PROSPECTIVE MODELS OF MORTALITY WITH FORCED DRIFT

APPLICATION TO THE LONGEVITY RISK FOR LIFE ANNUITIES

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SUMMARY

The anticipation of the future tendency of death rates is a delicate exercise and a bad anticipation of the drift has important financial consequences for a life annuity plan.

Instead of seeking to anticipate the future drift only starting from past information, we will build an model including as constraint: an evolution of the life expectancy at a fixed age (that it could be simply modelled by a linear evolution, the origin ordinate and the slope would become the parameters of the model).

The objective of this work will be to define a model of this type, to apply it to a life annuity plan and to show in what it can allow a better technical management of the mortality risk.

KEYWORDS: Prospective tables, extrapolation, adjustment, life annuities, stochastic mortality.

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1. INTRODUCTION

The prospective analyses of mortality result in anticipating the future trends of the death rates at the various ages. In the current traditional construction models of prospective tables, like the Lee-Carter model (see in particular LEE and CARTER [1992], LEE [2000], SITHOLE and al. [2000]) or the Poisson models (*cf.* BROUHNS and al. [2002] and PLANCHET and THEROND [2006] for a presentation and a discussion of these models), the drift of future mortality is anticipated starting from the last observations.

Even in admitting that it is legitimate to extend in the years to come the tendencies observed in the past (we will refer to CAREY and TULAPURKAR [2003] for analyses integrating the biological and environmental considerations, like GUTTERMAN and VANDERHOOF [1999] for a discussion on this point), several sources of uncertainty come to disturb the determination of the future tendency: the choice of the observation period, stochastic fluctuations of death rates, extraordinary events, *etc*. This uncertainty creates for the insurers of life annuity plans and pension plans, a systematic risk (not mutualisable) whose financial impact can be very important.

Thus, in France, the recent updating of the tables used by the insurers for the reserve of the life annuities illustrates the difficulties of this anticipation and the financial associated stakes. So, compared to TPG 1993[1] tables into force until the 12/31/2006, the new TGH 05 and TGF 05 tables which come into effect the 01/01/2007, lead to increases of provision sometimes higher than 20%, like illustrates it the following table:

Old	Generation	TPG 1993	Women	Women/TPG	Men	Men/TPG
50	1955	26,81647	28,40552	5,9%	26,75507	-0,2%
55	1950	24,26368	25,95575	7,0%	24,07474	-0,8%
60	1945	21,50832	23,30185	8,3%	21,25828	-1,2%
65	1940	18,53412	20,39677	10,0%	18,22126	-1,7%
70	1935	15,39467	17,28922	12,3%	15,08772	-2,0%
75	1930	12,25679	14,08680	14,9%	12,05698	-1,6%
80	1925	9,35194	10,96271	17,2%	9,12890	-2,4%
85	1920	6,88306	8,15548	18,5%	6,64827	-3,4%
90	1915	4,93310	5,89309	19,5%	4,73880	-3,9%
95	1910	3,46780	4,29408	23,8%	3,40109	-1,9%

Fig. 1 : *Comparison of the reserve coefficients of TPG 1993 and TGH/TGF 05*

In this context, it is expedient to try to measure the risk associated with this anticipation error and to quantify its impact in term of provisions for a shareholder plan.

[1] These tables are obtained on the basis of female population mortality on the period 1961-1987, used since July 1st 1993.

In practice the above observations lead to the conclusion that the annuity plan is subjected at a important risk of model. Thus, it is possible to reformulate the consequences of the passage of TPG 1993[2] tables with the TGH 05 tables evoked above by bringing together the anticipated evolution of the life expectancy at age 60 years in the two prospective models. We obtains as follows:

