Estimation Errors and SCR Calculation

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Abstract

The risk of not accurately estimating the amount of future losses is an essential issue in risk measurements. Sources of estimation risk include errors in estimation of parameters which can affect directly the VaR precision. Estimation risk in itself is related to operational risk in a way that the losses are arising from estimation errors.

In this working paper, we are interested in measuring the error induced on the SCR by the estimation error of the parameters.

Keywords: Estimation Risk, Solvency II, ORSA.

1 Introduction

Measuring the Value-at-Risk of the own funds is a central topic in insurance with the new Solvency II framework and finance regarding Basle II and III accords.

Many banks and financial institutions, develop models to compute the value-at-risk and resulting capital requirement, but we know that any model is by definition an imperfect simplification and in some cases a model will produce results that are bias due to parameter estimation errors. For instance, this point is illustrated in BOYLE AND WINDCLIFF [2004] for pension plans investment and in PLANCHET AND THEROND [2012] for the determination of Solvency capital in the Solvency II Framework. As a direct consequence of parameter estimation risk, the capital requirement may be underestimated.

Article 101 of the European directive states that the Solvency Capital Requirement (SCR), shall correspond to the Value-at-Risk of the basic own funds of an insurance or reinsurance undertaking subject to a confidence level of 99.5% over a one-year period.

We are interested in this working paper, in assessing the potential loss of accuracy from estimation error when calculating the SCR, we expand this analytical framework where an insurer must calculate a VaR to a confidence level of 99.5% on a distribution which we must estimate the parameters, now this estimation might lead to important differences in the numerical results. To be able to illustrate this situation we took the very particular simple case of the only market risk for an asset consisting of a zero coupon bond, we are thus led to study the distribution of $P(1, T-1)$ which is the value of the asset at $t = 1$ and we highlight the possible undervaluation of the Solvency Capital Requirement if a special attention is not given to the risk of parameter estimation. At the end, we are going to check the effect of adding another zero coupon bond on our Capital estimation.

2 Model Presentation

For sake of simplicity, we use in this work the classical one-factor interest rate Vasicek model (cf. Vasicek [1977]). The key point here is that we need closed form solutions for the zero-coupon bond price. Moreover, the distribution of the ZC price at a future time is known (and is log normal one). This will help us to compute very easily the quantile of the price distribution.

2.1 Simulating Interest rates: The Vasicek model

We consider an asset consisting of a zero-coupon bond now Vasicek model, assumes that $r(t)$ is an adapted process on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{0<t<T})$ that satisfies the following stochastic differential equation:

$$dr_t = k(\theta - r_t)dt + \sigma dW_t$$

where k, θ and σ are non-negative constants and r_t is the current level of interest rate. The parameter θ is the long run normal interest rate. The coefficient $k > 0$ determines the speed of pushing the interest rate towards its long run normal level, and W is a standard Brownian motion.

The solution of the $r(t)$ stochastic differential equation would generate (cf. Bayazit [2004]):

$$r_t = r_0 e^{-kt} + \theta(1 - e^{-kt}) + \sigma e^{-kt} \int_0^t e^{ku} dW_u \tag{1}$$

and for $0 \le s \le t \le T$, $r_t = r_s e^{-k(t-s)} + \theta(1 - e^{-k(t-s)}) + \sigma \int_s^t e^{-k(t-u)} dW_u$

If one wants to take the market price of risk λ into account, (for the purpose of this article and to minimize our uncertainty, we took a constant market price of risk $\lambda(t, r_t) = \lambda$, but in general a market price of risk has a more complex structure, more on this topic could be found in CAJA AND PLANCHET [2011]), which allows us to switch between the real universe $\mathbb{P}$ to the risk-neutral world $\mathbb{Q}$, then with the respect of the new probability measure $\mathbb{Q}$, we can rewrite the stochastic equation as follows (cf. Van Elen [2010]):

$$dr_t = k(\theta^* - r_t)dt + \sigma dW_t^*$$

The structure of this equation in the risk-neutral world is comparable to that in the real universe where, $\theta^* = \theta - \dfrac{\sigma \lambda}{k}$.

