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Introduction – Towards a “finance of insurance”

Valuation of the life insurance business: a general framework

  Asset-liability approach

  Fair valuation

A new toolkit for corporate governance

Selected bibliography

Valuation of the life insurance business: a general framework

Asset-Liability approach

The basic components of the life insurance business:

⊙ valuation date: t

⊙ payment dates: $k = t{+}1,\ t{+}2,\ \dots,\ n$

⊕ premiums: $\widetilde{X}_{t+1}, \widetilde{X}_{t+2}, \dots, \widetilde{X}_n$

⊖ benefits: $\widetilde{Y}_{t+1}, \widetilde{Y}_{t+2}, \dots, \widetilde{Y}_n$

⊕ cash-flows from the investment portfolio: $Z_{t+1}, Z_{t+2}, \dots, Z_n$
$\leftrightarrow$ Z_k generated from the investment of reserves
(solvency margin and additional shareholders funds can be also considered)

The payment streams:

$$\widetilde{\mathbf{X}} = \{\widetilde{X}_k\}\,, \quad \widetilde{\mathbf{Y}} = \{\widetilde{Y}_k\}\,, \quad \mathbf{Z} = \{Z_k\}$$

are vectors of random variables.

Sources of uncertainty

- actuarial uncertainty (denoted by $\tilde{}$):
  tipically, lifetime
- capital market uncertainty:
  nominal interest rates, real interest rates, stock market indexes etc.

! uncertainty concerning surrenders must be properly modelled.

• Assume that the value at time t of the investment portfolio:

$$A_t := V(t; \mathbf{Z}) ,$$

and the value at time t of the policy portfolio:

$$V_t := V(t; \widetilde{\mathbf{Y}} - \widetilde{\mathbf{X}}) ,$$

are properly defined.

The *surplus* Ω_t at time t is given by:

$$\Omega_t = A_t - V_t .$$

Ω_t is also referred to as the *embedded value.*

• Typical decomposition of Ω_t

A statutory reserve R_t is defined at time t; hence the surplus is derived as:

$$\Omega_t = M_t + E_t ,$$

where:

$$M_t = A_t - R_t ,$$

and:

$$E_t = R_t - V_t .$$

E_t is also referred to as the *Value of Business In Force* (VBIF).

The decomposition corresponds to the separation between the investment department and the actuarial department.

Traditionally the valuation of A_t and E_t involves different skills; different valuation methods are used.

e.g.:

- · mark-to-market valuation for A_t (observed prices + market model)
- · capital budgeting methods for E_t (Net Present Value of future profits, under scenario assumptions)

! methods can be inconsistent (the actual surplus is not correctly measured).

Moreover, usually:

- – $\widetilde{\mathbf{X}}$ and $\widetilde{\mathbf{Y}}$ are linked to $\mathbf{Z}$
  (partipating policies, unit linked, index linked policies)
- – the policies contain embedded options

Fair valuation

- · same valuation method for A_t and E_t (insurance liabilities are valuated as if they were traded securities)
- · arbitrage models for contingent claims pricing are adopted

$\Longrightarrow$ *New MATH for Life Actuaries* [Bühlmann, 2002]

$\Longrightarrow$ *Revolution* [Morgan Stanley, 2002]

• Under fair valuation we shall refer to:

$$V_t := V(t; \widetilde{\mathbf{Y}}) - V(t; \widetilde{\mathbf{X}}),$$

as the *stochastic reserve.*

⊙ *New MATH for Life Actuaries*

· the "point is that for dealing with the time value of money the classical actuarial tecnique is nowadays so far off economic reality that it needs to be fundamentally revised"

⊕ the new approach: from "compound interest" to the "numeraire" (mark-to-market valuation)

· managing financial risk: "if the Life Insurer invests exactly in the replicating portfolio then the financial risk is zero. The classical actuarial model – even under a matching assets strategy – can never achieve this since il does not account for contractual guarantees (most important in practice the guarantee of the technical interest rate)."

$\longrightarrow$ it is exactly this fact, which urges for a change in our cherished traditional actuarial model.

· the effect on Embedded Value calculation.

[Bühlmann, 2002]

⊙ *Revolution*

The proposals for International Accounting Standards (IAS), as set out in the most recent Draft Statement of Principle (DSOP), are:

· "IAS should be an *asset and liability approach*: profits are recognised in terms of the changes in the gap between insurance assets and liabilities"

· "both assets and liabilities should be valued on a *fair value* basis: market value for most categories of assets and a DCF approach for liabilities – the latter using a discount rate based on *market consistent assumption* ($\longleftarrow$ RAD method)

· "option pricing model possibly using stochastic techniques should be used to value guarantees built into contracts"

$\oplus$ a newly defined NAV or shareholders' funds figure which represents the difference between the fair value of the assets and the fair (realistic) value of the liabilities encoppassing the realistic value of contractual guarantees ($\longleftarrow$ Ω).

$\oplus$ clear implications for product design, investment policy, ...
$\longleftarrow$ profit test

[Morgan Stanley, 2002]

A new toolkit for corporate governance

- stochastic reserve
  - $\longrightarrow$ stochastic reserve *vs* statutory reserve
  - $\longrightarrow$ put decomposition (market value of guarantees)
  - $\longrightarrow$ stochastic duration (delta) of assets and liabilities
  - $\Longrightarrow$ asset-liability management
- embedded value
  - $\longrightarrow$ profit testing
  - $\longrightarrow$ embedded value earnings
- risk capital

methods, techniques

· arbitrage principle

· hedging argument

· stochastic calculus

· Monte Carlo simulation

[DF, M, 2001b, pp. 302-303]; [Bühlmann, 1987]

Selected bibliography

⊙ *How Actuaries Can Use Financial Economics*, [Smith, 1996]
financial economics can enhance the work of actuaries (p. 1058)
: the key question must be whether a general disregard by actuaries of the information contained in observed prices would have a detrimental impact on the soundness of our financial management
⟵ one message of financial economics is that market value is of crucial importance to virtually any financial problem (p. 1059)

⊙ *coherence*, *decision theory*, [de Finetti, 1970(74)]
decision theory ..., in the restricted sense to which is often reduced by 'objectivist' statisticians, considers only the question 'what decision is appropriate given the *accepted hypothesis* and not 'what decision is appropriate in the given state of uncertainty' (p. 216)

⊙ *market models*, *pricing*
[Black, Scholes, 1973]; [Cox, Ingersoll, Ross, 1985]; [M, 1995]

⊙ *asset-liability valuation, asset-liability management*
[Redington, 1952]; [Boyle, 1978; 1980; 1986]; [DF, 1995]

⊙ *guarantee valuation (management)* via *option pricing theory*
[Brennan, Schwartz, 1976; 1979b]; [Boyle, Schwartz, 1977].

⊙ *a new framework for corporate governance*
[Bühlmann, 1995; 2002]; [Brennan, 1993]; [Merton, 1989];
[DF, M, 2001a; 2001b].

Chapter 1 – The standard actuarial valuation framework

Endowment policy with constant sum insured

• An **endowment policy** with term n years for a life aged x provides for payment of the sum insured C_0

- at the end of the year of death if this occurs within the first n years (**term insurance**), otherwise
- at the end of the nth year (**pure endowment**).

Annual premiums: the policyholder pays a constant amount Π at the beginning of each of the n years as long as he or she is alive.

• *The actuarial uncertainty* is decribed by the r.v.:

T_x: the future lifetime of the insured having age x.

If $\mathbf{1}_{\{\mathcal{E}\}}$ is the indicator function of the event $\mathcal{E}$, the asset-liability streams generated by the policy at time 0 are given by:

$$\widetilde{Y}_k = \begin{cases} \mathbf{1}_{\{k-1<T_x\leq k\}}\, C_0\,, & k=1,2,\ldots,n-1\,, \\ \mathbf{1}_{\{T_x\geq n-1\}}\, C_0\ , & k=n\,, \end{cases}$$

$$\widetilde{X}_k = \begin{cases} \mathbf{1}_{\{T_x>k\}}\, \Pi\,, & k=1,2,\ldots,n-1\,, \\ 0\,, & k=n\,, \end{cases}$$

and by the cash-flow stream $\mathbf{Z}$ generated by the investment of Π at time 0.

• *Technical means*

Let $F(t)$ the p.d.f. of T_x, that is:

$$F(t) = \mathbf{P}(T_x \leq t) = \mathbf{E}(\mathbf{1}_{\{T_x \leq t\}}) , \quad t \geq 0 .$$

We shall denote by $\overline{Y}$ and $\overline{X}$ the expectation of $\widetilde{Y}$ and $\widetilde{X}$, resp. That is:

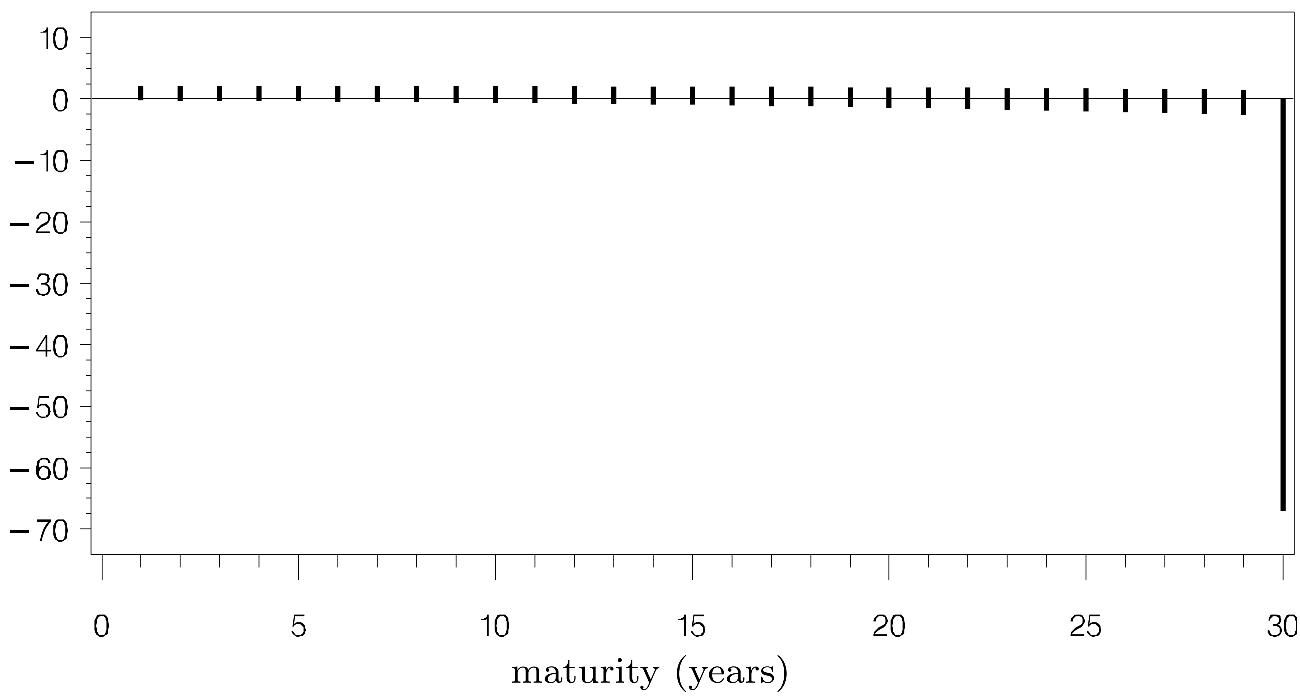
$$\overline{Y}_k = \begin{cases} \mathbf{P}(k-1 < T_x \leq k)\, C_0 , & k = 1, 2, \ldots, n-1 , \\ \mathbf{P}(T_x \geq n-1)\, C_0 , & k = n , \end{cases}$$

$$\overline{X}_k = \begin{cases} \mathbf{P}(T_x > k)\, \Pi , & k = 1, 2, \ldots, n-1 , \\ 0 , & k = n , \end{cases}$$

We shall refer to $\overline{Y}$ and $\overline{X}$ as the *technical mean* of $\widetilde{Y}$ and $\widetilde{X}$, resp.

Example. Endowment with $n = 30$, $x = 40$, $C_0 = 100$, $\Pi = 2,217$.

Technical means of premiums and benefits



Using standard actuarial notations:

$$\overline{Y}_k = \begin{cases} {}_{k-1}p_x \; q_{x+k-1} \, C_0 \,, & k = 1, 2, \ldots, n-1 \,, \\ {}_{n-1}p_x \, C_0 \,, & k = n \,, \end{cases}$$

$$\overline{X}_k = \begin{cases} {}_{k}p_x \, \Pi \,, & k = 1, 2, \ldots, n-1 \,, \\ 0 \,, & k = n \,. \end{cases}$$

Using a simplified notation:

$$\overline{Y}_k = \mathbf{P}_0(C_0; k) \, C_0 \,, \quad k = 1, 2, \ldots, n \,,$$

$$\overline{X}_k = \mathbf{P}_0(\Pi; k) \, \Pi \,, \quad k = 1, 2, \ldots, n \,,$$

where:

$$\mathbf{P}_t(x; s)$$

is the probability, at time t, that x will be paid at time $s \geq t$.

The standard actuarial valuation framework

Tipically, the current value of future payments is obtained

1) taking the expectation of future cash flows under a *conservative* probability measure $\mathbf{P}^{(1)}$,
2) discounting the expected cash flows under the *technical rate of interest* i.

- The technical rate i and the life tables which define the probability $\mathbf{P}^{(1)}$ are contractually specified.
- $\mathbf{P}^{(1)}$ and i are the basis of the *first order valuation*.

The value at time 0 of the stream $\widetilde{\mathbf{Y}}$ is given by:

$$R_0(\widetilde{\mathbf{Y}}) = C_0 \sum_{k=1}^{n} (1+i)^{-k}\, \mathbf{P}_0^{(1)}(C_0; k)\,.$$

The **net single premium** for $\widetilde{\mathbf{Y}}$ is defined as:

$$U := R_0(\widetilde{\mathbf{Y}})\,.$$

The **net annual premium** is fixed as the constant annual amount A (to be paid at the beginning of the year) for which expected benefits and premiums have same value at time 0 (the *Equivalence principle*):

$$A = \frac{U}{\sum_{k=0}^{n-1} (1+i)^{-k}\, \mathbf{P}_0^{(1)}(A; k)}\,.$$

Remark. The premium Π effectively charged (the *gross premium*) is greater than the net premium A; tipically:

$$\Pi - A = \text{ safety loading } + \text{ expense loading}$$

Since A is derived by conservative $\mathbf{P}^{(1)}$ and i, the net premium just includes an implicit risk loading.

