Applying Financial Economics to Life and Pension Insurance

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Ph.D. dissertation

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Contents

Preface

This dissertation is written during my three-year Ph.D. study at the Department of Accounting, Finance & Law at the University of Southern Denmark.

As part of the Ph.D. program I spent six months visiting the Anderson School of Management at University of California, Los Angeles. I want to thank the Ph.D. students at the Anderson School for making my stay there pleasant. A special thanks to Amit Goyal for his guidance and help. The stay at the Anderson School was financed through a grant from the Tuborg Foundation for which I am very grateful.

Many people have contributed with comments and suggestions to the papers contained in this dissertation. These were highly appreciated. In particular, I want to thank Anders Damgaard, Christian Riis Flor, Henrik Ramlau-Hansen, Martin Skovgaard Hansen, Bjarne Astrup Jensen, Jesper Lund, Monika Mertens, Claus Munk, Ragnar Norberg, Mogens Steffensen, and especially Peter Ove Christensen and my supervisor Kristian R. Miltersen.

Part III of this dissertation contains the paper "Portfolio Choice and Fair Pricing in Life Insurance Companies". It is the result of joint work with Martin Skovgaard Hansen, whom I very much enjoyed working with. I would like to thank Martin for all the fruitful discussions we have had during my three years at the Department of Accounting, Finance & Law. An earlier version of our paper was presented at the Laboratory of Actuarial Mathematics at Copenhagen University, Denmark in March 2001, at the *International Symposium on Financial Exposure in Life and Pension Insurance*, April 2001 in Bergen, Norway, and at the *Nordic Symposium on Contingent Claims*, May 2001 in Stockholm, Sweden.

The paper, "Minimum Rate of Return Guarantees: The Danish Case", is included as part IV of the dissertation. The paper is the outcome of joint work with Kristian R. Miltersen. I want to thank Kristian Miltersen for the opportunity to do research with him. I benefitted a great deal from this opportunity. Our paper was presented at the *Quantitative Methods in Finance 1999 Conference*, Sydney, Australia, The EDEN Doctoral Tutorial at *The European Finance Association's 26th Annual Meeting*, Helsinki, Finland, *The 10th International AFIR Colloquium*, Tromsø, Norway, the Danish–German Ph.D. seminar at the Johannes Gutenberg–Universität, Mainz, Germany in May 2000, and *The first World Congress of the Bachelier Finance Society, 2000*, Paris, France. The paper is forthcoming in the *Scandinavian Actuarial Journal*.

I also want to thank Lene Holbæk for assistance with the proof reading of various parts of the text. A thanks to the Department of Finance at the Copenhagen School of Business is also in order. They kindly provided me with an office that I could use after my move to Copenhagen. Finally, a special thanks to Steen, friends and family for their support. In particular, for their ability to never doubt that I would reach the goal. A goal that often seemed far out of reach.

Mette Hansen
Odense, March 2002

Preface to the printed edition

Compared to the examined edition some references have been updated, layout has been changed slightly, and a few typos and other minor corrections have been made.

I want to thank the members of my Ph.D.-commitee, Peter Ove Christensen (University of Southern Denmark), Jørgen Aase Nielsen (University of Aarhus), and Svein-Arne Persson (Norwegian School of Economics and Business Administration), for their comments to the dissertation and suggestions for future research.

Mette Hansen
Odense, June 2002

Part I

Introduction

Chapter 1

Introduction

In recent years the life and pension insurance business has attracted a great deal of attention to itself. The reason is that a large number of insurance companies worldwide have found it difficult to honour contracts which they have sold to their customers. The products are, for instance, sold as part of a retirement plan. The contracts are typically offered with a guaranteed minimum benefit. In general, the benefit is in some way linked to the financial market. The financial risk is mostly borne by the insurance company alone.[1] The insurance companies have therefore issued contracts that have one or several embedded options.[2] This does not pose a problem in itself since the companies can simply charge the customer a price which incorporates a premium for the option(s). The problem is that life insurance companies have neglected to collect an additional premium for the option(s). In principle, they have only charged a premium for the guarantee. This problem has been known in financial economics since the pioneering work of Brennan and Schwartz (1976). However, not much notice was taken until the nineties. The high interest level that prevailed during the seventies and eighties made the embedded options more or less valueless and therefore they did not give rise to any difficulties. In the early nineties the interest rate level fell, and the value of the option(s) increased substantially. In the end, this rendered several companies insolvent, cf. Briys and de Varenne (1997).