In particular, we can use the explicit formula seen previously with,

$$r_t = r_0 e^{-kt} + \theta^*(1 - e^{-kt}) + \sigma e^{-kt} \int_0^t e^{ku} dW_u^*.$$

A zero-coupon bond of maturity T is a financial security paying one unit of cash at a prespecified date T in the future without intermediate payments. The price at time $t \le T$ is denoted by $P(t, T)$.

The general case of a zero coupon bond at time t with maturity T is (cf. Bayazit [2004]):

$$P(t, T) = A(t, T)e^{-B(t, T)r(t)},$$

where,
$$
\begin{cases}
A(t, T) = exp\left[(\theta^* - \dfrac{\sigma^2}{2k^2})(B(t, T) - (T - t)) - \dfrac{\sigma^2}{4k}B(t, T)^2 \right] \\
B(t, T) = \dfrac{1 - e^{-k(T-t)}}{k}
\end{cases}
$$

2

2.2 Parameter Estimation

For the Vasicek model, we have $dr_t = k(\theta - r_t)dt + \sigma dW_t$ with the solution already calculated that generate $r_t = r_s e^{-k(t-s)} + \theta(1 - e^{-k(t-s)}) + \sigma e^{-kt} \int_s^t e^{ku} dW_u,$

now r_t is normally distributed with mean and variance:

$$\begin{cases} \mathbb{E}[r_t|\mathcal{F}_s] = r_s e^{-k(t-s)} + \theta(1 - e^{-k(t-s)}) \\ Var[r_t|\mathcal{F}_s] = \dfrac{\sigma^2}{2k}(1 - e^{-2k(t-s)}) \end{cases}$$

The properties of the integral of a deterministic function relative to a Brownian motion lead to the exact discretization (cf. Planchet & Thérond [2009]):

$$r_{t+\delta} = r_t e^{-k\delta} + \theta(1 - e^{-k\delta}) + \sigma \sqrt{\frac{1 - e^{-2k\delta}}{2k}} \epsilon,$$

where ϵ is a random variable that follows the standard normal distribution and δ is the discretization step.

To calibrate this short rate model, let's rewrite it in more familiar regression format: $y_t = \alpha + \beta x_t + \sigma_1 \epsilon_t$

Where, $y_t = r_{t+\delta}$, $\alpha = \theta(1 - e^{-k\delta})$, $\beta = e^{-k\delta}$, $x_t = r_t$ and $\sigma_1 = \sigma \sqrt{\dfrac{1 - e^{-2k\delta}}{2k}}$

The OLS regression provides the maximum likelihood estimator for the parameters: α, β and σ_1 (cf. Brigo $et\ al.$ [2007]).

One can for instance compare the formulas stated above with the estimators derived directly via maximum likelihood given our Log-Likelihood function: $\ln(\mathcal{L}) = \ln\left(\dfrac{1}{\sigma_1\sqrt{2\pi}}\right)^n - \dfrac{1}{2\sigma_1^2}\sum_{i=1}^n (r_i - r_{i-1}\beta - \theta(1 - \beta))^2$

which are of the form (cf. Brigo & Mercurio [2006]):

$$\hat{\beta} = \frac{n\sum_{i=1}^n r_i r_{i-1} - \sum_{i=1}^n r_i \sum_{i=1}^n r_{i-1}}{n\sum_{i=1}^n r_{i-1}^2 - \left(\sum_{i=1}^n r_{i-1}\right)^2}$$

$$\hat{\theta} = \frac{\sum_{i=1}^n [r_i - \hat{\beta}r_{i-1}]}{n(1 - \hat{\beta})}$$

$$\hat{\sigma}_1^2 = \frac{1}{n}\sum_{i=1}^n [r_i - \hat{\beta}r_{i-1} - \hat{\theta}(1 - \hat{\beta})]^2$$

Parameter estimation is an important stage in the simulation of trajectories of a continuous process because it can cause a bias as we will see in the next section.

