Credit Valuation

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Revision History		
Rev	**Date**	**Description**
1.0.0	March 22, 1984	Original Release
1.0.1	May 6, 1998	Reprint of original (format changes only)
1.1.0	July 9, 1999	Correction of typos in reprint version. Changes were made to equation (9) and equation (14). Edits to (9) are shown in bold: $$EL = (D_T + C_T) N\left(\frac{\log(D_T + C_T) - \log(A - F) - \mu T + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) - (A - F)e^{\mu T} N\left(\frac{\log(D_T + C_T) - \log(A - F) - \mu T \boldsymbol{-} \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) - C_T N\left(\frac{\log C_T - \log(A - F) - \mu T + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) + (A - F)e^{\mu T} N\left(\frac{\log C_T - \log(A - F) - \mu T \boldsymbol{-} \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)$$ The same edits were made to (14).

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The Philosophy of Credit Valuation

The Approach

Credit valuation is a necessary prerequisite to lending. It insures a desired quality of the asset portfolio, and results in loan pricing that corresponds to the risks assumed. It also provides means to reduce the likelihood of substantive losses through portfolio diversification.

Credit valuation is an objective and quantitative process. It should not depend on the judgement of a particular person or committee. Instead, it should be based on observable quantities, most particularly the market value of the borrower's assets. Credit risk should be measured in terms of probabilities and mathematical expectations, rather than assessed by qualitative ratings. When performed in this manner, we can refer to a credit valuation model.

A credit valuation model requires a theory that describes the causality between the attributes of the borrowing entity (a corporation) and its potential bankruptcy. This does not mean merely an empirical analysis that consists of examining a large number of different variables until a fit is found to the data. Statistical correlations among data do not necessarily signify causal relationships, and therefore provide no assurance of predictive power.

The credit model should be consistent with the modern financial theory, particularly with the theory of option pricing. The various liabilities of a firm are claims on the firm's value, which often take the form of options. The option pricing theory provides means to determine the value of each of the claims, and consequently allows one to price the firm's debt.

If the credit model provides a realistic description of the relationship between the state of the firm and the probability of default on its obligations, it will also reflect the development in the borrower's credit standing through time. This means that the model can be used to monitor changes and give an early warning of potential deterioration of credit. Obviously, this is only possible if the model is based on current, rather than historical, measurements. It also implies that the relevant variables are the actual market values rather than accounting values.

Pursuing this kind of approach to credit valuation means parting ways with some of the traditional credit analysis. Conventional analysis involves detailed examination of the company's operations, projection of cash flows, and assessment of the future earning power of the firm. Such analysis is not necessary. This is not because future prospects of the firm are not of primary importance – they most definitely are. It is because an assessment, based on all currently available information of the company's future, has already been made by the aggregate of the market participants, and reflected in the firm's current market value. Both current and prospective investors perform this analysis, and their actions set the price at an equilibrium value through the means of supply and demand. We do not assume that this assessment is accurate in the sense that its implicit forecasts of future prospects will be realized. We only assume that any one person or institution is unlikely to arrive at a superior valuation.

The most junior claim on the firm's assets is equity. If the future earnings of the corporation start looking better or worse than before, the stock price will be the first to reflect the changing prospects. Our challenge is to properly interpret the changing share prices.

It is also not essential to determine whether the firm will have enough cash flow for payment of interest and maturing debt. What is important is whether the market value of the company's assets (i.e., its business) will be adequate. If the assets of the firm have sufficient market value, the firm can easily raise cash it needs by selling off a portion of its assets. If the assets are not easily transferable, the firm can sell them indirectly, by issuing additional equity or additional debt. In any case, the firm's ability to pay its debt is dependent upon its future market value, rather than on its future cash position.

The Firm's Value

The value of a firm is the value of its business as a going concern. This value depends on the future prospects and profitability of the firm's business, its risks, and its standing relative to other investment opportunities existing in the economy. The firm's business constitutes its assets, and the present assessment of the future returns from the firm's business constitutes the current value of the firm's assets.

