Systematic risk of mortality on an annuity plan

Version 1.0

Frédéric PLANCHET / Marc JUILLARD / Laurent FAUCILLON

ISFA
WINTER & Associés

Motivations

With current accounting and solvency standards, an annuity (or pension) plan liabilities is evaluated as an mean with « prudent » parameters (mortality and discount rate).

New accouting (IFRS 4 Phase 2) and solvency (Solvency 2) projects lead to :

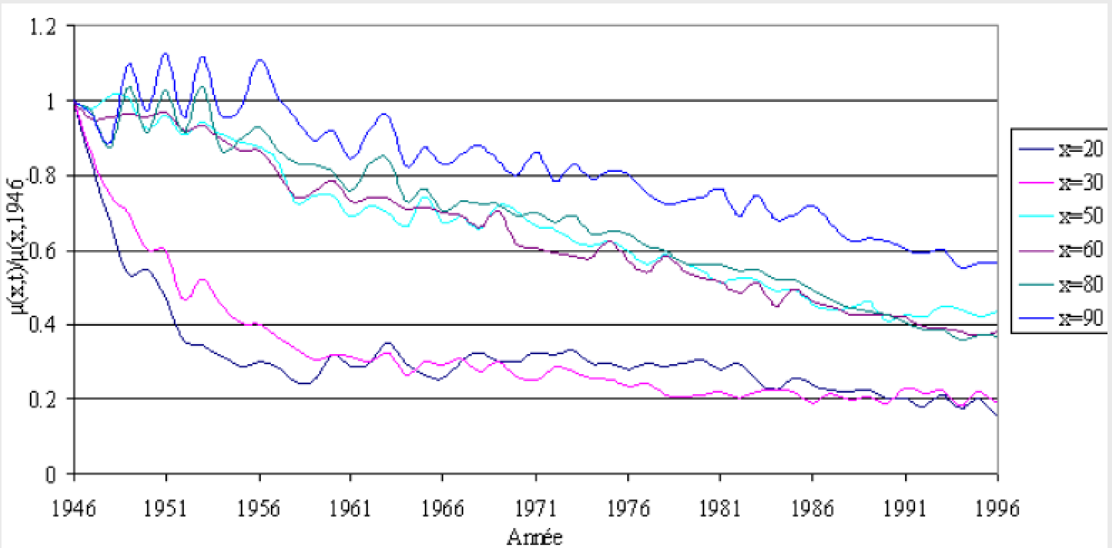
- compute a mean with « realistic » parameters ;
- add an explicit « risk margin ».

As a consequence it becomes necessary to identify and then quantify the different risk sources.

This presentation deals with the systematic mortality risk and propose a tractable model to compute the risk margin.

Systematic risk of mortality

Recent studies (CURRIE and al. [2004]) show that the evolution of the instantaneous death rate presents erratic variations at the different ages around the emerging tendency. These variations are not explained by sampling fluctuations; CAIRNS and al. [2004] gives a detailed analysis of this phenomenon. The same phenomenon is noticed in this study using the data from INED (*cf. infra* a specification of these data):



Systematic risk of mortality

These variations have a systematic impact on the fixed age individuals and are not mutualisable. They run therefore a potential important risk to an annuity plan whose technical balance is built on the mutualisation of the survival risk of its members.

It leads to look for a model able to explain these fluctuations around the underlying value and to draw consequences regarding the level of mathematical reserves that the plan has to be set in order to ensure its technical balance.

More precisely the part of non mutualisable risk among the global risk has to be quantify in order to value the risk that the plan has to face (the measure has to be determined).

Systematic risk of mortality

Stochastic mortality models are well adapted for this analysis. They suggest that the future mortality rate $\mu(x,t)$ is random and therefore is a stochastic process (as a function of t with a fixed x).

The mortality rate really observed given an age and a year is the realization of a random variable: the analogy with the methods of Bayesian adjustment can be noticed (about these methods, refer to TAYLOR [1992]).