Among other things this dissertation deals with ways of pricing various types of insurance contracts. That is, valuation techniques known from financial economics are applied to determine premiums that take into consideration the financial risk inherent in the contracts. Other issues such as how a company should invest given that it has issued contracts with guarantees, and how competition among life insurance companies affect the decisions of the companies, are considered.

[1] Disregarding pure equity-linked contracts where the entire investment risk is born the contract holder.

[2] There is at least one option in play, namely the so-called bonus option. Other options might be present. For instance, a policy holder typically has a surrender option, i.e. the option to buy back the policy at a certain price.

1.1 Some definitions

There exist many different types of life insurance and pension contracts in the world today. Three basic types of life insurance contracts that are considered in the dissertation are:

- **Term Life Insurance**. The contract has a fixed expiration date. It pays a certain (known in advance) amount at the time the insured dies if this occurs before expiration of the contract and nothing otherwise.

- **Pure Endowment Insurance**. Pays a predetermined amount only if the insured is alive at the expiration date of the contract.

- **Endowment Insurance**. Is a combination of the former two contracts, i.e. it pays out either at the time of death of the insured or at the expiration of the contract, whatever occurs first.

In addition to these three types of contracts, several of the models considered ignore mortality issues and hence consider only contracts with a known payout date. The insured is henceforth called *the customer*, and the insurer is most often called *the company*.

Consider a person of age x at the time of entering into a contract with the company. Let T_x denote the life expectancy of the person, that is, the time until death. Assume that the probability distribution for T_x is known, continuous, and has a density function. Let $F_x(\cdot)$ denote the probability distribution and let $f_x(\cdot)$ denote the density function assuming that it exists. Then, the death and survival probability in the usual actuarial notation are given by

$$Pr(T_x < t) = F_x(t) \equiv {}_tq_x \quad \text{and} \quad Pr(T_x \geq t) = 1 - F_x(t) \equiv {}_tp_x, \tag{1.1.1}$$

respectively. Often the probability that the person reaches age $x + n$ and dies within the following year is needed.[3] This probability is given by

$$Pr(n < T_x \leq n + 1) = {}_{n|1}q_x. \tag{1.1.2}$$

A hazard rate process for the time of death can be defined. This process is typically called the *Force of Mortality*. The date t force of mortality is defined by

$$\mu_{x+t} \equiv \frac{f_x(t)}{1 - F_x(t)} = -\frac{\partial}{\partial t} \ln({}_tp_x).$$

[3]The articles by Brennan and Schwartz (1976), Bacinello and Ortu (1993a), Bacinello and Ortu (1993b), and Bacinello and Ortu (1994) use this formulation of the death probability.

$\mu_{x+t}dt$ is interpreted as the probability of death taking place in $[t, t+dt]$, given that the person is x years old at date 0.

Often the so-called Makeham formula is used for parametrization of the force of mortality. Makeham's formula says that $\mu_{x+t} = A + Bc^{x+t}$ where $A > 0, B$, and c are constants, see Gerber (1997). In this case, the probability that the person reaches age $x+n$ can be written as ${}_np_x = e^{-\int_0^n \mu_{x+s}ds} = e^{-An-\frac{B}{\ln c}(c^{x+n}-c^x)}$.

1.2 Equity-linked and participating polices

In the survey in this dissertation, part II, the benefits or payout of the contracts are in some way connected to the financial market. In particular, two overall classes of payout are considered. The two classes of contracts will be called *equity-linked* and *participating*. Several variations within each class are analyzed. There seems to be some confusion with regard to which of the names should be used for different kinds of contracts. In many cases a contract that is said to be of the participating type is actually included in the description of a contract of the equity-linked type, see Report (1998). The main difference between the two classes of contracts lies in the distinction between contracts linking payout to a fixed reference and contracts liking payout to the company's own investment portfolio.

