Generalized Jackknife Moment estimators of the tail index

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1. The Generalized Jackknife Moment estimators

In *Statistical Extreme Value Theory* we are mainly interested in the estimation of *parameters of rare events*, the basic parameter being the *tail index* $\gamma = \gamma(F)$, directly related to the tail weight of the model *F*(.). The *tail index* γ is the shape parameter in the *Generalized Extreme Value* (*GEV*) distribution function (d.f.), which appears as the non-degenerate limiting d.f. of the sequence of maximum values, linearly normalized.

Whenever there is a non-degenerate limit of the sequence of normalized maximum values we say that *F* is in the *domain of attraction* of G_γ, and write $F \in D(G_\gamma)$. Denoting by $U(t) := F^{\leftarrow}(1-1/t)$, the quantile function, and for heavy tails ($\gamma > 0$), $F \in D(G_\gamma)$ iff $1 - F \in RV_{-1/\gamma}$ iff $U \in RV_\gamma$, where RV_α stands for the class of *regularly varying* functions at infinity with index of regular variation equal to α. These conditions characterize completely the first order behaviour of *F*(.) [Gnedenko (1943), de Haan (1970)], and the second order theory has been worked out in full generality by de Haan and Stadtmüller (1996) — for a large class of models there exists a function $A(t)$ of constant sign for large *t*, such that $\left[U(tx)/U(t) - x^\gamma\right]/A(t) \underset{t\to\infty}{\rightarrow} x^\gamma\left(x^\rho - 1\right)/\rho$, for every $x > 0$, where $\rho(\le 0)$ is a second order parameter.

In this paper we are going to work with the *Moment estimator* (Dekkers *et al*, 1989), a serious candidate to the *tail index* estimation,

$$(1) \qquad \gamma_{n,M}(k) := M_n^{(1)}(k) + 1 - \frac{1}{2}\left\{1 - \frac{\left(M_n^{(1)}(k)\right)^2}{M_n^{(2)}(k)}\right\}^{-1}, \; M_n^{(j)}(k) := \frac{1}{k}\sum_{i=1}^{k}\left\{\ln X_{n-i+1:n} - \ln X_{n-k:n}\right\}^j,$$

for which

$$(2) \qquad \gamma_{n,M}(k) \overset{d}{=} \gamma + \sqrt{\frac{\gamma^2+1}{k}}\, Z_n + \frac{2-\rho+(\gamma-2)(1-\rho)}{\gamma(1-\rho)^2} A(n/k) + o_p\left(A(n/k)\right),$$

Z_n asymptotically standard Normal (Dekkers and de Haan, 1993).

If we consider two intermediate levels $k_1 = k$, $k_2 = k/2$, $A(n/(k/2)) = 2^\rho A(n/k)(1+o(1))$, and $\frac{BIAS_\infty\left[\gamma_{n,M}(k)\right]}{BIAS_\infty\left[\gamma_{n,M}(k/2)\right]} = \frac{A(n/k)}{A(2n/k)} = 2^{-\rho}(1+o(1))$, the Generalized Jackknife random variable (Gray and Shucany, 1972) associated to the moment estimators at the two levels *k*/2 and *k* is

$$(3) \qquad \gamma_{n,M}^G(k) = \frac{\gamma_{n,M}(k) - 2^{-\rho}\gamma_{n,M}(k/2)}{1-2^{-\rho}}.$$

If we got to know ρ we would be able to remove completely the asymptotic bias, with the use of (3). In order to avoid the estimation of ρ, and since we are here going to work, with Fréchet and Cauchy parents, for which we have the second order parameter equal to –1 and –2, respectively, we shall consider the following two alternative estimators to the Moment estimator,

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(4) $\gamma^{G_F}_{n,M}(k) \;:=\; 2\gamma_{n,M}(k/2) - \gamma_{n,M}(k)$, and $\gamma^{G_C}_{n,M}(k) \;:=\; \frac{4}{3}\gamma_{n,M}(k/2) - \frac{1}{3}\gamma_{n,M}(k)$.

For a more general application of the *Generalized Jackknife* Theory to the estimation of parameters of *rare events*, see Gomes et al (1998).

2. Distributional properties of estimators for Fréchet and Cauchy parents

In Table 1, we show the *Mean Values* (E_1, E_2, E_3) and the *Mean Square Errors* (MSE_1, MSE_2, MSE_3) of $\gamma_n^{(1)}(k) \equiv \gamma_{n,M}(k)$, $\gamma_n^{(2)}(k) \equiv \gamma^{G_F}_{n,M}(k)$ and $\gamma_n^{(3)}(k) \equiv \gamma^{G_C}_{n,M}(k)$, respectively, at the optimal levels (levels where minimum *MSE* is attained) for a Fréchet model, with $\gamma = 1$, and for a Cauchy model, which is also attracted towards a GEV($\gamma = 1$). The simulation was based on 20 replicas with 5000 runs each, and the smallest *BIAS* and the smallest *MSE,* were put in *italic* and underlined.

F	n	E_1	E_2	E_3	MSE_1	MSE_2	MSE_3
R	200	1.0574	0.9840	*1.0316*	0.0262	0.0296	*0.0215*
É	500	1.0483	*0.9948*	1.0251	0.0134	0.0103	*0.00932*
C	1000	1.0410	*0.9921*	1.0228	0.00815	*0.00487*	0.00511
H	5000	1.0264	*0.9982*	1.0221	0.00268	*0.000942*	0.00151
E	10000	1.0215	*0.9992*	1.0223	0.00167	*0.000472*	0.00102
T	20000	1.0173	*0.9995*	1.02271	0.00105	*0.000236*	0.000780
C	200	*1.0147*	0.8116	0.8992	*0.0500*	0.1404	0.0849
A	500	*1.0292*	0.8680	0.9518	*0.0208*	0.0516	0.0256
U	1000	*1.0282*	0.9118	0.9663	*0.0113*	0.0275	0.0117
C	5000	1.0192	0.9607	*0.9862*	0.00293	0.00677	*0.00209*
H	10000	1.0155	0.9715	*0.9932*	0.00166	0.00378	*0.00101*
Y	20000	1.0122	0.9790	*0.9966*	0.000946	0.00212	*0.000500*

Notice that for Fréchet parents $\gamma^{G_F}_{n,M}(k)$ is the optimal estimator for sample sizes $n \geq 1000$, with a drastic reduction of *MSE,* relatively to the Moment estimator. For Cauchy parents $\gamma^{G_C}_{n,M}(k)$ takes the role of $\gamma^{G_F}_{n,M}(k)$, but the improvement is not spectacular.

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FRENCH RESUMÉ

Nous considérons des statistiques généralisées de Jackknife associés à l'estimateurs des Moments de l'index de variation régulière, évaluées en deux niveaux, et nous étudions le gain en efficience, dans le cas de distributions parente Fréchet et Cauchy.