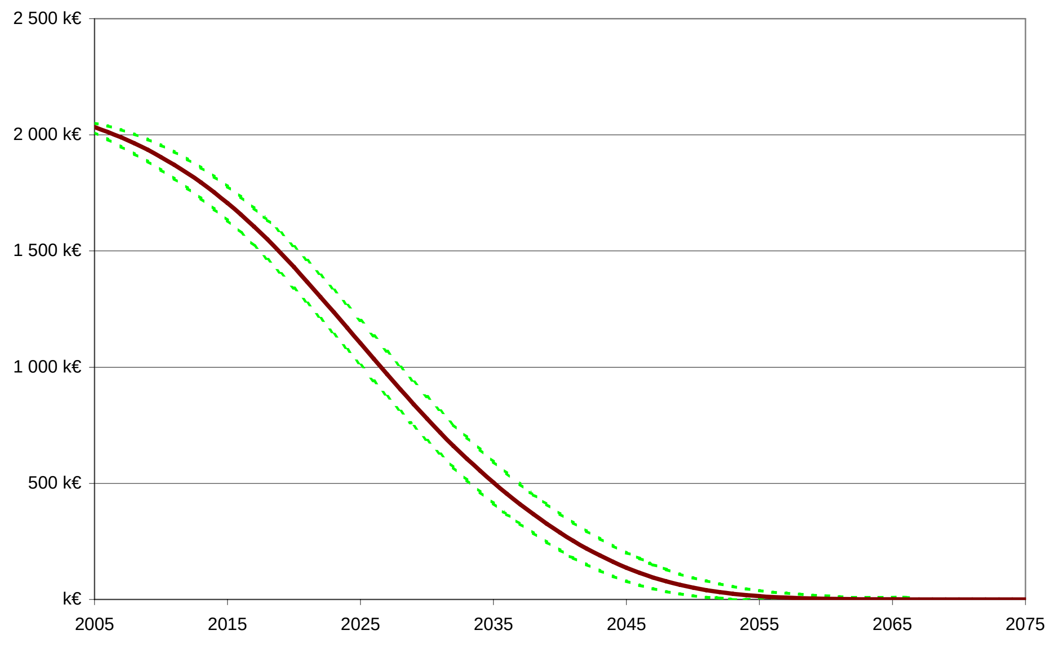
Systematic risk of mortality

Recent models of construction of prospective life tables as the Lee-Carter model (particularly refer to LEE and CARTER [1992], LEE [2000], SITHOLE and al. [2000]) or Poisson models (cf. BROUHNS and al. [2002] and PLANCHET and THÉROND [2006] for a presentation and an analysis of these models), are particular cases of stochastic models, even if they were originally built in order to make (temporal) extrapolations of the determinist surface . Once previous rates adjusted, the future mortality rates are deduced from the extrapolation of the temporal component (parametric or not) of a kept prospective models (Obviously this purely extrapolative approach could be criticised; for example consult GUTTERMAN and VANDERHOOF [1999] about these questions).

In this paper the Lee-Carter model is kept, it allows easily to build stochastic mortality surfaces, moreover it becomes a standard in order to build prospective life tables. The log-Poisson variant (cf. BROUHNS and al. [2002]) leads to very similar results.

Case of an annuity plan

We are interesting in analysing the consequences for an annuity plan. In the numerical applications, it is used a portfolio formed of 374 pensioners with an average age of 63.8 years at 31/12/2005. The average annual pension is up to 5.5 k€. The graphic *infra* shows the expected flows due to pensions as a function of time built from the life table named TV 2000 (a French table).



Case of an annuity plan

We need :

- L_0 the amount of the initial mathematical reserve,
- $\tilde{F}_t$ the (random) flow of pension which has to be paid at the date t,
- i the (discrete) discount rate for the mathematical reserve,
- J the set of individuals,
- $x(j)$ at the initial date the age of the individual j and r_j the amount of his annual pension.

The random value of the liabilities is analysed, let be the random variable :

$$\Lambda = \sum_{t=1}^{\infty} \tilde{F}_t (1+i)^{-t} = \sum_{t=1}^{\infty} \frac{1}{(1+i)^t} \sum_{j \in \boldsymbol{J}} r_j * \boldsymbol{1}_{]t;\infty[} \left(T_{x(j)}\right)$$

$$L_0 = E(\Lambda) = \sum_{t=1}^{\infty} F_t (1+i)^{-t}$$

Plan of the presentation

We have to :

- construct a reference prospective life table (Lee-Carter) ;
- construct a stochastic life tables generator ;
- get informations about the liabilities distribution (using simulations techniques).

The stochastic generator will be adapted to the modelized risk :

- fluctuations over the trend ;
- trend specification.