An equity-linked or unit-linked contract in its simplest form gives the customer (buyer of the contract) the value of a certain reference portfolio at the payout date. In a lot of countries the contracts are offered with a guarantee. That is, the customer is guaranteed a certain minimum payout. The survey in the present dissertation considers only contracts that are offered with a guarantee. In a participating policy the payout to the customer is tied to the return on the issuing company's *own* investment portfolio. It is often difficult to distinguish between the two cases since when modeling participating polices, an assumption of the company's investment portfolio being fixed is typically made, and this turns the policy into an equity-linked one. Most of the models grouped within the so-called participating class of polices in the survey can therefore be viewed as specific forms of equity-linked policies. In general, the contracts placed in the class of participating policies in the survey provide the customer with a guaranteed minimum payout and possibly some of the surplus that might be generated on the contract. This surplus is known as *bonus* once it is distributed to the customer. When the company's investment portfolio is modeled as a fixed reference portfolio, the main distinction between *equity-linked* and *participating* policies is the way bonus is distributed. The contracts grouped under participating policies typically have a more advanced rule for distributing bonus than the equity-linked contracts.

The guarantee element in the insurance contracts can be thought of as arising from the traditional actuarial practice for calculating premia where the value of contract benefits and hence premiums are based on assumptions of the future level of mortality rates, interest rates, and costs of handling the contract. These assumptions are set so that the company is on "the safe side" with respect to being able to honor the contract,[4] i.e. so that the reserve is large enough. Valuing the contract using more realistic assumptions, i.e. values of mortality, etc.[5] usually yields a surplus since the assumptions used to begin with were "on the safe side". The so-called contribution principle states that this surplus must be given back to the customers (and equity holders of the company) according to the way they have contributed to it. In case that surplus is negative, the insurance company has to cover the deficit and the customer receives no bonus. It is customary that the terms of an insurance contract cannot be altered during the life of the contract and therefore the contract actually provides the customer with guaranteed benefits based on the initial assumptions (the first order basis). In some cases, the guaranteed benefits are given as an average guaranteed rate of return on the customer's stake, in others the guaranteed benefits are given through a guarantee on each year's return.[6]

The primary difference between a customer and the life insurance company is that the customer himself cannot hedge mortality risk, whereas the insurance company is assumed to be able to. The company works under the assumption that it can apply the Law of Large Numbers and diversify mortality risk away by pooling together many customers with similar characteristics such as age and gender.[7]

An example

A small example of how the guarantee can arise from standard actuarial practice is provided.[8] Consider a single premium pure endowment equity-linked contract with maturity T. The customer pays D units of account initially for the contract. The company typically invests this amount in a reference portfolio. The company uses the *first* order basis assumptions for the rate of return on the reference portfolio and for the force of mortality to determine the reserves given the premium D. Let r_g and $\hat{\mu}_{x+t}$ be the first order rate of return and force of mortality, respectively. The company expects the reference portfolio to grow at a rate equal to r_g, and therefore

[4] The set of assumptions are also known as the *first order* or *technical basis*.

[5] That is, using what is also known as the *second order basis*.

[6] The reader who knows Danish is referred to Jacobsen (2002) for a discussion of the guarantee element from a legal point of view.

[7] In principle infinitely many customers are needed.

[8] A lot of insurance companies nowadays have become increasingly aware of the financial risk present when offering a policy with a minimum guarantee. They would therefore consider applying financial methods when determining the premium. However, this was not the case originally, which is what has caused a lot of problems for several companies since it seems that they have not collected any premium for the option that the guarantee gives rises to, c.f. Grosen and Jørgensen (2000b). The example is meant as a basic example of how the guarantee can arise from the traditional actuarial way of determining premia, reserves, etc.