The value of the firm's assets is different from the bottom line on the firm's balance sheet. The book asset value is a fairly arbitrary statement of the initial cost of the physical assets of the company and their depreciation. When the firm is bought or sold, the value traded is the ongoing business. The difference between the amount paid for that value and the amount of the book assets is usually accounted for as the "goodwill".

The value of the firm's assets can be measured by the price at which the total of the firm's liabilities can be bought or sold. The various liabilities of the firm are claims on its assets. The sum of the market value of the liabilities is the amount for which sole possession of the total of the firm's assets can be obtained (or disposed of) and that is exactly what the firm is worth. The market value of the individual liabilities is directly observable if the liabilities are publicly traded. Thus, the value of equity can be usually obtained by multiplying the share price by the number of shares outstanding. The various bond issues can often be valued as the current price per unit of face value times the total face amount of the issue. If the debt is privately placed, an approximate valuation can be achieved by pricing the debt at current interest rates. Current liabilities can be typically valued at their nominal amount, since they are usually immediately payable.

Although the sum of the market values of liabilities is a convenient way to determine the value of the assets, the asset value does not depend on the structure and composition of the liabilities. If the firm decides to raise additional equity to retire part of its debt, or to borrow in order to buy back some of its outstanding stock, the value of the firm's assets does not change. What changes is merely the division of the ownership of these assets. The same is true even in bankruptcy proceeding. Bankruptcy is a transfer of ownership from the stockholders to the holders of debt. If the firm is worth more as a going concern than its liquidation value, the debt holders will keep it going. If the debt holders do not want to run the firm, they can sell it to somebody who does.

Loan Default

We will start with a simple situation. Consider a corporation that at present has no debt, and wants to borrow. Assume the debt is in the form of a discount note issued by the company (such as commercial paper). How much risk does the buyer of the note (the lender) take, and how much should he pay for it? In other words, how does he value the credit?

In buying the note, the lender purchases a claim on the firm's assets, and thereby becomes a partial owner of the company. The value of the company's assets increases by the amount received on the note (the stock price itself does not change by the issuance of the debt). The new total value of the firm's assets is equal to the value of the stock and the value of the debt.

With time, the market value of the company's assets will change. (Perhaps not the book value, but we are not concerned with book values.) The value of assets will be changing as the market's perception of the future earning power of the company changes. These changes obviously involve considerable uncertainty. We can characterize these changes as a stochastic (random) process, subject to a probability law.

What concerns the lender is the market value of the firm's assets when the note matures. Two situations are possible. The asset value is at least that of the face value of the debt, or the asset value is less than the debt,

In the first situation, the stockholders will pay the debt. The total value of the company is sufficient for them to do so. If the firm does not have enough cash, the stockholders can raise it by selling a part of the assets at their market value. Moreover, it is in the interest of the stockholders to pay the loan, since otherwise the lenders would force the firm to bankruptcy and the stockholders would lose control of the firm (although not money, apart from bankruptcy costs). Since the borrower is both willing and able to repay the loan, the lender will realize no loss.

In the situation that the market value of the firm's assets falls below the amount due on the loan, the company cannot repay the lender. There is no way to raise the cash. No other lender would refinance the loan, because that would mean taking over the loss from the original lender. It is also not possible to raise additional equity, since the stock is worthless. The company has to declare bankruptcy. The stockholders get nothing, while the lenders take over the assets. The lenders will thus realize a loss equal to the difference between the face value of the debt and the market value of the assets.

The risk to the lender at the time he contemplates making the loan is that the second situation may arise. The probability of this situation is the probability that the asset value at the maturity of the loan will be less than the loan balance. If we can describe the process governing the changes in the asset value, this probability can be explicitly calculated. This calculation provides a measure of credit risk.