The **net premium reserve** at time k, given that the contract is currently in force, is given by:

$$R_k := R_k(\widetilde{\mathbf{Y}}) - R_k(\widetilde{\mathbf{X}}) \,, \qquad k = 0, 1, \ldots, n \,,$$

where:

$$R_k(\widetilde{\mathbf{Y}}) := C_0 \sum_{j=k+1}^{n} v^{(j-k)} \, \mathbf{P}_k^{(1)}(C_0; j) \,;$$

$$R_k(\widetilde{\mathbf{X}}) := A \sum_{j=k+1}^{n-1} v^{(j-k)} \, \mathbf{P}_k^{(1)}(A; j) \,,$$

being $v := 1/(1+i)$.

Remark. The *mathematical reserve* M_k at time k is defined considering also the premium paid at time $k < n$; that is:

$$M_k := R_k - A \,.$$

Obviously $M_0 = 0$ because de definition of A. ■

Using the technical means:

$$\overline{C}'_{k,j} := C_0 \, \mathbf{P}_k^{(1)}(C_0; j) \,,$$

$$\overline{A}'_{k,j} := A \, \mathbf{P}_k^{(1)}(A; j) \,,$$

one has:

$$R_k(\widetilde{\mathbf{C}}) = \sum_{j=k+1}^{n} \overline{C}'_{k,j} \, v^{(j-k)} \,,$$

$$R_k(\widetilde{\mathbf{A}}) = \sum_{j=k+1}^{n-1} \overline{A}'_{k,j} \, v^{(j-k)} \,.$$

The reserve as a budget constraint

The technical reserve R_k is a *regulatory requirement*: it is the level of funding the company must maintain at time k if the contract is currently in force.

- If $T_x \geq k - 1$, the recursive relation holds:

$$R_{k-1}\,(1+i) = q'_{x+k-1}\,C_0 + (1 - q'_{x+k-1})\,(R_k - A)$$

$\longrightarrow$ a (statutory based) equilibrium constraint:

if the contract is in force at time k–1, then the result of investing the reserve R_{k-1} for one year at the rate i must be equal to the expected value of the insurer's liabilities at the end of the year.

Participating endowment policy

- Both premiums and benefits are linked to the return of a reference investment portfolio (the "segregated fund")
- Premiums earned are invested into the reference fund
- A minimum guarantee is provided

Let:

- C_k : benefit (eventually) paid at time k;
- A_k : net annual premium (eventually) paid at time k;
- F_t : market value at time $t \geq 0$ of the reference fund;
- $I_k := F_k/F_{k-1} - 1$ annual rate of return of the reference fund at time k;
- i: technical interest rate.

• A typical indexation mechanism:

Given the initial values A_0 e C_0, premiums and benefits are given by:

$$A_k = A_{k-1}\,(1+\rho_k)\,, \quad C_k = C_{k-1}\,(1+\rho_k)\,,$$

being:

$$\rho_k := \frac{\max\{\beta\, I_k,\, i\} - i}{1+i}\,,$$

where the *participation coefficient*:

$$\beta \in [0,1]\,,$$

is fixed at time 0.

For $0 \le h \le k \le n$ we define the *readjustment factor* from h to k:

$$\Phi(h,k) := \prod_{j=h+1}^{k} (1+\rho_j) \;=\; v^{k-h} \prod_{j=h+1}^{k} \big(1+\max\{\beta\, I_j, i\}\big)$$

(being $\Phi(k,k)=1$).
Therefore the readjustment mechanism can be expressed as:

$$A_k = A_0\, \Phi(0,k)\,, \quad C_k = C_0\, \Phi(0,k)\,.$$

Under a more generale scheme, one can use different readjustment rules for premiums and benefits:

$$A_k = A_0\; \Phi^A(0,k)\,, \quad C_k = C_0\; \Phi^C(0,k)\,,$$

by suitably defining Φ^A and Φ^C.

The standard actuarial valuation of a participating policy

In an participating policy the random streams of premiums and benefits are:

$$\widetilde{Y}_k = \begin{cases} \mathbf{1}_{\{k-1<T_x\le k\}}\, C_k\,, & k=1,2,\ldots,n-1\,, \\ \\ \mathbf{1}_{\{T_x\ge n-1\}}\, C_k\,, & k=n\,, \end{cases}$$

$$\widetilde{X}_k = \begin{cases} \mathbf{1}_{\{T_x>k\}}\, A_k\,, & k=1,2,\ldots,n-1\,, \\ \\ 0\,, & k=n\,, \end{cases}$$

Now $\widetilde{Y}_k$ and $\widetilde{X}_k$ are affected by both actuarial and financial uncertainty. Hence the "actuarial expectations":

$$\widehat{Y}_k := \mathbf{P}_0(C_k;k)\, C_k\,,$$

$$\widehat{X}_k := \mathbf{P}_0(A_k;k)\, A_k\,,$$

are random variables, since the amounts C_k and A_k are not known at time 0.

Nonetheless, the standard valuation method applied at time k treats the contract in the same way as a non participating policy, assuming that the premium and the sum insured remain fixed at the current levels A_k and C_k, respectively.
The net premium reserve at time k is then given by:

$$\begin{aligned} R_k = \quad & C_k \sum_{j=k+1}^{n} v^{(j-k)}\, \mathbf{P}_k^{(1)}(C_j;j) \\ & - A_k \sum_{j=k+1}^{n-1} v^{(j-k)}\, \mathbf{P}_k^{(1)}(A_j;j)\,. \end{aligned}$$

Defining again the technical means:

$$\overline{C}'_{k,j} := C_k \,\mathbf{P}_k^{(1)}(C_j; j)\,,$$

$$\overline{A}'_{k,j} := A_k \,\mathbf{P}_k^{(1)}(A_j; j)\,,$$

one has:

$$R_k = \sum_{j=k+1}^{n} \overline{C}'_{k,j}\; v^{(j-k)} - \sum_{j=k+1}^{n-1} \overline{A}'_{k,j}\; v^{(j-k)}\,.$$

Remark. At time k, the technical mean:s

$$\overline{C}_{k,j} := C_k \,\mathbf{P}_k(C_j; j)\,, \qquad \overline{A}_{k,j} := A_k \,\mathbf{P}_k(A_j; j)\,, \qquad j > k\,,$$

are known; the actuarial expectations:

$$\widehat{C}_j := C_j \,\mathbf{P}_k(C_j; j))\,, \qquad \widehat{A}_j := A_j \,\mathbf{P}_k(A_j; j))\,, \qquad j > k\,,$$

are random variables (given the financial uncertainty affecting the $\Phi(k, j)$ factors). ■

The reserve as a budget constraint

Also for a participating policy the technical reserve R_k is a regulatory requirement.

- If $T_x \geq k - 1$, the recursive relation for the reserve is:

$$R_{k-1}\,(1+\rho_k)(1+i) = q'_{x+k-1}\, C_k + (1 - q'_{x+k-1})\,(R_k - A_k)\,.$$

The readjustment as an interest rate crediting

The readjustment of premiums and benefits is a profit-sharing mechanism; at each year end part of the return I_k in excess of i is credited to the policy.

Let us consider a single premium policy (a net premium $U = R_0$ is paid at time 0).

At the end of the year k the interest crediting mechanism:

$$C_k = C_{k-1} \, \frac{1 + \max\{\beta \, I_k, \, i\}}{1 + i}$$

allocates the annual return:

$$R_{k-1} \, I_k$$

of the reference fund between the insurance company and the policy's technical reserve, according the following participation rule

– the amount:

$$R_{k-1} \, \max\{\beta \, I_k, \, i\} = R_{k-1} \left[\beta \, I_k + \max\{i - \beta \, I_k, \, 0\}\right] ,$$

is credited to the policy;

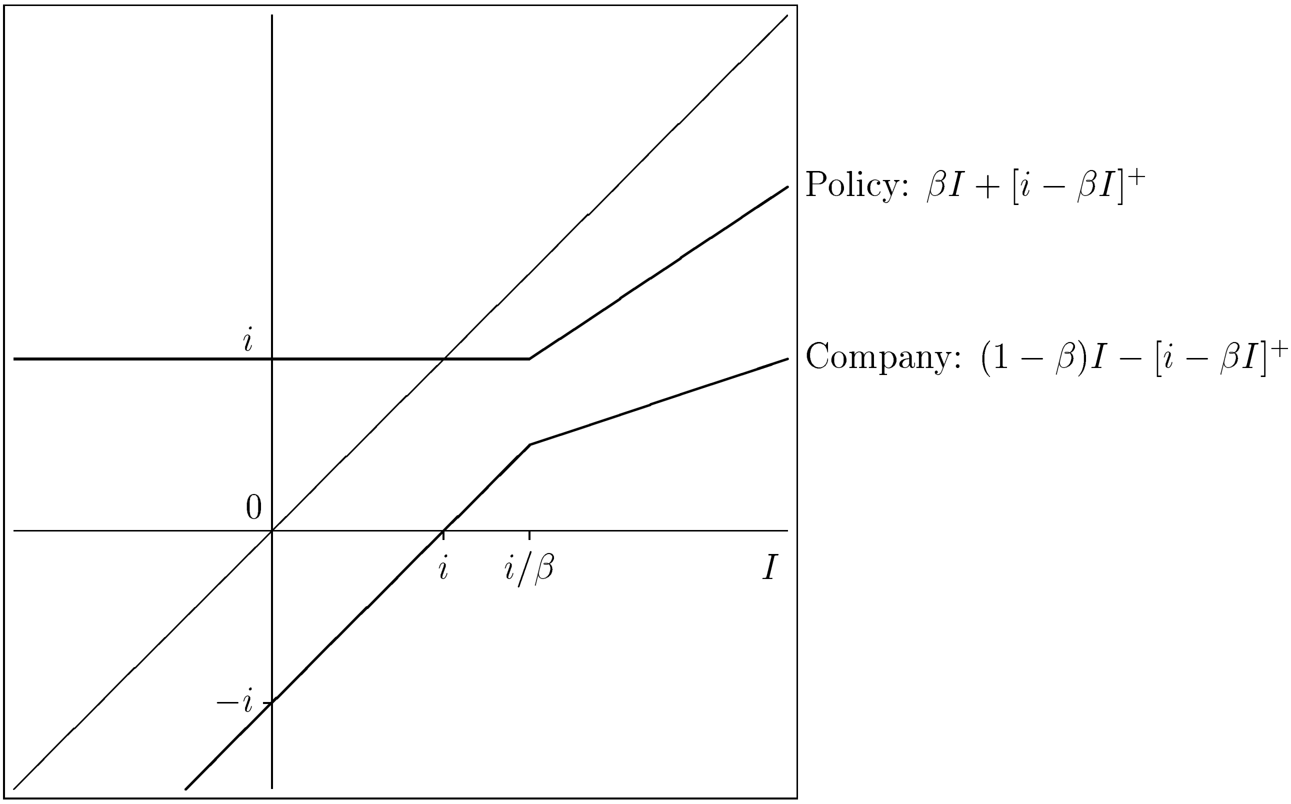
– the amount:

$$R_{k-1} \left[I_k - \max\{\beta \, I_k, \, i\}\right] = R_{k-1} \left[(1 - \beta) \, I_k - \max\{i - \beta \, I_k, \, 0\}\right] ,$$

is retained by the insurance company.

This allocation rule can be illustrated by the following payoff diagrams:

Allocation of the annual return



$\longrightarrow$ in each year the insurer

- is earning an *investment gain* $(1-\beta)\, I_k$, and
- is shorting an *interest rate put option* written on the reference fund.

The minimum guarantee option embedded in the policy can destroy value and is a potential hazard to the company solvency.

General participating mechanism

In the previous participating mechanism we have a direct linking to the reference fund return.
This is typical in some countries. In Italy virtually all the life insurance policies provide this kind of profit sharing rule, with the participation coefficient β contractually specified at the policy issuance; the mechanism can include a lag.

Remark. One can also find readjustment rules linked to inflation.
See [DF, M, 1994] for the valuation of inflation options embedded in Italian policies by an arbitrage model for real and nominal interest rates. ■

In many cases the linking of the policy payoffs to the reference portfolio is not so direct. The distribution of profits

· is driven by smoothing mechanism, lags, ...

· is not necessarily a binding agreement between the parties

$\longrightarrow$ bonus policies.

See [Grosen, Jørgensen, 2000] for an interesting representation of the interest rate option embedded in bonus policies.

In many participating policies the minimum guarantees do not represent a contractual constraint. The embedded option are rather determined by *reputation constraints* [DF, M, 1999].

Chapter 2 – Valuation principles

The general asset-liability framework
Asset-liability management
The valuation principle
Some properties of the value funcion
Technical means and valuation factors
Valuation of payment streams
Stochastic reserve
VBIF: the stochastic reserve approach
Decomposition of VBIF

The general asset-liability framework

The basic components of the life insurance business:

- ⊙ valuation date: t
- ⊙ payment dates: $k = t+1, t+2, \ldots, n$
- ⊕ premiums: $\widetilde{X}_{t+1}, \widetilde{X}_{t+2}, \ldots, \widetilde{X}_n$
- ⊖ benefits: $\widetilde{Y}_{t+1}, \widetilde{Y}_{t+2}, \ldots, \widetilde{Y}_n$
- ⊕ cash-flows from the investment portfolio: $Z_{t+1}, Z_{t+2}, \ldots, Z_n$

The payment streams:

$$\widetilde{\mathbf{X}} = \{\widetilde{X}_k\}\,, \quad \widetilde{\mathbf{Y}} = \{\widetilde{Y}_k\}\,, \quad \mathbf{Z} = \{Z_k\}$$

are vectors of random variables.

The payments $\widetilde{Y}_k$ and $\widetilde{X}_k$ are defined as:

$$\widetilde{Y}_k = \begin{cases} Y_k\,, & \text{with prob.} \quad \mathbf{P}_t(Y_k;k) \\ 0\,, & \text{with prob.} \quad 1 - \mathbf{P}_t(Y_k;k) \end{cases}$$

$$\widetilde{X}_k = \begin{cases} X_k\,, & \text{with prob.} \quad \mathbf{P}_t(X_k;k) \\ 0\,, & \text{with prob.} \quad 1 - \mathbf{P}_t(Xk;k) \end{cases}$$

where Y_k and X_k are the r.v.s:

$$Y_k = Y_t\ \Phi^Y(t,k)\,, \quad X_k = X_t\ \Phi^X(t,k)\,,$$

depending on financial uncertainty through the readjustment factors Φ.