Given the parameters estimated all what is left is to find the market price of risk λ which give us the right price of the zero coupon. Since $P(0,T)$ is usually known (given by the market at time $t = 0$), and $P(0,T)$ is a function of k, θ, σ and λ as seen previously, λ would be calculated as:

$$\lambda = \left(\left(\theta - \frac{\sigma^2}{2k^2}\right) - \frac{\left(\frac{\sigma^2}{4k}B(0,T) + r_0\right)B(0,T) + ln\big(P(0,T)\big)}{B(0,T) - T}\right)\frac{k}{\sigma}$$

2.3 Calculation of the SCR

Article 101 of the European directive (cf. CEIOPS [2009]) states that the Solvency Capital Requirement (SCR), shall correspond to the Value-at-Risk of the basic own funds of an insurance or reinsurance undertaking subject to a confidence level of 99.5% over a one-year period.

In our case the SCR would be calculated as follows:

$r_t = r_0 e^{-kt} + \theta^*(1 - e^{-kt}) + \sigma e^{-kt} \int_0^t e^{ku} dW_u^*$ is normaly distributed with mean and variance:

$$
\begin{cases}
\mathbb{E}[r_t | \mathcal{F}_0] = r_0 e^{-kt} + \theta^*(1 - e^{-kt}) \\
Var[r_t | \mathcal{F}_0] = \dfrac{\sigma^2}{2k}(1 - e^{-2kt})
\end{cases}
$$

In addition, $A(t,T)$ and $B(t,T)$ are deterministic in the the zero coupon price fundamental equation:

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)},$$

So, $P(t,T)$ would follow the LogNormal distribution since if X is normally distributed with mean μ_x and variance σ_x^2, ae^{-bX} would follow the Lognormal distribution with:
mean: $e^{ln(a)-b\mu_x+\frac{1}{2}b^2\sigma_x^2}$ and variance: $(e^{b^2\sigma_x^2} - 1)e^{-2b\mu_x+b^2\sigma_x^2}$.

So in our case,

$$\mathbb{E}\big[P(t,T)\big] = exp\left\{ ln(A(t,T)) - B(t,T)\left(r_0 e^{-kt} + \theta^*(1 - e^{-kt})\right) + \frac{1}{2}B(t,T)^2\left(\frac{\sigma^2}{2k}(1 - e^{-2kt})\right) \right\}$$

$$Var\big[P(t,T)\big] = \left(e^{B(t,T)^2\frac{\sigma^2}{2k}\left(1-e^{-2kt}\right)} - 1\right)exp\left\{ -2B(t,T)\left(r_0 e^{-kt} + \theta^*(1 - e^{-kt})\right) + B(t,T)^2\frac{\sigma^2}{2k}(1 - e^{-2kt}) \right\}$$

To reiterate, since we have $P(t,T) \hookrightarrow \mathcal{LN}\left(\mu_L = ln(A(t,T)) - B(t,T)\mathbb{E}[r_t|\mathcal{F}_0], \sigma_L^2 = B(t,T)^2 Var[r_t|\mathcal{F}_0] \right)$,

if we denote x_p^{LN} our quantile, p the critical value and Φ the cdf of a standardized gaussian random variable we would have:

$$x_p^{LN} = VaR_p(P(t,T)) = F_0^{-1}(p) = exp(\sigma_L \Phi^{-1} \underbrace{(p)}_{1-\alpha} + \mu_L)$$

At the end, the Solvency Capital Requirement (SCR) of our zero coupon bond would be:

$$SCR = P(0,T) - x_p(e^{-R_1}P(1, T - 1))$$

In practice, $p = 0.5\%$ and R_1 is the spot rate for $t = 1$.