A reasonable specification of the behavior of the asset value is that the change in market value over an interval of time is independent of its past changes, and has an expected component and a random component. The magnitude of both the expected and random components is proportional to the asset value (that is, it is the same for each dollar of assets). This type of process is variously referred to as a logarithmic Wiener process, or a proportional Brownian motion, or a geometric random walk.

The probability of default calculated under this assumption depends on the following quantities: the initial value of assets; the expected rate of return on assets; the variability of the asset value; the face value of the debt; and the loan term.

The higher is the initial asset value in relation to the loan amount; the lower is the probability of default on the loan. If the company borrows little relative to the market value of its equity, the loan is comparatively safe. If the company levers itself considerably (in market value terms), the riskiness of the loan is high.

The default probability also depends highly on the variability of the asset value. If the assets grow more or less along the firm's expected growth path, the loan carries little risk even with relatively high leverage. If, on the other hand, the asset value fluctuates wildly, the likelihood of default on the loan is considerable.

As to the length of the loan term, typically the default probability will increase with the term. In effect, more things can go wrong with the company over a long interval than over a short one. For very long loans, however, the probability of default may start decreasing again, as the long-term asset growth asserts itself over the fluctuations.

The probability of default does not in itself provide a measure of the magnitude of the possible loss. It only characterizes the occurrence of loss, rather than the dollar amount. This latter quantity can be measured by the expected loss. Naturally, we care about both the probability of default and the size of loss.

The expected value of a quantity is defined as the average of the possible values of that quantity, each value being weighted by the probability of its occurrence. The expected loss is therefore the probability weighted mean dollar amount of the difference between the face value of the loan and the actual receipts by the lender.

The same considerations that led to a formula for the probability of default also allow deriving an equation for the expected loss. In the example of a commercial borrower whose liabilities consist of equity and one class of debt, the formula for the expected loss turns out to depend on the same quantities as the probability of default; namely the current market value of total assets, the expected asset return, the variability of the asset value, the face value of debt, etc. The expected loss is given as the difference of two terms: the first term is the loan face value multiplied by the probability of default. This would be the expected loss if default meant losing the entire loan. As it is, there is a recovery, equal to the assets of the bankrupt firm, and the formula for the expected loss has a second term subtracted from the first, which represents the expected amount recovered.

Debt Structure

The financial structure of most corporations is more complicated than the one with which we have dealt so far. The liabilities will include current liabilities (such as accounts payable, provisions for taxes, etc.), debt of various terms, and equity. The whole structure of liabilities needs to be considered in valuing the company's credit from the viewpoint of a particular lender.

The first question to address is determining the hierarchy of the claims on the firm's assets. In other words, the priority and subordination of the claims in the event of dissolution of the firm has to be considered. From the viewpoint of a particular lender, the relevant distinction is between the claims that take precedence over that lender's claim, claims that are at par, and claims that are subordinated to the lender's claim. This last category includes the firm's equity.

It is obvious that we need to talk about valuation of the borrower's credit for a given lender, not for the lenders in general. Depending on the standing of the lender's claim in the hierarchy of debt, a company may be a good credit risk, or a poor one, even though the probability of bankruptcy is the same for everybody. As a matter of fact, the same event can improve the firm's credit for one lender and make it worse for another lender. For instance, issuing additional debt reduces the expected loss for holders of claims with a higher priority, while it increases the expected loss for holders of claims subordinated to the new debt.

In general, the credit standing of a commercial borrower from the viewpoint of a particular lender improves whenever debt with lower priority is added, or debt with higher priority retired. It deteriorates with decreasing the total amount of more junior debt and with increasing the total amount of more senior debt. Lower priority debt, like equity, is a protection for the lender; the corresponding assets provide a cushion between the value of total assets and the face value of his claim.

In addition to categorizing liabilities of a firm by their priority, it is necessary to distinguish among them on the basis of their term. The firm goes bankrupt if its assets are less than the face value of debt that is due at that time; if the value of the assets is less than the amount of debt which is not yet due, the firm can, and will, continue operating. A lender must therefore determine which of the firm's liabilities mature within the term of his claim.

