Indexed Annual Premiums (indexed premiums and benefits)
- SP: Single Premium (indexed benefits)
- RSP: Recurring Single Premium (indexed benefits)
- TI: Term Insurance

RAD vs RNP method: total VBIF

(thousand of Euro)

policy type	technical rate	technical reserve	RAD total VBIF	total VBIF (a)	RNP base value (b)	put value (b-a)
CAP_1	4%	1,620,000	164,408	117,536	170,873	53,337
CAP_2	2.5-3%	93,948	25,506	27,256	32,018	4,763
CAP_2	4%	502,756	69,118	47,722	80,231	32,509
IAP_1	4%	939,297	102,314	56,080	106,472	50,392
IAP_2	2,5-3%	35,427	12,754	13,877	16,224	2,347
IAP_2	4%	148,743	27,654	19,692	31,858	12,164
SP	2,5-3%	97,315	3,224	1,257	3,240	1,982
SP	4%	44,319	1,480	-420	1,222	1,640
RSP	2,5-3%	322,632	55,881	62,950	80,079	17,128
RSP	4%	79,811	5,917	644	5,868	5,223
TI	4%	6,405	5,832	6,358	6,358	0
group	2,5-3%	284,550	7,198	1,957	9,643	7,685
group	4%	824,794	30,462	-378	65,194	65,572
portfolio		**5,000,000**	**511,749**	**354,533**	**609,278**	**254,746**

legend

- CAP: Constant Annual Premiums (indexed benefits)
- IAP: Indexed Annual Premiums (indexed premiums and benefits)
- SP: Single Premium (indexed benefits)
- RSP: Recurring Single Premium (indexed benefits)
- TI: Term Insurance

Chapter 9 – Unit-linked and index-linked policies

Unit-linked endowment policy

Similarities with participating policies

The standard valuation framework

Reserve and sum insured

Unit-linked policies with minimum guarantee

Put decomposition

Stochastic reserve and VBIF

Profits from management fees

Surrenders

Index-linked endowment policy

Similarities with participating (and u-l) policies

The standard valuation framework

Financial risk

Stochastic reserve and VBIF

Unit-linked endowment policy

• Given an investment fund, let F_t be the market value at time t of one unit of the fund.

A unit-linked endowment policy with term n years for a life aged x provides for payment of

- a number N^{D} of units at the end of the year of death if this occurs within the first n years (**term insurance**),

otherwise

- a number N^{L} of units at the end of the nth year (**pure endowment**).

If the policy is **single premium**, the insured pays a lump sum U at time 0.

• The insured benefit in case of death at time k is:

$$C_k^{\mathrm{D}} = N^{\mathrm{D}} F_k \,;$$

if the insured is alive at time n the benefit is:

$$C_n^{\mathrm{L}} = N^{\mathrm{L}} F_n$$

$\longrightarrow$ the insured sums are contractually defined in **stochastic units**.

• Typically the management of the reference fund is under the insurer control.

Similarities with participating policies

Let:

- C_k : benefit (eventually) paid at time k;
- F_t : market value of the reference fund;
- $I_k := F_k/F_{k-1} - 1$: annual rate of return of the fund at time k.

The benefits at time k are given by:

$$C_0 = N F_0 \,,$$
$$C_k = C_{k-1}\,(1 + I_k)\,, \quad k = 1, 2, \ldots, n\,.$$

For $0 \le h \le k \le n$, we can define the *readjustment factors*:

$$\Phi(h,k) := \prod_{j=h+1}^{k} (1 + I_j) = \frac{F_k}{F_h}$$

(being $\Phi(k,k) = 1$).

Hence:

$$C_k = C_0\,\Phi(0,k)\,.$$

The standard valuation framework

- At time 0 we have the liability stream:

$$\widetilde{\mathbf{C}} = \left\{\widetilde{C}_k\,;\ k = 1, 2, \ldots, n\right\};$$

where:

$$\widetilde{C}_k = \begin{cases} C_k\,, & \text{with prob.} \quad \mathbf{P}_0(C_k; k) \\ 0\,, & \text{with prob.} \quad 1 - \mathbf{P}_0(C_k; k) \end{cases}$$

- The net single premium is given by:

$$U = C_0^{\mathrm{D}} \sum_{k=1}^{n} \mathbf{P}_0^{(1)}(C_0^{\mathrm{D}}; k) \;+\; C_0^{\mathrm{L}}\, \mathbf{P}_0^{(1)}(C_0^{\mathrm{L}}; n)\,,$$

or:

$$U = F_0\, \overline{N}_0\,,$$

where:

$$\overline{N}_0 := N^{\mathrm{D}} \sum_{k=1}^{n} \mathbf{P}_0^{(1)}(C_0^{\mathrm{D}}; k) \;+\; N^{\mathrm{L}}\, \mathbf{P}_0^{(1)}(C_0^{\mathrm{L}}; n)\,.$$

$\longrightarrow$ first order basis: probability $\mathbf{P}^{(1)}$ and technical rate $i = 0$.

