QUANTIFICATION OF THE SYSTEMATIC RISK OF MORTALITY ON AN ANNUITY PLAN

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ABSTRACT

The aim of this paper is to propose a realistic and operational model to quantify the systematic risk of mortality included in a liabilities of retirement. The presented model is built on the basis of the model of Lee-Carter. The stochastic prospective tables thus built make it possible to project the evolution of the random mortality rates in the future and to quantify the systematic risk of mortality.

KEYWORDS: Prospective tables, extrapolation, adjustment, life annuities, stochastic mortality.

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1. INTRODUCTION

For a decade life tables used for the reserving and the pricing of life annuities try to take into account the underlying drift of mortality, and it conveys a regular increase of the life expectancy. Thus it can be seen in France an increase of the life expectancy at birth of a term per year for about thirty years. This necessity of using prospective life tables which consider this phenomenon of drift has been taken into account by the legislator and « generational » tables have been approved ten years ago. Constructed from the mortality of female population between 1961 and 1987, these tables have been used for pricing and reserving life annuities contract (immediate or differed) since 1st July 1993. In the process of updating new tables might arrive in 2006.

However recent studies (CURRIE and al. [2004]) show that the evolution of the instantaneous death rate presents erratic variations at the different ages around the emerging tendency. These variations are not explained by sampling fluctuations; CAIRNS and al. [2004] gives a detailed analysis of this phenomenon. The same phenomenon is noticed in this study using the data from INED (*cf. infra* a specification of these data):

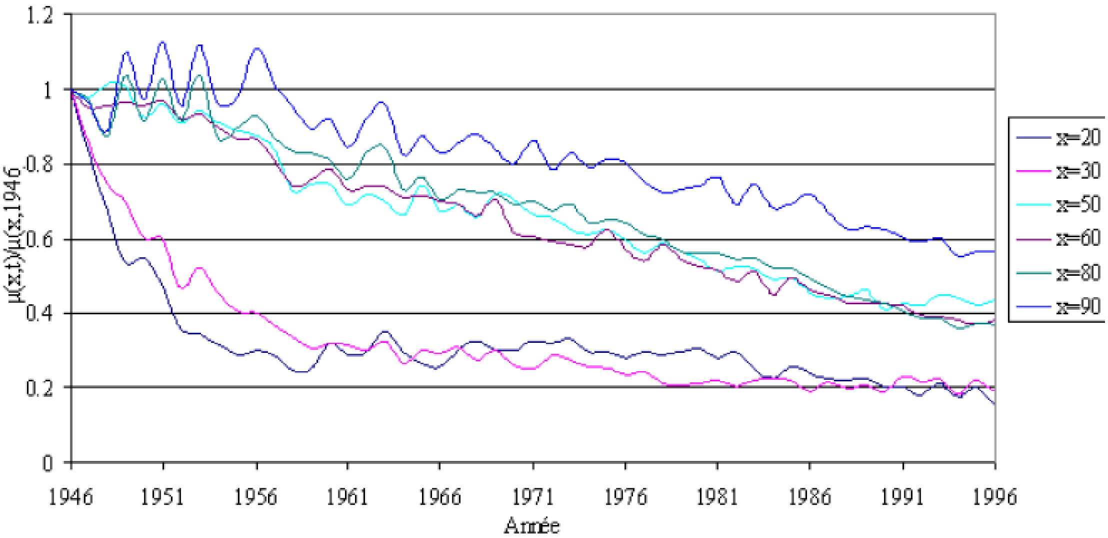

Fig. 1 : *Death rate fluctuation around the long term trend*

These variations have a systematic impact on the fixed age individuals and are not mutualisable. They run therefore a potential important risk to an annuity plan whose technical balance is built on the mutualisation of the survival risk of its members.

It leads to look for a model able to explain these fluctuations around the underlying value and to draw consequences regarding the level of mathematical reserves that the plan has to be set in order to ensure its technical balance. More precisely the part of non mutualisable risk among the global risk has to be quantify in order to value the risk that the plan has to face (the measure has to be determined).

Stochastic mortality models are well adapted for this analysis. They suggest that the future mortality rate $\mu(x,t)$ is random and therefore $\mu(x,t)$ is a stochastic process (as a function of t with a fixed x). The mortality rate really observed given an age and a year is the realization of a random variable: the analogy with the methods of Bayesian adjustment can be noticed (about these methods, refer to TAYLOR [1992]).

Stochastic modellings of mortality phenomena are numerous in the articles. Models of financial inspiration arising from the pricing problem of mortality derivatives are not considered in this paper. The reader interested in these approaches can consult DAHL [2004] and SCHRAGER [2004]. It can just noticed that the adjustment of these models to mortality data is not as simple as that, the not taken into account of dependences between ages is problematic with the studied problem (these models suggest an evolution phase according to time for each age without taking into account correlations between close ages).

Several standard models as Bayesian adjustments and the Kimeldorf-Jones model (KIMELDORF and JONES [1967]) are *de facto* stochastic models. They are however essentially built with the prospect of the adjustment of the instantaneous mortality rates, without considering the prospective dimension which is essential for the analysis of a pension plan liabilities.

Recent models of construction of prospective life tables as the Lee-Carter model (particularly refer to LEE and CARTER [1992], LEE [2000], SITHOLE and al. [2000]) or Poisson models (*cf.* BROUHNS and al. [2002] and PLANCHET and THÉROND [2006] for a presentation and an analysis of these models), are particular cases of stochastic models, even if they were originally built in order to make (temporal) extrapolations of the determinist surface $\mu(x,t)$. Once previous rates adjusted, the future mortality rates are deduced from the extrapolation of the temporal component (parametric or not) of a kept prospective models (Obviously this purely extrapolative approach could be criticised; for example consult GUTTERMAN and VANDERHOOF [1999] about these questions).

In this paper the Lee-Carter model is kept, it allows easily to build stochastic mortality surfaces, moreover it becomes a standard in order to build prospective life tables. The log-Poisson variant (*cf.* BROUHNS and al. [2002]) leads to very similar results.

After building a set of prospective life tables on national data thanks to this model, it is used to quantify the random component of the systematic risk and to evaluate the liabilities level of the pension plan.

The data used in the numerical applications are taken from PLANCHET and THÉROND [2004].

The present paper uses a study done in the framework of « l'Institut de Science Financière et d'Assurances » (Insurance and Financial Science Institute, a French school of actuaries) from the Lyon 1 university by FAUCILLON and al. [2006].

1.1. CHARACTERISTICS OF THE PORTFOLIO

Afterwards in the numerical applications, it is used a portfolio formed of 374 pensioners with an average age of 63.8 years at 31/12/2005. The average annual pension is up to 5.5 k€. The graphic *infra* shows the expected flows due to pensions as a function of time built from the life table named TV 2000 (a French table).