This can lead to a very complicated situation if the structure of debt by priority and by term takes the most general form. In a simple situation when all debt matures at the same time, the holder of a claim is not concerned about any subordinated claims. His loss may only come if the company's assets at the maturity of the debt are less than the total of his loan and all debt with a higher priority. Moreover, the lender only needs to consider the possible value of the company's assets as of the date his loan is due.

If, however, different claims mature on different dates, claims that mature early may trigger a bankruptcy even if they are junior to the lender's claim. His loan may still be paid in full, if the firm's assets at that time exceed the total of his and the more senior debt. It is no longer sufficient, however, to consider only the more senior claims; and it is no longer sufficient for the lender to be concerned about the value of the firm's assets on the maturity date of his claim only.

Fortunately, from the viewpoint of the provider of short-term credit to a commercial borrower, the situation is relatively simple. It is reasonable to assume that the more senior claims (such as employee wages and benefits, and provisions for taxes) are also short-term; and that debt at par with ours is either similarly short (bank revolving credit, etc.) or, as with notes and bonds, matures after the term of our debt.

In this case, default occurs if, on the maturity date of our loan, the market value of the borrower's assets is less than the maturing debt amount (the total short-term obligations). The probability of default is then given by a similar formula to the one obtained in the case of a single class of debt; except that the face value of debt in that formula is replaced by the value of the short-term debt only. In other words, the term debt is treated like equity.

The expected loss amount, however, needs now to be calculated by a different formula than in the simple case of one type of debt. If, on the maturity of our loan, the market value of the firm's assets exceeds the total maturing debt, there is no loss. If the assets are less than the total maturing debt but more than the higher priority debt, the loss is equal to the maturing debt amount less the value of assets. Finally, if the assets are less than the higher priority debt, the loss is complete and we recover nothing. This is a more complex loss function than in the case of one class of debt. Nevertheless, it is still possible to derive a formula for the expected loss – it is just a more complicated equation.

Here it may seem that if the value of the firm's assets is less than the total maturing debt, the amount received by the short-term lender would be further decreased by payments to the holders of the long-term debt. Indeed, if the firm were forced into bankruptcy, the long-term debt would become payable and the short-term lender would only receive a proportional part of the remaining assets. This, however, can be avoided. The short-term lender should in this situation renew a partial credit to the firm, equal to the exact difference between the amount due and the value of the firm's assets. This will keep the firm from going bankrupt and prevent the long-term lenders from collecting on their claim. The loss to the short-term lender will thus be limited to the same amount as if the long-term debt was a subordinated claim. From our viewpoint, long-term debt is as good as capital.

Capital Flows

An explicit consideration must be paid to flows of value from the firm to its owners (stockholders as well as holders of debt). Unlike other cash flows, payments to owners are not reflected in the current market value of the firm, since they do not change the total owners' wealth. For example, if the company decides to double its dividends, or to accelerate repayment of its outstanding debt, the total current value of the firm will not change. In contrast, if taxes double, the firm's value will decline. Now, although changes in policy concerning payments to owners do not affect the firm's total value, they do affect the distribution of value between the different classes of claims. Thus, an extra dividend will transfer some value from the lenders to the stockholders. Consequently, payments to owners, such as dividends and interest on debt, need to be included in the lender's credit valuation.

When considering short-term lending, it is a reasonable, indeed conservative, approximation to assume that the total dividends expected to be paid during the term of the loan are paid at the beginning date. This means that the market value of the firm's assets is reduced by the total expected dividend payout. Similarly, interest on existing debt expected to be paid during the term of our loan is taken to reduce the initial assets.