- Similarly, the net premium reserve at time $k = 0, 1, \ldots, n$ is defined by:

$$R_k = F_k\, \overline{N}_k\,,$$

where:

$$\overline{N}_k =: N^{\mathrm{D}} \sum_{j=k+1}^{n} \mathbf{P}_k^{(1)}(C_j^{\mathrm{D}}; j) \;+\; N^{\mathrm{L}}\, \mathbf{P}_k^{(1)}(C_n^{\mathrm{L}}; n)\,.$$

$\Longrightarrow$

- The policy can be "hedged" by the insurer by purchasing $\overline{N}_0$ units at time 0;

· the hedging strategy is a replicating strategy, because the portfolio purchased at time 0 replicates (on the average) the future liabilities;

· the hedging strategy is a static strategy;

· the hedging strategy is not completely riskless, because of mortality uncertainty;

· If $N^{\mathrm{D}} = N^{\mathrm{L}} = N$ the hedging strategy is a riskless strategy. Both financial and actuarial risk are eliminated from the policy.

Reserve and sum insured

The reserve at time k can be expressed as:

$$
\begin{aligned}
R_k &= C_k^{\mathrm{D}} \sum_{j=k+1}^{n} \mathbf{P}_k(C_j^{\mathrm{D}}; j) \,+\, C_n^{\mathrm{L}}\, \mathbf{P}_k(C_n^{\mathrm{L}}; n) \\
&= (C_k^{\mathrm{D}} - C_k^{\mathrm{L}}) \sum_{j=k+1}^{n} \mathbf{P}_k(C_j^{\mathrm{D}}; j) \\
&\quad + C_k^{\mathrm{L}} \sum_{j=k+1}^{n} \mathbf{P}_k(C_j^{\mathrm{D}}; j) \,+\, C_k^{\mathrm{L}}\, \mathbf{P}_k(C_n^{\mathrm{L}}; n)\,.
\end{aligned}
$$

Since:

$$
\sum_{j=k+1}^{n} \mathbf{P}_k(C_j^{\mathrm{D}}; j) + \mathbf{P}_k(C_n^{\mathrm{L}}; n) = \sum_{j=k+1}^{n} {}_{j-1/1}q_x \,+\, {}_n p_x = 1\,,
$$

one has:

$$
R_k = C_k^{\mathrm{L}} + (C_k^{\mathrm{D}} - C_k^{\mathrm{L}}) \sum_{j=k+1}^{n} \mathbf{P}_k(C_j^{\mathrm{D}}; j)\,.
$$

• If $N^{\mathrm{D}} = N^{\mathrm{L}} = N$, then: $C_k^{\mathrm{D}} = C_k^{\mathrm{L}} = C_k = N\,F_k\,,\ \forall\, k$; hence:

$$
R_k = C_k
$$

$\longrightarrow$ the reserve at time k is equal to the current value C_k of the $N = U/F_0$ units purchased at time 0
$\longrightarrow$ the contract is not exposed to mortality risk.

• If $N^{\mathrm{D}} > N^{\mathrm{L}}$ the reserve is greater than C_k^{L}.

Unit-linked policies with minimum guaratee

Assume that the insured benefits C_k cannot be lower than a floor value NM_k fixed at time 0:

$$C_k = \max\{NF_k\,,\ NM_k\}\,.$$

e.g.:

$$M_k := F_0\,(1+g)^k\,,$$

with g: a minimum guaranteed annual return

$\longrightarrow$ *maturity guarantee.*

Remark. Tipically the reference fund F has a substantial equity component. Thus also negative values of g can be of interest. ■

The insured sum can also be expressed as:

$$C_n = NF_0\ \max\left\{\frac{F_n}{F_0}\,,\ (1+g)^n\right\}\,,$$

hence:

$$C_n = C_0\ \Phi(0,n)\,,$$

where:

$$C_0 = NF_0\,,$$

and:

$$\Phi(0,n) = \max\left\{\frac{F_n}{F_0}\,,\ (1+g)^n\right\}\,.$$

Put decomposition

Since C_n can be written as:

$$C_n = N\,F_n + N\,\max\{M_n - F_n\,,\,0\}\,,$$

the policy is equivalent to a u-l policy with sum insured $N\,F_n$ and without minimum guarantee, plus a contract providing at time n the payoff:

$$P_n := N\,\max\{M_n - F_n\,,\,0\}\,.$$

This is the payoff of a portfolio of N *european put options* on the price of the unit, with exercise date n and strike price M_n.

$\Longrightarrow$ In principle, the insurer can eliminate financial risk by purchasing the put options.

Remark. The expression:

$$C_n = C_0\,\max\left\{\frac{F_n}{F_0}\,,\,(1+g)^n\right\}\,,$$

can be written as:

$$C_n = C_0\,\max\left\{\prod_{k=1}^{n}\frac{F_k}{F_{k-1}}\,,\,\prod_{k=1}^{n}(1+g)\right\}\,.$$

The payoff of a policy with *annual guarantees* can be expressed instead as:

$$C_n = C_0\,\prod_{k=1}^{n}\max\left\{\frac{F_k}{F_{k-1}}\,,\,(1+g)\right\}\,.$$

Of course in a multiple period contract there is a significant difference between the two guarantees. ■