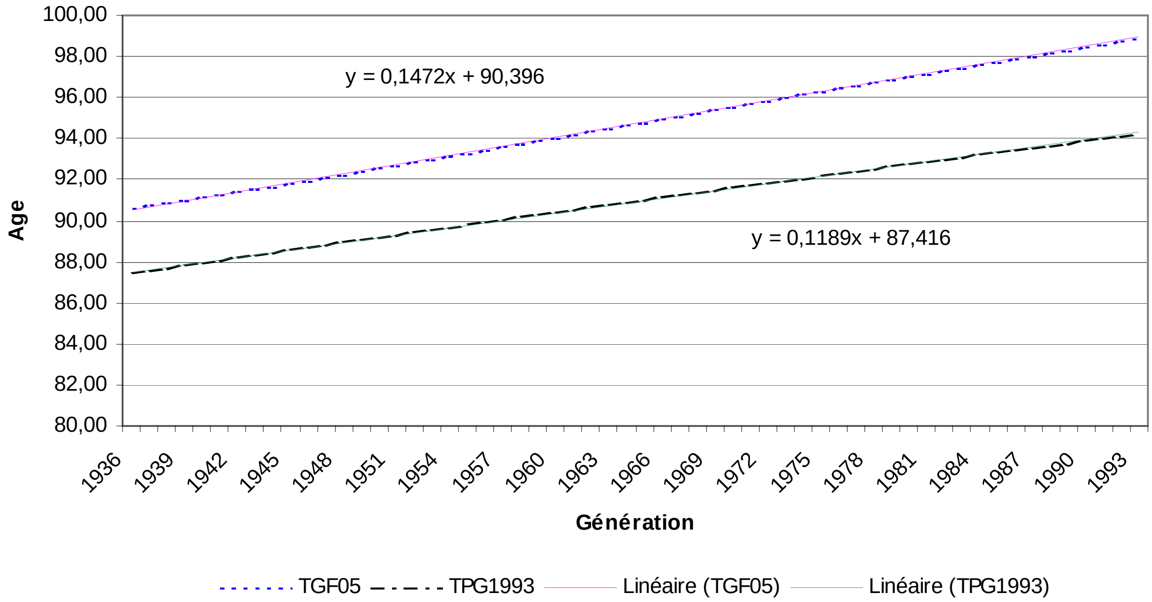

Fig. 2 : *Anticipated evolution of the life expectancy at age 60 years*

We observe that not only the absolute levels differ appreciably, but also that the speed of growth of the life expectancy at age 60 years was underestimated in 1993: whereas the TPG 1993 anticipate an increase of 1,4 months a year, TGF 05 tables lay down a drift of 1,8 months per year , that is to say 23 % more. Moreover, we note that anticipations of the life expectancy follow linear trends, which we will use thereafter.

This illustrates the difficult anticipation of the drift tendency of future mortality starting from historical data[3]. In this context, we propose to use as parameter of control of the model, in fixing it like *a priori* constraint: the life expectancy at a given age (60 years) and his future trend. Such an approach allows to quantify for example the impact on the charge plan of a 0,1 months/year error on the speed of drift of this expectancy and to integrate explicitly the impact indicators on the engagement evaluation of the plan for an error of model.

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[2] The TPG 1993 are female tables.

[3] The reference populations used in two series tables differ, but we would obtain the same conclusions in using the INSEE prospective tables instead of TGF 05 tables.

We use for this point, in the present study the Lee-Carter model (see in particular LEE and CARTER [1992], LEE [2000], SITHOLE and al. [2000]) to build a mortality surface $\mu(x,t)$. After an adjustment of the past rates, death rates for the future years result from the extrapolation of the temporal component in integrating the constraint posed *a priori*. We can note that the use of the log-Poisson alternative (*cf.* BROUHNS and al. [2002]) would lead to very close results, that will not be taken again here.

The numerical applications of this work are taken again results obtained in EL HORR and al. [2007].

2. THE MORTALITY MODEL

2.1. RECALLS ON THE LEE-CARTER MODEL

The selected model to build the prospective tables is adapted from Lee-Carter model (LEE and CARTER [1992]). We recall that the suggested modelling for the instantaneous death rate in Lee-Carter is the following one:

$$\ln \mu_{xt} = \alpha_x + \beta_x k_t + \varepsilon_{xt}, \tag{1}$$

in supposing the random variables ε_{xt} are independent, identically distributed according to a $N\left(0,\sigma^2\right)$ law and we dispose of a historical record $t_m \le t \le t_M$. The question of the parameter adjustment of the model is not tackled here. The interested reader will refer to the many references on the subject (quoted for example in PLANCHET and THÉROND [2006]).

Once the mortality surface adjusted on the data, the (k_t) series has to be modelled to extrapolate the future rates; for this, we use in general a very simple modelling that is the base of a linear regression in supposing an affine trend:

$$k_t^* = at + b + \gamma_t, \tag{2}$$

with (γ_t) a Gaussian white noise of variance σ_γ. Thus, we obtain estimators $\hat{a}$ and $\hat{b}$ that allow to build the projected surfaces while using simply $k_t^* = \hat{a}t + \hat{b}$.