3 Estimation Risk

To show the estimation risk we are going to take the very particular simple case of the only market risk for an asset consisting of a zero coupon bond, we are thus led to study the distribution of $P(1, T - 1)$ which is the value of the asset at $t = 1$ and then we will add to it one more zero coupon bond and check how estimation risk affects the Solvency Capital Requirement if a special attention is not given to the risk of parameter estimation.

3.1 Case of one zero coupon bond

For the case of one zero coupon bond, we are going to follow the below steps to calculate our Solvency Capital requirement and show the estimation error behind:

1. We fix the Vasicek parameters set $\vartheta = (k_0, \theta_0, \sigma_0, \lambda_0)$ by applying the Maximum Likelihood estimation technique stated previously in section 2.2.

2. Let $\omega = (k, \theta, \sigma)$, given the asymptotic normality of MLE, we have: $\hat{\omega} \hookrightarrow \mathcal{N}(\underbrace{(k_0, \theta_0, \sigma_0)}_{\omega_0}, \Sigma)$, where

   Σ is calculated using the inverse of fisher information matrix: $\Sigma = \left(-\mathbb{E}\left[\dfrac{\partial^2 \ln \mathcal{L}}{\partial \omega \partial \omega'} \right] \right)^{-1}_{\omega = \omega_0}$.

3. We estimate the market price of risk λ since $P(0,T)$ is known at $t = 0$, which allows us to switch between the real measure $\mathbb{P}$ to the risk-neutral measure $\mathbb{Q}$ see section 2.2;

4. Calculate the Solvency capital Requirement (SCR) of the Zero-Coupon bond at 0.5%, and its Relative difference $\epsilon = \dfrac{S\hat{C}R - SCR}{SCR}$;

5. Repeat steps 2 to 5 on a number N of times (in our example, we took $N = 10^5$ iterations) to be able to analyze the empirical cumulative distribution function of $\epsilon = (\epsilon_1, ..., \epsilon_N)$, where $\epsilon_i = \dfrac{S\hat{C}R_i - SCR}{SCR}$.

As for the calculation of Fisher's inverse information Matrix:

$$\Sigma = \left(-\mathbb{E}\left[\frac{\partial^2 \ln \mathcal{L}}{\partial \omega \partial \omega'} \right] \right)^{-1}_{\omega = \omega_0},$$

we used the log-likelihood function: $\ln(\mathcal{L}) = \ln\left(\dfrac{1}{\sigma_1 \sqrt{2\pi}} \right)^n - \dfrac{1}{2\sigma_1^2} \sum\limits_{i=1}^{n} (r_i - r_{i-1}\beta - \theta(1 - \beta))^2$ see appendix A for the full support.

3.2 Estimation Risk of a Portfolio consisting of two Zero-Coupon bonds

In this section, we will be interested in capturing the effect of parameter estimation when we add another zero-coupon bond to our portfolio; we took the two zero coupon bonds maturing respectively at T_1 and T_2 months.

Let P_0 be the Price at $t = 0$ of our Portfolio, $P_0 = P(0, T_1) + P(0, T_2)$ with the only difference that P will not follow a LogNormal Distribution in that case.
Following the same estimation technique of parameters from above, we explain how we computed the market price of risk λ which give us the right price of the Portfolio.

Now with $P(0,T) = e^{\left(\theta - \frac{\lambda \sigma}{k} - 1/2 \frac{\sigma^2}{k^2}\right)\left(\frac{1 - e^{-Tk/12}}{k} - T/12\right) - 1/4 \frac{\sigma^2\left(1 - e^{-Tk/12}\right)^2}{k^3} - \frac{\left(1 - e^{-Tk/12}\right)r0}{k}}$, already illustrated above we would have:

$$
\begin{aligned}
P_0 &= P(0, T_1) + P(0, T_2) \\
P_0 &= e^{\left(\theta - \frac{\lambda \sigma}{k} - 1/2 \frac{\sigma^2}{k^2}\right)\left(\frac{1 - e^{-T_1 k/12}}{k} - T_1/12\right) - 1/4 \frac{\sigma^2\left(1 - e^{-T_1 k/12}\right)^2}{k^3} - \frac{\left(1 - e^{-T_1 k/12}\right)r0}{k}} + \\
&\quad e^{\left(\theta - \frac{\lambda \sigma}{k} - 1/2 \frac{\sigma^2}{k^2}\right)\left(\frac{1 - e^{-T_2 k/12}}{k} - T_2/12\right) - 1/4 \frac{\sigma^2\left(1 - e^{-T_2 k/12}\right)^2}{k^3} - \frac{\left(1 - e^{-T_2 k/12}\right)r0}{k}}
\end{aligned}
$$

We can solve this equation numerically and find λ by using the dichotomy method.

3.2.1 The sum of lognormal variables

We often encounter the sum of lognormal variables in financial modeling, a lot of methods are used to approximate the sum into a lognormal distribution but in our study, we are going to apply Fenton-Wilkinson approach (cf. Fenton [1960]) the most used for its simplicity. We place ourselves in the

case where we have l lognormal distributions $X_i = e^{Y_i}$ where $Y_i \hookrightarrow \mathcal{N}(\mu_i, \sigma_i)$, the approach states that the sum of l lognormal distributions $S = \sum_{i=1}^{l} e^{Y_i}$ can be approximated by a lognormal distribution e^Z, $Z \hookrightarrow \mathcal{N}(\mu_Z, \sigma_Z^2)$ where,

$$
\begin{aligned}
\mu_Z &= 2\ln(m_1) - 1/2\ln(m_2) \\
\sigma_Z^2 &= \ln(m_2) - 2\ln(m_1)
\end{aligned}
$$

and,

$$m_1 = \mathbb{E}(S) = e^{\mu_Z + \sigma_Z^2/2} = \sum_{i=1}^{l} e^{\mu_{Y_i} + \sigma_{Y_i}^2},$$

$$m_2 = \mathbb{E}(S^2) = e^{2\mu_Z + 2\sigma_Z^2} = \sum_{i=1}^{l} e^{2\mu_{Y_i} + 2\sigma_{Y_i}^2} + 2\sum_{i=1}^{l-1}\sum_{j=i+1}^{l} e^{\mu_{Y_i} + \mu_{Y_j}} e^{1/2(\sigma_{Y_i}^2 + \sigma_{Y_j}^2 + 2\rho_{ij}\sigma_{Y_i}\sigma_{Y_j})},$$

are the first and second moment of S (cf. El Faouzi & Maurin [2006])[1].

As a result, and by applying the same previous steps as in section 3.1, where $SCR = P - x_p \bigg(\underbrace{P(12, T-12) + P(12, T'-12)}_{P_1} \bigg)$,

with x_p our quantile estimated, P_1 is the price of our Portfolio at $t = 12$ months and typically in the Solvency II context $p = 0.5\%$, we were able to estimate our Capital Requirement.

3.3 Numerical Results

In this section, we apply the preceding theoretical discussion of our estimation technique to the problem at hand. From the Federal Reserve (FR), we took our interest rates, dated from January 1982 till August 2008 ($n = 320$ months), and compared our results with smaller data dated from July 2001 till August 2008, which gives us 86 months overall. We estimated the model parameters of Vasicek then we applied the asymptotic normality of Maximum Likelihood to generate various outcomes. This simulation example is intended to indicate how parameter estimation can affect directly the Solvency Capital Requirement for one zero coupon bond of maturity 120 months and two zero coupon bonds of respective maturity of 60 and 120 months. Now since the estimation of the SCR is executed on simulated values, the simulations and the estimation of the SCR has to be effected on a large size sample (we took $N = 10^5$ simulations).