Stochastic reserve and VBIF

• The stochastic reserve at time t (for a single premium policy) is given by:

$$V_t = V(t; \widetilde{\mathbf{C}}) .$$

• Correspondingly, the VBIF at time t is defined by:

$$E_t = R_t - V_t .$$

⊙ Under our assumption, we have (for a policy with $C_k^{\mathrm{D}} = C_k^{\mathrm{L}} = C_k$):

$$V_t = \sum_{k=t+1}^{n} \mathbf{P}_t(C_k; k)\, V(t; C_k) .$$

· If the policy does not provide minimum guarantees, i.e. $C_k = NF_k$:

$$V(t; C_k) = NF_t ,$$

since, by the no-arbitrage principle:

$$V(t; F_k) = F_t .$$

Thus, given that $\sum_{k=t+1}^{n} \mathbf{P}_t(C_k; k) = 1$, one has:

$$V_t = NF_t = C_t = R_t ,$$

and:

$$E_t = 0 .$$

Profits from management fees

Assume that at each year end the fund pays to the insurer management fees determined as a fraction f of the current NAV; at time k the value F_k^* of the fund is now:

$$F_k^* = F_k\,(1-f)^k\,,$$

where F_k is the value of an analogous fund without management fees.

The sum insured is now $C_k = NF_k^*$; hence:

$$V(0;C_k) = N\,(1-f)^k\,V(0;F_k) = NF_0\,(1-f)^k = R_0\,(1-f)^k\,.$$

Therefore the stochastic reserve at time 0 is:

$$\begin{aligned} V_0 &= \sum_{k=1}^{n} \mathbf{P}_0(C_k;k)\,V(0;C_k) \\ &= R_0 \sum_{k=1}^{n} \mathbf{P}_0(C_k;k)\,(1-f)^k < R_0\,. \end{aligned}$$

The VBIF is given by:

$$E_0 = R_0 - V_0 = R_0\left[1-\sum_{k=1}^{n} \mathbf{P}_0(C_k;k)\,(1-f)^k\right].$$

Remark. For a policy without embedded options the VBIF is independent of the fund investment strategy. ■

Remark. If the policy provides minimum return guarantees the value of the embedded put option is subtracted from the VBIF.
The put price is generally depending on the investment strategy. ■

Surrenders

• When applied to unit-linked policies Assumption 1 can result to be critical.

• In a policy without embedded options the redemption at time k causes a loss for the insurer equal to the current value E_k of the residual VBIF
$\longrightarrow$ the value E_k provides a benchmark for defining appropriate penalties (contractually specified at time 0 as a fraction of the NAV F_k^* at time k).

• To avoid serious hedging problems, minimum guarantees should not be provided in case of surrender.

Index-linked endowment policy

• Let us refer to a capital market index F_t.

Let $\Phi^{\mathrm{L}}(0,k)$ and $\Phi^{\mathrm{D}}(0,k)$ be fixed functions of F_j, $j=1,2,\dots,k$; that is:

$$\Phi^{\mathrm{L}}(0,k) = \Phi^{\mathrm{L}}(F_1, F_2, \dots, F_k)\,,$$

$$\Phi^{\mathrm{D}}(0,k) = \Phi^{\mathrm{D}}(F_1, F_2, \dots, F_k)\,.$$

An i-l endowment with term n years for a life with age x provides for payment of

· the benefit $C_0^{\mathrm{D}}\,\Phi^{\mathrm{D}}(0,k)$ at the end of the year of death if this occurs within the first n years (**term insurance**),

otherwise

· the benefit $C_0^{\mathrm{L}}\,\Phi^{\mathrm{L}}(0,n)$ at the end of year n (**pure endowment**),

where the initial benefits C_0^{D} and C_0^{L} are fixed at time 0.

Single premium: the insured pays a lump sum U at time 0.

• Some elementary examples:

$$\Phi^{\mathrm{L}}(0,k) = \Phi^{\mathrm{D}}(0,k) = \frac{F_k}{F_0}\,,$$

$$\Phi^{\mathrm{L}}(0,k) = \Phi^{\mathrm{D}}(0,k) = \frac{\left(\sum_{j=1}^{k} F_j\right)/k}{F_0}\,,$$

$$\Phi^{\mathrm{L}}(0,k) = \Phi^{\mathrm{D}}(0,k) = \max\left\{\frac{F_k}{F_0},\ (1+g)^k\right\}\,.$$

Similarities with participating (and u-l) policies

Let:

- C_k : benefit (eventually) paid at time k;
- F_t : market value of the reference index;
- $I_k := F_k/F_{k-1} - 1$: annual rate of return of the index at time k.

Given the initial sum insured C_0, the benefits at time k are given by:

$$C_k = C_0 \, \Phi(0,k) \,, \quad k = 1, 2, \ldots, n \,.$$

where the function:

$$\Phi(0,k) = \Phi(F_1, F_2, \ldots, F_k) \,,$$

is contractually fixed at time 0.

Remark. It is relevant to observe that in the i-l policies the reference index is observed on the market and cannot be influenced by the insurer. ■

The standard valuation framework

- At time 0 we have the liability stream:

$$\widetilde{\mathbf{C}} = \left\{ \widetilde{C}_k \, ; \; k = 1, 2, \ldots, n \right\} ;$$

where:

$$\widetilde{C}_k = \begin{cases} C_k \, , & \text{with prob.} \quad \mathbf{P}_0(C_k; k) \\ 0 \, , & \text{with prob.} \quad 1 - \mathbf{P}_0(C_k; k) \end{cases}$$

- The net single premium is given by:

$$U = C_0^{\mathrm{D}} \sum_{k=1}^{n} (1+i)^{-k} \, \mathbf{P}_0^{(1)}(C_0^{\mathrm{D}}; k) \, + \, C_0^{\mathrm{L}} \, (1+i)^{-n} \, \mathbf{P}_0^{(1)}(C_0^{\mathrm{L}}; n) \, ,$$

$\longrightarrow$ first order basis: probability $\mathbf{P}^{(1)}$ and technical rate i.