In the sequel, after having briefly presented the closing method of the proposed table, we present an adaptation of this model that takes into account a constraint that we sets *a priori* on the future trend of death rates.

2.2. CLOSING OF THE TABLE

The estimate of the gross death rates is in general possible only up to an age limit relatively far away from the maximum survival age. In practice, the estimated gross amounts have a great instability to the high ages because of the weak number of available persons. Thus , we have seldom good quality data beyond 90-95 years. Consequently, we resort to a closing method allowing to supplement the table before carrying out the adjustment.

The various closing methods of table will not be here detailed and the interested reader will consult, on this subject, PLANCHET and THÉROND [2006] or DENUIT and QUASHIE [2005].

We retains in this study a simple model in which death rates at the great ages, up to 120 years, are extrapolated while being based on the following formula:

$$q_x = a \times \mathbf{exp}\left(bx\right), \qquad (3)$$

where a and b are the real numbers determined by the constraint $q_{120} = 1$ and by the connection at the rates q_x for the ages lower than $x_0 = 86$, age to which we begin extrapolation.

Furthermore, we recalls that in the context of the evaluation of the annuity plan engagement, considering the average age of the shareholders, the final selected closing method has only a relative importance (see PLANCHET and THÉROND [2006] for a quantification of this impact).

2.3. GENERAL PRESENTATION OF THE MODEL

The usual formulation of the Lee-Carter model $\mathbf{ln}\,\mu_{xt} = \alpha_x + \beta_x k_t$ rests on a model implicitly described in continuous time. However, for the need of numerical applications, it proves to be necessary to make a hypothesis allowing to boil down to the observations, by discrete way. The traditional hypothesis consists in supposing the constancy of the hazard function on each square of the Lexis diagram, which leads to $\mu_{xt} = -\mathbf{ln}\left(1-q_{xt}\right)$.

Initially, in order to avoid this hypothesis (which is undeniable for the high ages in particular), the model is written directly in discrete time by using the “logits” of death rates:

$$\mathbf{ln}\frac{q_{xt}}{1-q_{xt}} = \alpha_x + \beta_x k_t. \qquad (4)$$

As the inverse transformation of the logistic function is $y \to \dfrac{e^y}{1+e^y}$ it is equivalent to pose:

$$q_{xt} = \frac{\mathbf{exp}\left(\alpha_x + \beta_x k_t\right)}{1 + \mathbf{exp}\left(\alpha_x + \beta_x k_t\right)}. \tag{5}$$

This approach has the advantage of proposing an explicit parameterisation of the death rates q_{xt}. With regard to the parameter estimate of the model, the process is strictly identical to that proposed by Lee-Carter, in replacing systematically $\mathbf{ln}\,\mu_{xt}$ by $\mathbf{ln}\frac{q_{xt}}{1-q_{xt}}$. Moreover, this formulation is better adapted to the recognition of a constraint on the life expectancy evolution (by generation) at an age given for the prospective part; indeed we have the following statement:

$$e_{xt} = \sum_{h>0} \prod_{k=0}^{h-1} \left(1 - q_{x+k,t+k}\right), \tag{6}$$

and we have an explicit expression for q_{xt} according to the parameters (α, β, k). We can also notice that:

$$q_{xt} = 1 - \frac{e_{xt}}{1 + e_{x+1,t+1}}. \tag{7}$$

Having a complete prospective surface is equivalent to determine the values of $\left(k_t\ ; t \geq t_M\right)$. We have $\left(e_{xt}\ ; t \geq t_M\right)$ for a fixed age *x*; we wants to deduce the values of $\left(k_t\ ; t \geq t_M\right)$.

At this stage, it remains to specify the age x_0 selected as “selected age” for the integration of the expert judgement in the model, as well as the shape of the future trend of the life expectancy at this age, namely the shape of $t \rightarrow e_{x_0 t}$, $t \geq t_M$.

Here, we will notice that by “continuity” of $\left(k_t\right)$, the value k_{t_M+1} should not be “too far” from k_{t_M} , which induces a constraint on $t \rightarrow e_{x_0 t}$, $t \geq t_M$. In the (most frequent) cases where the general shape of $\left(k_t\right)$ is linear, we can imagine simply to introduce the expert judgement through a break of slope on the straight line $t \rightarrow k_t$, $t \geq t_M$. However, this approach will not be privileged because of the character not immediately apprehensive of $\left(k_t\right)$, the residual life expectancy is a easier and more intuitive concept.