Asset-liability management

At time t:

• determine the value of assets:

$$V(t;\mathbf{Z}) + V(t;\widetilde{\mathbf{X}})$$

and the value of liabilities:

$$V(t;\widetilde{\mathbf{Y}})\,.$$

• control the financial equilibrium between $V(t;\mathbf{Z}+\widetilde{\mathbf{X}})$ and $V(t;\widetilde{\mathbf{Y}})$.

"To provide a meaningful analytical framework to value the liabilities one needs to understand the process by which the assets are valued.
...
The conventional actuarial approach is to estimate the expected value of the future cash flows; ... . These expected cash flows are then discounted at an *appropriate* interest rate.
However, if the contract is traded in an efficient secondary market the market place furnished a unique market price at any instant. This market price reflects the consensus of expectations of the market at that instant and is somewhat more objective than the actuarial value.
Many of the contracts valued by actuaries are not traded in efficient secondary markets, but the pricing of these contracts should be closely related to the returns available in the financial markets where the corresponding assets are typically invested."
[Boyle, 1986, pp. 144–5]

$\Longrightarrow$ provide a **market based** valuation process using a stochastic pricing model based on the **no-arbitrage principle**: fair valuation.

Since the reference portfolio is tipically invested in bonds and equities, we need (at least) a two-factor model market model.

Objectives:

For an outstanding portfolio of policies:

- determine the market value of the insurance portfolio (the *stochastic reserve*)
- determine the market value of the guarantees
- compare the stochastic reserve with the statutory reserve
- determine the sensitivity of the stochastic reserve to changes in the market interest rates and in the stock market index
- determine the market value of the reference portfolio
- analyze the exposure to financial risk of the overall portfolio (assets and outstanding policies)

• The fundamental assumption in our valuation framework is that uncertainty concerning eliminations can be separated by financial uncertainty and then eliminated by taking expectations. Payoffs exposed to financial uncertainty are then priced using the principles of contingent claims analysis.

For example, for any $t \leq k$, given the stochastic sum payable at time k:

$$\widetilde{Y}_k := \mathbf{1}_{\{\mathcal{E}_k\}} \, C_t \, \Phi(t,k) \, ,$$

with C_t known at time t, we assume the property:

$$V\big(t; \widetilde{Y}_k\big) = \mathbf{P}_t(\mathcal{E}_k) \; C_t \; V\big(t; \Phi(t,k)\big) \, ,$$

where $V\big(t; \Phi(t,k)\big)$ is derived under the no-arbitrage principle in perfect markets.

This valuation principle can be expressed more explicitly as the consequence of separated assumptions.

The valuation principle

• At time t, let us refer to a general kind of payoff due at time k, defined as:

$$\widetilde{Y}_k := \mathbf{1}_{\{\mathcal{E}_k\}} \, Y_k \, ,$$

where $\mathcal{E}_k$ is the event:

"The policy has not yet been surrendered
and Y_k must be paid at time k".

We may also assume:

$$Y_k := C_t \, \Phi(t, k) \, ,$$

where C_t is observed at time t and $\Phi(t, k)$ is exposed only to financial uncertainty.

• **Assumption 0**

Our main assumption on the price functional $V\big(t; \widetilde{Y}_k\big)$ is inspired by the fundamental results in contingent claims analysis and by some recent developments concerning the premium calculation principles in insurance.

We assume that the value at time t of $\widetilde{Y}_k$ is given by:

$$V\big(t; \widetilde{Y}_k\big) = \mathbf{E}_t^J \big[\mathbf{1}_{\{\mathcal{E}_k\}} \, Y_k \; \chi(t, k)\big] \, ,$$

where:

· $\mathbf{E}_t^J$ is the expectation operator taken with respect to a specified probability measure $\mathbf{J}$, conditional on the information at time t;

· $\chi(t, k)$ is the stochastic discount factor over the time interval $[t, k]$.

Remark. The probability $\mathbf{J}$ is a multivariate measure different from the "natural" probability measure $\mathbf{P}$. The $\mathbf{J}$ measure can be interpreted as a distorted probability measure which allows to properly take into account risk aversion while preserving the linearity property of the value functional; thus $\mathbf{J}$ can be referred to as a *risk-adjusted* *measure*. As concerning premium calculation in insurance one can refer to [Wang, 1996]. ■

• Assumption 1

We assume a form of independence between elimination uncertainty and financial uncertainty; precisely:

$$V\big(t;\widetilde{Y}_k\big) = \mathbf{E}_t^H\big[\mathbf{1}_{\{\mathcal{E}_k\}}\big]\ \mathbf{E}_t^Q\big[Y_k\ \chi(t,k)\big]\,.$$

In a broad sense, this separation property can be referred to as independence between actuarial and capital market uncertainty.

Remark. Assumption 1 is natural if surrenders are not allowed. It is a good approximation if surrenders are strongly discouraged by proper penalties ($\longrightarrow$ redemptions mainly determined by personal consumption plans). ■

Remark. As concerning the expectation $\mathbf{E}_t^Q\big[Y_k\ \chi(t,k)\big]$, the probability measure **Q** is the *equivalent martingale measure* well known in arbitrage pricing in finance; it is also referred to as *risk-neutral measure*.
! The risk-neutral measure **Q** is determined by market data.
Both the discount factor $\chi(t,k)$ and the r.n. probability must be specified under an appropriate stochastic model. Once the sources of market uncertainty are specified in the model, χ and the r.n. measure are the same for all the securities which depend on these risk factors
$\Longrightarrow$ if the model is calibrated in order to match the observed price of traded securities, it can be applied to non-traded securities, providing coherent pricing. ■

Remark. If Y_k is a deterministic benefit, i.e. $Y_k = C_t$, one obtains:

$$V\big(t; \widetilde{Y}_k\big) = C_t\, \mathbf{E}_t^H\big[\mathbf{1}_{\{\mathcal{E}_k\}}\big]\, v(t,k)\,,$$

where $v(t,k) = \mathbf{E}_t^Q\big[\chi(t,k)\big]$ is the risk-free discount factor prevailing on the market at time t.
The quantity $C_t\, \mathbf{E}_t^H\big[\mathbf{1}_{\{\mathcal{E}_k\}}\big]$ can be considered a *certainty equivalent* of $C_t\, \mathbf{1}_{\{\mathcal{E}_k\}}$, derived under a transformed probability measure $\mathbf{H}$.
Under risk aversion the difference:

$$\mathbf{E}_t^H\big[\mathbf{1}_{\{\mathcal{E}_k\}}\big] - \mathbf{E}_t\big[\mathbf{1}_{\{\mathcal{E}_k\}}\big]$$

is positive and determines the risk loading embedded in the premium calculation.
For the relations between this approach and the expected utility theory see Wang [1996]. ■

• Assumption 2

The risk-adjusted probability $\mathbf{H}$ concerning eliminations coincides with the natural probability $\mathbf{P}$; hence:

$$\mathbf{E}_t^H\big[\mathbf{1}_{\{\mathcal{E}_k\}}\big] = \mathbf{E}_t\big[\mathbf{1}_{\{\mathcal{E}_k\}}\big] = \mathbf{P}_t(Y_k; k)\,.$$

This assumption implies that a risk premium for uncertainty concerning eliminations is not required.

Remark. Assumption 2 could be relaxed by properly modelling elimination uncertainty and by defining suitable risk measures.
Other possible approach: *risk capital* for mortality risk. ■

Some properties of the value function

The no-arbitrage and perfect market assumptions imply some fundamental properties.

• Linearity:

For any a, b constant:

$$V(t;\, a\,X + b\,Y) = a\,V(t;X) + b\,V(t;Y)\,.$$

• "Reinvestment security theorem" [M, 1991]; [DF, M, S, 1993].

For $t \leq T \leq s$, let $I_{T,s}$ be the rate of return of a frictionless investment between the dates T and s.
If:

X_T is a random sum payable at time T,

and if:

$X_T\,(1 + I_{T,s})$ is payable at time s,

then:

$$V\big(t;\, X_T\,(1 + I_{T,s})\big) = V\big(t;\, X_T\big)\,.$$

Technical means and valuation factors

Under assumptions 1 and 2, one has:

$$\begin{aligned} V\big(t; \widetilde{Y}_k\big) &= \mathbf{P}_t(Y_k; k)\, \mathbf{E}_t^Q\big[Y_k\ \chi(t,k)\big] \\ &= \mathbf{P}_t(Y_k; k)\, V\big(t; Y_k\big)\,. \end{aligned}$$

Since $Y_k = C_t\, \Phi(t,k)$:

$$V\big(t; \widetilde{Y}_k\big) = \mathbf{P}_t(Y_k; k)\, C_t\ \mathbf{E}_t^Q\big[\Phi(t,k)\ \chi(t,k)\big]\,.$$

Let us define

• the *technical mean* of $\widetilde{Y}_k$ at time t:

$$\overline{Y}_{t,k} := \mathbf{E}_t\big[\mathbf{1}_{\{\mathcal{E}_k\}}\big]\, C_t = \mathbf{P}_t(C_k; k)\, C_t\,,$$

and

• the *valuation factor* over $[t,k]$:

$$u(t,k) := V\big(t; \Phi(t,k)\big) = \mathbf{E}_t^Q\big[\Phi(t,k)\ \chi(t,k)\big]\,.$$

Then:

$$V\big(t; \widetilde{Y}_k\big) = \overline{Y}_{t,k}\ u(t,k)\,.$$

Remarks

- The factor $u(t,k)$ is the time t price of a stochastic (indexed) zero coupon bond maturing at time k and with *face value* 1.
- The outstanding benefits is priced as a stochastic zcb with maturity k and face value $\overline{Y}_{t,k}$.
- The face value $\overline{Y}_{t,k}$ is determined by actuarial assumptions on the probability $\mathbf{P}$.
- The price $u(t,k)$ must be given by an arbitrage pricing model.
- Under the perfect market assumptions, the payoff Y_k can be **replicated** by a dynamic trading strategy and $C_t\, u(t,k)$ is the time t price of the replicating portfolio. ■

Valuation of payment streams

At time t, consider the payment streams generated by an outstanding policy (or by a ptf. of o.p.):

$$\widetilde{Y}_k := \mathbf{1}_{\{\mathcal{E}_k\}}\, C_t\, \Phi^Y(t,k)\,,$$

$$\widetilde{X}_k := \mathbf{1}_{\{\mathcal{E}_k\}}\, \Pi_t\, \Phi^X(t,k)\,,$$

for $k = t{+}1, t{+}2, \ldots, n$.

Under our assumptions:

$$V\big(t; \widetilde{\mathbf{Y}}\big) = \sum_{k=t+1}^{n} C_t\, \mathbf{P}_t(C_k; k)\, V\big(t; \Phi^Y(t,k)\big)\,,$$

$$V\big(t; \widetilde{\mathbf{X}}\big) = \sum_{k=t+1}^{n} \Pi_t\, \mathbf{P}_t(\Pi_k; k)\, V\big(t; \Phi^X(t,k)\big)\,.$$

That is:

$$V\big(t; \widetilde{\mathbf{Y}}\big) = \sum_{k=t+1}^{n} \overline{Y}_{t,k}\; u^Y(t,k)\,,$$

$$V\big(t; \widetilde{\mathbf{X}}\big) = \sum_{k=t+1}^{n} \overline{X}_{t,k}\; u^X(t,k)\,,$$

where $\{\overline{Y}_{t,k}\}$ and $\{\overline{X}_{t,k}\}$ are the technical means of benefits and premiums at time t, and:

$$u^Y(t,k) = V\big(t; \Phi^Y(t,k)\big)\,, \quad u^X(t,k) = V\big(t; \Phi^X(t,k)\big)\,,$$

are the correspondig valuation factors.

Remark. The outstanding benefits and premiums are priced as a portfolio of n stochastic zcbs with maturity $k = t{+}1, t{+}2, \ldots, n$ and face value $\overline{Y}_{t,k}$ and $\overline{X}_{t,k}$, resp. ■

Stochastic reserve

We define the *stochastic reserve* at time t as:

$$V_t = V\big(t; \widetilde{\mathbf{Y}}\big) - V\big(t; \widetilde{\mathbf{X}}\big)\,,$$

that is:

$$\begin{aligned} V_t = & \sum_{k=t+1}^{n} \mathbf{P}_t(Y_k; k)\ \ C_t\ \ V\big(t; \Phi^Y(t,k)\big) \\ & - \sum_{k=t+1}^{n} \mathbf{P}_t(X_k; k)\ \ \Pi_t\ \ V\big(t; \Phi^X(t,k)\big)\,. \end{aligned}$$

or:

$$V_t = \sum_{k=t+1}^{n} \overline{Y}_{t,k}\ \ u^Y(t,k) - \sum_{k=t+1}^{n} \overline{X}_{t,k}\ \ u^X(t,k)\,.$$

If the probabilities $\mathbf{P}$ are "first order" probabilities and if net premiums A_t are considered, then V_t is the *stochastic net premium reserve*.

• The stochastic reserve V_t represents the market price at time t of the **equivalent portfolio**, that is the portfolio of traded securities which replicates the stochastic liabilities $\overline{Y}_k - \overline{X}_k$ of the insurer.

Remark. Under the perfect market assumptions the trading strategy replicating the technical means $\overline{Y}_k - \overline{X}_k$ is a riskless strategy.
The replication of $\widetilde{Y}_k - \widetilde{X}_k$ involves a residual component of (non financial) risk. ∎

VBIF: the stochastic reserve approach

The value embedded at time t in the outstanding policy portfolio, the VBIF E_t, can be obtained by observing that:

- the market value of the future net liabilities $\widetilde{Y} - \widetilde{X}$ generated by the policies is given by V_t
- the net liability stream $\widetilde{Y} - \widetilde{X}$ is backed by the technical reserve which current level is R_t

$\rightarrow$ the VBIF can be derived as the difference between the time t value $V(t; R_t) = R_t$ of the assets backing the liabilities and the time t value $V(t; \widetilde{Y} - \widetilde{X})$ of these liabilities.