If the policy is fully indexed the technical interest rate is set equal to 0.

- The net premium reserve at time $k = 0, 1, \ldots, n$ is defined by:

$$R_k =: \; C_k^{\mathrm{D}} \sum_{j=k+1}^{n} (1+i)^{-(j-k)} \mathbf{P}_k^{(1)}(C_j^{\mathrm{D}}; j) + C_n^{\mathrm{L}} \; (1+i)^{-(n-k)} \mathbf{P}_k^{(1)}(C_n^{\mathrm{L}}; n) \, .$$

Financial risk

To meet solvency requirements the insurer purchases a portfolio of assets backing the contract. At each date $t \in [0, n]$ the market value A_t of the asset portfolio cannot be lower than the technical reserve:

$$A_t \geq R_t \,.$$

• *Classical scheme.* The insurer is involved in a replicating investment strategy providing the result $A_t \geq R_t$ for any t.
[Brennan, Schwartz, 1976]

Remark. If the Φ functions include minimum guarantees the replicating strategy is a dynamic hedging strategy, as prescribed by the option pricing theory. ■

• *Scheme with underlying security.* At time 0 let us consider a stochastic zcb with maturity n and terminal payoff:

$$Y_n := \Phi(0, n) \,.$$

Assume that the zcb is traded on the market at the price Q_t an assume that:

$$Q_0 = 1, \text{ at time } 0$$

$$Q_t = \Phi(0, t), \text{ for each } t < n$$

$\Rightarrow$ the equality $A_t = R_t$ is guaranteed if at time 0 the insurer purchases $A_0 = C_0$ units of this zcb.

▽ In actual contracts the equality $Q_t = \Phi(0,t)$ is obtained "by definition" since the price Q_t is used as the reference index; that is the Φ function is defined as:

$$\Phi(0,t) := \frac{Q_t}{Q_0}, \quad \forall\, t\,.$$

→ In a policy written on an underlying security the insurer is not faced with investment risk.

→ If the issuer of the underlying security is defaultable the policy involves counter-party risk. This default risk can be faced by the insurer or by the policyholder, depending on the specific contractual clauses.

→ If the price of the underlying security is determined on a non efficient market, the insurer can incur in losses in case of redemption if the price Q_t is greater than the fair value of the security. This surrender risk can be reduced by stipulating a buy-back ageement with the bond issuer.

→ Typically the underlying security of the i-l policy is a *structured bond* which includes minimum return guarantees. If the price Q_t is not efficiently determined the insurer needs an appropriate pricing model in order to control possible deviations of Q_t from its fair value.

Stochastic reserve and VBIF

The usual definitions apply to i-l policies.

⊙ The stochastic reserve at time t (for a single premium policy) is given by:

$$V_t = V(t; \widetilde{\mathbf{C}}) \,.$$

⊙ Correspondingly, the VBIF at time t is defined by:

$$E_t = R_t - V_t \,.$$

Appendix – An elementary model for arbitrage pricing

The derivative contract
Single period binomial model
The hedging (or replication) argument
The risk-neutral valuation
Valuing a life insurance liability
Valuing the investment gain
Example

The derivative contract

Let us consider at time t a stochastic zcb with maturity $T > t$ and payoff D_T.
Assume that D_T is a function:

$$D_T := g(F_T)\,,$$

where F is the market price of a traded security (or of a portfolio of traded securities)
$\longrightarrow$ the price F_t can be observed on the market at time t.

The zcb D is a *contingent claim* or a *derivative contract*; the portfolio F can be referred to as *the underlying* of this contract.

The valuation problem is to derive the time t value of the derivative contract, that is the price:

$$D_t = V(t; D_T)\,.$$

Single period binomial model

Let $t = 0$ and $T = 1$ and assume (slightly changing notations) the following binomial evolution of the undrlying price.

0 1

	uF	with prob. p
F		
	dF	with prob. $1 - p$

$\rightarrow$ stochastic growth factor φ, with possible values u or d.
Let $u > d$.

Assume there exists a riskless investment opportunity (the riskless bond) with interest rate r in $[0, 1]$

$\rightarrow$ deterministic growth factor: $m := 1 + r$.

We suppose that F pays no dividends and we make the usual perfect market assumptions; that is:

– no transaction costs, no taxes;

– short sales are allowed;

– the agents are price taker and prefer more to less;

– the securities are infinitely divisible;

– riskless arbitrage opportunities are precluded.