Since these payments decrease the initial asset value, the probability of default and the expected loss increase. These payments are withdrawals of capital from the firm and as such change the relative value of the different claims. It should be noted that if the stockholders vote themselves additional dividends that have not been anticipated, they transfer wealth from the debt holders to themselves. It is important for the creditor not to underestimate the dividend payments.

Loan Pricing

The purpose of credit valuation is for loan pricing. Pricing a loan means determining the current value of the loan as a function of its risks. A loan is an asset that can be bought and sold like any other asset. A lender has no economic reason to refuse making the loan if the price is right. If the riskiness of the loan does not suit the lender's preferences, he can sell the loan to somebody whose preferences it does fit.

Of course, determining the interest rate to be charged on a given loan (which is what is usually meant by loan pricing) is the same thing as determining the present value of the loan payments. It is just more convenient in view of the general theory of asset pricing to obtain the value of the loan first and then derive the interest rate from it.

It would seem that a loan should be priced at the present value of the expected payoff (that is, the face amount less the expected loss), using the risk-free rate as the discount rate. Indeed, by subtracting the expected loss from the face amount a provision is made for the possibility of default, and discounting this amount to present at the risk-free rate then simply accounts for the time value of money.

This, however, is not correct. If it were, then the expected rate of return on the loan would be the risk-free rate, while risky assets in general earn higher than the risk-free rate. In particular, the assets of the firm to which the loan is made may be earning a rate of return whose expected value is higher than the risk-free rate. Since the loan is a claim on these assets, sharing the risks associated with these assets, it should also share the higher expected return.

The exact answer to the pricing of the loan is provided by the option pricing theory. The option pricing theory is in turn a special case of the theory of pricing derivative assets, that is assets whose value depends solely on the value of another, underlying asset. This is the situation at hand: the value of the loan is a function of the value of the firm's assets on which the loan is a claim.

It turns out, on the basis of this theory, that the value of the loan cannot be determined from the knowledge of the expected loss alone. As a matter of fact, it cannot be determined even from knowing the whole probability distribution of the loss. What is needed is the joint probability distribution of the loss together with the value of the underlying assets of the firm.

The equation for the value of the loan provided by the derivative-asset-pricing theory has a curious form. The loan value is equal to the present value of the expected payoff, discounted by the risk-free rate, with the expected payoff calculated as if the firm's assets earned the risk-free rate rather than its actual expected rate. In other words, we can take the formula for the expected loan loss, but substitute in it the risk-free rate for the expected asset rate of return. This hypothetical expected loss is subtracted from the loan face value, and the difference discounted to present at the risk-free rate. This provides the correct loan price.

If the expected rate of return on the assets of the firm is higher than the risk-free rate (which in general it will be), the premium to be charged on the loan over the risk-free rate will actually be higher than the expected loss. This extra increment above the expected loss is a compensation for the variance of loss, or more accurately, for a component of that variance that is related to the systematic factors in the economy. The possible deviation of the loss from its expected value is in part due to factors specific to the firm, and in part due to more general factors, such as the market in general. It is this second source of variance that carries compensation to the lender beyond the amount of the expected loss itself.

Portfolio Diversification

Portfolio diversification is a means of reducing the probability of large losses. Even if the expected loss on an individual loan is small, the loan can still result in a large loss. If this loan is a part of a portfolio, such a loss is a smaller percentage of the total assets. The portfolio can only incur a large loss if a number of loans in the portfolio realize losses simultaneously. This is less likely than the default on a single loan.

Diversification does not reduce the expected loss. The expected loss on a portfolio is the average of the expected losses on the individual loans, weighted by their relative proportions in the portfolio. If each loan in the portfolio had an expected loss of .1%, the expected loss on the portfolio would still be .1%.

What changes is the certainty of that loss. With a single loan, there may be no loss, but there may also be a big or total loss. In other words, there is a large dispersion of the possible loss amount around its expected, or mean, value. With a diversified portfolio, the dispersion of the portfolio loss around its expected value is much smaller.