That is:

$$E_t = R_t - V_t \, .$$

Remark. The difference $E_t = R_t - V_t$ (if positive) is not immediately available to the insurer, but will be progressively delivered in the future as profits emerging during the life of the policies; however the present value of these profits, by no-arbitrage, must be equal to E_t. ■

Remark. Since the annual profits emerging during the life of the policies are also referred to as *technical gains*, E_t can also be interpreted as the market value, at time t, of the future technical gains. ■

Decomposition of VBIF

- *Different bases for actuarial valuations*

  – *first order* (using conservative probability $\mathbf{P}^{(1)}$);
  – *second order* (using realistic probability $\mathbf{P}^{(2)}$);
  – *third order* (realistic probability $\mathbf{P}^{(3)}$ including surrenders);
  – considering expense-loaded premiums Π;

Let us define:

$V_t^{(1)}$: net premium stochastic reserve on first order basis
$V_t^{(2)}$: net premium stochastic reserve on second order basis
$V_t^{(3)}$: net premium stochastic reserve on third order basis
$\widehat{V}_t^{(3)}$: gross premium stochastic reserve on third order basis

- *Components of the technical gains*

The VBIF:

$$\widehat{E}_t^{(3)} := R_t - \widehat{V}_t^{(3)}$$

is the market value of total technical gains (gross of expenses).

It can be decomposed as:

$R_t - V_t^{(1)}$: *investment gain*,

$V_t - V_t^{(2)}$: *mortality gain*,

$V_t^{(2)} - V_t^{(3)}$: *surrender gain*,

$V_t^{(3)} - \widehat{V}_t^{(3)}$: *value of loadings*.

Remark. Considering the first order technical means of benefits and net premiums and recalling the definition of R_t, the investment gain is given by:

$$E_t = \sum_{k=t+1}^{n} \overline{C}'_{t,k} \ \left[v^{k-t} - u^Y(t,k)\right] - \sum_{k=t+1}^{n} \overline{A}'_{t,k} \ \left[v^{k-t} - u^X(t,k)\right].$$

Chapter 3 – Basic finance of life insurance contracts

The financial structure of a policy
The embedded options
Put decomposition
Call decomposition
The minimum guarantees in a multiperiod contract
The return of the reference fund

The financial structure of a policy

Let us consider at time t an n-year life insurance contract with $\tau = n - t$ years to maturity.

Let us assume that the contract is *a single premium pure endowment* and avoid actuarial uncertainty assuming $\mathbf{P}_t(Y_n; n) = 1$;
let Y_t be the current value of the sum insured.
$\longrightarrow$ stochastic zcb with time to maturity τ and face value Y_t.

The terminal payoff Y_n of the contract is given by:

$$Y_n = Y_t \prod_{k=t+1}^{n} (1 + \rho_k) \,,$$

where:

$$\rho_k := \frac{\max\{\beta\, I_k,\, i\} - i}{1 + i} \,.$$

Recalling that the traditional net premium reserve is given by:

$$R_t = Y_t \, (1 + i)^{-\tau} \,,$$

the terminal value of the contract can be expressed as:

$$Y_n = R_t \prod_{k=t+1}^{n} \big(1 + \max\{\beta\, I_k,\, i\}\big) \,.$$

Interpretation. The terminal payoff is the $\beta\%$ of the result of the investment of the contractual reserve, with a minimum guaranteed return i at each year.

The embedded options

In the simple case $\beta = 1$ and $n = t + 1$ (one period contract):

$$Y_{t+1} = R_t \max\left\{\frac{F_{t+1}}{F_t},\ 1 + i\right\},$$

that is:

$$Y_{t+1} = N_t \max\{F_{t+1},\ M_{t+1}\},$$

where:

$$N_t = \frac{R_t}{F_t}$$

is the number of units of the reference fund purchased at time t with the amount R_t, and:

$$M_{t+1} := F_t\,(1 + i)\,,$$

is the minimum guaranteed terminal payoff.

Since:

$$\max\{x, y\} = x + \max\{y - x,\, 0\} = y + \max\{x - y,\, 0\}\,,$$

two meaningful decompositions of Y_n can be considered.

Put decomposition

$$Y_{t+1} = N_t \left(F_{t+1} + \max\{M_{t+1} - F_{t+1},\ 0\} \right).$$

Y_{t+1} is the sum of an investment component (the *base component*):

$$B_{t+1} = N_t\, F_{t+1}\,,$$

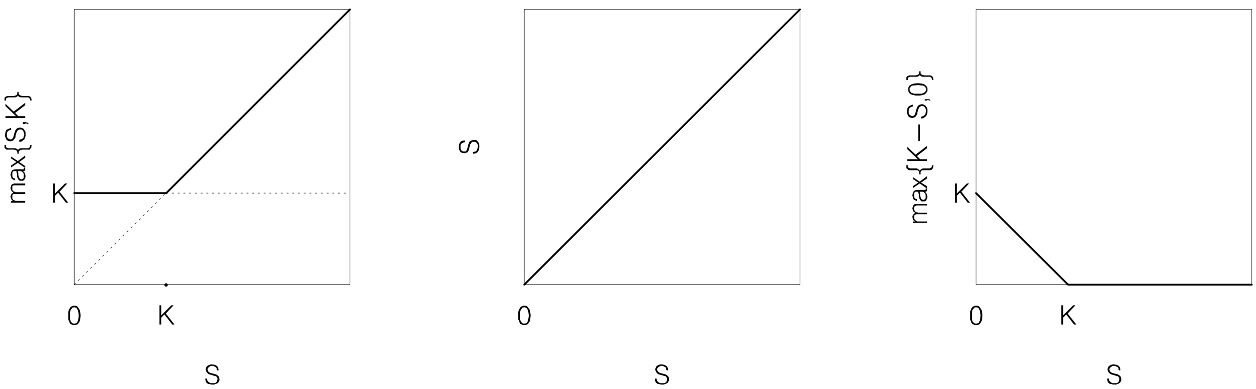
and of N_t protective european put options on the market value of the fund (the *put component*):

$$P_{t+1} = N_t\ \max\{M_{t+1} - F_{t+1},\ 0\}\ .$$

The time t value of the contract (the *stochastic reserve* V_t), is then given by:

$$V(t; Y_{t+1}) = V(t; B_{t+1}) + V(t; P_{t+1})\,,$$

i.e. as the sum of the *base value* B_t and the *price* P_t *of the minimum guarantee*.



Call decomposition

$$Y_{t+1} = N_t \left(M_{t+1} + \max\left\{F_{t+1} - M_{t+1},\ 0\right\} \right).$$

The payoff of the contract is the sum of a *guaranteed component*:

$$G_{t+1} = N_t\, M_{t+1} = R_t\,(1+i)\,,$$

and a *call component*:

$$C_{t+1} = N_t\ \max\left\{F_{t+1} - M_{t+1},\ 0\right\}\ .$$

Hence the stochastic reserve can be expressed as:

$$V(t; Y_{t+1}) = V(t; G_{t+1}) + V(t; C_{t+1})\,,$$

i.e. as the *present value G_t of the guaranteed terminal payoff* plus the price C_t of the call option which give the value of the excess return of the fund over the minimum guarantee.

The minimum guarantees in a multiperiod contract

Let us consider the more general case of a τ-year contract with participation coefficient β.
In a multiperiod setting the options embedded in the contract are *ratchet*-type options (*cliquet* options).
Nonetheless we can recover the put decomposition by comparing the contract with an analogous contract without minimum guarantee.
Defining the *base component* of Y_n:

$$B_n = R_t \prod_{k=t+1}^{n} \left(1 + \beta\, I_k\right),$$

the *value of minimum guarantees* is defined as:

$$P_t = V(t; Y_n) - V(t; B_n) = V_t - B_t .$$

The call decomposition can be also recovered by comparing the contract with an analogous contract with detrministic (i.e. completely guaranteed) payoff. Defining the *guaranteed component*:

$$M_n = R_t \prod_{k=t+1}^{n} \left(1 + i\right) = R_t\,(1+i)^{\tau} .$$

the call component is given by:

$$C_t = V(t; Y_n) - V(t; M_n) = V_t - G_t .$$

The return of the reference fund

The fund return in the period $[k-1, k]$:

$$I_k := \frac{F_k - F_{k-1}}{F_{k-1}} \, .$$

A critical point for determining the price $V(t)$ of the insurance contracts is the determination of the characteristics of the stochastic process $\{F_t\}$ representing the time t value of the reference fund.

Obviously F_t is strongly influenced by the market value of the securities composing the fund; however in many practical situations both F_t and I_t are contractually defined by accounting rules which can bias the true mark-to-market valuation.

Deferring the discussion of these possible distorsions, we shall assume here that F_t is the market value of the reference fund.

Chapter 4 – The valuation model. Value and risk measures

The valuation model

If the benefits are not indexed to real interest rates, price and (financial) risk of the outstanding policies can be derived by a a 2-factor arbitrage model.

We make the usual continuous-time perfect-market assumptions and we model the sources of uncertainty as diffusion processes.

$\longrightarrow$ diffusion model with two state variables

- *Interest rate uncertainty*
  Cox-Ingersoll-Ross type model.

State variable: r_t, the instantaneous nominal interest rate (spot rate).

The spot rate follows a mean-reverting square-root process:

$$dr_t = \alpha\,(\gamma - r_t)\,dt + \rho\,\sqrt{r_t}\;dZ_t^r\;,$$

where:

$\alpha > 0$: speed of adjustment (mean reversion coefficient);

$\gamma > 0$: long term rate;

$\rho > 0$: volatility parameter.

Properties:

$\oplus$ negative interest rates are precluded

$\oplus$ the variance of the interest rate is proportional to the level of interest rates

$\oplus$ the transition density for r_t is non-central chi-squared

$\oplus$ there is a steady state (Gamma) distribution for r_t (the steady state mean is $\mathbf{E}_t[r_\infty | r_t] = \gamma$)

$\ominus$ the long term rate is constant

$\ominus$ returns for different maturities are perfectly correlated

$\ominus$ the term structure of interest rates is endogenously given

The market price of interest rate risk is endogenously specified as the function:

$$q(r_t, t) = \pi \sqrt{r_t}/\rho\,, \quad \pi \text{ constant}\,.$$

$\oplus$ In *Dynamic Financial Analysis* (DFA) applications the CIR model is considered sufficiently "flexible, simple, well-specifed, realistic"

[Kaufmann, Gadmer, Klett, 2001]; [Rogers, 1995]

- *Stock price uncertainty*
  Black-Scholes type model.

State variable: S_t, the stock index.

The stock index follows a geometric brownian motion:

$$dS_t = \mu\, S_t\, dt + \sigma\, S_t\, dZ_t^S\,,$$

where:

μ: istantaneous expected return;

$\sigma > 0$: volatility parameter.

Properties:

$\oplus$ negative prices are precluded

· the transition density for S_t is lognormal

· the log-return process $\{\log(S_t/S_0)\}$ is a brownian motion $\Longrightarrow$ normality assumption on returns

$\ominus$ the volatility is constant

- The two sources of uncertainty are correlated:

$$\mathbf{Cov}[dZ_t^r,\, dZ_t^S] = \rho^{rS}\, dt\,.$$

Hedging argument and valuation equation

By the Markov property, the time t price of a traded security (or of a ptf of traded securities) is a function of the state variables:

$$V = V(t; r_t, S_t)\,.$$

Under the usual perfect market conditions the no-arbitrage argument leads to the general valuation equation:

$$\begin{aligned}&\frac{1}{2}\rho^2 r\,\frac{\partial^2 V}{\partial r^2} + \frac{1}{2}\sigma^2 S^2\,\frac{\partial^2 V}{\partial S^2} + \rho^{rS}\,\rho\,\sigma\,\sqrt{r}\,S\,\frac{\partial^2 V}{\partial r \partial S}\\&+ [\alpha(\gamma - r) + \pi\, r]\,\frac{\partial V}{\partial r} + rS\,\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} = r\,V\,.\end{aligned}$$

The valuation equation is valid for any (portfolio of) security; it must be solved under the appropriate boundary conditions.

The risk-neutral probability

The price V is not directly determined by the natural probability measure. It is derived by the risk-neutral measure $\mathbf{Q}$ which is obtained by substituting the original drifts coefficients:

$$\alpha\,(\gamma - r_t)\,, \qquad \mu\,S_t\,,$$

of $\{r_t\}$ e $\{S_t\}$, resp., by the risk-adjusted drifts:

$$\alpha\,(\gamma - r_t) + \pi\,r_t\,, \quad r_t\,S_t\,.$$

Remark. The risk-adjusted drift for $\{r_t\}$ can be written as:

$$\widehat{\alpha}\,(\widehat{\gamma} - r_t)\,,$$

where:

$$\widehat{\alpha} = \alpha - \pi\,, \qquad \widehat{\gamma} = \gamma\,[\alpha/(\alpha - \pi)]\,.$$

The probability $\mathbf{Q}$ is also referred to the *equivalent martingale measure.* ■

The endogenous term structure

The price $v(t,s)$ of the deterministic unit zcb with maturity s is obtained by solving the general valuation equation under the terminal condition $v(s,s)=1$
$\longrightarrow$ closed form solution:

$$v(t,t+\tau) = A(\tau)\, e^{-r_t\, B(\tau)}, \quad \tau > 0\,.$$

The yield curve is immediately derived as:

$$i(t,t+\tau) = \left[\frac{1}{v(t,t+\tau)}\right]^{\frac{1}{\tau}} - 1\,.$$

Remark. The functions $A(\tau)$ and $B(\tau)$ only depend on the parameters $\widehat{\alpha}$, $\widehat{\gamma}$ and ρ. ∎

Integral expression of prices

A fundamental martingale property provides the following expression for the arbitrage price V at time t of a random amount Y_s payable at time $s \geq t$:

$$V(t; Y_s) = \mathbf{E}_t^Q \big[Y_s \, \chi(t, s) \big] \, ,$$

where:

$$\chi(t, s) := e^{-\int_t^s r_u \, du} \, ,$$

is the stochastic discount factor over the interval $[t, s]$ and $\mathbf{E}_t^Q$ is the conditional expectation operator under the risk-neutral measure $\mathbf{Q}$.

$\oplus$ In particular:

$$v(t, s) = \mathbf{E}_t^Q \big[\chi(t, s) \big] \, .$$

• Using the integral expression the arbitrage price of complex securities can be easily obtained by Monte Carlo procedures.

[Boyle, Broadie, Glasserman, 1997]

Measures of basis risk

The (financial) risk inherent to the sources of uncertinty is expressed as sensitivity to the state variables.