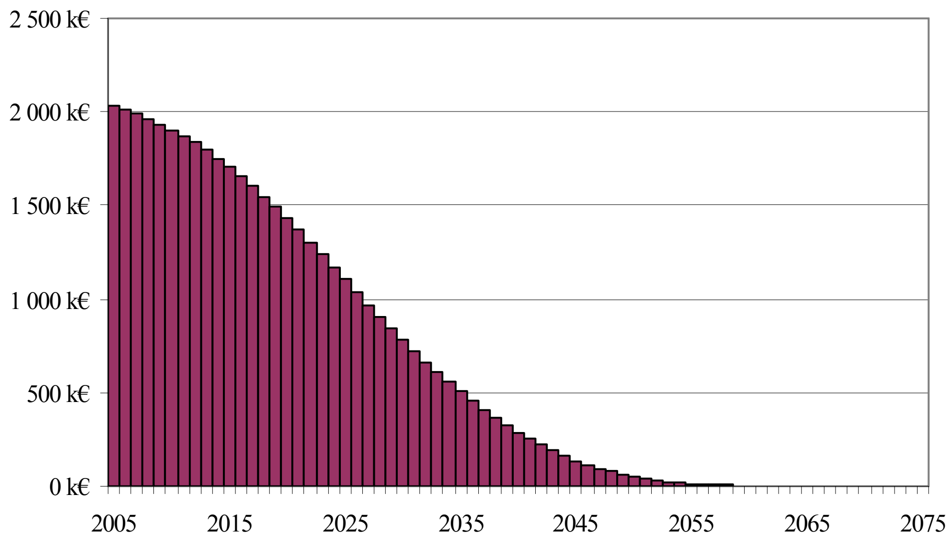

Fig. 2 : *Expected flows of pensions*

With a discount rate of 2.5 % for the reserves, the initial mathematical reserve is up to 37.7 M€. It can be noticed in PLANCHET and THÉROND [2004] that calculating with the female moment table TV2000 leads to a 32.8 M€ reserve. This incidentally reveals the significance of a prospective approach of the mortality in order to evaluate the level of the liabilities but without excessively under estimate it.

1.2. NOTATIONS

In this paper the following notations are used:

- ✓ L_0 the amount of the initial mathematical reserve,
- ✓ $\tilde{F}_t$ the (random) flow of pension which has to be paid at the date t,
- ✓ i the (discrete) discount rate for the mathematical reserve,
- ✓ **J** the set of individuals,
- ✓ $x(j)$ at the initial date the age of the individual j and r_j the amount of his annual pension.

1.3. PROBLEM

The portfolio only contains pension in process of paying, supposed non reversible. In 0, the insurer estimates the sequence of probable flows due to claims, flows are noted $\left(F_t\right)_{t \geq 1}$:

$$F_t = \mathbf{E}\left[\tilde{F}_t \mid \Phi_0\right], \tag{1}$$

With:

$$\tilde{F}_t = \sum_{j \in \mathbf{J}} r_j * \mathbf{1}_{]t;\infty[}\left(T_{x(j)}\right), \tag{2}$$

where $T_{x(j)}$ means the (random) date of death of the $x(j)$ aged head. Under classical notations of life insurance (*cf.* PETAUTON [1996]):

$$F_t = \sum_{j \in \mathbf{J}} r_j * \frac{l_{x(j)+t}}{l_{x(j)}}. \tag{3}$$

From this estimate it determines the mathematical reserve L_0:

$$L_0 = \sum_{t=1}^{\infty} F_t \left(1+i\right)^{-t}. \tag{4}$$

Afterwards the random value of the liabilities is analysed, let be the random variable:

$$\Lambda = \sum_{t=1}^{\infty} \tilde{F}_t \left(1+i\right)^{-t} = \sum_{t=1}^{\infty} \frac{1}{\left(1+i\right)^t} \sum_{j \in \boldsymbol{J}} r_j * \mathbf{1}_{]t;\infty[}\left(T_{x(j)}\right) \tag{5}$$

This random variable is such as $E\left(\Lambda\right) = L_0$. The reader could refer to MAGNIN and PLANCHET [2000] in order to have a detailed analysis of the law of Λ when the mortality is known (possibly with stochastic interest rates).

First properties of Λ are studied with the classical hypothesis of a known future mortality (determinist case) thanks to prospective tables, second they are studied in a stochastic mortality context. In the last case stochastic mortality surfaces Π have to be considered, and given a surface, the conditional law of the liabilities $\Lambda | \Pi$.

2. CASE OF A DETERMINIST MORTALITY

2.1. THE MORTALITY MODEL

The choice to model instantaneous rates of withdrawal (hazard rate) is made ; the link with the (annual) discrete data available is done under the classic hypothesis of constancy of hazard rate on each square of the Lexis diagram (*cf.* PLANCHET and THEROND [2006]), it leads to assume $\mu_{xt} = -\mathbf{ln}\left(1 - q_{xt}\right)$.

The Lee-Carter model is retained in order to build the prospective life tables. This is a method of extrapolation of past trends, initially used with American data it has quickly become a standard (see the original article LEE and CARTER [1992]). These authors proposed the following modelling to the instantaneous death rate:

$$\mathbf{ln}\,\mu_{xt} = \alpha_x + \beta_x k_t + \varepsilon_{xt}, \tag{6}$$

assuming ε_{xt} as independent random variable, identically distributed with a law $N\left(0,\sigma^2\right)$. The main idea of the model is to assess a parametric (determinist) structure added by a random phenomenon to the series (double suffixed by x and t) of logarithm of instantaneous death rates. The retained optimization criteria is the maximization of the variance part explained by the model, it comes to minimize the variance of errors.

The α_x parameter is understood as the average value of $\mathbf{ln}\left(\mu_{xt}\right)$ during time. Moreover it is checked that $\frac{d\,\mathbf{ln}\left(\mu_{xt}\right)}{dt} = \beta_x \frac{dk_t}{dt}$ and so it can be deduced that the β_x coefficient means the sensitivity of the instantaneous death at the age x comparing to the global trend k_t, as $\frac{d\,\mathbf{ln}\left(\mu_{xt}\right)}{dk_t} = \beta_x$. The Lee-Carter model assumes particularly the constancy during time of this sensibility.

Constraints on parameters have to be added in order to make the model identifiable. The following constraints are generally used:

$$\sum_{x=x_m}^{x_M} \beta_x = 1 \text{ and } \sum_{t=t_m}^{t_M} k_t = 0 . \tag{7}$$

Thus the parameters are obtained thanks to the criterion of (non linear) least squares:

$$\left(\hat{\alpha}_x, \hat{\beta}_x, \hat{k}_t\right) = \mathbf{arg\,min} \sum_{x,t} \left(\mathbf{ln}\, \mu^{*}_{xt} - \alpha_x - \beta_x k_t\right)^2 \tag{8}$$

So this optimization program has to be solved under the constraints of identifiability. The number of parameters which need to be estimated is high, it is equal to $2 \times \left(x_M - x_m + 1\right) + t_M - t_m - 1$. Algorithms of solution are not explained in this paper, the interested reader can see the numerous articles describing this model, for example BROUHNS and DENUIT [2002]. A detailed presentation is also proposed in PLANCHET and THÉROND [2006]. A discussion of the limits of this model is proposed in LELIEUR and PLANCHET [2006] and in SERANT [2005].