the customer can expect at least a benefit of $e^{r_g}D$ at date T (if alive). According to the first order basis the company's reserves toward the customer's contract at date T (just before payout is made) should be ${}_T\hat{p}_x e^{r_g T} D$, where ${}_T\hat{p}_x = e^{-\int_0^T \hat{\mu}_{x+s} ds}$. The return on the reference portfolio is typically higher than r_g, which yields some bonus to the customer. Moreover, bonus might arise from the assumptions on mortality. In particular, more customers than expected could die before maturity. Let $S(t)$ denote the date t value of one unit of the reference portfolio that the customer's premium is invested in. Hence, the customer's payout is based on $\frac{D}{S(0)}$ units of the reference portfolio. The reserves according to the *second* order basis are then given by ${}_T p_x \frac{D}{S(0)} S(T)$, where ${}_T p_x = e^{-\int_0^T \mu_{x+s} ds}$ and μ_{x+s} is the second order force of mortality. If the second order reserves are larger than the first order reserves the customer receives the second order reserves, otherwise the customer gets the first order reserves. The difference between the second and first order reserves (if positive) is bonus to the customer. It should now be clear that the customer is granted an option by the company since, given that he is alive at date T, he receives $\max(\frac{D}{S(0)} S(T), e^{r_g T} D) = e^{r_g T} D + \max(\frac{D}{S(0)} S(T), 0)$. Thus he receives a guaranteed amount plus a call option on his part of a reference portfolio.

Most often, the bonus arising from differences in first and second order forces of mortality is very small compared to the bonus arising from the assumptions concerning the rates of return. In the following it is therefore assumed that the first and second order forces of mortality are equal. i.e. ${}_T\hat{p}_x = {}_T p_x$.[9]

Assumption 1.2.1. The first and second order forces of mortality are equal.

This assumption simplifies the valuation procedure since when bonus only arises from differences in first and second order rates of return, one can simply use the probability distribution ${}_t p_x$ $(F_x(t))$ from (1.1.1).

1.2.1 Valuation of a deterministic benefit

In this section financial theory is applied in the calculation of the single premium of either a pure endowment or term insurance contract with maturity T. Let $Y_\tau = Y$, $\tau \leq T$ be the deterministic benefit or payout to the customer at the date of the insurance payout event. That is, τ is either the time of death or maturity depending on the insurance type. For a pure endowment, payout occurs at maturity if the customer is alive. Hence, there is a certain positive probability that payout occurs at date T, and zero probability that it occurs prior to T. In the case of a term insurance there are

[9] Milevsky and Promislow (2001) use the framework of Duffie and Singleton (1999) to allow for a stochastic force of mortality. Duffie and Singleton (1999) value defaultable corporate bonds based on a hazard rate model for the default process. The valuation of pure endowments is analogous to the valuation of defaultable zero-coupon bonds with the force of mortality as the default process and a recovery of zero since there is no payout in the event of death (i.e. default). The main observation made by Milevsky and Promislow (2001) is that the stochastic mortality can be hedged.

positive probabilities attached to payout Y occurring at a date τ between initiation and maturity T. Moreover, recall that the time of death for a single customer is stochastic, whereas the company more or less knows how a large pool of customers die assuming the customers have independent and identical distributed times of death and given assumption 1.2.1.[10]

Assume that the continuously compounded interest rate is constant and equal to r. Let Q denote the equivalent martingale measure or risk neutral probability measure known from standard financial economics. Assume that Q is unique. Moreover, assume that financial risks and mortality risk are independent. This assumption is typically made and it seems reasonable that a person's time of death does not, for instance, depend on a stock market index.[11]

If payment for the contract is made in the form of a single premium, then this premium must equal the market value of the payout. Let $V_t(\cdot)$ denote the market value operator,[12] then the single premiums are given by (1.2.1) and (1.2.2).

Pure endowment

$$ {}_Tp_x V_0(Y_T) = {}_Tp_x E^Q[e^{-rT}Y] = {}_Tp_x e^{-rT} Y \tag{1.2.1} $$

Term insurance

$$ \int_0^T V_0(Y_t) f_x(t) dt = \int_0^T E^Q[e^{-rt}Y_t] f_x(t) dt = \int_0^T e^{-rt} Y f_x(t) dt \tag{1.2.2} $$

The single premium for an endowment insurance is equal to the sum of (1.2.1) and (1.2.2). For the term insurance contract it is used that the size of the payout is independent if the payout date, i.e. $Y_t = Y$ for all $t \leq T$.