- Interest rate risk:

$$\Omega^r(t;Y_s) := -\frac{\partial V(t;Y_s)}{V(t;Y_s)\,\partial r}\,.$$

For the deterministic zcb:

$$\Omega^r(t;1_s) := -\frac{\partial v(t,s)}{v(t,s)\,\partial r} = B(s-t)\,.$$

· The *stochastic duration* $D(t;Y_s)$ can be defined as the maturity of the deterministic zcb with the same risk of Y_s.
Hence (for $t=0$):

$$D(0;Y_s) = B^{-1}\big(\Omega^r\big)\,.$$

For a deterministic paiment stream $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$:

$$D(0;\mathbf{x}) = B^{-1}\left[\frac{\sum_{k=1}^{n} B(t_k)\,x_k\,v(0,t_k)}{\sum_{k=1}^{n} x_k\,v(0,t_k)}\right]\,.$$

Remark. The Macaulay duration is given by:

$$D^{\mathrm{Mc}}(0;\mathbf{x}) = \frac{\sum_{k=1}^{n} t_k\,x_k\,v(0,t_k)}{\sum_{k=1}^{n} x_k\,v(0,t_k)}\,.$$

■

- Stock price risk:

$$\Omega^S(t;Y_s) := \frac{\partial V(t;Y_s)}{V(t;Y_s)\,\partial S}\,.$$

$$\mathrm{Delta} = \frac{\partial V(t;Y_s)}{\partial S}\,.$$

Value-at-Risk

Assume there are given:

· a portfolio with value $V(t)$;
· a probability p^* ("confidence" level);
· a portfolio (ideal) unwinding period θ.

VaR: is the maximum loss in portfolio's value in period θ with probability p^*.

⊕ In the two-factor model: VaR induced by r_t, VaR induced by S_t.

• When the value function $V(r_t, S_t, t)$ is monotonic both with respect to r and S, the VaRs can be directly obtained from percentiles of the probability distributions of r and S (*underlying percentile* method).

• if V is monotonic decreasing with respect to r, the interest rate VaR is given by:

$$V(r_t, t) - V(r_t^*,\, t{+}\theta)\,,$$

where r_t^* is the "underlying percentile": $\mathbf{P}(r_{t+\theta} \leq r_t^* | r_t) = p^*$;
$\mathbf{P}$ is the non central chi-squared distribution of the CIR model.

• if V is monotonic increasing with respect to S, the stock-market VaR is:

$$V(S_t, t) - V(S_t^*,\, t{+}\theta)\,,$$

where S_t^* is implicitly defined by: $\mathbf{P}(S_{t+\theta} > S_t^* | S_t) = p^*$,
$\mathbf{P}$ is the BS's model lognormal distribution.

⊕ VaR has to be evaluated trough the distributions describing the natural probabilities.

⊕ When θ is very small, one usually neglects in VaR calculations the deterministic price changes (time decay); zero-mean increments of r and S are furthermore assumed.

$\oplus$ In alm schemes, separate VaR calculations on asset and liabilities do not suffices:
$\longrightarrow$ VaR calculation on the net portfolio.
In particular, one does not know *a priori* the sign of the net value sensitivity to downward movements of interest rates; hence one cannot identify before calculations the direction of adverse market moves.

Risk Capital

Extending VaR definition to "long" horizons θ (e.g. one year), the "maximum potential loss with probability p^*" notion becomes a useful quantity in strategic planning (risk capital, economic capital).

In this setting p^* can be chosen with respect to a target rating level. e.g. for an "A" rating: $\theta = 1$ year, $p^* = 0.9993$.

Due to the length of time horizon θ, the effects of time decay and of expected changes in market variables cannot be neglected.
$\longrightarrow$ definition as "unexpected adverse deviation from expected value (best estimate)".

$\oplus$ In alm schemes, as for VaR, separate RC analysis of asset and liabilities, is not enough;
in particular, only after netting one can identify the sign of value sensitivity to downward movements of interest rates, hence identifying the direction of adverse market moves.

Interest Rate Risk Capital (hedging)
1-year 99.93% maximum losses
{market values}
million Euro

	a.m.m.	asset	investments	premiums	liability	asset-liability
life	+350	+255	+157	+98	-331	**-76**
life	-240	-190	-117	-73	+244	**+54**
life	0	{+6.500}	{+4.000}	{+2.500}	{-6.000}	**{+500}**
life netting	-240	-190	-117	-73	+244	**+54**

Chapter 5 – Applying the valuation model

Calibration of the valuation model

• The risk-neutral parameters must be estimated in order that for traded securities the theoric prices provided by the model are as close as possible to the observed prices.

• In the CIR model closed form expressions are available for the price of coupon bonds and of simple types of interest rate options usually traded on the market.

$\longrightarrow$ at time t the parameters $\widehat{\alpha}, \widehat{\gamma}$ and ρ can be estimated by minimizing the s.s.e. between model and market prices, over a given cross-section of observed prices.

A typical cross-section:
LIBOR *interest rate swaps* of different maturities + a sample of *interest rate caps*.

Remark. In practical applications the spot rate r_t (not observable) is often included in the estimation procedure. ■

• The implementation of the calibration procedure provides a complete specification of the risk-neutral measure **Q** which is appropriate at time t for the valuation of non traded contracts.

Remark. It is worth to stress that the expected rate of return μ of the stock index and the (natural) expectations on the future interest rates do not enter into the calibration procedure, so they are irrelevant for the no-arbitrage valuation. ■

[DF, M, 1991b]; [P, 1999]

Computation of the valuation factors

Using the integral expression we have:

$$u^X(t,k) = \mathbf{E}_t^Q \left[\Phi^X(t,k)\, e^{-\int_t^k r_u\, du} \right],$$

$$u^Y(t,k) = \mathbf{E}_t^Q \left[\Phi^Y(t,k)\, e^{-\int_t^k r_u\, du} \right].$$

$\longrightarrow$ the valuation factors can be derived by numerical methods.

- Monte Carlo simulation:
  - define a discrete time equivalent of the risk-neutral s.d.e.s:

$$dr_t = \widehat{\alpha}\,(\widehat{\gamma} - r_t)\, dt + \rho\, \sqrt{r_t}\, dZ_t^r ,$$

$$dS_t = r_t\, S_t\, dt + \sigma\, S_t\, dZ_t^S ,$$

    with $\mathbf{Cov}[dZ_t^r,\, dZ_t^S] = \rho^{rS}\, dt$;
  - generate a (discrete) sample-path for r and S, from the (exogenous) starting values r_t and S_t to r_n and S_n;
  - calculate along the paths the annual values of I_k and derive the corresponding values of $\Phi(t,k)$;
  - calculate along the paths the (discrete equivalent of the) discount factors $\exp\big(-\int_t^k r_u\, du\big)$;
  - compute $\Phi(t,k)\exp\big(-\int_t^k r_u\, du\big)$ for each k and save these discounted values;
  - iterate the procedure N times and derive the valuation factors $u(t,k)$ as the average of the N discounted values.

- using displaced values of the starting values r_t and S_t one can obtain numerical derivatives of prices which can be used to compute the relevant risk measures.

Term structures of valuation rates

One can define the valuation rates:

$$j^X(t,k) = \left[\frac{1}{u^X(t,k)}\right]^{\frac{1}{k-t}} - 1\,, \qquad j^Y(t,k) = \left[\frac{1}{u^Y(t,k)}\right]^{\frac{1}{k-t}} - 1\,.$$

In general it is interesting to compare these "term structures" :

• with the risk-free term structure:

$$i(t,k) := \left[\frac{1}{v(t,k)}\right]^{\frac{1}{k-t}} - 1\,,$$

prevailing on the market at time t,

• with the "flat term structure" corresponding to the technical interest rate i.

The return of the reference fund

The fund return in the period $[k-1, k]$:

$$I_k := \frac{F_k - F_{k-1}}{F_{k-1}} .$$

A critical point for determining the price $V(t)$ of the insurance contracts is the determination of the characteristics of the stochastic process $\{F_t\}$ representing the market value of the reference fund.

Since the fund can be composed by bonds and equities, we assume:

$$F_t := \alpha\, S_t + (1-\alpha)\, W_t\,, \quad 0 \le \alpha \le 1\,,$$

where:

S_t is a stock index,

W_t is a bond index.

- The stock index

The process $\{S_t\}$ can be modelled as a geometric brownian motion, as in the Black and Scholes model.

- The bond index

W_t must be chosen as similar as possible to the results of a trading strategy which is considered feasible by the fund manager.
A possible choice is to model $W(t)$ as the cumulated results of a buy–and–sell strategy, with a fixed trading horizon Δt, of coupon bonds with a fixed duration $D \ge \Delta t$.

The results of the valuation procedure significantly depend on the assumptions on D and Δt.

For a short–term roll–over strategy (e.g. $\Delta t = D = 3$ months), W_t displays smooth sample paths with high dispersion in the long run.

For a buy–and–sell strategy of coupon bonds with mid/long duration ($\Delta t = 3$ months, $D = 4, 10$ years), the sample paths display greater local volatility, but a reduced long–run dispersion.

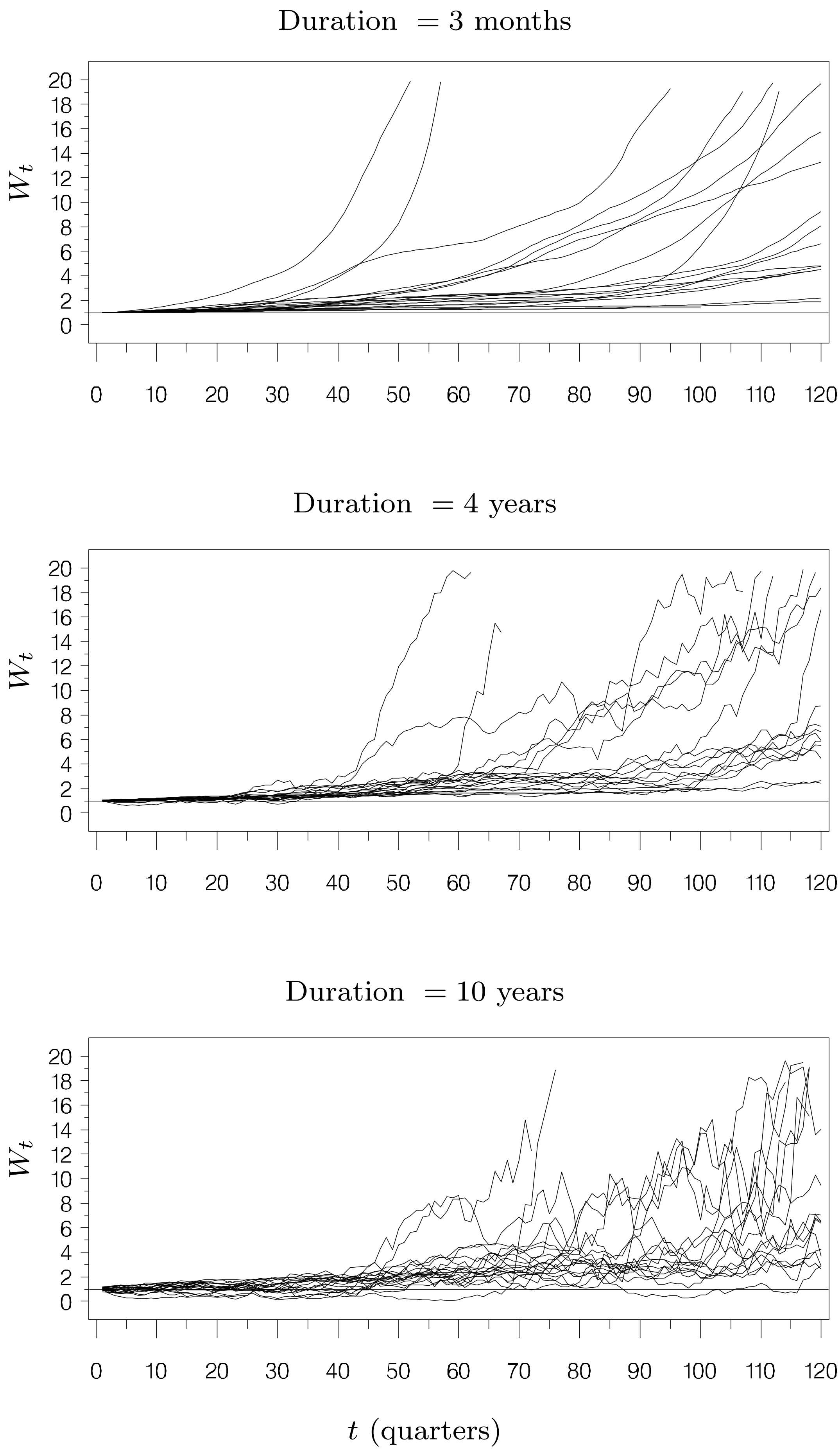
This behavior is consistent with the empirical findings that short term rates are more volatile than long–term rates.

Remark. This effect could be enhanced by the strong degree of mean reversion which characterizes the 1–factor CIR model.

• Sample paths of W_t under the 1–factor CIR model.

Same parameters

Buy–and–sell strategy of zcb's with maturity D, with trading horizon $\Delta t = 3$ months.