Once the mortality surface adjusted on past data, the $\left(k_t\right)$ series has to be modelled in order to extrapolate future rates. To do that an ARIMA[1] model is usually used but any other modelling of temporal series can be used. However and considering the trend obtained on available data:

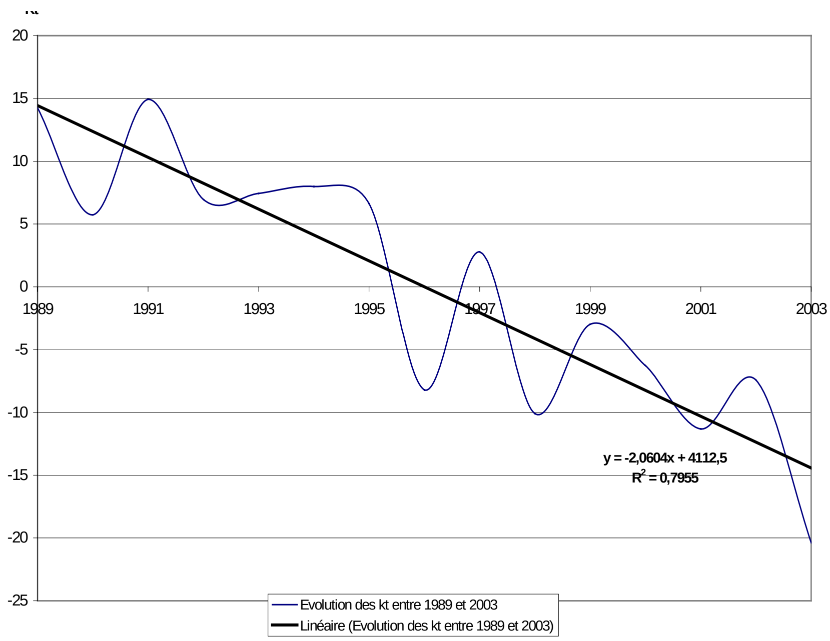

Fig. 3 : *Evolution of the parameter k(t) during time*

[1] By using the Box and Jenkins approach.

the easiest way to model the trend is a linear regression of its coefficients assuming an affine trend:

$$k_t^* = at + b + \gamma_t, \tag{9}$$

where (γ_t) is a Gaussian white noise of variance σ_γ. Thus the estimators $\hat{a}$ and $\hat{b}$ are obtained and it allows to build projected surfaces in simply using $\hat{k}_t = \hat{a}t + \hat{b}$. This model is subsequently used.

It has to be pointed out that the goal of this paper is to quantify the amount of risk due to the incertitude on the future mortality in the global risk to which a pension plan is exposed. So the choice of a simple but operational stochastic model has been done. If the limits of the Lee-Carter model can have an impact on the absolute level of the calculated liabilities, it has to be smallest on the distribution between the mutualisable part and the non-mutualisable part of the mortality risk. This point is broached in the conclusion.

According to the current distribution per age of the studied group it can be noticed that the closing method of the table is not decisive in calculating the value of the liabilities. It can be more generally considered that the importance of the choice of a closing method can be modulated in the case of a pension plan. On this point PLANCHET and THÉROND [2006] can be seen. The problem of the closing method of tables is also discussed in DENUIT and QUASHIE [2005] with a slightly different point of view.

2.2. NUMERICAL APPLICATIONS

In this section the obtained results are introduced. First these calculated with the proposed family of prospective life tables, and second the consequences on the value of the pension plan liabilities.

2.2.1. Prospective life tables

Prospective life tables used in this study are built from the instantaneous mortality tables provided by INED[2] in MESLE and VALLIN [2002]; the link between discrete death rate and its continuous version allows to obtain the hazard rates which have to be assessed from the basic data:

[2] These tables are available on http://www.ined.fr/publications/cdrom_vallin_mesle/Tables-de-mortalite/Tables-du-moment/Tables-du-moment-XX.htm

$$\hat{\mu}_{xt} = -\mathbf{ln}\left(1-\hat{q}_{xt}\right). \tag{10}$$

The assessment on the historical data leads the following mortality surface:

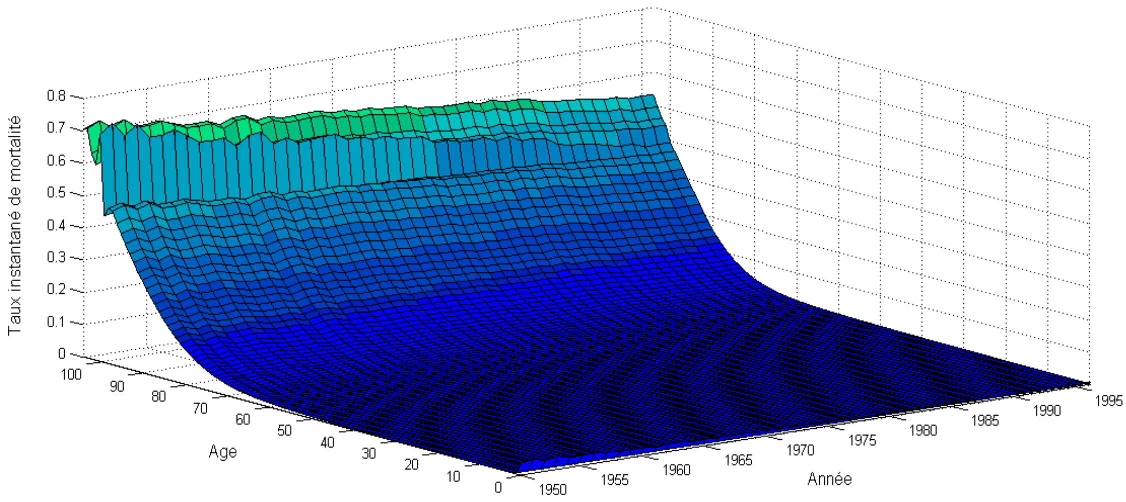

Fig. 4 : *Fitted mortality surface*

Concerning the prediction part of the model, the following results are obtained with our data: $\hat{b} = 4059{,}94439$, $\hat{a} = \text{-}2{,}05775$ and $\hat{\sigma}_{\gamma} = 3{,}9388782$.

2.2.2. The plan liabilities

Beyond moments of order one and two of the Λ distribution which can be explicitly obtained (cf. MAGNIN and PLANCHET [2000]), the liabilities distribution of the plan is the main topic of this section. The selected method consists in simulating the lifespan of pensioners, $T_{x(j)}, j \in J$ and then calculating:

$$\Lambda = \sum_{t=1}^{\infty} \frac{1}{(1+i)^t} \sum_{j \in \boldsymbol{J}} r_j * \mathbf{1}_{]t;\infty[}\left(T_{x(j)}\right). \tag{11}$$

on the basis of the $T_{x(j)}, j \in J$ realizations obtained. The $\lambda_1, ..., \lambda_n$ realizations of Λ are so obtained and the empiric distribution of the liabilities can be determined. This distribution is represented below (in the case of 20 000 drawings):