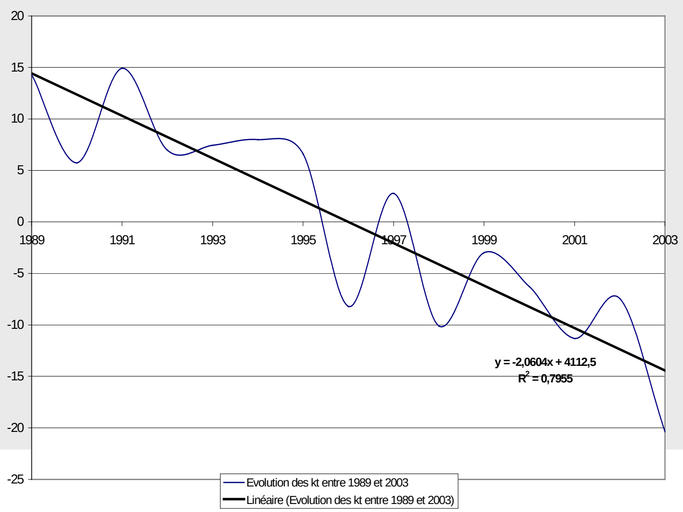
Deterministic mortality model

Lee-Carter model : $\ln \mu_{xt} = \alpha_x + \beta_x k_t + \varepsilon_{xt}$

Parameters are obtained thanks to the criterion of (non linear) least squares :

$$\left(\hat{\alpha}_x, \hat{\beta}_x, \hat{k}_t\right) = \arg\min \sum_{x,t} \left(\ln \mu^*_{xt} - \alpha_x - \beta_x k_t\right)^2$$

Once the mortality surface adjusted on past data, the (k_t) series has to be modelled in order to extrapolate future rates.

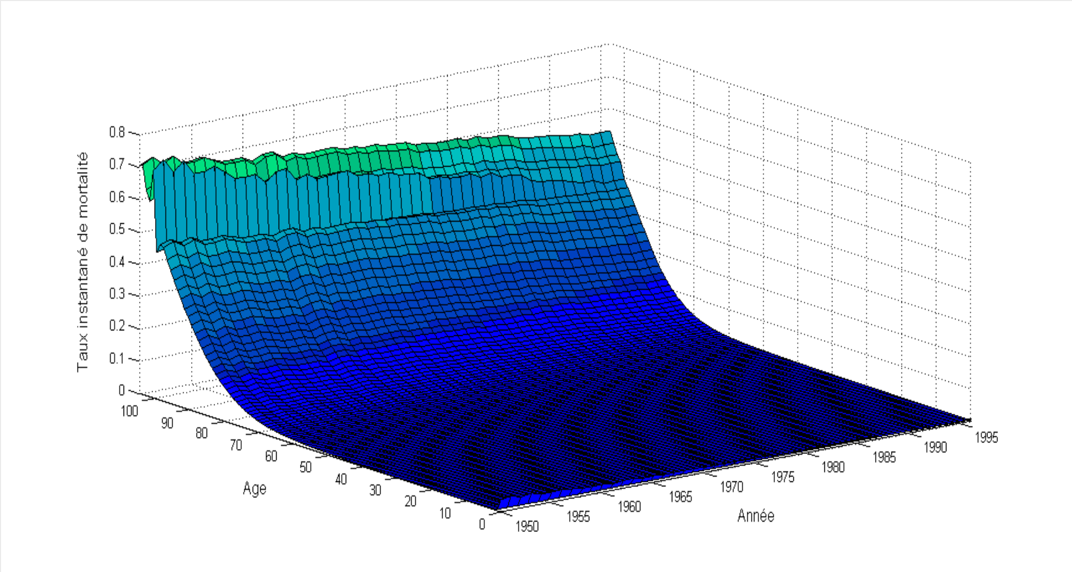


$$k^*_t = at + b + \gamma_t$$

$$\gamma_t \approx N\left(0, \sigma^2_\gamma\right)$$

Deterministic mortality model

Prospective life tables used in this study are built from the instantaneous mortality tables provided by INED in Mesle and Vallin [2002]. The assessment on the historical data leads the following mortality surface :



Stochastic mortality model : fluctuations over the trend

The regression equation, which allows to obtain the trend of future mortality, is used:

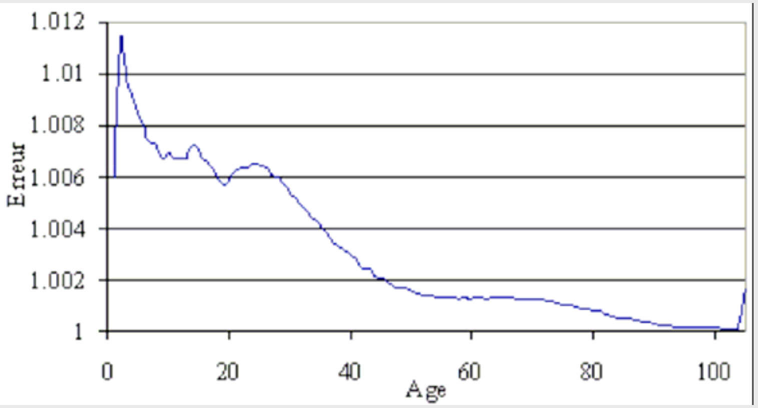
$$k_t^* = at + b + \gamma_t$$

Realizations of instantaneous rate of withdrawal are so obtained via :

$$\mu_{xt}^* = \mathbf{exp}\left(\alpha_x + \beta_x k_t^*\right)$$

With this specification, the stochastic model has a tendency to overestimate the rate of withdrawal.

$$E\left(\mu_{xt}^*\right) = \mu_{xt}\,\mathbf{exp}\left(\frac{\beta_x^2 \sigma_\gamma^2}{2}\right) > \mu_{xt}$$



Stochastic mortality model : fluctuations over the trend

As a consequence a "bias corrected" version of the proposed model will be used. It is defined by :

$$\mu_{xt}^{*} = \mathbf{exp}\left(\alpha_x - \frac{\beta_x^2 \sigma_\gamma^2}{2} + \beta_x k_t^{*} \right)$$

This version of the model satisfies by construction :

$$E\left(\mu_{xt}^{*}\right) = \mu_{xt}$$

So it is more in accordance with the purpose to « disturb » the surface of mortality, but under the hypothesis that this surface properly defines the future trend of the instantaneous rate of death.

Systematic risk measure

The variance of the sum of the future discounted flows Λ is considered as a measure of risk. The following result is obtained by conditioning with the mortality surface Π and using the equation of the variance decomposition:

$$\boldsymbol{V}\left[\Lambda\right]=\boldsymbol{E}\left[\boldsymbol{V}\left(\Lambda\middle|\Pi\right)\right]+\boldsymbol{V}\left[\boldsymbol{E}\left(\Lambda\middle|\Pi\right)\right]$$

The second term of the right hand side of the expression above represents the systematic risk linked to the pension plan; the first one represents the technical risk, *i.e.* the mutualisable risk of mortality. In practice the part of the variance explained by the component of the systematic risk is considered as an indicator, defined as below :

$$\omega\left(\sigma_\gamma\right)=\frac{\boldsymbol{V}\left[\boldsymbol{E}\left(\Lambda\middle|\Pi\right)\right]}{\boldsymbol{V}\left[\Lambda\right]}$$

Lifespan simulation

It is really important to efficiently simulate the lifespan; so the « inversion method » is used in a discrete context, it leads on the fact that defining the variable *T* by :

$$\sum_{i=0}^{j-1} p_i \leq U < \sum_{i=0}^{j} p_i \rightarrow T_x = x + j$$

with :

$$p_i = q_{x+i} \times \prod_{j=0}^{i-1} \left(1 - q_{x+j}\right)$$

Compared to the direct approach which consists in fixing the lifespan of each pensioner by a drawing on each period and then comparing the results with the death rate corresponding to the pensioner age, this approach divides by about 20 the simulation time. This is an essential optimization in order to keep an operational nature to the model in a context of stochastic mortality, *infra* examined.