An ideally diversified portfolio would have no deviation of the actual loss from the expected amount. It would be like playing the statistical odds in an infinite population. Since the expected loss is a probability weighted average of the possibilities, and in such an ideal situation the frequencies of the occurrence of each possibility conform to their probabilities, the portfolio loss would be guaranteed to be no more, or less, than the expected value. Some loans in the portfolio would realize losses larger than those expected and some would realize no losses or losses smaller than expected. These individual deviations would average out.

In reality, an ideally diversified portfolio is not possible. For one thing, it would take an infinite number of loans in the portfolio to achieve this. More importantly, however, it would necessitate that there is a sufficient degree of independence among the individual loans. It would be necessary that an occurrence of larger than expected losses on some loans does not substantially decrease the likelihood of smaller than expected losses on other loans. Now, the loss on a loan results from a decline of the assets of the borrowing firm below the face value of the loan. The changes in the value of assets among firms in the economy are correlated, that is, tend to move together. There are factors common to all firms, such as their dependence on economy in general. Such common factors affect the asset values of all companies, and consequently the loss experience on all loans in the portfolio. This common, or systematic, risk cannot be diversified away. Only the risks that are specific for the individual companies, unrelated from one to another, can be reduced by diversification.

What this means is that even a very large portfolio of loans will have a substantial likelihood of a loss which is larger, or smaller, than that expected. There is a limit to the extent to which the variation of the actual loss from the expected loss can be reduced. This limit is the systematic portfolio risk. A well-diversified loan portfolio will have only this systematic risk, with very little of the specific risk. The goal of diversification is to bring the riskiness of the portfolio close to this minimum.

This goal can be achieved by ensuring that the loans in the portfolio are not unduly concentrated in any one segment of the market, such as a particular industry or particular type of firms. The less the companies in the portfolio have in common, the lower is the probability of large portfolio losses. The degree of diversification can be measured quantitatively by the variance of loss (variance characterizes the degree to which a quantity can deviate from its expected value), and this measure should be minimized subject to the portfolio requirements and constraints.

Summary

The approach to credit valuation presented here differs in many aspects from traditional credit analysis. It does not involve judgmental evaluation of the company's operations and prospects. Instead, it is based on an explicit economic theory of bankruptcy and default, applied within the context of the modern financial theory. It thus relies on a belief in market values, and the efficiency of the market to reflect all available information in security prices.

The model considers the borrower's credit to be a function of the value of his assets. For a corporate borrower, the assets are the firm's ongoing business. The market value of the firm's assets can be determined from the market price of the company's stock.

A default on a loan occurs if the value of the firm at the maturity of the loan is less than the amount due. Given a description of the firm's value as a stochastic process, the probability of default on the loan can be calculated. This probability depends on the initial market value of the firm, the total amount of debt and the hierarchy of debt, dividends and interest expense, the expected rate of return on assets, the variability of the asset value, and the loan term. The expected loss on the loan can also be calculated from these quantities.

The loan is priced to compensate the lender for the expected loss and for the systematic component of the variance of loss. The pricing formulas are derived from the theory of option pricing.

Portfolio diversification, although it does not reduce the expected loss, decreases the variance of the possible loss around its expected value. The limit to diversification is given by the amount of systematic (non-diversifiable) risk. This risk arises from dependence of the individual companies on the total economy.

The Credit Valuation Model

Definitions

Define a firm as an entity consisting of its assets (its ongoing business). Let the claims on these assets consist of current liabilities, short-term debt, long-term debt (bonds) and equity. Denote by:

A – market value of total assets
C – market value of current liabilities
D – market value of short-term debt
B – market value of bonds
S – market value of equity.

By definition,

$$A = C + D + B + S \tag{1}$$

Let T be the term to maturity of the short-term debt, and denote the face value of the short-term debt by D_T. Assume that the current liabilities are also payable at time T, the amount due being denoted by C_T, and that the term of the long debt is greater than T. We will assume further that the current liabilities constitute a claim senior to the short-term debt.