A first consequence: to prevent arbitrage the following inequalities must hold:

$$u > m > d\,.$$

The hedging (or replication) argument

Correspondingly to the evolution of F, we have the derivative price evolution:

0 1

$$F \quad \begin{array}{ll} F_u = uF & \text{with prob. } p \\ F_d = dF & \text{with prob. } 1-p \end{array}$$

$$D \quad \begin{array}{ll} D_u = g(uF) & \text{with prob. } p \\ D_d = g(dF) & \text{with prob. } 1-p \end{array}$$

Let us consider a portfolio containing Δ units of F and the amount B in riskless bond.
The price evolution of this portfolio is given by:

$$F\,\Delta + B \quad \begin{array}{ll} uF\,\Delta + mB & \text{with prob. } p \\ dF\,\Delta + mB & \text{with prob. } 1-p \end{array}$$

In order that the portfolio replicates the contingent claim payoff the following equalities must hold:

$$\begin{cases} uF\,\Delta & + & mB & = & G_u \\ dF\,\Delta & + & mB & = & G_d \end{cases}$$

Solving these equations, we obtain:

$$\Delta = \frac{D_u - D_d}{(u-d)\,F}\,,$$

and:

$$B = \frac{u\,D_d - d\,D_u}{(u-d)\,m}\,.$$

For these values of Δ anf B the portfolio exactly replicates the terminal value of D (the *equivalent portfolio*, or *replicating portfolio*).

To avoid arbitrage the price of this ptf must be equal to the price of the derivative (the "low of one price"); that is:

$$\begin{aligned} D &= F\,\Delta + B = \\ &= \frac{D_u - D_d}{u-d} + \frac{u\,D_d - d\,D_u}{(u-b)\,m} = \\ &= \frac{1}{m}\left(\frac{m-u}{u-d}\,D_u + \frac{u-m}{u-d}\,D_d\right)\,. \end{aligned}$$

This equation can be rewritten as:

$$D = \frac{1}{m}\Big[q\,D_u + (1-q)\,D_d\Big]\,.$$

where:

$$q := \frac{m-d}{u-d}$$

- The value of the derivative D is independent on the natural probability p.

- The value of the derivative does not depend on investor's attitude toward risk.

The risk-neutral valuation

The contingent claim price can be expressed as discounted the expectation:

$$D_0 = \frac{1}{1+r} \, \mathbf{E}_0^Q \left[D_1 \right]$$

where $\mathbf{E}_t^Q$ is the expectation operator with respect to the probability q, which is referred to as *risk-neutral* probability.

The expected return of F (with respect to the natural probability p) is given by:

$$\begin{aligned} E_F := & \frac{\mathbf{E}_0[F_1]}{F_0} - 1 \\ = & (u-1)\, p + (d-1)\, (1-p)\,. \end{aligned}$$

If the expectation is taken with respect to q one has:

$$\begin{aligned} \frac{\mathbf{E}_0^Q[F_1]}{F_0} - 1 &= (u-1)\, q + (d-1)\, (1-q) \\ &= (u-1)\, \frac{m-d}{u-d} + (d-1)\, \frac{u-m}{u-d} \\ &= m - 1 = r\,. \end{aligned}$$

Valuing a life insurance liability

Single premium pure endowment maturing at time 1, with current sum insured C_0 an technical rate i.
The policy is participating, with reference return $I_1 := F_1/F_0 - 1$ and participation coefficient β: hence the benefit a time 1 is:

$$Y_1 = C_0 \, \frac{1 + \max\{\beta\, I_1,\, i\}}{1+i}\,,$$

or:

$$Y_1 = R\left[1 + \max\{\beta\, I_1,\, i\}\right],$$

where $R := C_0/(1+i)$.

We have:

0	1	
F	$F_u = uF$	with prob. p
	$F_d = dF$	with prob. $1-p$
I	$I_u = u - 1$	with prob. p
	$I_d = d - 1$	with prob. $1-p$
Y	$Y_u = R\,[1 + \max\{\beta\,(u-1),\, i\}]$	
	$Y_d = R\,[1 + \max\{\beta\,(d-1),\, i\}]$	

Using the expression:

$$Y_1 = R\left(1 + \beta\, I_1\right) + R\, \max\{i - \beta\, I_1,\, 0\}\,,$$

we can value separately the linear component (the "base" component):

$$L_1 = R\left(1 + \beta\, I_1\right),$$

and the put component:

$$P_1 = R\, \max\{i - \beta\, I_1,\, 0\}\,.$$

• For the base component we have:

$$L_u = R\,[1 + \beta\,(u-1)] = R\,[(1-\beta) + \beta\, u]\,,$$
$$L_d = R\,[1 + \beta\,(d-1)] = R\,[(1-\beta) + \beta\, d]\,.$$

Hence we find:

$$\begin{aligned} L &= \frac{1}{m}\left[q\, L_u + (1-q)\, L_d\right] \\ &= R\,\frac{1}{m}\left[(1-\beta) + \beta\,[q\, u + (1-q)\, d]\right], \end{aligned}$$

and:

$$\Delta_L = \frac{L_u - L_d}{(u-d)\, F} = \beta\,\frac{R}{F}\,.$$

• For the put component we have:

$$P = \frac{1}{m}\left[q\, P_u + (1-q)\, P_d\right],$$

and:

$$\Delta_P = \frac{P_u - P_d}{(u-d)\, F}\,,$$

where:

$$P_u = R\, \max\{i - \beta\,(u-1),\, 0\}\,,$$
$$P_d = R\, \max\{i - \beta\,(d-1),\, 0\}\,.$$

Since $P_u \le P_d$, then $\Delta_P \le 0$.