2.4. A SIMPLE SPECIFICATION OF THE MODEL

The simplest specification that we can imagine for $t \to e_{x_0 t}$, $t \geq t_M$, and overall in phase with the past observations, is a linear evolution of the future life expectancy. Besides, this shape is in phase with what we note on the usual prospective tables (TGP 1993, TGH/TGF 05, etc). Thus, we can define:

$$e_{x_0}(t) = a \times t + b \, , \tag{8}$$

a and b being the 2 parameters of our model fixed in advance by the expert judgement.

In the general case, once the shape of $t \to e_{x_0 t}$, $t \geq t_M$ is fixed, the determination of $t \to k_t$, $t \geq t_M$ through the relation $e_{xt} = \sum_{h>0} \prod_{k=0}^{h-1} \left(1 - q_{x+k,t+k}\right)$ is not simple because it is delicate to fix an *a priori* horizon to limit the considered number of (k_t). Thus, we propose here to use approximate coefficients, obtained while forcing (k_t) to have a polynomial form:

$$k(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3, \, t \geq t_M \, . \tag{9}$$

With this hypothesis, we have a relation with the following form: $e_x(t) = \varphi_{xt}(a_0, a_1, a_2, a_3)$ and thus we boil down to seek $\theta = (a_0, a_1, a_2, a_3)$ minimizing the standard deviation between the “expert judgement” values for the life expectancy to the selected age and the prediction of these values by the model, namely seeking to solve the program:

$$\underset{(a_0, a_1, a_2, a_3)}{Min} \sum_{t=t_M}^{t_M + h} \left(a + bt - \varphi_{x_0 t}\left(a_0, a_1, a_2, a_3\right) \right)^2 \tag{10}$$

with:

$$\sum_{h>0} \prod_{k=0}^{h-1} \left(1 - \frac{\mathbf{exp}\left(\alpha_{x+k} + \beta_{x+k}\left(a_0 + a_1(t+k) + a_2(t+k)^2 + a_3(t+k)^3\right)\right)}{1 + \mathbf{exp}\left(\alpha_{x+k} + \beta_{x+k}\left(a_0 + a_1(t+k) + a_2(t+k)^2 + a_3(t+k)^3\right)\right)} \right) = \varphi_{xt}\left(a_0, a_1, a_2, a_3\right) \tag{11}$$

The resolution of the above program numerically does not pose a particular problem. We will note simply that it is advisable to project the death rates on a horizon much more important than the horizon of projection of the residual life expectancies: for instance, to estimate the life expectancy at age 65 years in 2050, it is necessary to have the death rates until 2105 on the hypothesis of a maximum survival age of 120 years.

3. NUMERICAL APPLICATION

The numerical illustration proposed here is structured in three parts: initially, we justify the relevance of the use of the logit death rates instead of the hazard function, then we analyse the behaviour of our model for its prospective component by introducing the expert judgement. Lastly, we show the application that we can make of this model for the quantification of the longevity risk carried by a life annuity plan.

3.1. TRANSITION FROM STANDARD LEE-CARTER MODEL TO LOGISTIC LEE-CARTER MODEL

The prospective table used in this study is built starting from the current tables provided by the INED[4] in MESLE and VALLIN [2002]. The adjustment on historical data of the "logistic" Lee-Carter model leads to the following mortality surface:

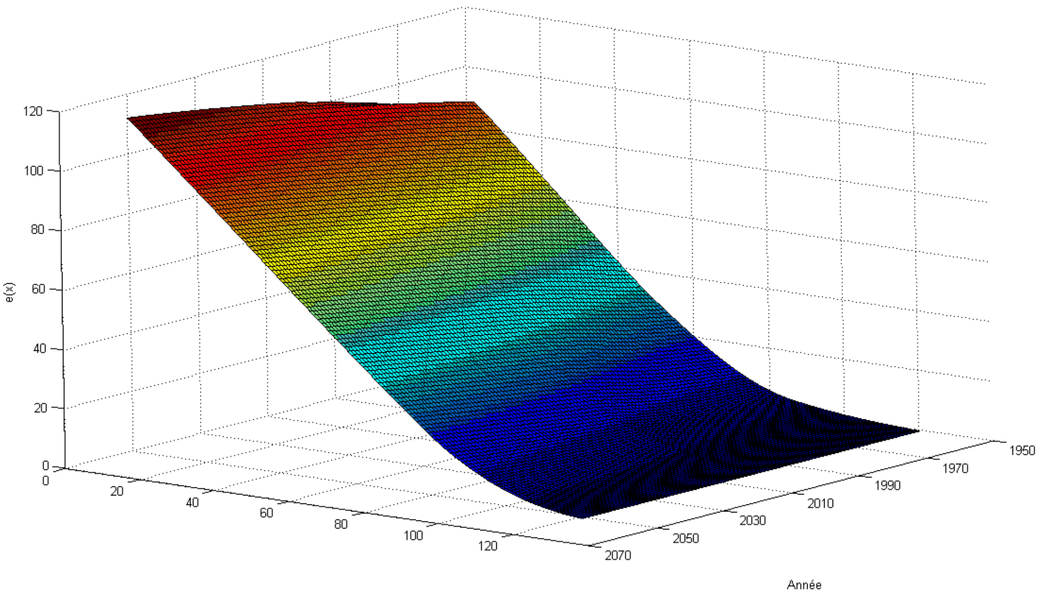

Fig. 3 : *Adjusted mortality surface (Lee-Carter on the logits)*

The proximity of the two models (standard Lee-Carter and logistic Lee-Carter) is illustrated by the comparison of the various parameters, carried out below:

[4] This tables are available on http://www.ined.fr/publications/cdrom_vallin_mesle/Tables-de-mortalite/Tables-du-moment/Tables-du-moment-XX.htm

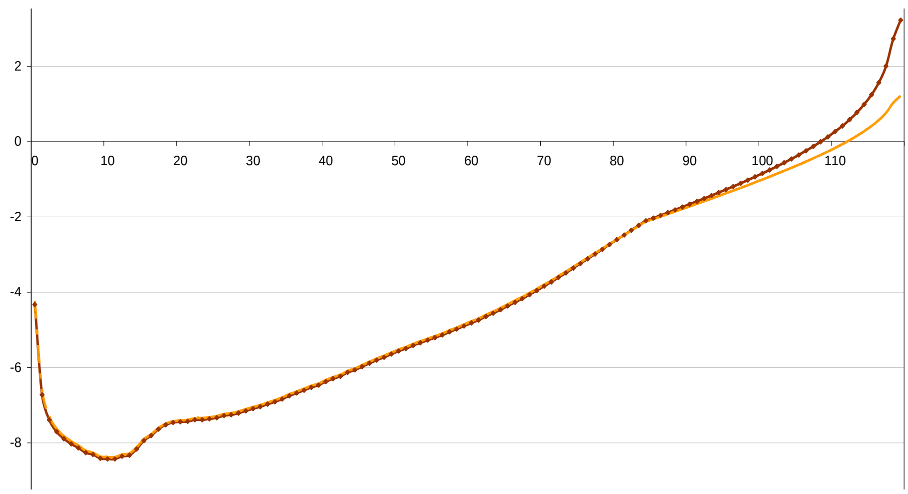

Fig. 4 : *Comparison of the estimates of* α_x

As expected, we note a gap that becomes more pronounced at the high ages and a great proximity of the other values; the situation is similar with the parameter β_x :

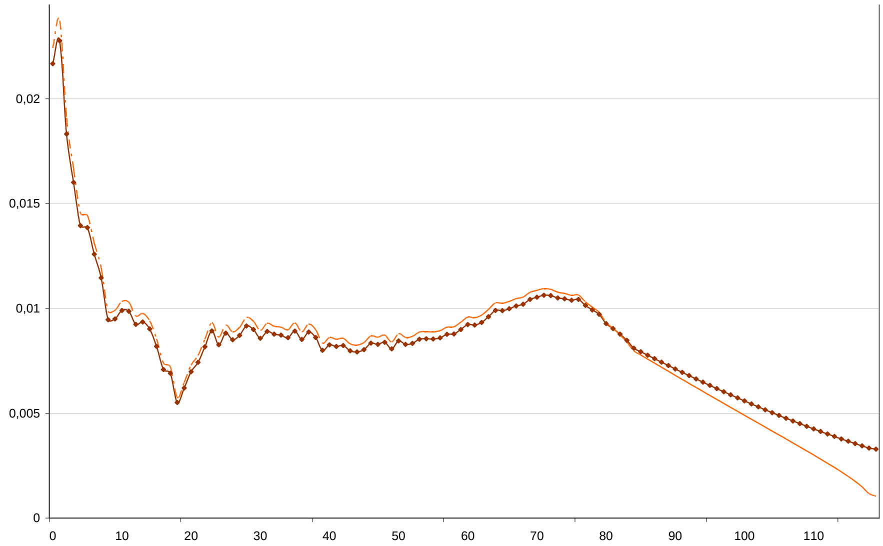

Fig. 5 : *Comparison of the estimates of* β_x

Concerning the temporal component, the differences between the two models are insignificant:

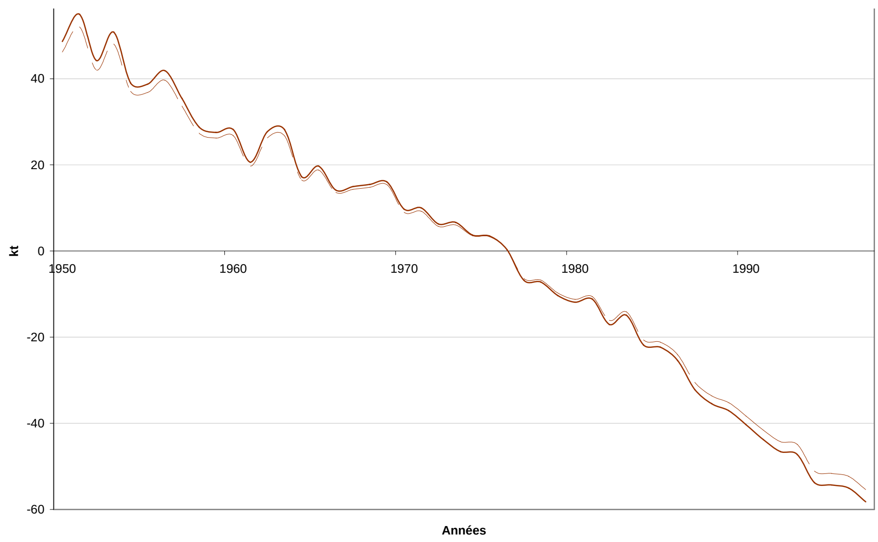

Fig. 6 : *Comparison of the estimates of* k_t

From now on, we use the version of the model on the basis of logits death rates. The results hereafter illustrate the behaviour of the model controlled by the expert judgement.

3.2. ANALYSE OF THE PROSPECTIVE COMPONENT

In order to simplify the expression of the expert judgement, on the one hand, and to ensure the continuity of the evolution of the life expectancy, on the other hand, we oblige $e_{64}\left(t_M\right)$ to be equal to the value resulting from the initial adjustment, is approximately 28 years on our data.

Then, it remains to fix $e_{64}\left(t\right)$ for an *a priori* given date *t*. We retain $t = 2050$ as horizon of expression of the expert judgement, that is to say a prospective opinion at approximately 50 years. Thus, two situations are compared: $e_{64}\left(2050\right) = 38$ (which will be our situation of reference) and $e_{64}\left(2050\right) = 48$.

First of all, we take interest in the impact of integration of the expert judgement in the projection of the temporal parameter:

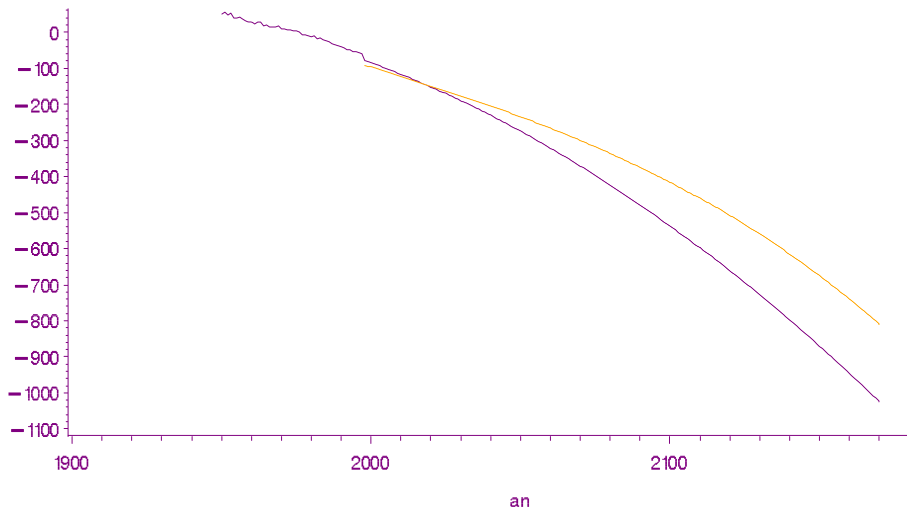

Fig. 7 : *Comparison of the estimates of k_t in the constrained model and the non constrained model*