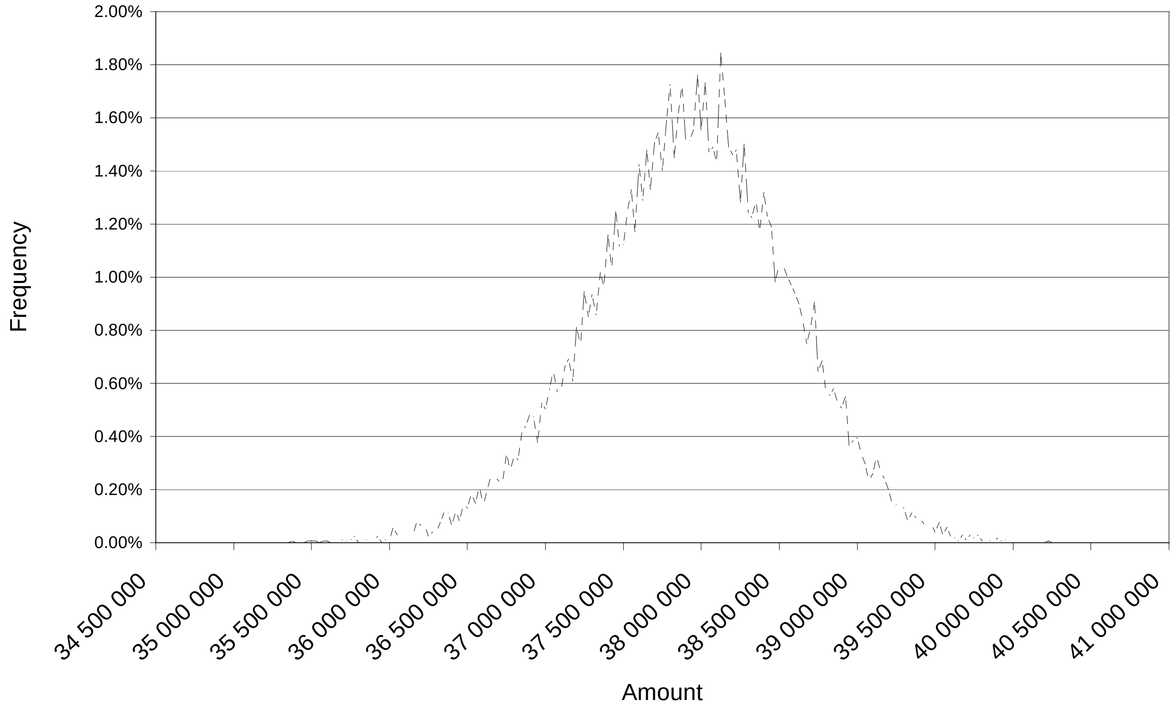

Fig. 5 : *Empiric distribution of the liabilities*

The global liabilities L_0 is approach by $\overline{\lambda} = \frac{1}{N}\sum_{n=1}^{N}\lambda_n$. The variance of the liabilities is estimated by $\frac{1}{N-1}\sum_{n=1}^{N}\left(\lambda_n - \mathrm{L}_0\right)^2$; more precisely the empiric variation coefficient is the main concern :

$$cv = \frac{\sqrt{\frac{1}{N-1}\sum_{n=1}^{N}\left(\lambda_n - \mathrm{L}_0\right)^2}}{\frac{1}{N}\sum_{n=1}^{N}\lambda_n}, \tag{12}$$

This coefficient gives an indicator of the liabilities dispersion and in a way a kind of its « dangerousness ». The following value of these indicators are summarized:

	Determinist
Expectation	37 973 994
Root mean square error	631 525
Lower bound of the confidence interval	36 700 000
Upper bound of the confidence interval	39 175 000
Variation coefficient	1,66%

In spite of the quite little size of the portfolio it can be noticed that the liabilities is evaluated with a relative precision of more or less 6.1% (ratio of the half-size of the confidence interval to the expectation). With this point of view the demographic risk is well controlled.

Remark: It is really important to efficiently simulate the lifespan; so the « inversion method » is used in a discrete context, it leads on the fact that defining the variable T by:

- ✓ T=0 if $U < p_0$
- ✓ T=1 if $p_0 \le U < p_0 + p_1$
- ✓ ….
- ✓ T=j if $\sum_{i=0}^{j-1} p_i \le U < \sum_{i=0}^{j} p_i$

Where U is an uniform random variable on $[0,1]$, thus T is distributed by the law $(p_n)_{n \geq 0}$. The link between the p_i probabilities and the death rates previously determined is given by $p_i = q_{x+i} \times \prod_{j=0}^{i-1} \left(1 - q_{x+j}\right)$. Compared to the direct approach which consists in fixing the lifespan of each pensioner by a drawing on each period and then comparing the results with the death rate corresponding to the pensioner age, this approach divides by about 20 the simulation time. This is an essential optimization in order to keep an operational nature to the model in a context of stochastic mortality, *infra* examined.

3. CASE OF A STOCHASTIC MORTALITY

A component of systematic risk is now joined to the model through an uncertainty about the future mortality. So the surface built in the previous section defines the reference trend and the observed mortality is supposed to fluctuate around it.

3.1. THE MORTALITY MODEL

The regression equation, which allows to obtain the trend of future mortality, is used:

$$k_t^* = at + b + \gamma_t , \tag{13}$$

and realizations of future mortality are obtained by doing selection with the law of residual $\left(\gamma_t\right)$, $N\left(0, \sigma_\gamma^2\right)$. The variable k_t^* is as $E\left(k_t^*\right) = k_t$. Realizations of instantaneous rate of withdrawal are so obtained *via*:

$$\mu_{xt}^{*} = \mathbf{exp}\left(\alpha_x + \beta_x k_t^{*}\right). \tag{14}$$

In this approach it can be noticed that $E\left(\mu_{xt}^{*}\right) = E\,\mathbf{exp}\left(\alpha_x + \beta_x k_t^{*}\right) = \mathbf{exp}\left(\alpha_x + \beta_x k_t + \frac{\beta_x^2 \sigma_\gamma^2}{2}\right)$

and so:

$$E\left(\mu_{xt}^{*}\right) = \mu_{xt}\,\mathbf{exp}\left(\frac{\beta_x^2 \sigma_\gamma^2}{2}\right) > \mu_{xt}. \tag{15}$$

So the stochastic model has a tendency to overestimate the rate of withdrawal. But according to the following graph, this bias is very low in practice and so few penalizing for the model because few significant at high ages:

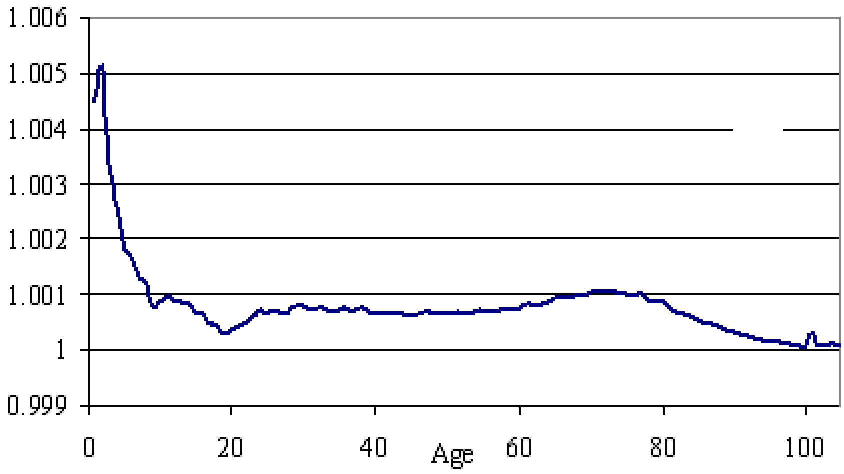

Fig. 6 : *Bias analysis as a function of the age*

By the way, it can be remarked that it is not direct to calculate the bias so introduced in the discrete rate q_x. Actually Jensen inequality ($f\left(EX\right) \leq Ef\left(X\right)$) applied to $f\left(x\right) = -\left(1 - e^{-x}\right)$ leads to $E\left(q_x^{*}\right) \leq 1 - \mathbf{exp}\left(-E\mu_x^{*}\right)$, what does not allow to conclude simply about the way of the bias. However it shows that the link between q_x and μ_x does not amplify the bias, and so the bias on the discrete rate must also be low.