As so, and with our fixed values parameters shown in table 1 below, we have been able to estimate our Solvency Capital Requirement and its relative difference $\epsilon = (\epsilon_1, ...\epsilon_N)$, where $\epsilon_i = \dfrac{S\hat{C}R_i - SCR}{SCR}$. With the given Inverse Fisher information matrix for $n = 320$:

$$
\begin{bmatrix}
0.00004655 & -0.00004988 & -2.7924 \times 10^{-13} \\
-0.00004988 & 0.0001488 & 8.7048 \times 10^{-13} \\
-2.7924 \times 10^{-13} & 8.7048 \times 10^{-13} & 1.5591 \times 10^{-8}
\end{bmatrix}
$$

and for $n = 86$:

$$
\begin{bmatrix}
0.0002959 & -0.0003975 & -1.1661 \times 10^{-12} \\
-0.0003975 & 0.0009364 & 2.69943 \times 10^{-12} \\
-1.1661 \times 10^{-12} & 2.69943 \times 10^{-12} & 3.2427 \times 10^{-8}
\end{bmatrix}
$$

---

[1]Corrections of the formulas given in this paper have been made regarding the second moment of S: $\mathbb{E}(S^2)$

Fixed Values		
n	320	86
$r_0 =$	1.75%	1.75%
$k =$	0.219	0.153
$\theta =$	3.388%	0.971%
$\sigma =$	1.104%	0.8189%
$\lambda =$	−0.4189	−0.9954
$P(0, 120) =$	0.678	0.678
$P(0, 60) =$	0.854	0.854
$P =$	1.532	1.532

Table 1: Vasicek parameter model

Now we are interested in the particular case of $\epsilon < 0$ and more particularly where $\epsilon < \epsilon_{max}$ where the SCR is underestimated by more than 3%. For that, table 2 compares the results regarding, $\mathbb{P}(\epsilon \leq -0.03)$, for our one and two zero coupon bonds of maturity 60 and 120 months.

$n = 320$		$n = 86$	
P(0,120)	P(0,60)+P(0,120)	P(0,120)	P(0,60)+P(0,120)
45.3%	42.9%	48.4%	47.8%

Table 2: Underestimation of the SCR

We can easily check that whichever the case treated, on average, 45% of the simulations underestimated our solvency capital requirement by more than 3%, ($\mathbb{P}(\epsilon \leq -3\%) = 45\%$).
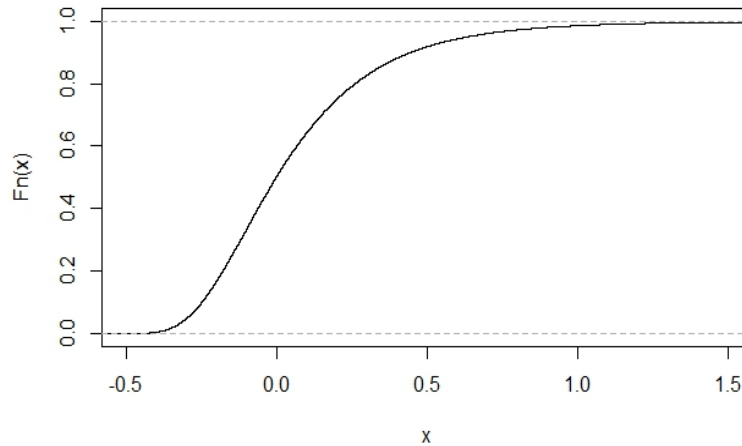

Figure 1: Empirical Cumulative Distribution Function of ϵ

Figure 1, shows us the empirical cumulative distribution function of ϵ of our Portfolio composed of one zero coupon bond maturing in 120 months, for $n = 320$. We notice that, 45.3% of our cases underestimated the Solvency Capital Requirement by more than 3% given a probability $\mathbb{P}(\epsilon < -3\%) = 45.3\%$.