Valuing the investment gain

Referring to the same policy, we consider the investment gain of the insurer at time 1, given by:

$$G_1 = R\left[I_1 - \max\{\beta\, I_1,\, i\}\right],$$

which can be written as:

$$G_1 = R\,(1-\beta)\,I_1 - R\,\max\{i - \beta\, I_1,\, 0\}\,.$$

Thus G_1 can be written as the difference between the linear component:

$$H_1 = R\,(1-\beta)\,I_1\,,$$

and the put component:

$$P_1 = R\,\max\{i - \beta\, I_1,\, 0\}\,.$$

- The linear component is now given by:

$$H_u = R\,[(1-\beta)\,(u-1)]\,,$$
$$H_d = R\,[(1-\beta)\,(d-1)]\,.$$

Hence we find:

$$\begin{aligned} H &= \frac{1}{m}\left[q\,H_u + (1-q)\,H_d\right] \\ &= R\,\frac{1}{m}\,(1-\beta)\left[q\,u + (1-q)\,d - 1\right], \end{aligned}$$

and:

$$\Delta_H = \frac{H_u - H_d}{(u-d)\,F} = (1-\beta)\,\frac{R}{F}\,.$$

Example

Let:

$$u = 1.1\,,\ d = 1/u\,,\ r = 5\%\,,\ F = 10 \quad (\text{“market parameters”})\,;$$

$$C_0 = 102,\ i = 2\%\ \text{(hence } R = 100),\ \beta = 0.8 \quad (\text{“policy features”})\,.$$

Under the binomial scheme:

$$F = 10 \qquad \begin{matrix} F_u = 11 \\ F_d = 9.09091 \end{matrix}$$

$$I \qquad \begin{matrix} I_u = 1.1 - 1 = 0.1 \\ I_d = 0.909091 - 1 = -0.091 \end{matrix}$$

$$Y \qquad \begin{matrix} Y_u = 100 \times (1 + \max\{0.8 \times 0.1,\ 0.02\}) = 108 \\ Y_d = 100 \times (1 + \max\{0.8 \times -0.091,\ 0.02\}) = 102 \end{matrix}$$

The risk-neutral probability is:

$$q = \frac{m - d}{u - d} = \frac{1.05 - 0.909091}{1.1 - 0.909091} = 0.7381\,,$$

and the value of the insurance liability is:

$$\begin{aligned} V(0; Y_1) &= \frac{\mathbf{E}_0^Q(Y_1)}{1 + r} = \frac{1}{m}\Big[q\,Y_u + (1 - q)\,Y_d\Big] \\ &= \frac{1}{1.05}\Big[0.7381 \times 108 + (1 - 0.7381) \times 102\Big] \\ &= \frac{1}{1.05}\,106.4286 = 101.361\,. \end{aligned}$$

The composition of the replicating portfolio is:

$$\Delta = \frac{Y_u - Y_d}{(u-d)\,F} = \frac{108 - 102}{11 - 9.09091} = 3.1429\,,$$

and:

$$B = \frac{u\,Y_d - d\,Y_u}{(u-d)\,m} = \frac{1.1 \times 102 - 0.909091 \times 108}{(1.1 - 0.909091) \times 1.05} = \frac{14.018}{0.2005} = 69.932\,.$$

Hence in order to hedge the liability Y_1, the insurer must allocate the amount $V = 101.361$ investing:

- ⊙ $10 \times 3.1429/101.361 = 31\%$ of V in the reference fund,

and:

- ⊙ $69.932/101.361 = 69\%$ of V in riskless bonds.

Valuation of the components

- **Base component**

$$L_u = R\,[1 + \beta\,(u-1)] = 100 \times (1 + 0.8 \times 0.1) = 108\,,$$

$$L_d = R\,[1 + \beta\,(d-1)] = 100 \times (1 + 0.8 \times -0.091) = 92.7273\,.$$

Hence we find:

$$\begin{aligned} L &= \frac{1}{m}\left[q\,L_u + (1-q)\,L_d\right] \\ &= \frac{1}{1.05}\left[0.7381 \times 108 + (1 - 0.7381) \times 92.7273\right] \\ &= 99.0476\,. \end{aligned}$$

and:

$$\Delta_L = \beta\,\frac{R}{F} = 0.8 \times \frac{100}{10} = 8\,.$$

- **Put component**

$$\begin{aligned} P_u &= R\,\max\{i-\beta\,I_u,\,0\} \\ &= 100\times\max\{0.05-0.8\times 0.1,\,0\}=0\,, \\ P_d &= R\,\max\{i-\beta\,I_d,\,0\} \\ &= 100\times\max\{0.05-0.8\times -0.091,\,0\}=9.27273\,. \end{aligned}$$

Thus the put value is:

$$\begin{aligned} P &= \frac{1}{m}\Big[q\,P_u+(1-q)\,P_d\Big] \\ &= \frac{1}{1.05}\Big[0.7381\times 0+(1-0.7381)\times 9.27273\Big] \\ &= 2.31293\,. \end{aligned}$$

with delta:

$$\Delta_P=\frac{P_u-P_d}{(u-d)\,F}=\frac{0-2.31293}{11-9.09091}=-4.8571\,.$$

In fact one can obtain:

$$V=L+P=99.0476+2.31293=101.361\,.$$

$$\Delta=\Delta_L+\Delta_P=8-4.8571=3.1429\,.$$

The investment gain

The investment gain generated by the policy at time 1 can have the values,

$$\begin{aligned} G_u &= R\left[I_u - \max\{\beta\, I_u,\, i\}\right] \\ &= 100 \times \left[0.1 - \max\{0.8 \times 0.1,\, 0.05\}\right] = 2\,, \\ G_d &= R\left[I_d - \max\{\beta\, I_d,\, i\}\right] \\ &= 100 \times \left[-0.091 - \max\{0.8 \times -0.091,\, 0.05\}\right] = -11.0909\,. \end{aligned}$$