Assume that the total asset value follows a stochastic process described by the equation

$$dA = \mu A dt + \sigma A dz, \quad t > 0 \tag{2}$$

where μ and σ^2 are the instantaneous mean and variance, respectively, of the rate of return on assets, and dz is an increment of the Wiener process. Let F be the total amount of dividends and bond interest over the term T, assumed to be prepaid at time $t = 0$,

$$dA(0) = -F \tag{3}$$

It follows from equations (2) and (3) that the logarithm of the total asset value at time $t > 0$ is normally distributed with the mean

$$E\left(\log A(t) \middle| A(0) = A\right) = \log\left(A - F\right) + \mu t - \frac{1}{2}\sigma^2 t \tag{4}$$

and variance

$$\mathrm{Var}\left(\log A(t) \middle| A(0) = A\right) = \sigma^2 t \tag{5}$$

Loan Default

The short-term loan is in default if the value $A(T)$ of the assets at the maturity of the loan is less than the amount payable,

$$A(T) < D_T + C_T$$

Denote by p the probability of default,

$$p = P\left[A(T) < D_T + C_T \middle| A(0) = A\right] \tag{6}$$

In evaluating this probability, we have

$$p = P\left[\log A(T) < \log(D_T + C_T) \middle| A(0) = A\right]$$

and therefore

$$p = N\left(\frac{\log(D_T + C_T) - \log(A - F) - \mu T + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) \tag{7}$$

where N is the cumulative normal distribution function.

The loan loss L on the short-term debt is given by:

$$\begin{aligned} L &= 0 && \text{if } A(T) \geq D_T + C_T \\ &= D_T + C_T - A(T) && \text{if } C_T \leq A(T) < D_T + C_T \\ &= D_T && \text{if } A(T) < C_T \end{aligned} \tag{8}$$

The expected loss is then

$$EL = \int_{C_T}^{D_T + C_T} (D_T + C_T - a) f(a) da + \int_0^{C_T} D_T f(a) da$$

where f is the probability density of $A(T)$ given $A(0) = A$. Evaluating the integral yields the following equation for the expected loss:

$$
\begin{aligned}
EL = & \left(D_T + C_T\right) N\left(\frac{\log\left(D_T + C_T\right) - \log\left(A - F\right) - \mu T + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) \\
& -\left(A - F\right) e^{\mu T} N\left(\frac{\log\left(D_T + C_T\right) - \log\left(A - F\right) - \mu T - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) \\
& -C_T N\left(\frac{\log C_T - \log\left(A - F\right) - \mu T + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) \\
& +\left(A - F\right) e^{\mu T} N\left(\frac{\log C_T - \log\left(A - F\right) - \mu T - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)
\end{aligned}
\tag{9}
$$

The sum of the first two terms in this equation is the expected loss on the combined claim comprised of the current liabilities and the short term debt. It can be interpreted as the face value of this claim multiplied by the probability of default (which is the expected loss on this claim if there were no recoveries) less the recovered amount. The negative of the sum of the third and fourth terms is the expected loss on the current liabilities alone. It is similarly given as the difference of the expected gross loss less the expected recovery. The expected loss on the short-term loan is thus the difference between the expected loss on the combined claim minus the expected loss on the current liabilities.

Loan Pricing

The various liabilities of the firm are claims on the firm's total assets. The value of each of these claims is a function of the value of the total assets. An asset whose value depends solely on the value of another, underlying, asset is called a derivative asset. Each of the firm's liabilities is thus a derivative of the total asset value.

The theory of derivative asset pricing (cf. Black and Scholes, 1973, and Merton, 1974) states that if the value A of an asset follows the equation

$$dA = \mu A dt + \sigma A dz$$

then the value of a derivative asset D satisfies the partial differential equation

$$D_t + rAD_A + \frac{1}{2}\sigma^2 A^2 D_{AA} - rD = 0 \tag{10}$$

where r is the riskless rate of interest and subscripts denote partial derivatives.