7

4 Conclusion

In this article, we have highlighted the possible underestimation of the Solvency Capital Requirement, by taking into consideration the very simple case of an asset consisting of one and two zero coupon bonds. Applying the Vasicek model, enabled us to illustrate the direct consequence of parameter estimation risk on the capital requirement. For example BOYLE AND WINDCLIFF [2004] shows a similar consequence by working on the pension plans investment.

In practice, we do not know the true value of our parameters and estimated values are usually treated as correct, yet if we take into consideration the impact of parameter estimation risk, then our capital requirement have a 50% chance to be underestimated by more then 3% as the study shows us.

So it would be more appropriate in reality to privilege simple and prudent models and avoid complexing them, in a way, to prevent more estimation errors that might have severe influences in some cases on our Solvency Capital Requirement.

Moreover, such simplified models should be used to comply to the Own Risk Solvency Assessment (ORSA) required by Solvency II.

References

AHMAD R. & WILMOTT P. [2006], *The Market Price of Interest-rate Risk: Measuring and Modelling Fear and Greed in the Fixed-income Markets* , Wilmott magazine

BAYAZIT D. [2004], *Yield Curve Estimation and Prediction with Vasicek Model*, A Thesis submitted to the Graduate School of Applied Mathematics of the Middle East Technical University

BOYLE P. & WINDCLIFF H. [2004], *The 1/n pension investment puzzle*, North American Actuarial Journal 8

BRIGO D. & MERCURIO F. [2006], *Interest Rate Models: Theory and Practice - With Smile, Inflation and Credit*, New York NY, Springer verlag

BRIGO D. & DALESSANDRO A. & NEUGEBAUER M. & TRIKI F. [2007], *A Stochastic Process Toolkit for Risk Management*, Fitch Solutions, and Department of Mathematics, Imperial College, London

CAJA A. & PLANCHET F. [2011], *La mesure du prix de marché du risque: quels outils pour une utilisation dans les modèles en assurance ?*, Cahier de Assurances et gestion des risques Vol.78 (3/4)

CEIOPS [2009], *CEIOPS' Advice for Level 2 Implementing Measures on solvency II*, CEIOPS-DOC-48/09, www.eiopa.europa.eu

EL FAOUZI N.E & MAURIN M. [2006], *Sur la somme de variables log-normales: application à la fiabilité de temps de parcours routiers*, working paper, INRETS

FENTON L. [1960], *The sum of lognormal probability distribution in scatter transmission system*, IEEE Trans. Communication Systems, Vol CS-8 pp. 56-57, March 1960

PLANCHET F. & JACQUEMIN J. [2003], *L'utilisation de méthodes de simulation en assurances*, Bulletin Français d'Actuariat, vol. 6, No. 11, 3-69

PLANCHET F. & THÉROND P-E. [2005], *Simulation de trajectoires de processus continus*, Belgian Actuarial Bulletin, Vol. 5, No. 1

PLANCHET F. & THÉROND P-E. [2012], *Risque de modèle et détermination du capital économique dans le projet Solvabilité 2*, International Review of Applied Financial Issues and Economics

VAN ELEN E. [2010], *Term structure forecasting does a good fit imply reasonable simulation results?*, Tilburg School of Economics and Management Tilburg University

VASICEK O. [1977], *An Equilibrium Characterization of the Term Structure*, Journal of Financial Economics, Volume 5, 177-188

A The Inverse of Fisher Information Matrix:

The calculation of the Inverse Fisher Information Matrix: $\Sigma = \left(-\mathbb{E}\left[\dfrac{\partial^2 \ln \mathcal{L}}{\partial \omega \partial \omega'}\right]\right)^{-1}_{\omega=\omega_0}$, was done by the use of the Log-Likelihood function:

$$\ln(\mathcal{L}) = \ln\left(\frac{1}{\sigma_1 \sqrt{2\pi}}\right)^n - \frac{1}{2\sigma_1^2} \sum_{i=1}^{n} (r_i - r_{i-1}\beta - \theta(1-\beta))^2$$