The ratio of the two mortality surfaces thus obtained for ages 60-120 is represented hereafter:

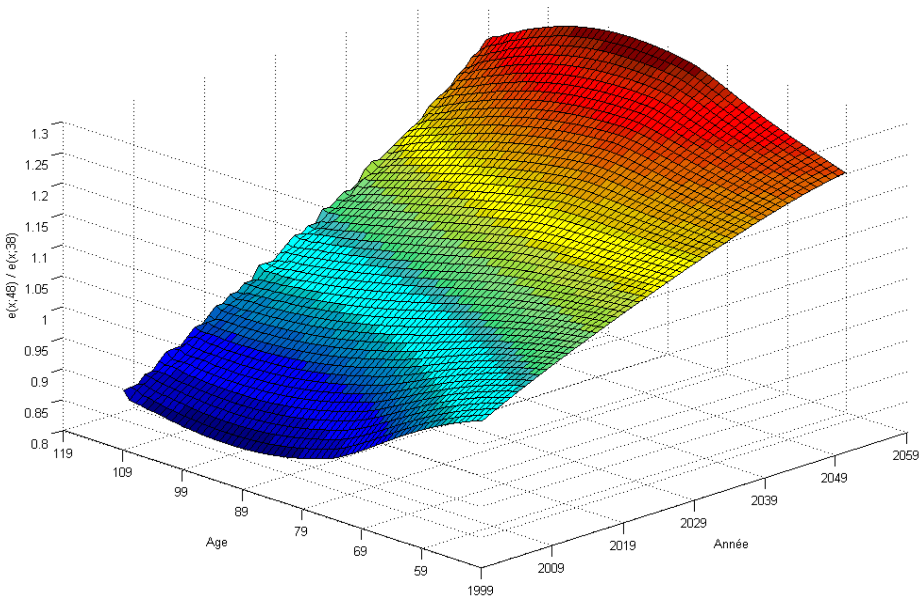

Fig. 8 : *Adjusted mortality surface (Lee-Carter on the logits)*

Initially, we notice that the formulated expert judgement at a given age is reflected on the whole ages. It is a consequence of the very structuring character (and thus very constraining) of the Lee-Carter model, in which the only data of $t \rightarrow k_t, t \geq t_M$ determines entirely the prospective surface for all the ages.

We note, which is not intuitive, that to anticipate a more important increase in the life expectancy at age 64 years leads in short run to revise downwards the estimate of the life expectancy at the higher ages during a few years. Of course, in the long run, the life expectancy at all the concerned ages becomes higher than the expectancy that was anticipated in the reference model.

This compared evolution of the life expectancies in the two situations is the consequence of a long run projection of the instantaneous death rates. Below, the figure presents the evolution of the ratio of the death rates on the two working hypotheses:

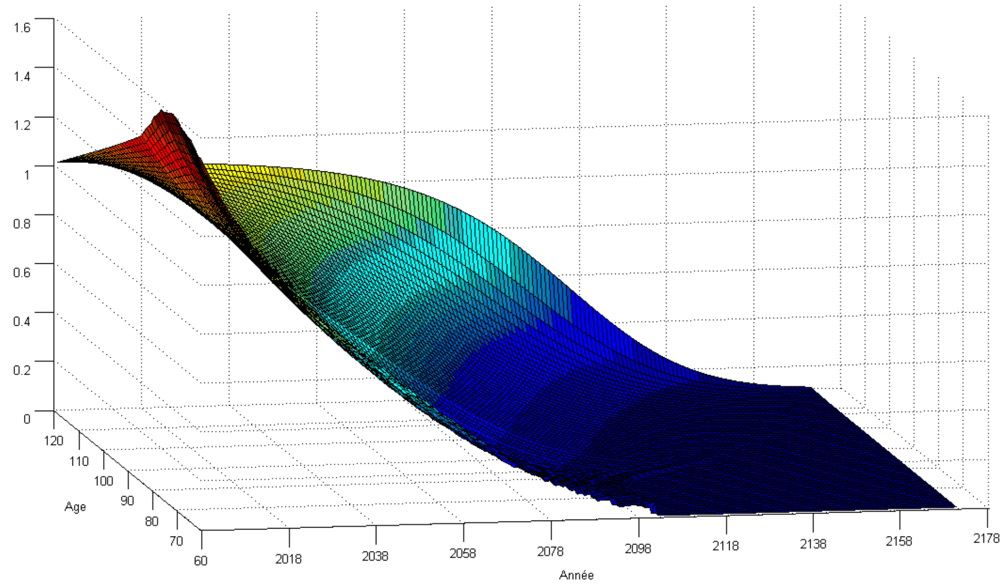

Fig. 9 : *Adjusted mortality surface (Lee-Carter on the logits)*

We find in a "reversed" mode, the characteristics of preceding surface. We observe that the gap of life expectancy anticipated at age 64 years at a horizon of approximately 50 years (+ 26 %) implies a decrease much stronger death rates at all the ages (beyond 60 years). Thus, we observe abatements of more 90 % after one century.