It can be observed that the volatility σ_γ is a parameter allowing to simply control the degree of uncertainty inherent to the future mortality. The determinist model analyzed in the previous section is identified in the particular case $\sigma_\gamma = 0$.

Even if it seems to be few penalizing at the proximity of the volatility σ_γ evaluated on the available data, the bias generated on the rate of withdrawal via the simulation method of the mortality surfaces could be embarrassing in situations with stronger volatility: this point will be illustrated with a numerical application. As a consequence a “bias corrected” version of the proposed model will be used. It is defined by:

$$\mu_{xt}^* = \mathbf{exp}\left(\alpha_x - \frac{\beta_x^2 \sigma_\gamma^2}{2} + \beta_x k_t^* \right). \tag{16}$$

This version of the model satisfies $E\left(\mu_{xt}^*\right) = \mu_{xt}$ by construction. So it is more in accordance with the purpose to « disturb » the surface of mortality, but under the hypothesis that this surface properly defines the future trend of the instantaneous rate of death.

3.2. RISK DISTRIBUTION ANALYSIS FOR A PENSION PLAN

The variance of the sum of the future discounted flows Λ is considered as a measure of risk. The following result is obtained by conditioning with the mortality surface Π and using the equation of the variance decomposition:

$$\boldsymbol{V}\left[\Lambda\right] = \boldsymbol{E}\left[\boldsymbol{V}\left(\Lambda|\Pi\right)\right] + \boldsymbol{V}\left[\boldsymbol{E}\left(\Lambda|\Pi\right)\right]. \tag{17}$$

The second term of the right hand side of the expression below represents the systematic risk linked to the pension plan; the first one represents the technical risk, *i.e.* the mutualisable risk of mortality. In practice the part of the variance explained by the component of the systematic risk is considered as an indicator, defined as below:

$$\omega\left(\sigma_\gamma\right) = \frac{\boldsymbol{V}\left[\boldsymbol{E}\left(\Lambda|\Pi\right)\right]}{\boldsymbol{V}\left[\Lambda\right]}. \tag{18}$$

$\omega\left(\sigma_\gamma\right)$ an increasing function of σ_γ ; when the size of the group tends towards infinity, $\omega\left(\sigma_\gamma\right)$ converges on 1. In other words all the variance is explained by the systematic component in a perfectly mutualised group. Direct calculus of $\omega\left(\sigma_\gamma\right)$ is not easy, so an approach by simulations is used: thanks to a drawing with a law $N\left(0,\sigma_\gamma^2\right)$ a mortality surface is first generated (in practice, Gaussian realizations are obtained by inversion of the distribution function in using the Moro approximation, as explained in PLANCHET et al. [2005]); second the survival of pensioners (« the liabilities ») is simulated conditionally to this mortality hypothesis.

More positively, if $\lambda_{n,m}$ is the realization of Λ resulting by the n-th trajectory of the mortality and the m-th trajectory of the liabilities, let's note :

$$\overline{\lambda}_n = \frac{1}{M}\sum_{m=1}^{M}\lambda_{n,m} \text{ and } \overline{\overline{\lambda}} = \frac{1}{N}\sum_{n=1}^{N}\overline{\lambda}_n = \frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\lambda_{n,m}. \tag{19}$$

The following quantities are unbiased and convergent estimators of $\boldsymbol{E}\left[\boldsymbol{V}\left(\Lambda|M\right)\right]$ and $\boldsymbol{V}\left[\boldsymbol{E}\left(\Lambda|M\right)\right]$ respectively:

$$\hat{\boldsymbol{E}}\left[\boldsymbol{V}\left(\Lambda|M\right)\right] = \frac{1}{N}\sum_{n=1}^{N}\frac{1}{M-1}\sum_{m=1}^{M}\left(\lambda_{n,m} - \overline{\lambda}_n\right)^2, \tag{20}$$

$$\hat{\boldsymbol{V}}\left[\boldsymbol{E}\left(\Lambda|M\right)\right] = \frac{1}{N-1}\sum_{n=1}^{N}\left(\overline{\lambda}_n - \overline{\overline{\lambda}}\right)^2. \tag{21}$$

Number of simulations is empirically standardized by stopping the algorithm when the results are stabilized (difference between two consecutive results $< 10^{-3}$).

3.3. NUMERICAL APPLICATIONS

Numerical applications are proposed at two levels: first the taking into account of the stochastic mortality will be analyzed as an impact on the distribution of the pension liabilities; second the evolution of the stochastic part of the variance in the global variance will be determined as a function of the volatility of the mortality surface (so the uncertainty linked to the assessing of this one).

3.3.1. Liabilities analysis

First, thanks to the estimated value of the volatility of the mortality surface ($\sigma_\gamma = 3{,}94$), the empiric distribution of the liabilities is determined, it is represented below with the reference distribution previously obtained in the determinist (with 20 000 drawings):