Empirical estimators

Risk measure :

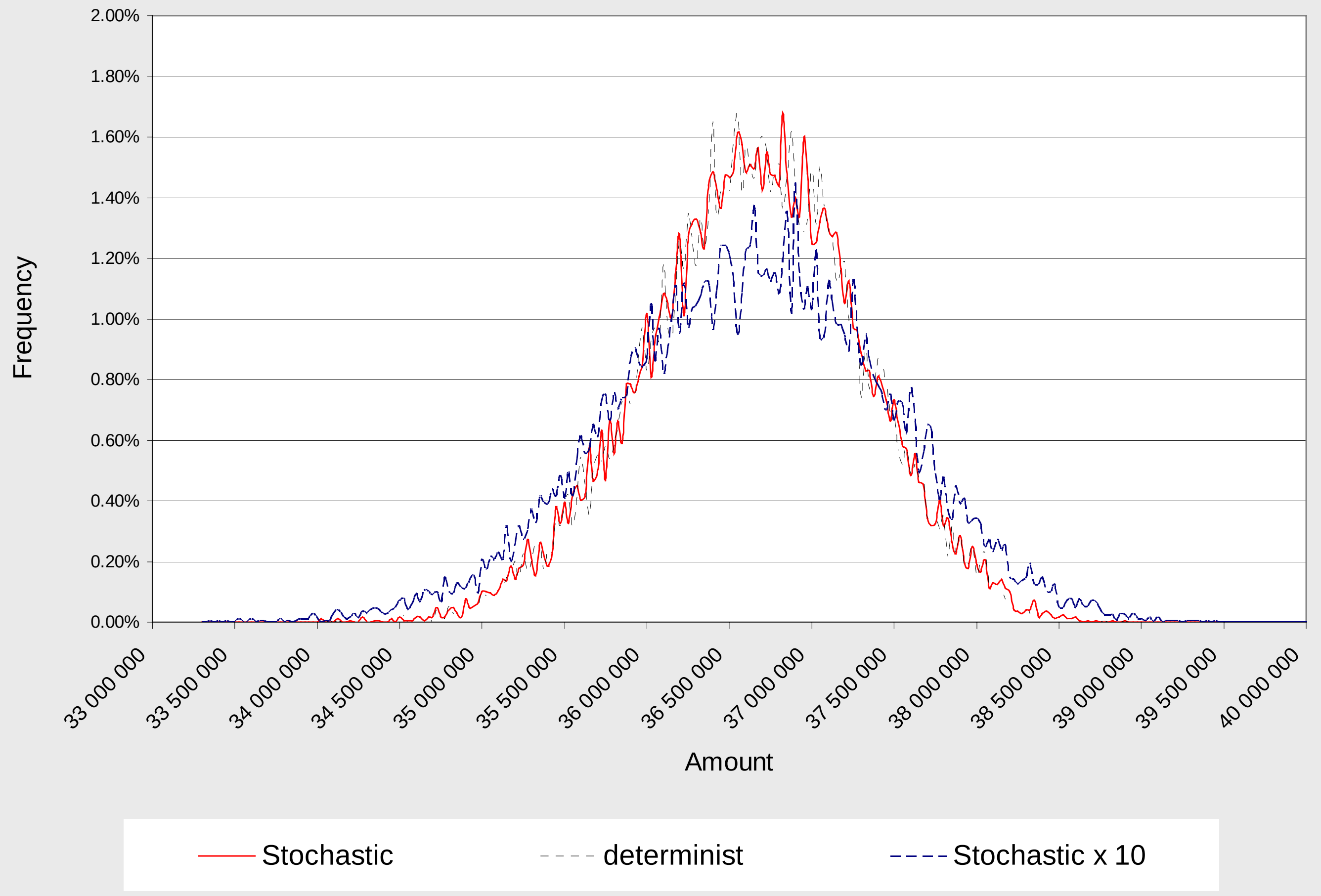
$$\bar{\lambda}_n = \frac{1}{M}\sum_{m=1}^{M}\lambda_{n,m} \qquad \bar{\bar{\lambda}} = \frac{1}{N}\sum_{n=1}^{N}\bar{\lambda}_n = \frac{1}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M}\lambda_{n,m}$$

$$\hat{\boldsymbol{E}}\left[\boldsymbol{V}\left(\Lambda|M\right)\right] = \frac{1}{N}\sum_{n=1}^{N}\frac{1}{M-1}\sum_{m=1}^{M}\left(\lambda_{n,m}-\bar{\lambda}_n\right)^2 \qquad \hat{\boldsymbol{V}}\left[\boldsymbol{E}\left(\Lambda|M\right)\right] = \frac{1}{N-1}\sum_{n=1}^{N}\left(\bar{\lambda}_n-\bar{\bar{\lambda}}\right)^2$$