This aspect gives an idea of the constraint, which represents a profit of residual life expectancy at 64 years on the annual death rates. In other words, a weak variation of annual death rates does not really impact the residual life expectancy. In other words, the engagement of an annuity plan is not very sensitive to instantaneous phenomena affecting mortality over one year.

3.3. APPLICATION TO A LIFE ANNUITY PLAN

An annuity plan is mainly confronted with a financial risk and, in second manner, at a risk of bad anticipation of the mortality of the shareholders. This longevity risk (see PLANCHET and al. [2006]) must, in the context of the reform project of the prudential rules "Solvabilité 2", be quantified. The model that we propose here, allows to propose an evaluation of this risk, we illustrate this point in the continuation of this paragraph.

We use for this illustration a portfolio constituted by 374 female shareholders with an average age of 63,8 years at 12/31/2005. The average annual annuity figures up to 5,5 k€. The Figure 2 (see below) will present the expected flows dues to pensions as a function of time built from the mortality table named TV 2000.

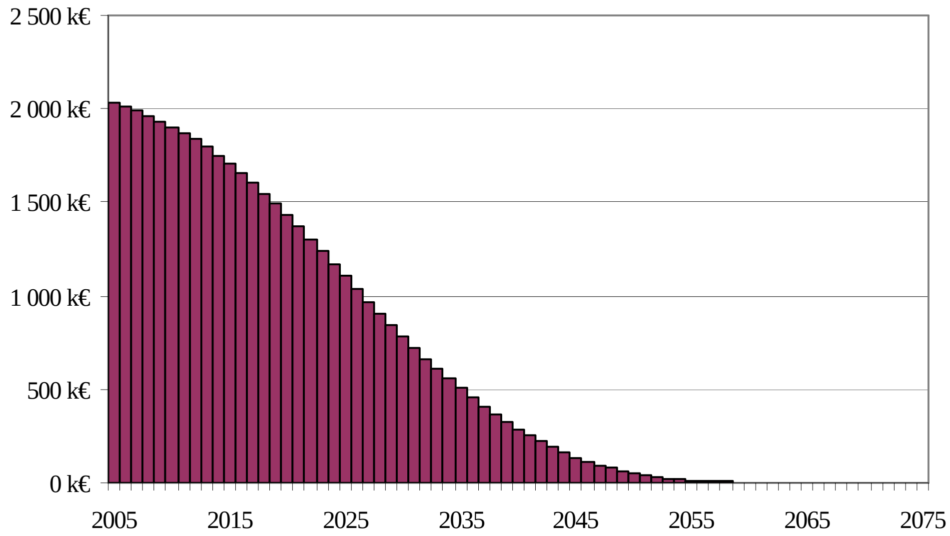

Fig. 10 : *Expected flows of pensions*

In reasoning in the act to simplify at a null discount rate (*i.e.* the capital of one € of pension is exactly equal to the life expectancy at this age), we find that the engagement of the plan passes from 67 M€ in the reference situation $e_{64}\left(2050\right)=38$ to 70,9 M€ in supposing a higher target value of 10 years. This gap of a little less than 6 % can be associated with the longevity risk.

We will notice incidentally that the gap of 10 years over 50 years of projection is coherent with the gap of almost 2 years observed over 10 years during the actualisation of the regulatory tables in France (see the introduction of this study).

4. CONCLUSION

Whereas the bad anticipation of the future life expectancy at the service ages of pensions constitutes a main risk for the annuity plans, the suggested model provides an operational tool that is simple to implement and that allows to measure the sensitivity of the engagement of the plan, in accordance with various hypotheses of evolution of this life expectancy.

In particular, the suggested model provides a simple and justifiable framework to quantify a specific capital requirement for the longevity risk carried by the annuity plan, in a way more readable and more robust than the stochastic mortality models.

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