Duration = 3 months
W_t
20
18
16
14
12
10
8
6
4
2
0
0 10 20 30 40 50 60 70 80 90 100 110 120
Duration = 4 years
W_t
20
18
16
14
12
10
8
6
4
2
0
0 10 20 30 40 50 60 70 80 90 100 110 120
Duration = 10 years
W_t
20
18
16
14
12
10
8
6
4
2
0
0 10 20 30 40 50 60 70 80 90 100 110 120
t (quarters)

• The effect of the assumptions on W_t can be illustrated by computing the price of the protective puts in the contract with payoff:

$$Y_n = 100 \prod_{k=1}^{n} (1 + \rho_k)\,,$$

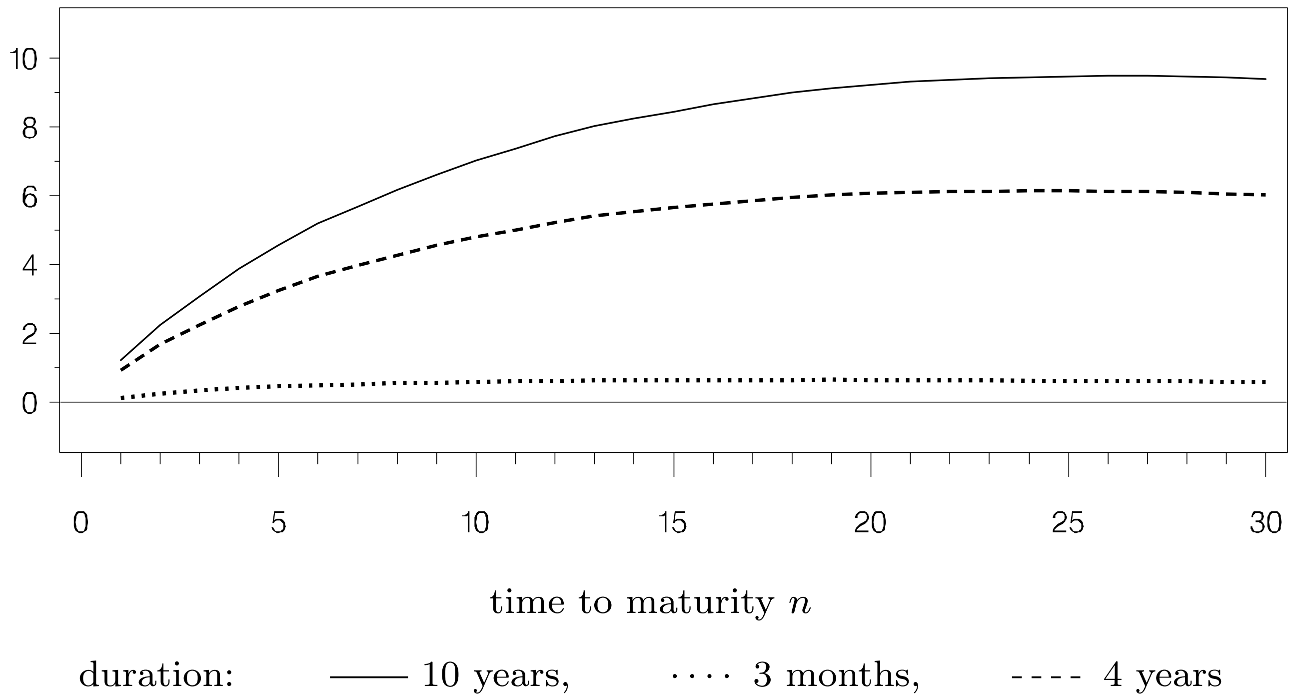
where:

$$\rho_k := \frac{\max\{0.8\, I_k,\, 0.03\} - 0.03}{1.03}$$

and $I_k = W_k / W_{k-1} - 1$, for long maturities ($\Delta t = 3$ months, $k \leq 30$ years).

Value of ratchet protective puts

$C = 100$, valuation date = 30/12/1999



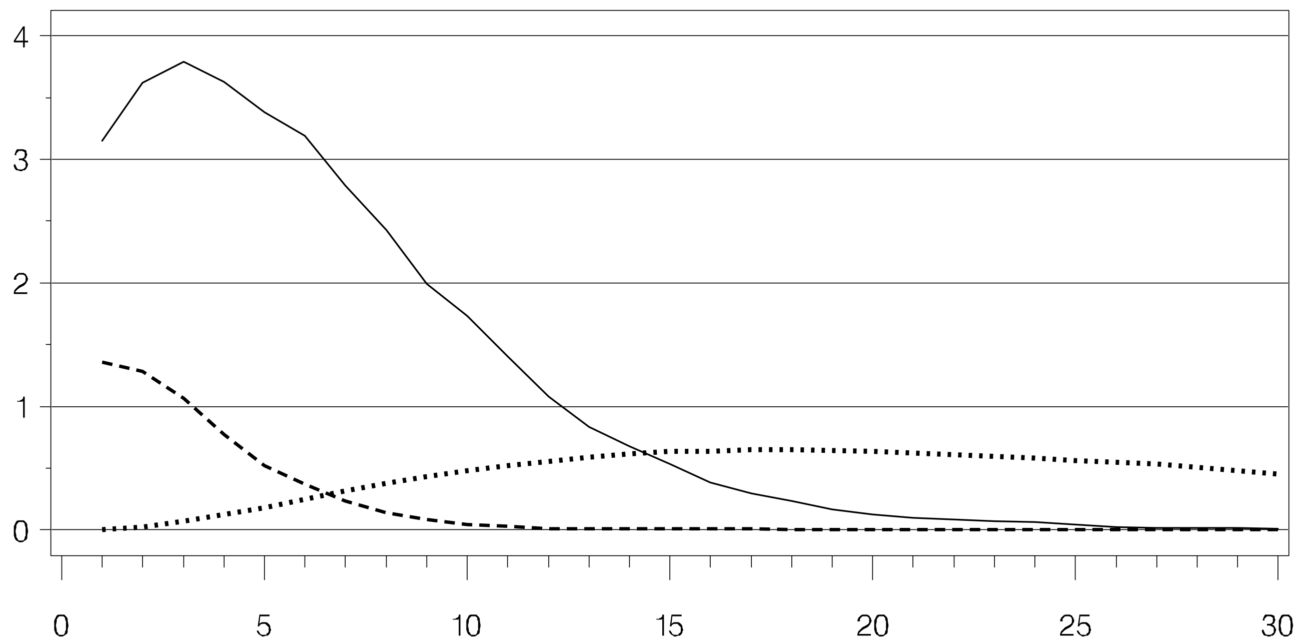
• Price of maturity guarantees

Without the ratchet mechanism (and with $\beta = 1$) the payoff reduces to:

$$Y_k = 100 \max\left\{1.03^k - \frac{W_k}{W_0}, 0\right\}.$$

Value of non ratchet protective puts

$C = 100$, valuation date = 30/12/1999



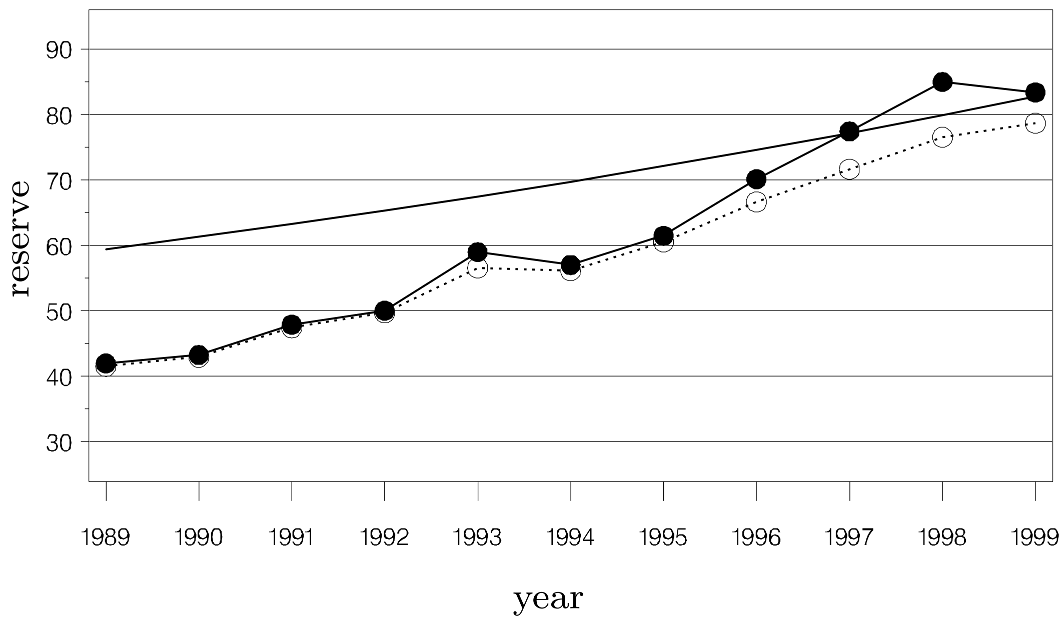
time to maturity n

duration: —— 10 years, ···· 3 months, ---- 4 years

Valuation during the life of the policy

- single premium pure endowment (issued at the end of 1989):
  $n = 15$ ys, $x = 40$ ys, $C_0 = 100$, $i = 3\%$, $\mathbf{P}'$: SIM81;
  participation coefficient: $\beta = 80\%$.
- characteristics of the underlying:
  $\alpha = 20\%$, $D = 4$ ys, $\Delta t = 1$ mth, $\sigma = 20\%$.
- valuation under changing market coditions:

year	ytm	R	V	E^I	$E^I\%$	P	$P\%$
29/12/89	15	64,19	45,36	18,83	29,3	0,50	1,1
31/12/90	14	66,11	46,67	19,44	29,4	0,37	0,8
31/12/91	13	68,10	51,56	16,54	24,3	0,52	1,0
31/12/92	12	70,14	53,70	16,44	23,4	0,37	0,7
31/12/93	11	72,24	63,02	9,22	12,8	2,60	4,1
30/12/94	10	74,41	60,74	13,67	18,4	0,96	1,6
29/12/95	9	76,64	65,39	11,25	14,7	0,98	1,5
31/12/96	8	78,94	74,39	4,55	5,8	3,66	4,9
31/12/97	7	81,31	81,74	-0,43	-0,5	6,05	7,4
30/12/98	6	83,75	89,02	-5,27	-6,3	8,78	9,9
30/12/99	5	86,26	86,96	-0,70	-0,8	4,88	5,6



—— technical (R), ●–●–● stochastic (V), ○··○··○ base ($V - P$)

Cost of the embedded put and expected fund's return

At the policy issuance the minimum return guarantees are far below the current level of market returns.

Remark. Article 18 of the Third EU Life Insurance Directive requires that interest rate guarantees do not exceed 60% of the rate on government debt. ■

In many cases at the issue the insurer can adopt a conservative investment strategy which strongly reduces the value of the embedded option.
However these strategies usually imply a lower level of expected return on the reference fund

$\longrightarrow$ trade-off between the cost of the put option and the expected return on the investment.

- Single premium pure endowments (with $\mathbf{P}_t(Y_n; n) = 1$):
  $n = 10$ years, $\beta = 80\%$, $i = 3\%$ and 4%.

⊙ Issue date: 31/12/1998

Bond component: 10 years gvt bond (BTP), coupon 4.2470%

Stock component: $\sigma = 20\%$

Parameter values for natural expectation: $\gamma = 0.05$, $\mu = 0.12$.

⊙ Issue date: 31/12/2000

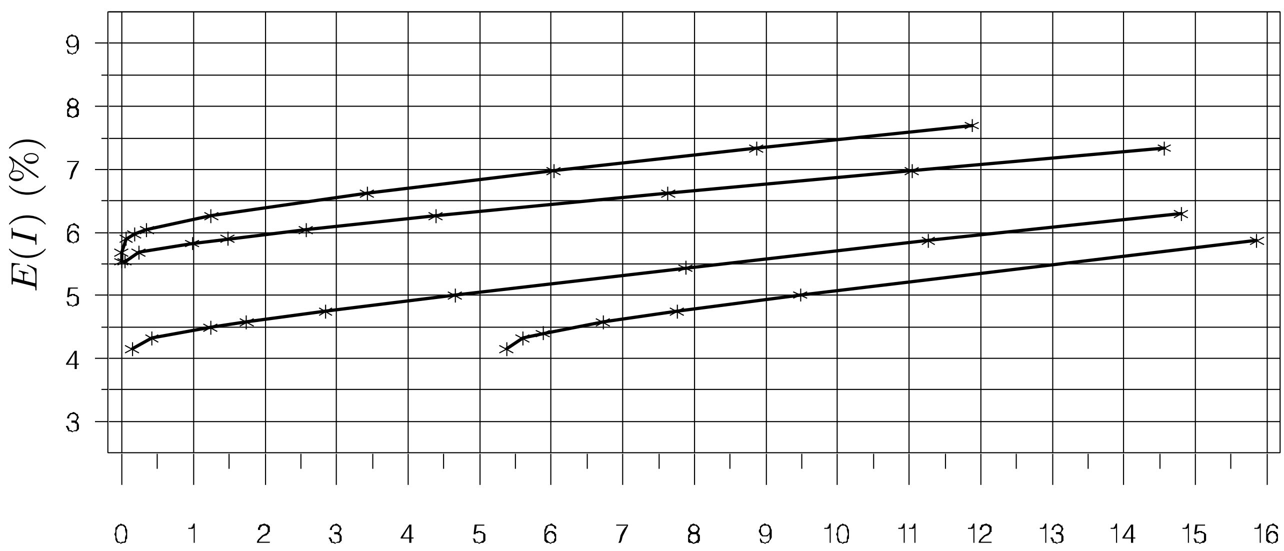
Bond component: 10 years gvt bond (BTP), coupon 5.5086%

Stock component: $\sigma = 20\%$

Parameter values for natural expectation: $\gamma = 0.06$, $\mu = 0.12$.