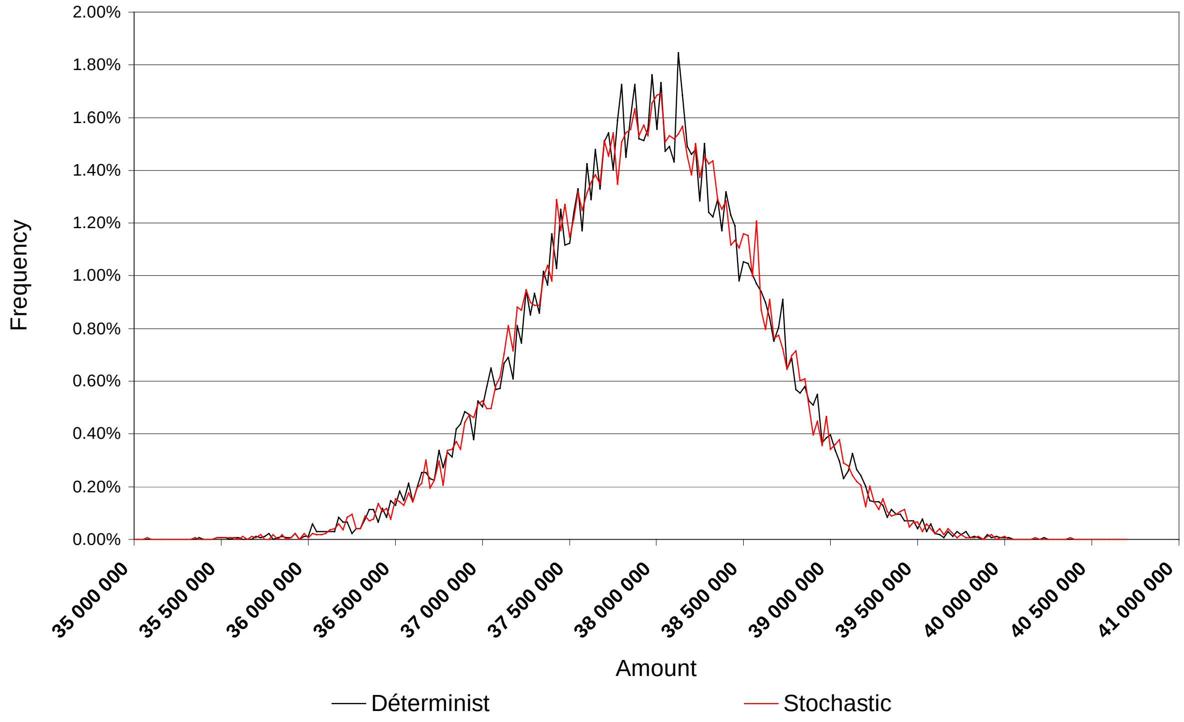

Fig. 7 : *Empiric distribution of the liabilities : determinist approach VS stochastic approach*

It can be noticed that the variation coefficient of the liabilities distribution is evaluated at 1.66 % in each case (determinist mortality and stochastic mortality): though it can be expected that the taken into account of this additional factor of risk will noticeably increase the dispersion, and so the « dangerousness » of the liabilities, even though near stability can be noticed in this case. Main results are summarized below:

	Determinist	**Stochastic**
Expectation	37 973 994	37 939 862
Root mean square error	631 525	628 266
Lower bound of the confidence interval	36 700 000	36 650 000
Upper bound of the confidence interval	39 175 00	39 100 000
Variation coefficient	1.66%	1.66%

It can be noticed in particular that the expectation of the liabilities is the same in the determinist and the stochastic case, it illustrates the low bias caused by the random selection of the mortality tables *via* k_t. In the present case the reference data's used in the assessment of the reference mortality fluctuate slightly, this leads to minor risk linked to this phenomenon from sampling risk viewpoint. This fact is all the more pronounced since the studied population is quite small.

In a « value at risk » (*VaR*) approach, the quartile at 75 % of the liabilities distribution is determined at 38.3 M€ in the stochastic case, which is nearly the same result as the determinist case.

These results are of course the direct consequences of the really weak estimated volatility of the k_t series; in an increasing case of the volatility, the liabilities distribution including the systematic risk rapidly diverges from the reference situation built by the determinist model. As an example, results with a tenfold increased volatility are presented below (number of drawings is still fixed to 20 000):

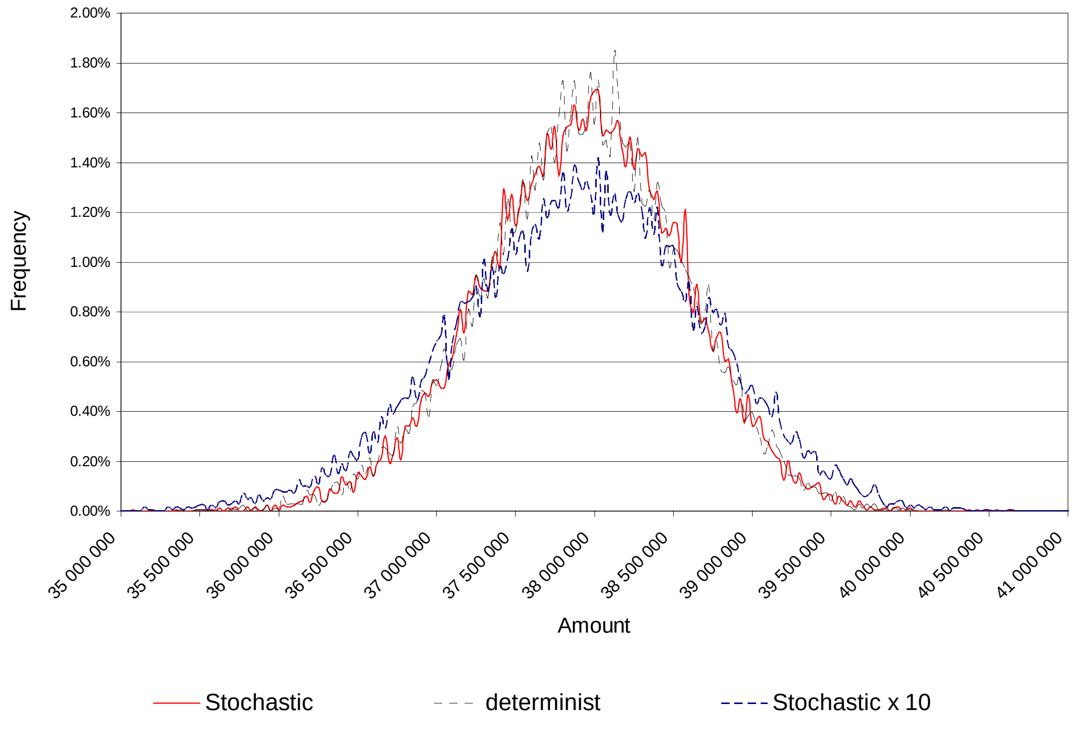

Fig. 8 : *Empiric distribution of the liabilities : determinist approach VS stochastic approach*

It can be noticed that the variation coefficient of the liabilities distribution is estimated to 2.06 % in the case of a taking into consideration of a stochastic mortality with a high volatility, against 1.66 % in the reference situation: as it can be expected the model is very sensitive to the volatility of the process generating surfaces of stochastic mortality. Main results are summarized below:

	Determinist	**Stochastic**	**Stochastic x10**
Expectation	37 973 994	37 939 862	37 925 061
Root mean square error	631 525	628 266	778 276

Lower bound of the confidence interval	36 700 000	36 650 000	35 600 000
Upper bound of the confidence interval	39 175 00	39 100 000	39 375 000
Variation coefficient	1.66%	1.66%	2.06%

Of course, other things being equal, the impact of the taking into account of the non-mutualisable part of the risk on the liabilities distribution is all the more significant since the sampling fluctuations are small, and since the group is big. In order to show this point, the liabilities distribution was calculated on a fictitious group built by adding 100 times the same group (the reference one), such a group is composed by 37 400 pensioners. The following graph is so obtained:

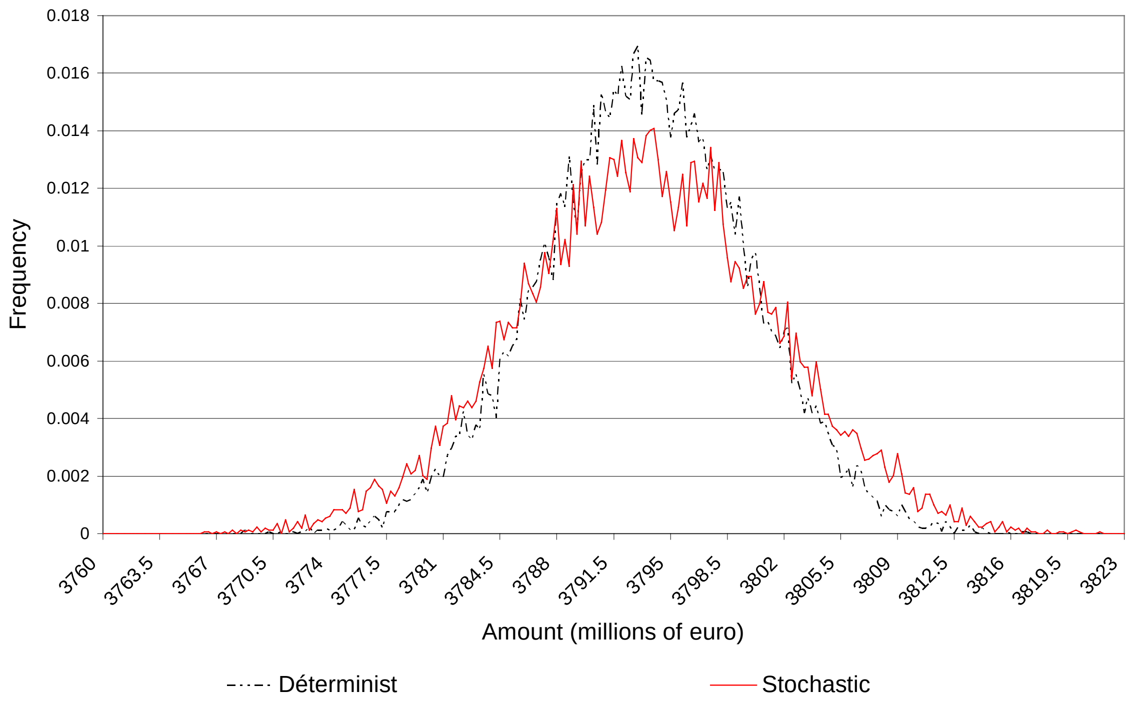

Fig. 9 : *Empiric distribution of the liabilities : size of the group x100*

It can be seen on the graph a flattening of the liabilities distribution. It reveals at first sight a more risky situation for the plan when the non-mutualisable part of risk is taken into consideration. Main results are summarized below:

	Determinist	**Stochastic**
Expectation	3 793 577 165	3 793 566 113
Root mean square error	6 324 333	7 771 749
Quartile at 5%	3 777 750 000	3 780 750 000
Quartile at 95%	3 803 750 000	3 806 250 000
Variation coefficient	0.167%	0.205%

It can be remarked that the liabilities expectation is simply multiplied by 100 in comparison with the reference situation. Nevertheless it can be observed a small difference between the determinist approach and the stochastic one: actually the root mean square error increases by 23% when the non-mutualisable risk s taken into account.

It can however be noticed that the demographic risk remains well control, even with a systematic risk factor. The confidence interval at 95 % of the liabilities level is (in billion euros) [3778;3808], what gives a precision level less than 1 % in the liabilities measure.

In a « value at risk » approach (*VaR*), the quartile at 75 % of the liabilities distribution is determined at 3 798 M€ in the stochastic case against 3 797 M€ in the determinist one: one's again it can be noticed that the absolute impact on the plan is very low.

3.3.2. Importance of the bias correction

As soon as the volatility of the underlying mortality process becomes significant, the correction of the bias becomes essential. The following graph illustrates this point; it represents the empirical distribution with volatility 10 times, then 20 times of the reference volatility of the model without bias correction:

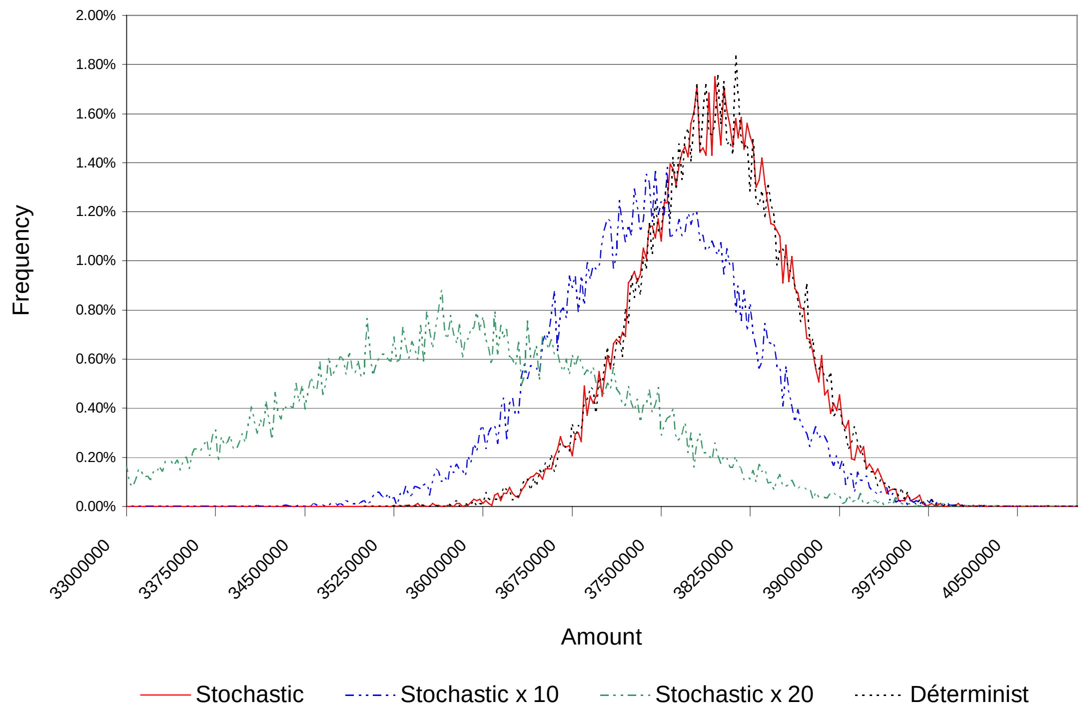

Fig. 10 : *Empiric distribution of the liabilities : determinist approach VS stochastic approach*

It can be noticed that the distribution including the systematic risk is moved on the left, because of the bias on the withdrawal rates, and so this distribution leads to a reduced assessment of the liabilities.

3.3.3. Explained part of variance analysis

As it has been shown by the last results, the evolution of the liabilities is a function of the volatility of the underlying stochastic mortality process, so it has to be quantified. The estimators *supra* presented allow to obtain the evolution graph of the systematic part of the risk in the global risk $\omega\left(\sigma_{\gamma}\right)$ as a function of the volatility σ_{γ} of the life tables generator.