When applied to pricing of the short-term loan, the value D of the loan is subject to equation (10) together with the boundary condition at $t = T$

$$D(T) = D_T - L \tag{11}$$

or, from equation (8),

$$\begin{aligned} D(T) &= D_T && \text{if } A(T) \geq D_T + C_T \\ &= A(T) - C_T && \text{if } C_T \leq A(T) < D_T + C_T \\ &= 0 && \text{if } A(T) < C_T \end{aligned} \tag{12}$$

The solution of equation (10) with the boundary condition (12) can be given as

$$D = \left(D_T - Q\right)e^{-rT} \tag{13}$$

where

$$\begin{aligned} Q = &\left(D_T + C_T\right)N\left(\frac{\log\left(D_T + C_T\right) - \log\left(A - F\right) - rT + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) \\ &-\left(A - F\right)e^{rT}N\left(\frac{\log\left(D_T + C_T\right) - \log\left(A - F\right) - rT - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) \\ &-C_T N\left(\frac{\log C_T - \log\left(A - F\right) - rT + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) \\ &+\left(A - F\right)e^{rT}N\left(\frac{\log C_T - \log\left(A - F\right) - rT - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) \end{aligned} \tag{14}$$

It may be noted that the quantity Q is given by a formula formally identical to that for the expected loss in equation (9), except that the expected rate of return on assets μ is replaced by the risk-free rate r. The quantity Q can be interpreted as the price of the loan loss. If all market participants were risk-neutral, the expected rate of return on all assets would be equal to the riskless rate r. In that case, $\mu = r$ and consequently $Q = EL$. In the risk-neutral world, the loan would then be priced as the present value of the expected amount received at the maturity, which is the loan face value less the expected loss. In general, however, the expected rate of return on assets will not be equal to the risk-free rate. If $\mu > r$, then $Q > EL$. The loan is priced in such a way that the return on the loan fully compensates the lender for the expected loss. In addition, however, the lender receives a compensation for the variance of the loss, to the extent that the expected return on assets carries a compensation for the asset variability. If the expected rate of return on assets in excess of the risk-free rate is proportional to the asset beta, the expected excess rate of return on the loan will be proportional to the beta of the loan. Indeed, the loan beta is

$$\beta_L = \frac{A}{D} D_A \beta \tag{15}$$

where β is the asset beta. Since equation (10) is equivalent to the condition

$$\mu_L - r = \frac{A}{D} D_A (\mu - r) \tag{16}$$

where μ_L is the expected rate of return on the loan, it follows that the expected excess return on the loan is equal to

$$\mu_L - r = \beta_L \frac{\mu - r}{\beta} \tag{17}$$

In other words, the loan return carries a compensation for the systematic portion of the variance of the possible loss, in addition to the expected value of the loss.

Denote by i the interest rate charged on the loan, that is,

$$D = D_T e^{-iT} \tag{18}$$

The difference between the loan interest rate and the riskless rate, which can be called the rate premium, is equal to

$$i - r = -\frac{1}{T} \log\left(1 - \frac{Q}{D_T}\right)$$

by virtue of equations (13) and (18). This is expressed in terms of continuously compounded rates. The same result can be stated, perhaps more intuitively, in terms of simple rates of interest. If R and I are the simple riskless rate and the simple loan interest rate, respectively, over the term T,

$$1 + RT = e^{rT}$$
$$1 + IT = e^{iT}$$

then the simple interest rate premium can be written as

$$I - R = \frac{1}{T}\frac{Q}{D} \tag{19}$$

Again, if Q were equal to the expected loss (as it would be in a risk-neutral world), the loan interest rate would exceed the risk-free rate simply by the amount of the expected loss taken as a percentage of the amount advanced, and annualized. If investors are risk-averse, the rate premium would in addition to the expected loss include a premium for the systematic risk of the loan.