$\mathbf{E}(I)$ vs put

30/12/1998 (3% and 4%) and 29/12/2000 (3% and 4%) – $0\% \leq \alpha \leq 30\%$



More details about market effects

Single premium pure endowments (with $\mathbf{P}_t(Y_n; n) = 1$)

XY policy – **valuation date 02/01/1990**
Contractual details

NITERAZ	PASSI	MAT	ITEC	SMIN	IMIN	BETA	C0	QAZ	PASORDET	DURRIF
10000	12	20	0.04	0	0.04	0.8	100	0.2	1	4

XY policy - valuation date 02/01/1990
Parameters of stochastic processes

R0	DI	FI	NI	RO
0.13190	0.26428	0.25859	22.8793	0.054243

S0	SIGMAS	RORS	ALFA	GAMMA	PAI
100	0.25	-0.1	0.67318	0.05	0.42027

XY policy - valuation date 02/01/1990
Valuation factors and rates

mat.	valuation factor	valuation rate	market rate
1	0.93997	6.38598	14.1095
2	0.88357	6.38456	14.1085
3	0.82994	6.41070	14.1014
4	0.78074	6.38332	14.0912
5	0.73503	6.35023	14.0796
6	0.69188	6.33136	14.0677
7	0.65107	6.32233	14.0562
8	0.61259	6.31732	14.0453
9	0.57602	6.32084	14.0352
10	0.54202	6.31595	14.0260
11	0.50982	6.31598	14.0176
12	0.47958	6.31503	14.0100
13	0.45114	6.31418	14.0031
14	0.42424	6.31605	13.9969
15	0.39914	6.31437	13.9913
16	0.37549	6.31324	13.9862
17	0.35293	6.31801	13.9816
18	0.33219	6.31373	13.9774
19	0.31271	6.30938	13.9736
20	0.29421	6.30831	13.9702

XY policy - valuation date 02/01/1990
Benefit valuation

mat.	technical rate discounting	stochastic model	base value	put value	market rates discounting
1	96.1538	93.9973	93.8487	0.14864	87.6351
2	92.4556	88.3574	88.0549	0.30247	76.8005
3	88.8996	82.9935	82.5434	0.45009	67.3174
4	85.4804	78.0739	77.4931	0.58078	59.0190
5	82.1927	73.5035	72.8164	0.68708	51.7559
6	79.0315	69.1882	68.3970	0.79118	45.3965
7	75.9918	65.1071	64.2249	0.88226	39.8261
8	73.0690	61.2587	60.2933	0.96539	34.9446
9	70.2587	57.6016	56.5716	1.02998	30.6654
10	67.5564	54.2021	53.1207	1.08136	26.9130
11	64.9581	50.9819	49.8543	1.12766	23.6216
12	62.4597	47.9583	46.7916	1.16675	20.7341
13	60.0574	45.1143	43.9257	1.18861	18.2005
14	57.7475	42.4245	41.2183	1.20620	15.9771
15	55.5265	39.9136	38.6900	1.22354	14.0257
16	53.3908	37.5494	36.3181	1.23124	12.3130
17	51.3373	35.2926	34.0574	1.23514	10.8096
18	49.3628	33.2194	31.9901	1.22933	9.4899
19	47.4642	31.2709	30.0462	1.22468	8.3314
20	45.6387	29.4209	28.2069	1.21404	7.3143

XY policy - valuation date 02/01/1990
Sensitivity and duration

mat.	sensitivity	stochastic duration	delta
1	1.65866	2.15533	0.14522
2	1.79867	2.40614	0.13652
3	1.90886	2.61564	0.12828
4	1.99323	2.78407	0.12068
5	2.05598	2.91429	0.11359
6	2.10523	3.01966	0.10694
7	2.14368	3.10397	0.10064
8	2.17338	3.17037	0.09470
9	2.19632	3.22247	0.08904
10	2.21387	3.26280	0.08380
11	2.22729	3.29392	0.07882
12	2.23767	3.31817	0.07416
13	2.24550	3.33656	0.06976
14	2.25178	3.35138	0.06563
15	2.25635	3.36221	0.06173
16	2.25988	3.37058	0.05809
17	2.26256	3.37695	0.05459
18	2.26423	3.38091	0.05138
19	2.26548	3.38391	0.04835
20	2.26672	3.38685	0.04549

XY policy - **valuation date 31/12/1996**
Contractual details

NITERAZ	PASSI	MAT	ITEC	SMIN	IMIN	BETA	C0	QAZ	PASORDET	DURRIF
10000	12	20	0.04	0	0.04	0.8	100	0.2	1	4

XY policy - valuation date 31/12/1996
Parameters of stochastic processes

R0	DI	FI	NI	RO
0.060784	0.20892	0.19962	9.09257	0.060943

S0	SIGMAS	RORS	ALFA	GAMMA	PAI
100	0.25	-0.1	0.33770	0.05	0.14738

XY policy - valuation date 31/12/1996
Valuation factors and rates

mat.	valuation factor	valuation rate	market rate
1	0.96235	3.91276	6.52909
2	0.92479	3.98674	6.75505
3	0.88694	4.08039	6.95058
4	0.85070	4.12508	7.12043
5	0.81570	4.15833	7.26857
6	0.78146	4.19555	7.39825
7	0.74814	4.23229	7.51220
8	0.71593	4.26562	7.61270
9	0.68457	4.30068	7.70164
10	0.65462	4.32807	7.78062
11	0.62574	4.35409	7.85101
12	0.59806	4.37702	7.91393
13	0.57139	4.39925	7.97036
14	0.54581	4.41971	8.02114
15	0.52152	4.43565	8.06696
16	0.49817	4.45135	8.10843
17	0.47552	4.46958	8.14608
18	0.45437	4.47991	8.18036
19	0.43424	4.48809	8.21165
20	0.41485	4.49739	8.24028

XY policy - valuation date 31/12/1996
Benefit valuation

mat.	technical rate discounting	stochastic model	base value	put value	market rates discounting
1	96.1538	96.2346	94.9711	1.26346	93.8711
2	92.4556	92.4792	90.1115	2.36766	87.7452
3	88.8996	88.6938	85.3706	3.32318	81.7430
4	85.4804	85.0704	80.9553	4.11518	75.9470
5	82.1927	81.5699	76.8002	4.76966	70.4105
6	79.0315	78.1457	72.8095	5.33622	65.1654
7	75.9918	74.8142	68.9907	5.82347	60.2276
8	73.0690	71.5931	65.3411	6.25201	55.6022
9	70.2587	68.4567	61.8428	6.61391	51.2860
10	67.5564	65.4618	58.5653	6.89657	47.2708
11	64.9581	62.5742	55.4307	7.14352	43.5445
12	62.4597	59.8055	52.4602	7.34526	40.0931
13	60.0574	57.1392	49.6553	7.48395	36.9012
14	57.7475	54.5815	46.9773	7.60417	33.9530
15	55.5265	52.1517	44.4559	7.69575	31.2325
16	53.3908	49.8167	42.0686	7.74813	28.7241
17	51.3373	47.5524	39.7685	7.78390	26.4129
18	49.3628	45.4370	37.6540	7.78302	24.2845
19	47.4642	43.4241	35.6508	7.77331	22.3251
20	45.6387	41.4850	33.7369	7.74809	20.5221

XY policy - valuation date 31/12/1996
Sensitivity and duration

mat.	sensitivity	stochastic duration	delta
1	1.39679	1.62637	0.10114
2	1.71541	2.08229	0.09732
3	1.96993	2.47832	0.09345
4	2.16646	2.80683	0.08969
5	2.32018	3.07959	0.08604
6	2.44294	3.30857	0.08251
7	2.54277	3.50280	0.07902
8	2.62383	3.66626	0.07567
9	2.68946	3.80264	0.07239
10	2.74225	3.91509	0.06927
11	2.78486	4.00776	0.06621
12	2.81944	4.08424	0.06333
13	2.84762	4.14745	0.06052
14	2.87082	4.20008	0.05784
15	2.88922	4.24224	0.05528
16	2.90369	4.27565	0.05284
17	2.91562	4.30335	0.05045
18	2.92509	4.32548	0.04821
19	2.93276	4.34345	0.04605
20	2.93919	4.35857	0.04400

XY policy - **valuation date 30/12/1998**
Contractual details

NITERAZ	PASSI	MAT	ITEC	SMIN	IMIN	BETA	C0	QAZ	PASORDET	DURRIF
10000	12	20	0.04	0	0.04	0.8	100	0.2	1	4

XY policy - valuation date 30/12/1998
Parameters of stochastic processes

R0	DI	FI	NI	RO
0.028358	0.084769	0.035853	1.57251	0.059225

S0	SIGMAS	RORS	ALFA	GAMMA	PAI
100	0.25	-0.1	0.055157	0.05	0.068220

XY policy - valuation date 30/12/1998
Valuation factors and rates

mat.	valuation factor	valuation rate	market rate
1	0.98251	1.78055	3.03641
2	0.96394	1.85319	3.19426
3	0.94383	1.94564	3.34962
4	0.92349	2.00981	3.50218
5	0.90273	2.06778	3.65161
6	0.88105	2.13316	3.79764
7	0.85892	2.19637	3.94003
8	0.83645	2.25741	4.07855
9	0.81346	2.32045	4.21304
10	0.79036	2.38051	4.34334
11	0.76731	2.43709	4.46934
12	0.74413	2.49339	4.59096
13	0.72092	2.54912	4.70817
14	0.69797	2.60170	4.82094
15	0.67534	2.65145	4.92929
16	0.65282	2.70117	5.03325
17	0.63019	2.75329	5.13290
18	0.60849	2.79832	5.22831
19	0.58748	2.83913	5.31957
20	0.56664	2.88090	5.40681

XY policy - valuation date 30/12/1998
Benefit valuation

mat.	technical rate discounting	stochastic model	base value	put value	market rates discounting
1	96.1538	98.2506	95.5822	2.6684	97.0531
2	92.4556	96.3942	91.2913	5.1028	93.9050
3	88.8996	94.3831	87.0593	7.3237	90.5885
4	85.4804	92.3490	83.0958	9.2532	87.1369
5	82.1927	90.2728	79.3273	10.9455	83.5834
6	79.0315	88.1048	75.6625	12.4423	79.9604
7	75.9918	85.8918	72.1209	13.7710	76.2992
8	73.0690	83.6454	68.6873	14.9581	72.6290
9	70.2587	81.3463	65.3571	15.9892	68.9766
10	67.5564	79.0364	62.1870	16.8494	65.3661
11	64.9581	76.7309	59.1334	17.5975	61.8191
12	62.4597	74.4132	56.1959	18.2173	58.3539
13	60.0574	72.0916	53.3937	18.6979	54.9861
14	57.7475	69.7969	50.6852	19.1117	51.7282
15	55.5265	67.5342	48.1097	19.4244	48.5903
16	53.3908	65.2820	45.6462	19.6358	45.5796
17	51.3373	63.0192	43.2464	19.7728	42.7015
18	49.3628	60.8488	41.0194	19.8293	39.9589
19	47.4642	58.7476	38.8970	19.8506	37.3531
20	45.6387	56.6637	36.8558	19.8079	34.8840

XY policy - valuation date 30/12/1998
Sensitivity and duration

mat.	sensitivity	stochastic duration	delta
1	1.43915	1.42749	0.068134
2	2.02386	2.00225	0.067025
3	2.59153	2.55838	0.065823
4	3.11803	3.07306	0.064552
5	3.62114	3.56435	0.063245
6	4.10384	4.03561	0.061894
7	4.56845	4.48949	0.060505
8	5.01767	4.92891	0.059092
9	5.45201	5.35462	0.057611
10	5.87727	5.77248	0.056131
11	6.27844	6.16789	0.054592
12	6.66650	6.55170	0.053097
13	7.04037	6.92293	0.051562
14	7.40167	7.28319	0.050021
15	7.74526	7.62736	0.048500
16	8.07645	7.96070	0.046995
17	8.40227	8.29029	0.045454
18	8.70567	8.59881	0.043959
19	9.00325	8.90304	0.042497
20	9.29604	9.20405	0.041074

XY policy - **valuation date 07/09/2001**
Contractual details

NITERAZ	PASSI	MAT	ITEC	SMIN	IMIN	BETA	C0	QAZ	PASORDET	DURRIF
10000	12	20	0.04	0	0.04	0.8	100	0.2	1	4

XY policy - valuation date 07/09/2001
Parameters of stochastic processes

R0	DI	FI	NI	RO
0.032150	0.25967	0.25635	18.9307	0.041231

S0	SIGMAS	RORS	ALFA	GAMMA	PAI
100	0.25	-0.1	0.32182	0.05	0.068787

XY policy - valuation date 07/09/2001
Valuation factors and rates

mat.	valuation factor	valuation rate	market rate
1	0.97642	2.41530	3.64537
2	0.94980	2.60885	3.96559
3	0.92079	2.78884	4.23773
4	0.89139	2.91617	4.46984
5	0.86179	3.01952	4.66857
6	0.83209	3.11102	4.83939
7	0.80264	3.19062	4.98681
8	0.77373	3.25865	5.11456
9	0.74518	3.32203	5.22573
10	0.71776	3.37179	5.32289
11	0.69093	3.41816	5.40815
12	0.66488	3.45969	5.48329
13	0.63973	3.49596	5.54979
14	0.61533	3.52940	5.60889
15	0.59199	3.55687	5.66161
16	0.56943	3.58218	5.70884
17	0.54738	3.60836	5.75131
18	0.52660	3.62712	5.78965
19	0.50671	3.64280	5.82437
20	0.48737	3.65905	5.85594

XY policy - valuation date 07/09/2001
Benefit valuation

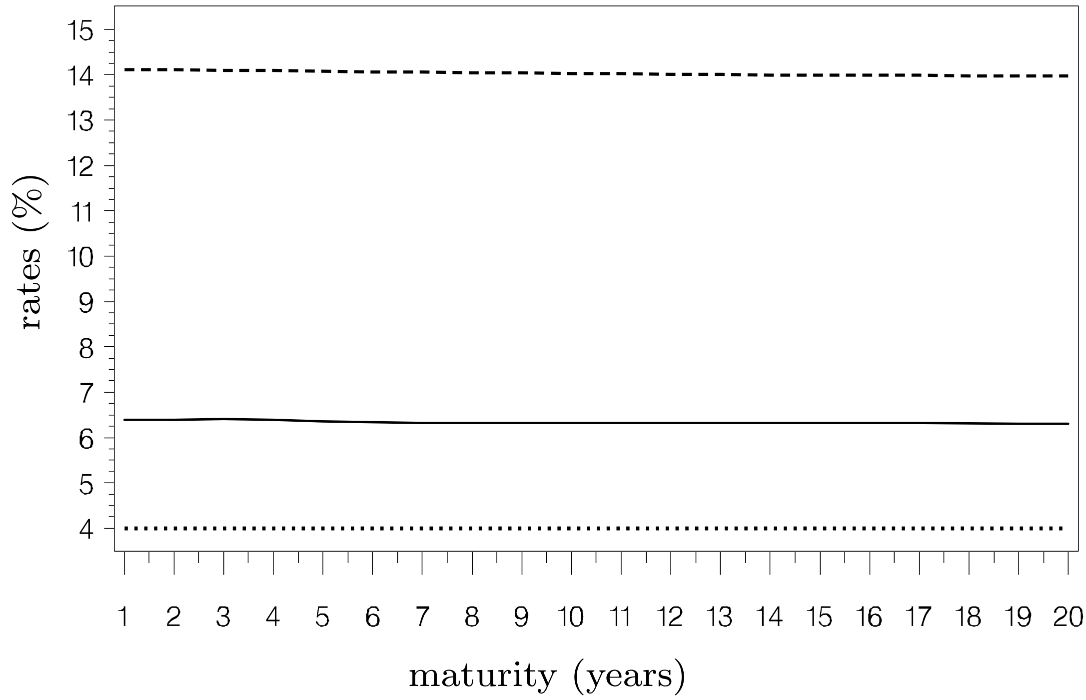
mat.	technical rate discounting	stochastic model	base value	put value	market rates discounting
1	96.1538	97.6417	95.4396	2.2021	96.4828
2	92.4556	94.9796	90.9859	3.9937	92.5168
3	88.8996	92.0793	86.5990	5.4803	88.2928
4	85.4804	89.1385	82.4636	6.6750	83.9530
5	82.1927	86.1792	78.5256	7.6536	79.6010
6	79.0315	83.2088	74.7295	8.4793	75.3101
7	75.9918	80.2636	71.0836	9.1800	71.1307
8	73.0690	77.3729	67.5809	9.7920	67.0961
9	70.2587	74.5184	64.2059	10.3126	63.2270
10	67.5564	71.7761	61.0430	10.7331	59.5350
11	64.9581	69.0932	58.0016	11.0916	56.0253
12	62.4597	66.4884	55.1018	11.3866	52.6982
13	60.0574	63.9728	52.3561	11.6167	49.5512
14	57.7475	61.5330	49.7250	11.8080	46.5794
15	55.5265	59.1993	47.2364	11.9629	43.7766
16	53.3908	56.9428	44.8789	12.0639	41.1357
17	51.3373	54.7380	42.5911	12.1468	38.6493
18	49.3628	52.6597	40.4814	12.1782	36.3097
19	47.4642	50.6706	38.4783	12.1924	34.1092
20	45.6387	48.7366	36.5483	12.1883	32.0402