Now by deriving the Log-Likelihood function we obtain the following:

$$\frac{\partial^2}{\partial \beta^2} \ln(\mathcal{L}) = -1/2 \frac{2n\theta^2 + \sum_{i=1}^{n} 2 r_{i-1}^2 - 4 r_{i-1}\theta}{\sigma_1^2}$$

$$\frac{\partial^2}{\partial \theta^2} \ln(\mathcal{L}) = -1/2 \frac{2n - 4n\beta + 2n\beta^2}{\sigma_1^2}$$

$$\frac{\partial^2}{\partial \sigma_1^2} \ln(\mathcal{L}) = \frac{n}{\sigma_1^2} - 3 \frac{n\theta^2 - 2n\theta^2\beta + n\theta^2\beta^2 + \sum_{i=1}^{n} r_i^2 - 2 r_i r_{i-1}\beta - 2 r_i\theta + 2 r_i\beta\theta + r_{i-1}^2\beta^2 + 2 r_{i-1}\beta\theta - 2 r_{i-1}\beta^2\theta}{\sigma_1^4}$$

$$\frac{\partial^2}{\partial \theta \partial \beta} \ln(\mathcal{L}) = -1/2 \frac{-4n\theta + 4n\theta\beta + \sum_{i=1}^{n} 2 r_i + 2 r_{i-1} - 4 r_{i-1}\beta}{\sigma_1^2}$$

$$\frac{\partial^2}{\partial \sigma_1 \partial \beta} \ln(\mathcal{L}) = \frac{-2n\theta^2 + 2n\theta^2\beta + \sum_{i=1}^{n} -2 r_i r_{i-1} + 2 r_i\theta + 2 r_{i-1}^2\beta + 2 r_{i-1}\theta - 4 r_{i-1}\beta\theta}{\sigma_1^3}$$

$$\frac{\partial^2}{\partial \sigma_1 \partial \theta} \ln(\mathcal{L}) = \frac{2n\theta - 4n\theta\beta + 2n\theta\beta^2 + \sum_{i=1}^{n} -2 r_i + 2 r_i\beta + 2 r_{i-1}\beta - 2 r_{i-1}\beta^2}{\sigma_1^3}$$

$$\frac{\partial^2}{\partial \beta \partial \theta} \ln(\mathcal{L}) = -1/2 \frac{-4n\theta + 4n\theta\beta + \sum_{i=1}^{n} 2 r_i + 2 r_{i-1} - 4 r_{i-1}\beta}{\sigma_1^2}$$

$$\frac{\partial^2}{\partial \beta \partial \sigma_1} \ln(\mathcal{L}) = \frac{-2n\theta^2 + 2n\theta^2\beta + \sum_{i=1}^{n} -2 r_i r_{i-1} + 2 r_i\theta + 2 r_{i-1}^2\beta + 2 r_{i-1}\theta - 4 r_{i-1}\beta\theta}{\sigma_1^3}$$

$$\frac{\partial^2}{\partial \theta \partial \sigma_1} \ln(\mathcal{L}) = \frac{2n\theta - 4n\theta\beta + 2n\theta\beta^2 + \sum_{i=1}^{n} -2 r_i + 2 r_i\beta + 2 r_{i-1}\beta - 2 r_{i-1}\beta^2}{\sigma_1^3}$$

We can easily check that the matrix is symmetric, since:

$$\frac{\partial^2}{\partial \theta \partial \beta} \ln(\mathcal{L}) = \frac{\partial^2}{\partial \beta \partial \theta} \ln(\mathcal{L}), \ \frac{\partial^2}{\partial \sigma_1 \partial \beta} \ln(\mathcal{L}) = \frac{\partial^2}{\partial \beta \partial \sigma_1} \ln(\mathcal{L}) \text{ and } \frac{\partial^2}{\partial \sigma_1 \partial \theta} \ln(\mathcal{L}) = \frac{\partial^2}{\partial \theta \partial \sigma_1} \ln(\mathcal{L}).$$