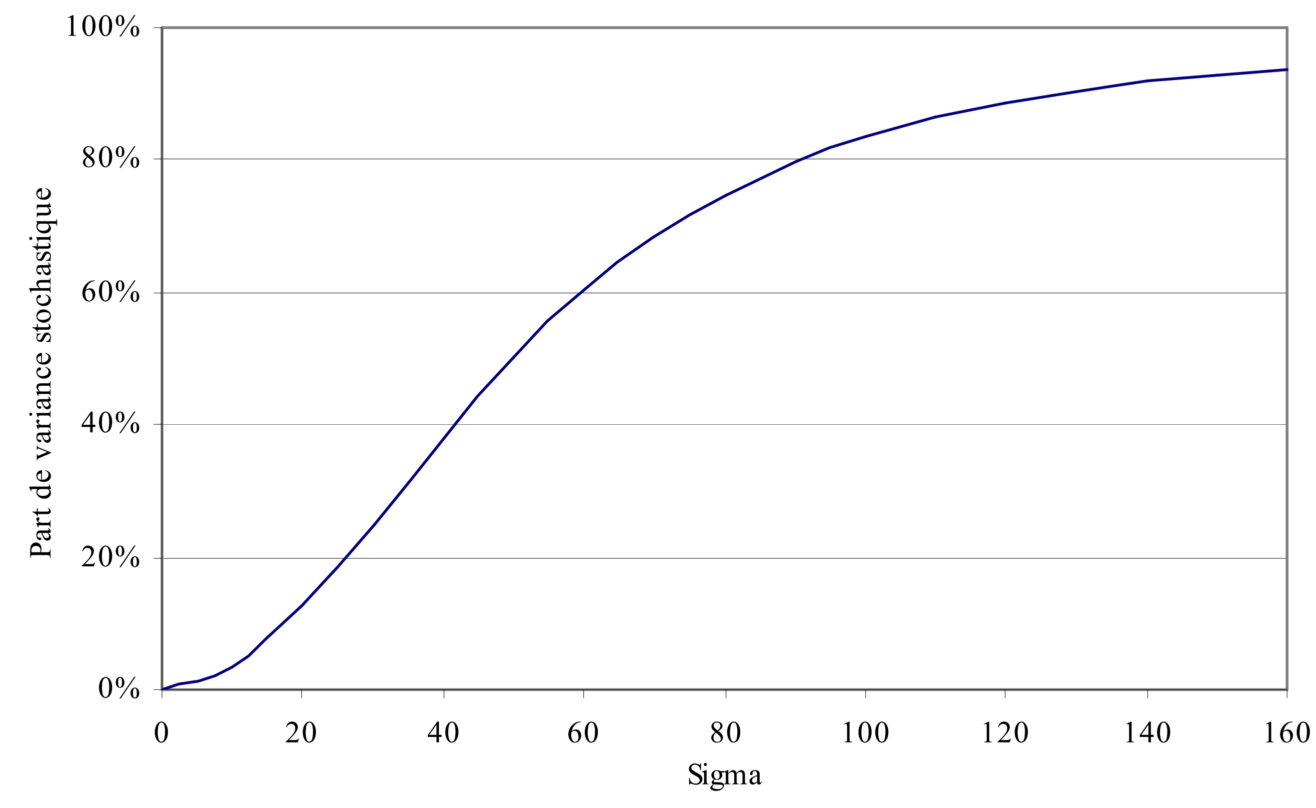
Empirical distribution fonction :

$$\hat{\mathrm{F}}(\lambda) = \frac{1}{NM}\sum_{m=1}^{M}\sum_{n=1}^{N}1_{]-\infty,\lambda_{n,m}]}(\lambda)$$

Numerical results : liabilities distribution



Numerical results : variance analysis



It can be noticed that the increase of the curve is slow at first, and speeds up for high volatility value. With our data we have $\hat{\sigma}_\gamma \approx 3{,}94$

Stochastic mortality model : trend risk

In fact the considered plan is subject to two distinct risks :

- the first, studied *supra*, is a consequence of the random fluctuations of the future mortality rate around the trend defined by the prospective tables.

- The second is linked to the uncertainty on this trend.

More precisely this uncertainty has two origins: the error due to the choice of the model, and the uncertainty linked to the determining of interpolation coefficients.

It is delicate to value *a priori* the model error ; a first approach consists for example in testing the sturdiness of the proposed model by doing the assessments on different periods. These estimations could lead to future mortality estimations sensitively divergent.