XY policy - valuation date 07/09/2001
Sensitivity and duration

mat.	sensitivity	stochastic duration	delta
1	1.12773	1.32842	0.071163
2	1.53663	1.94795	0.069274
3	1.83923	2.47922	0.067219
4	2.05673	2.91131	0.065066
5	2.21927	3.26869	0.062900
6	2.34178	3.56154	0.060754
7	2.43483	3.79970	0.058612
8	2.50559	3.99110	0.056538
9	2.55971	4.14414	0.054459
10	2.60042	4.26337	0.052461
11	2.63172	4.35758	0.050485
12	2.65581	4.43167	0.048623
13	2.67471	4.49081	0.046800
14	2.68926	4.53697	0.045046
15	2.70002	4.57147	0.043354
16	2.70815	4.59774	0.041718
17	2.71445	4.61821	0.040087
18	2.71922	4.63379	0.038568
19	2.72297	4.64606	0.037105
20	2.72583	4.65545	0.035704

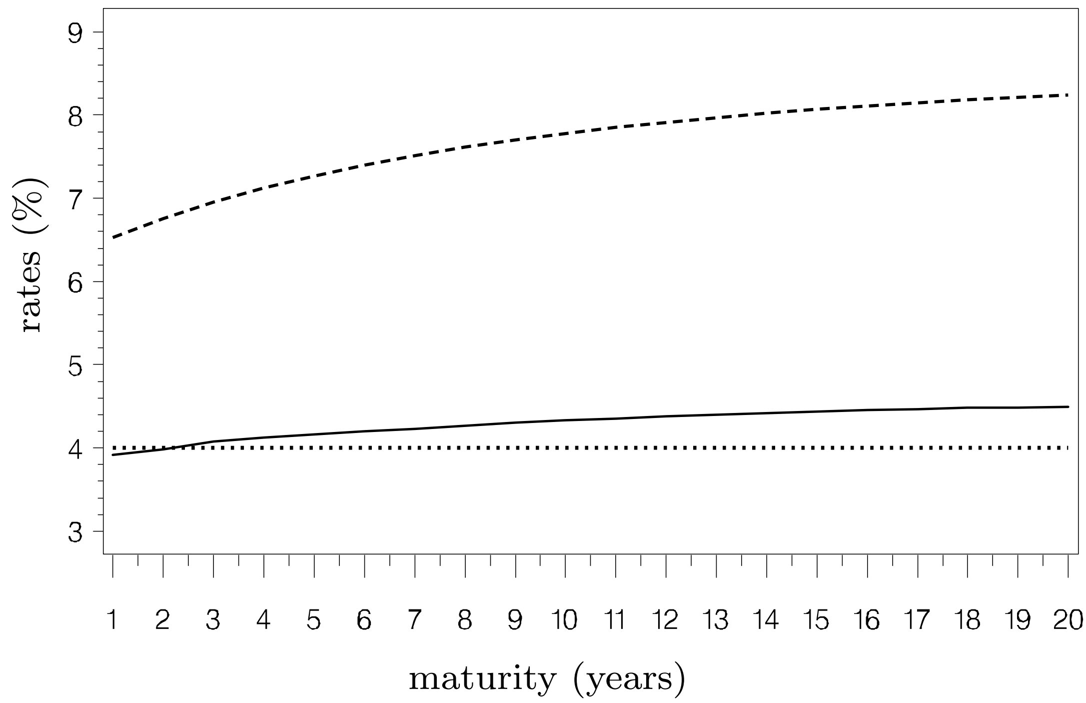
XY policy – Valuation rates

$\beta = 80\%,\ i = 4\%,\ t = 02/01/1990$



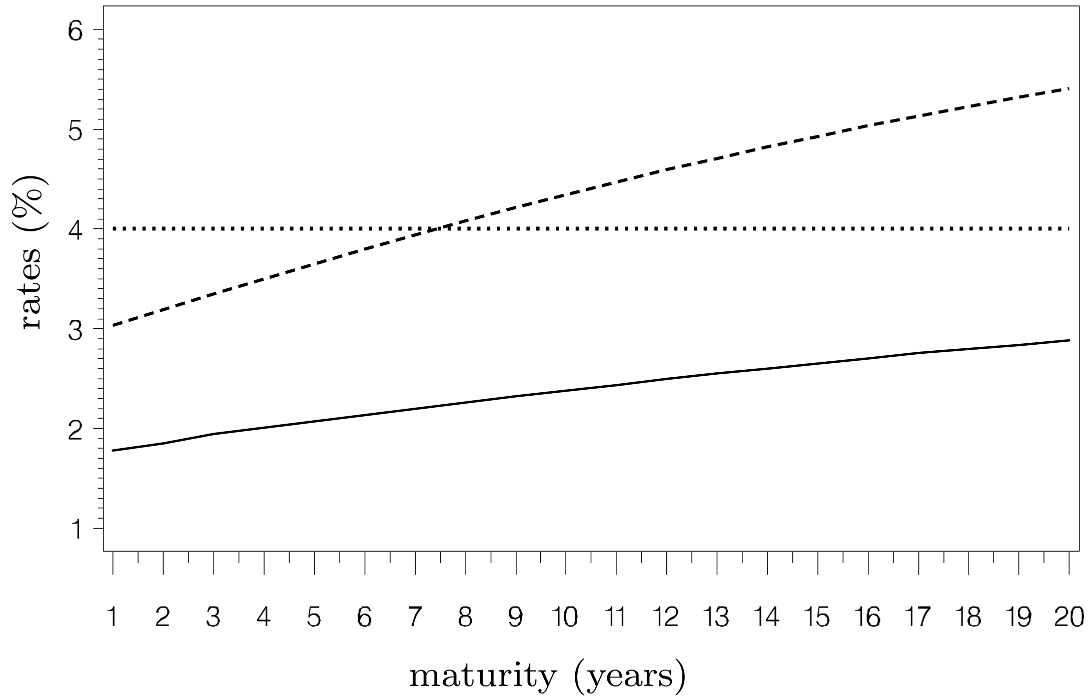
XY policy – Valuation rates

$\beta = 80\%,\ i = 4\%,\ t = 31/12/1996$



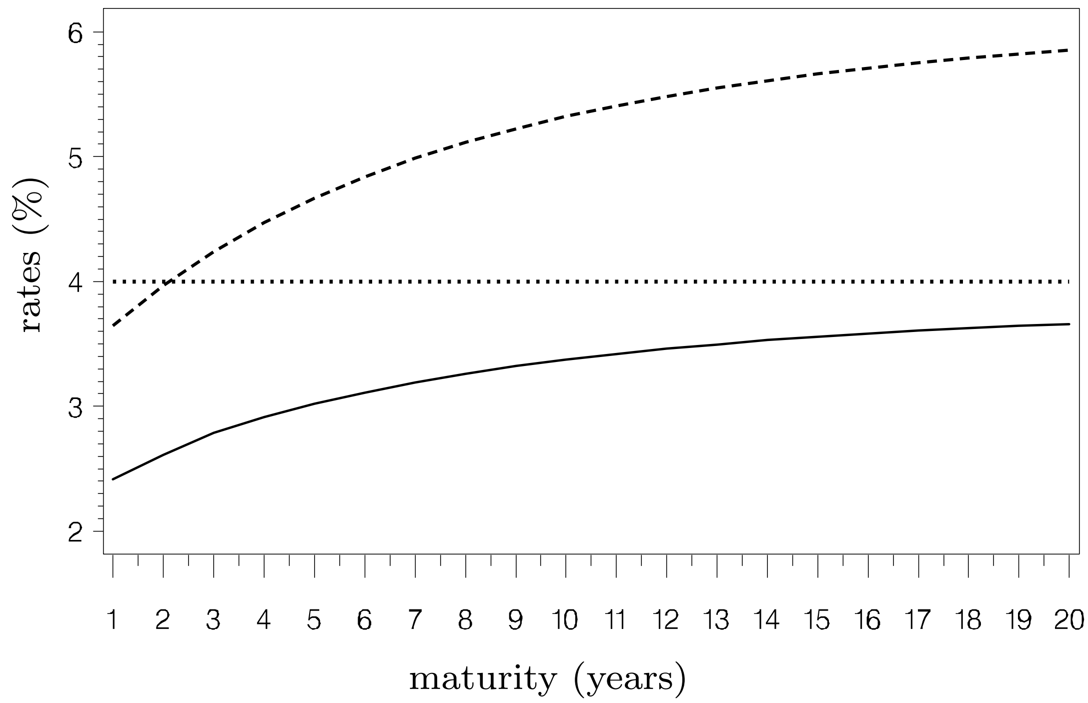
XY policy – Valuation rates

$\beta = 80\%,\ i = 4\%,\ t = 30/12/1998$



XY policy – Valuation rates

$\beta = 80\%,\ i = 4\%,\ t = 07/09/2001$



Chapter 6 – Other embedded options

Surrender options
Guaranteed annuity conversion options

Surrender options

Roughly speaking the right to redeem a policy is the right, for the policyholder, to sell the contract to the issuer before maturity at a specified price

$\longrightarrow$ american style put option.

If surrenders are not adequately penalized and if they are rationally exercised these options may have significant value.

We are dealing with long term american put options which are intrinsically interest rate sensitive
$\longrightarrow$ use of numerical pricing procedures.

See [DF, M, C, M, 1998] for the valuation by the CIR model of the american put options implicit in the savings bonds issued by the Italian government.

In many practical situations the american options embedded in financial contracts turn out to be not rationally exercised.
See [Schwartz, Torous, 1989] referring to mortgage-backed securities, and [Brennan, Schwartz 1979a] referring to Canadian savings bonds.

This seems to be also the case for many life insurance policies, where redemptions appears to be essentially driven by the evolution of personal consumption plans

$\longrightarrow$ surrenders can be modelled by "actuarial" methods using experience-based elimination tables.

Guaranteed annuity conversion options

Let us consider at time 0 a pure endowment with term n years and terminal benefit:

$$C_n := C_0 \, \Phi(0, n) \, .$$

In addition, the policy provides the policyholder the right to exchange at time n the cash payment C_n with an annuity with term m years and annuity rate g fixed at time 0.

Thus the policyholder is provided by a **conversion option** between the cash payment C_n at time n and a payment stream $\mathbf{c}$ of constant annual payments:

$$c := C_n / \ddot{a}_{\overline{n}|} \, , \quad \text{where :} \quad \ddot{a}_{\overline{n}|} := \sum_{k=0}^{m-1} (1+g)^{-k} \, ,$$

payable at the dates: $n, n+1, \ldots, n+m-1$.
The option is of european type, since it can be exercised at time n.

The value at time n of the annuity is:

$$A_n := V(n; \mathbf{c}) = c \, W_n \, , \quad \text{where :} \quad W_n := \sum_{k=0}^{m-1} v(n, \, n+k) \, .$$

Therefore the payoff of the policy is:

$$\widetilde{Y}_n := \mathbf{1}_{\{T_x \geq n\}} Y_n \, ,$$

where:

$$Y_n := \max\{C_n, \, A_n\} \, ;$$

Remark. Both the C_n and A_n are stochastic at time 0. The uncertainty on C_n depends on the value of the readjustment factor between 0 and n; the uncertainty on A_n concerns the zcb prices $v(n, \, n+k)$ prevailing at time n. ■

One can also write:

$$Y_n := C_n \max\{1, W_n/\ddot{a}_{\overline{n}|}\}\ , \quad \text{or :} \quad Y_n := c \max\{\ddot{a}_{\overline{n}|}, W_n\}\,;$$

The financial payoff Y_n can be written as:

$$Y_n := C_n \max\{1, W_n/\ddot{a}_{\overline{n}|}\} = C_n + C_n \max\{W_n/\ddot{a}_{\overline{n}|} - 1, 0\}\,;$$

hence the contract can be considered as a non convertible pure endowment plus an european call-option-type contract written on the stochastic ratio $W_n/\ddot{a}_{\overline{n}|}$ exercisable at time n with strike price 1.

Under our assumptions, for the option component $\widetilde{O}_n$ we have:

$$V(0;\widetilde{O}_n) = \mathbf{P}_0(Y_n;n)\,V(0;O_n)\,,$$

with:

$$V(0;O_n) = \mathbf{E}_0^Q\big[C_n \max\{W_n/\ddot{a}_{\overline{n}|} - 1, 0\}\;\chi(0,n)\big]\,.$$

One has:

$$V(0;O_n) = C_0\,\mathbf{E}_0^Q\big[\Phi(0,n)\,\max\{W_n/\ddot{a}_{\overline{n}|} - 1, 0\}\;\chi(0,n)\big]\,,$$

or:

$$V(0;O_n) = \frac{C_0}{\ddot{a}_{\overline{n}|}}\,\mathbf{E}_0^Q\big[\Phi(0,n)\,\max\{W_n - \ddot{a}_{\overline{n}|}, 0\}\;\chi(0,n)\big]\,,$$

Remark. Since $v(n,n+k) = \mathbf{E}_n^Q[\chi(n,n+k)]$, by the properties of conditional expectations:

$$V(0;O_n) = \frac{C_0}{\ddot{a}_{\overline{n}|}}\,\mathbf{E}_0^Q\left[\Phi(0,n)\max\left\{\sum_{k=0}^{m-1}\chi(0,n{+}k) - \ddot{a}_{\overline{n}|}\chi(0,n), 0\right\}\right].$$

If Φ is deterministic $\longrightarrow$ the typical payoff of a european option on coupon bond (see [DF, M, 1991a] for the solution in the 1-factor CIR model). ∎