Therefore the value of the investment gain is negative:

$$\begin{aligned} G &= \frac{1}{m}\left[q\, G_u + (1-q)\, G_d\right] \\ &= \frac{1}{1.05}\left[0.7381 \times 2 + (1 - 0.7381) \times -11.0909\right] \\ &= -1.36054\,. \end{aligned}$$

with delta:

$$\Delta_G = \frac{G_u - G_d}{(u-d)\, F} = \frac{2 + 11.0909}{11 - 9.09091} = 6.8571\,.$$

- For the **linear component** we have:

$$\begin{aligned} H_u &= R\,[(1-\beta)\, I_u] = R\,[(1 - 0.8) \times 0.1] = 2\,, \\ H_d &= R\,[(1-\beta)\, I_d] = R\,[(1 - 0.8) \times -0.091] = -1.818218. \end{aligned}$$

Hence one obtains:

$$\begin{aligned} H &= \frac{1}{m}\left[q\, H_u + (1-q)\, H_d\right] \\ &= \frac{1}{1.05}\left[0.7381 \times 2 + (1 - 0.7381) \times -1.818218\right] \\ &= 0.95238\,. \end{aligned}$$

with:

$$\Delta_H = (1-\beta)\,\frac{R}{F} = 0.2 \times 10 = 2\,.$$

Performing the valuation by $G = H - P$ we get:

$$G = H - P = 0.95238 - 2.31293 = -1.36054\,.$$

The retained interest H is not sufficient to offset the cost of the minimum guarantee.

Remark. The difference:

$$E := R - V = 100 - 101.361 = -1.361\,,$$

is the (investment component of) the VBIF generated by the contract. ■

Remark. For a participation coefficient $\beta = 0.6$ one would obtain:

$$V = 99.9546\,,\ L = 98.0952\,,\ P = 1.8594\,,\ H = 1.90476\,,$$

$$E = R - V = 100 - 99.9546 = 0.0454\,.$$ ■

References

Black, F., Scholes, M., *The Pricing of Options and Corporate Liabilities*, Journal of Political Economy, 81(1973), 3.

Boyle, P.P., *Immunization under Stochastic Models of the Term Structure*, Journal of the Institute of Actuaries 105(1978), 429.

Boyle, P.P., *Recent Models of the Term Structure of Interest Rates with Actuarial Applications*, Transactions of the 21st Congress of Actuaries, 1980, 95–104.

Boyle, P.P., *Recent Research on the Risk Return Relationship in Financial Economics*, in Goovaerts, M., de Vylder, F., Haezendonck, J. (eds.) *Insurance and Risk Theory*, Dordrecht, Reidel, 1986.

Boyle, P.P., Schwartz, E.S., *Equilibrium Prices of Guarantees Under Equity-Linked Contracts*, Journal of Risk and Insurance 46(1977), 4.

Boyle, P.P., Broadie, M., Glasserman, P., *Monte Carlo Methods for Security Pricing*, Journal of Economics Dynamics and Controll, 21(1997), 1267–1321.

Brennan, M.J., *Aspects of insurance, intermediation and finance*, Geneva Papers on Risk and Insurance, 18(1993), 7-30.

Brennan, M.J., Schwartz, E.S., *The Pricing of Equity-Linked Life Insurance Policies with an Asset Value Guarantee*, Journal of Financial Economics 3(1976), 3.

Brennan, M.J., Schwartz, E.S., *Savings Bonds: Theory and Empirical Evidence*, New York, Monograph Series in Finance and Economics, 4, 1979a.

Brennan, M.J., Schwartz, E.S., *Alternative Investment Strategies for the Issuers of Equity Linked Life Insurance Policies with an Asset Value Guarantee*, Journal of Business, 52(1979b), 1.

Bühlmann, H., *Actuaries of the Third Kind?*, Astin Bulletin, 17(1987), 2.

Bühlmann, H., *Life insurance with stochastic interest rates*, in *Financial Risk in Insurance*, G. Ottaviani ed., Berlin, Springer, 1995.

Bühlmann, H., *New MATH for Life Actuaries*, Working paper, May 2002.

Castellani, G., De Felice, M., Moriconi, F., *Price and Risk of Variable Rate Bonds: An Application of the Cox, Ingersoll, Ross Model to Italian Treasury Credit Certificates*, Research Group on "Models of the term Structure of Interest Rates", Working paper n. 1, September 1989.

Castellani, G., De Felice, M., Moriconi, F., *Asset-Liability Management. Semi-deterministic and Stochastic Approach*, Transactions of 24th International Congress of Actuaries, Montreal, 1992.

Copeland, T., Antikarov, V., *Real Options. A Practitioner's Guide*, New York, Texere, 2001.

Cox, J.C., Ingersoll, J.E., Ross, S.A., *A Theory of the Term Structure of Interest Rates*, Econometrica, 53(1985), 2.