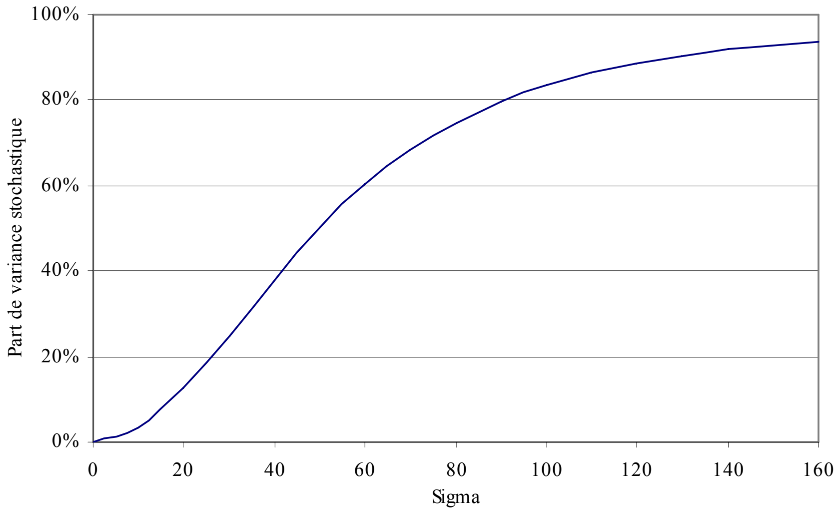

Fig. 11 : *Evolution of the part of systematic risk in the global risk as a function of σ_{γ}*

It can be noticed that the increase of the curve is slow at first, and speeds up for high volatility value.

For a fixed σ_{γ}, the part of the variance explained by the stochastic component of the mortality increases in the same way as the size of the portfolio ; the following graph is so obtained (the size of the portfolio is expressed in number of times it was multiplied by the size of the reference portfolio) :

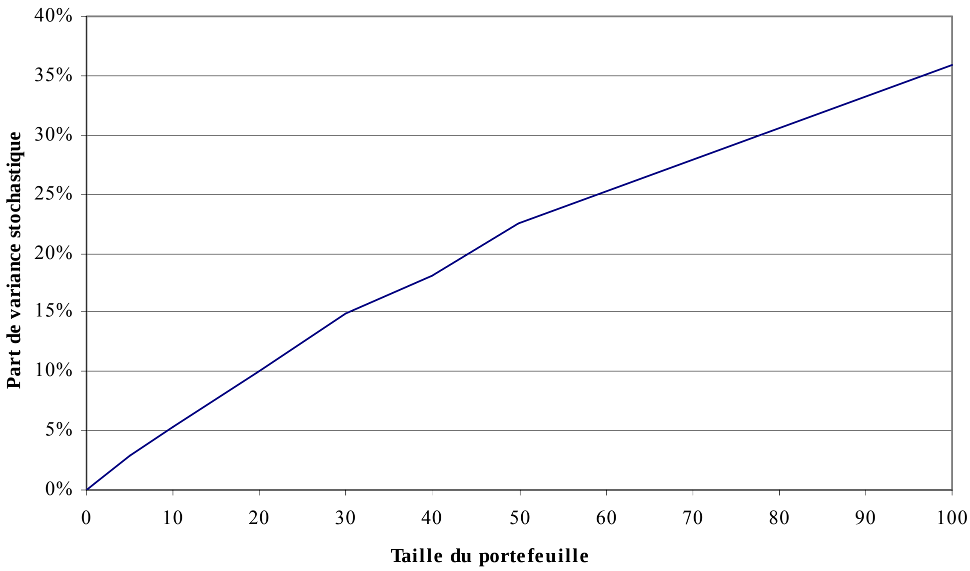

Fig. 12 : *Evolution of the part of systematic risk in the global risk as a function of the portfolio size*

As it can be observed, this source of risk has a relatively slow growth (linear one) in the global variance, for example a threshold of a third is reached for a plan of more than 300 000 pensioners; the sampling fluctuations are very low and so the global variance is small for plans with such a size.

3.4. MEASURE OF RISK ON THE TREND

In fact the considered plan is subject to two distinct risks: the first, studied in this article, is a consequence of the random fluctuations of the future mortality rate around the trend defined by the prospective tables. The second is linked to the uncertainty on this trend.

More precisely this uncertainty has two origins: the error due to the choice of the model, and the uncertainty linked to the determining of interpolation coefficients.

It is delicate to value *a priori* the model error; a first approach consists for example in testing the sturdiness of the proposed model by doing the assessments on different periods. These estimations could lead to future mortality estimations sensitively divergent.

The imprecision linked to the interpolation's coefficients themselves is easier to quantify. Actually in this model:

$$k_t^* = at + b + \gamma_t \tag{22}$$

the coefficients a and b are valued thanks to the standard least squares method, an it leads to assume a couple $\left(\hat{a},\hat{b}\right)$ as a Gaussian vector. So several extrapolation lines can be simulated and thus taken into consideration this variability source in the model.

This last point exceeds the scope of this paper and will be the main topic of a future and specific study. However it is important to keep on mind the existence of this second source of risk, potentially more significant for the plan: The gap between the slope of the different extrapolation lines leads to more and more divergent survival rates during time.

4. CONCLUSION

It has been implemented in this paper an operational model in order to quantify the part of the systematic risk of mortality in the liabilities of a pension plan. The obtained results tend to show that it is an insignificant risk for a population under a few dozen of thousands, seeing that sampling fluctuations explain the most part of the variance.

Concerning large portfolios the absolute level of risk remains low, even if the systematic risk becomes significant to a relative point of view seeing that the mutualisable risk decreases with the size of the portfolio. Finally the size of the confidence interval at 95 % concerning the estimated reserve is decreasing with the size of the under-risk population, even with the systematic risk.

The main limit of the proposed model is a direct consequence of a limit due to the Lee-Carter model (or Poisson models): actually death rates at different ages are supposed totally correlated, the random variable $\left(k_t\right)$ depends only on time[3] and the instantaneous withdrawal rates are obtained thanks to the formula $\ln \mu_{xt} = \alpha_x + \beta_x k_t$. In practice, this is contradicted by the figure 1.

However the approach proposed in this paper can easily be adapted to the whole models which are based on a temporal trend modelling by using a temporal series (with possibly several parameters). An introduction of other same type models (and among others the approach using the logits of death rates) is done in PLANCHET [2005] and in SERANT [2005].

It is useful to specify that such an approach gives an operational context in order to satisfy with the requirements of the future Solvency II measures (and in the second stage of the IFRS 4 context, but the accounting standards are less demanding with this subject).

[3] This component is modelized by an ARIMA process.

If these approaches allow to quantify the risk which is linked to the intrinsically random nature of the future mortality around a supposed predictable trend, it can be emphasized that they don't give any answers to a potential specification error on the reference trend. So for example it can be noticed that prospective models built by the past have already under estimate the life expectancy of the pensioners, and thus the level of the pension plan reserves. So only extrapolated models seem to be limited to this point of view.

Faced with uncertainty about the evolution of the future mortality, it can be considered introducing in the model an exogenous constraint which allows us to add an opinion *a priori* about the future mortality, regardless of the past trend. An idea in progress of investigating, which is the main point of a future paper, consists in imposing a constraint on the life expectancy for a reference age, and in controlling its evolution in an exogenous way.

A second idea in progress consists in modelling the error on the trend evaluation, then in quantifying the impact of this error on the liabilities distribution.

Finally and when the interest in the retirement problems is increasing, these models give an other tool in order to measure and so to manage the technical balance of a plan.

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