Let Q^t, $t \in [0, T]$ denote the t-forward probability measure.[13] The assumptions that financial risks and mortality risk are independent makes it possible to handle stochastic interest rates in the usual fashion, that is, discount payout with zero coupon bonds.

[10] The company does not exactly know how the group of customers die since the Law of Large Numbers argument demands infinitely many customers. However, it is assumed that the group of customers is large enough for the Law of Large Numbers to apply.

[11] Disregarding very tragic and luckily very rare events such as the terror actions on September 11, 2001.

[12] The date t market value of a random payout Z, is given by $V_t(Z) = E_t^Q[e^{-\int_t^T r(s)} Z]$, where $E_t^Q[\cdot]$ denotes the date t conditional expectation under Q, and $r(s)$ is the instantaneous risk free interest rate at date s.

[13] The t-forward measure was introduced by Jamshidian (1987) and Geman (1989). See also Geman, El Karoui, and Rochet (1995) for the change of numeraire technique applied to the pricing of a number of different options.

With a stochastic interest rate, the single premia are given by

$$ {}_Tp_x V_0(Y) = {}_Tp_x P(0,T) E^{Q^T}[Y] = {}_Tp_x P(0,T) Y \quad \text{for a pure endowment,} \tag{1.2.3} $$

$$ \int_0^T V_0(Y_t) f_x(t) dt = \int_0^T P(0,t) E^{Q^t}[Y] = \int_0^T P(0,t) Y f_x(t) dt \quad \text{for a term insurance,} \tag{1.2.4} $$

where $P(0,s)$ denotes the date 0 zero coupon bond price of a bond expiring at date s. Note, that the assumption of a deterministic benefit is not used until the last equalities of (1.2.1)-(1.2.4).

Remark 1.2.2. In the case where the customer pays a periodic premium, the present value of the premiums (both with respect to mortality and financial risk) must equal the values above, i.e. in (1.2.1)-(1.2.4) depending on the contract type. That is, the present value of the benefit must equal the present value of the premium payments just as in the single premium case. The expression for the periodic premium for the different types of contracts depends on how the premium is paid. Typically, it is paid once a year in advance. □

Remark 1.2.3. Brennan and Schwartz (1976), Bacinello and Ortu (1994) and others determined the single premium using the formulation of the probability of death from (1.1.2). That is, they work with a discrete probability distribution. Only the year the customer dies matters and not the exact time. It is assumed that the payout from the company is made in the end of the year in which the customer dies. With this formulation the single premium in (1.2.2) for the term insurance when interest rates are constant is replaced by, $\sum_{t=1}^{T} {}_{t-1|1}q_x e^{-rt} Y$. □

1.2.2 Valuation of a stochastic benefit

The same principles as above apply in the case where the benefit or payout is stochastic as long as the uncertainty arises from the uncertainty on the financial market, for instance, payout being a function of a traded stock index. The premiums are given by the second equality signs in (1.2.1)-(1.2.4).

In the survey, i.e. in part II, mortality risk is not taken into account. It should be clear from the above how to incorporate mortality risk under the given assumptions.

1.3 Outline of the dissertation

Part II consists of chapters 2 and 3 which together comprise a brief literature survey. A basic set-up for pricing equity-linked contracts with a guarantee is presented in chapter 2. The model builds primarily on Bacinello and Ortu (1994). Several different variations of the basic model are discussed and should give the reader an overview of some of the work done within this particular area of financial economics.[14] Chapter 3 deals with participating policies as they were defined in section 1.2. The work done in this area is difficult to analyze within a general framework, and therefore two models illustrating different aspects of participating policies are surveyed separately. The main difference between the models is whether the guarantee is an average rate of return guarantee or an annual rate of return guarantee. Moreover, there is a difference in the way bonus is distributed to the customer. Firstly, a model based on Briys and de Varenne (1994) is presented. The focus is on average rate of return guarantees also known as maturity guarantees. Secondly, a model based on Miltersen and Persson (2000) is considered. This serves as an example of how to collect premium for annual rate of return guarantees. A model by Grosen and Jørgensen (2000b) applying a different way of distributing bonus is also discussed.