De Felice, M., *Immunization Theory: an Actuarial Perspective on Asset–Liability Management*, in *Financial Risk in Insurance*, G. Ottaviani ed., Berlin, Springer, 1995.

De Felice, M., Moriconi, F., *La teoria dell'immunizzazione finanziaria. Modelli e strategie*, Bologna, Il Mulino, 1991(a).

De Felice, M., Moriconi, F., *Uno schema per la valutazione e la gestione dei titoli del debito pubblico*, in "Ricerche applicate e modelli per la politica economica", Roma, Banca d'Italia, 1991(b).

De Felice, M., Moriconi, F., *Un modello per la progettazione e la valutazione di polizze indicizzate e rivalutabili*, Rapporto per l'INA, Roma, giugno 1994.

De Felice, M., Moriconi, F., *Il controllo del rischio finanziario nell'attività assicurativa*, Associazione Amici della Scuola Normale Superiore, Pisa, 1995.

De Felice, M., Moriconi, F., *Fare e formare in finanza*, Banca Impresa Società, 16(1997), 3.

De Felice, M., Moriconi, F., *Definizione dei benchmark, misurazione delle performance e valutazione dei "costi" nei fondi pensione con minimo garantito*, CONSOB Quaderni di Finanza n. 36, ottobre 1999, 68–109.

De Felice, M., Moriconi, F., *Finanza dell'assicurazione sulla vita. Principî per l'asset-liability management e per la misurazione dell'embedded value*, Gruppo di ricerca su "Modelli per la finanza matematica", Working paper n. 40, giugno 2001a.

De Felice, M., Moriconi, F., *L'assicurazione tra probabilismo e "corporate governance"*, Assicurazioni, 68(2001b), 3-4.

De Felice, M., Moriconi, F., Salvemini, M.T., *Italian trasury credit certificates (CCTs): theory, practice, and quirks*, BNL Quarterly Review, 185(1993), 127-168.

De Felice, M., Moriconi, F., Cancellara, A., Mariano, A., *Il sistema di calcolo della Cassa depositi e prestiti per la progettazione e il controllo dei Buoni Postali*, Roma, Cassa depositi e prestiti, Quaderni monografici, n. 2, 1998.

de Finetti, B., *Teoria delle probabilità*, Torino, Einaudi, 1970; english version: *Theory of Probability*, Chichester, Wiley, 1974.

Gerber, H.U., *Life Insurance Mathematics*, Berlin, Springer, 1997[3].

Grosen, A., Jørgensen, P.L., *Fair valuation of life insurance liabilities: The impact of interest rate guarantees, surrender options, and bonus policies*, Insurance: Mathematics and Economics, 26(2000), 1.

Kaufmann, R., Gadmer, A., Klett, R., *Introduction to Dynamic Financial Analysis*, Astin Bulletin 31(2001), 1.

Merton, R.C., *On the application of the continuous-time theory of finance to financial intermediation and insurance*, Geneva Papers on Risk and Insurance, 14(1989), 225-261.

Morgan Stanley, *Revolution*, Equity Research, January 31, 2002.

Moriconi, F., *Immunizzazione semi-deterministica di portafogli di CCT*, Roma, Gruppo di ricerca su “Modelli per la finanza matematica”, Working paper n. 5, novembre 1991.

Moriconi, F., *Un modello stocastico bivariato per tassi di interesse nominali e reali*, Roma, Gruppo di ricerca su “Modelli per la finanza matematica”, Working paper n. 16, giugno 1994.

Moriconi, F., *Analyzing Default–Free Bond Markets by Diffusion Models*, in *Financial Risk in Insurance*, G. Ottaviani ed., Berlin, Springer, 1995.

Pacati, C., *Una stima di massima verosimiglianza del modello di Cox, Ingersoll e Ross univariato*, Roma, Gruppo di ricerca su “Modelli per la finanza matematica”, Working paper n. 30, ottobre 1998.

Pacati, C., *Estimating the Euro Term Structure of Interest Rates*, Roma, Gruppo di ricerca su “Modelli per la finanza matematica”, Working paper n. 32, March 1999.

Pacati, C., *Valutazione di portafogli di polizze vita con rivalutazione agli ennesimi*, Roma, Gruppo di ricerca su “Modelli per la finanza matematica”, Working paper n. 38, aprile 2000a.

Pacati, C., *Strutture per scadenza dei tassi lordi e strutture per scadenza dei tassi netti*, Siena, Quaderni del Dipartimento di Economia Politica n. 309, Università di Siena, novembre 2000b.

Redington, F.M., *Review of the principles of life office valuations* (*with discussion*), Journal of the Institute of Actuaries, 78(1952), 350.

Rogers, L.C.G., *Which Model for Term-Structure of Interest Rates Should One Use?*, Mathematical Finance, 65, 1995, 93-116.

Schwartz, E.S., Torous, W.N., *Prepayment and the valuation of mortgage-backed securities*, Journal of Finance, 44(1989), 2.

Smith, A.D., *How actuaries can use financial economics*, British Actuarial Journal, 2, 5(1996), 1057-1193.

Society of Actuaries, *Professional Actuarial Specialty Guide*, AA-1-98, 1998 <www.soa.org/library/aa-1-98.pdf>.

Wang, S., *Premium calculation by transforming the layer premium density*, Astin Bulletin, 26(1996), 1.