Stochastic mortality model : trend risk

We turn back to the equation :

$$k_t^* = at + b + \gamma_t$$

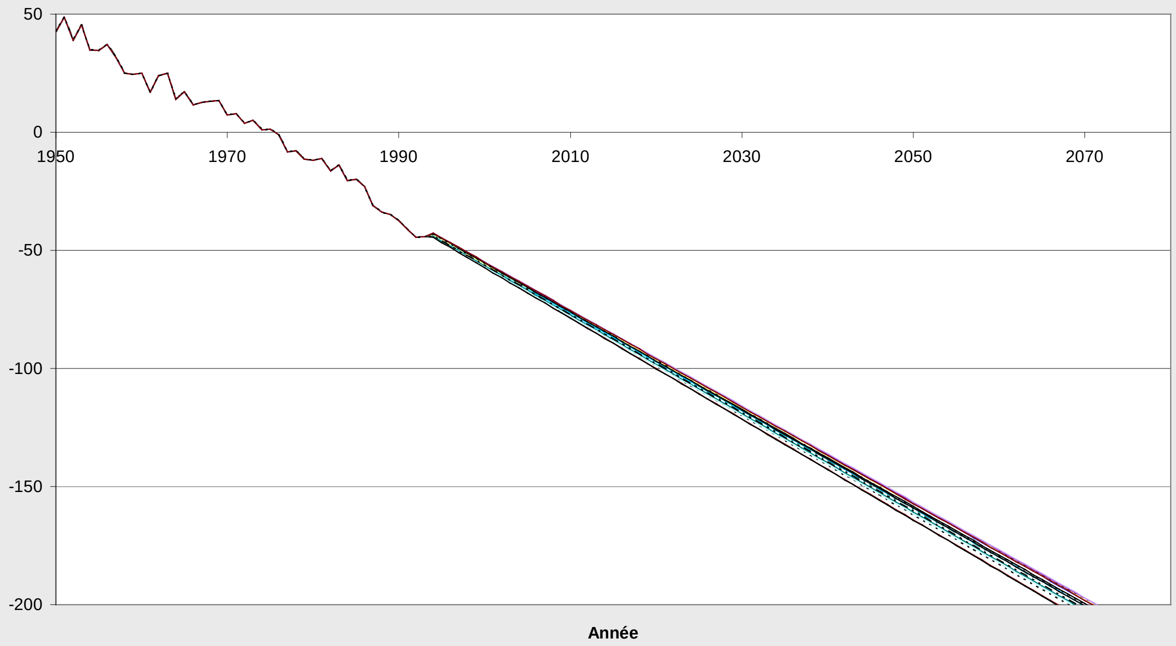
the coefficients a and b are valued thanks to the standard least squares method, an it leads to assume a couple $\left(\hat{a}, \hat{b}\right)$ as a Gaussian vector. So several extrapolation lines can be simulated and thus taken into consideration this variability source in the model.

$$\mu_{xt}^* = \mathbf{exp}\left(\alpha_x - \frac{\beta_x^2 \sigma_t^2}{2} + \beta_x k_t^*\right)$$