The work done by the author of this dissertation falls within the participating class of contracts. More specifically, the following contracts are considered:

i) Contracts with a simple bonus distribution, an average rate of return guarantee (binding/non-binding) and an asset portfolio that changes dynamically over time. That is, an extension of the Briys and de Varenne (1994) framework. This is, to the best of the author's knowledge, the first model that considers a participating policy in the strict sense of the word, that is, a model where the company can change the investment portfolio over time. An optimal portfolio choice problem is solved in connection with the problem of setting the terms of such a contract fairly. The work is formalized in the paper: "Portfolio Choice and Fair Pricing in Life Insurance Companies" and constitutes part III of the dissertation.

ii) Contracts with an annual guaranteed rate of return and a relatively advanced bonus distribution. Bonus is distributed throughout the life time of the contract and also as terminal bonus (if positive). The fair contract terms are determined. Moreover, an examination of what happens when heterogeneous customers share a bonus reserve is provided. An investigation of these issues does not seem to have been presented anywhere before. The work is found in part IV which is a version of the paper: "Minimum rate of return Guarantees: The Danish Case".

iii) Contracts that provide a minimum rate of return guarantee and possibly some

[14] See for example Steffensen (2001) and the reference therein for work done with more focus on actuarial aspects.

bonus. A promise of a certain rate of return (equal to or above the minimum rate of return guarantee) is given to the customers. This rate of return is only a promise and is not guaranteed. Competition between companies is modeled in a one period framework and serves as the determinant of the level of the promised rate of return. Portfolio choice is also considered though only a static framework is used. The impact of competition between companies does not seem to have been analyzed previously. The work is presented in part V which contains a version of the paper: "Competition among Life Insurance Companies: The driving force of high policy rates?".

All the papers considered in this dissertation except for the third paper by the author deal with fair contracts. It is assumed that the pension and life insurance market is competitive, and that the terms or price of a contract is set at the competitive price. This is not the case in the third paper since the life insurance market is no longer assumed to be competitive. Instead the life insurance market is characterized by a Cournot model of duopoly.[15] The third paper of the dissertation is therefore quite different from the two others. The paper tries to model competition between two life insurance companies. Competition is used as a possible explanation for the high policy rates, i.e. total annual rates of return, that are typically offered to customers in a life insurance company. That is, to the holders of interest rate guarantees.

The articles by the author of the dissertation are included in article form in parts III, IV, and V, respectively. A few extensions to Hansen and Miltersen (1999) are considered in an appendix to IV.

[15]See for instance Gibbons (1992) for an introduction to the Cournot model of duopoly.

Part II

Survey

Chapter 2

Equity-linked policies with a guaranteed benefit

In this section a framework for pricing life insurance contracts that have a payout which is linked to a certain reference portfolio and are equipped with a guaranteed minimum benefit is presented. The guarantee ensures the holder of the contract a minimum benefit at the time payout is due. The type of contract is often referred to as an *equity-linked* contract. The payout date depends on the type of insurance—typically pure endowment, term insurance, or endowment insurance are considered. Mortality risk is ignored in the following, but can be implemented through the approach from section 1.2.1. That is, using that mortality risk and financial risk are assumed to be independent, and that the issuing company is assumed to be able to diversify mortality away by pooling together a large group of similar customers. In other words that a Law of Large Numbers argument can be applied as it is usual when calculating premiums in life insurance.

The payment for this particular type of contract is typically composed of two parts. A single (or periodic) deposit which accumulates at the same rate as the reference portfolio,[1] and a single (or periodic) premium for the guarantee. One can think of the first part as the premium which the company has typically charged for such a contract, i.e. a premium calculated using only traditional actuarial principles. Historically, it seems to be the case that the companies have neglected the value of the option that arises from issuing the contract with a guarantee. The second part of the premium is the value of this option found by using a standard no-arbitrage argument known from financial economics.[2] Later, a different way of collecting payments for a contract is

[1] That is, it is *linked* to the reference portfolio.