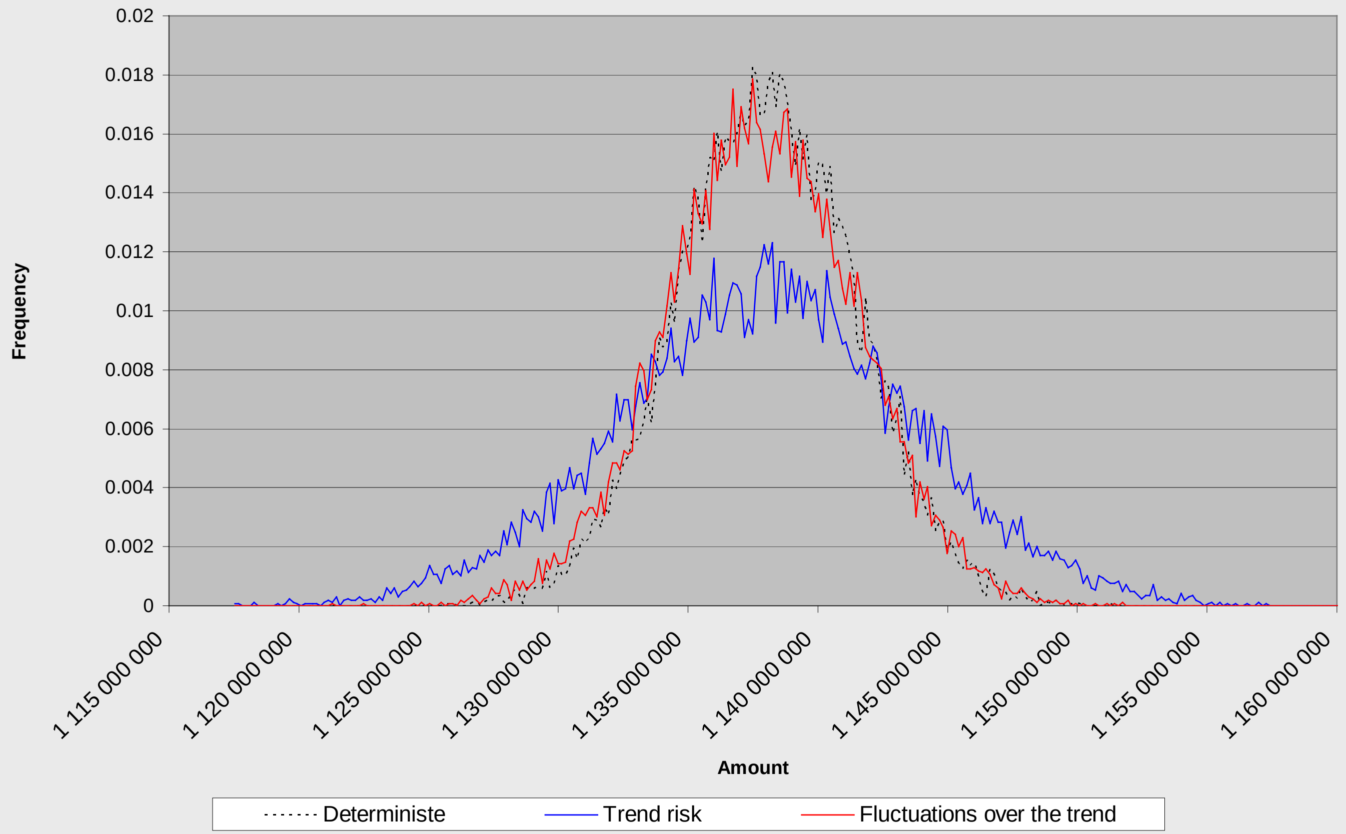
$$\sigma_\tau^2 = \frac{12\sigma_\gamma^2}{T\left(T^2-1\right)}\left(\tau^2 - \tau\left(T+1\right) + \frac{\left(T+1\right)\left(2T+1\right)}{2}\right) \qquad \tau = t - t_m + 1$$

Numerical results : trend simulation

This leads to :



Numerical results : liabilities distribution



NB : population size x30

Numerical results : liabilities distribution

Taking into account the « trend specification » risk leads to :

- increase the mathematical reserve to 0,30 % ;
- double the variation scope of the mathematical reserve (1,25% > 2,50%).

	Déterministe	Risque de dérive	Fluctuation autour de la tendance
Espérance	1 138	1 138	1 138
Ecart-type	3	6	4
Borne inférieure de l'intervalle de confiance	1 140	1 122	1 131
Borne supérieure de l'intervalle de confiance	1 145	1 149	1 145
Coefficient de variation	0,30%	0,50%	0,32%
Plage de variation	0,37%	2,37%	1,25%
VaR 75%	1 140	1 142	1 140
Marge de risque	0,20%	0,32%	0,21%

NB : results depends of the population's size, but is not very important.

Conclusion

We have proposed models to evaluate to two distinct risks :

- the consequence of the random fluctuations of the future mortality rate around the trend defined by the prospective tables.

- The uncertainty on this trend.

Our results seems to indicate that the first one has low impact in term of liabilities measure ; the second one is more important if the population is important ;

Question : take this risk into account in a Solvency 2 and IFRS 4 perspective ?

> More important risk : model specification (*cf.* TPG 1993 and TGH/TGF 05)

Contact

Frédéric PLANCHET

ISFA

WINTER & Associés

fplanchet@winter-associes.fr

WINTER & Associés

http://www.winter-associes.fr

43-47 avenue de la Grande Armée

F-75116 Paris

18, avenue Félix Faure

F-69007 Lyon