[2] At the date when contracts were originally issued it was typically the case that interest rates were so high that the guarantees, i.e. the options, were so far out-of-the money that they were valueless for all practical purposes. However, as interest rates and the guaranteed rate of return approach each other, the options become valuable and cannot be ignored when valuing contracts. See for example Grosen and Jørgensen (2000b) for a brief discussion of the subject.

	Single premium	**Periodic premium**
Constant interest rates	Brennan and Schwartz (1976) Boyle and Schwartz (1977) Brennan and Schwartz (1979) Persson and Aase (1994) Bacinello and Ortu (1993a) Ekern and Persson (1996) Grosen and Jørgensen (1997) Hipp (1996)	Brennan and Schwartz (1976) Boyle and Schwartz (1977) Brennan and Schwartz (1979) Persson and Aase (1994) Bacinello and Ortu (1993a)
Stochastic interest rates	Bacinello and Ortu (1994) Bacinello and Ortu (1993b) Persson and Aase (1997) Miltersen and Persson (1999) Lindset (2001)	Bacinello and Ortu (1994) Nielsen and Sandmann (1995)

Table 2.1: Papers discussed in chapter 2.

discussed. This alternative way of pricing is concerned with setting terms of a contract in such a way that the contract is what if often known as *fair*. That is, the company receives a payment stream for a contract that exactly covers the cost of issuing the contract, i.e. the terms of the contract are set such that there is zero-profit. This is not significantly different from the "up-front" premium approach, which is used in this chapter, since the up-front premium is determined such that the total premium for the contract equals the value of the payout to the customer.

The papers briefly discussed in this chapter analyze contracts offered with some form of guaranteed benefit. Payment for the contract is in most cases determined in the form of a single up-front premium, however, the case of a periodic premium is also considered. Some of the models operate with a constant interest rates environment while others allow for stochastic interest rates. The papers can roughly be categorized according to table 2.1. The model presented below is based primarily on Bacinello and Ortu (1994).

2.1 The model

Assumptions:

1. The life insurance market and the financial market are competitive.

2. The financial market is frictionless, complete, and free of arbitrage.

3. Several risky assets are traded on the financial market. In particular a certain

reference portfolio to which returns on the insurance contracts are linked.[3]

4. A bank account and zero-coupon bonds of all maturities are traded in the economy.

Assumption 2 implies that there exists a unique equivalent martingale probability measure under which discounted prices of traded asset are martingales.[4] Let Q denote this martingale probability measure also known as the *risk neutral probability measure.*

Notation:

T**:** Maturity date of the contract.

$S(t)$**:** Date t value of one unit of the reference portfolio.

$m(t)$**:** Number of units, at date t, of the reference portfolio in the customer's portfolio or account by the company.

$X(t)$**:** Date t value of the customer's portfolio or account, i.e. $X(t) = m(t)S(t)$.

$G(t)$**:** The minimum guaranteed amount available to the customer at the payout date, given that payout occurs at date t.[5]

$r(t)$**:** Risk free short term interest rate at date t. Continuously compounded, possibly stochastic.

$B(t)$**:** Date t value of one unit invested in the bank account at date 0. Thus,

$$B(t) = e^{\int_0^t r(s)ds}.$$

$P(t,T)$**:** Date t value of a zero coupon bond expiring at date T.

$Y(t)$**:** Payout to the customer at a known payout date t.[6]

D**:** Amount deposited initially by the customer in the single premium case and deemed to be invested in the reference portfolio.

[3] For a discussion of the existence of insurance companies that act mainly as financial intermediaries see Brennan (1993).

[4] Even though the martingale methodology is not used in papers by Brennan and Schwartz (1976), Boyle and Schwartz (1977), and Brennan and Schwartz (1979) this way of pricing is used in the present set-up. Here, one must remember that the martingale methodology was not known at time the papers were written since it was not introduced until Harrison and Kreps (1979) and Harrison and Pliska (1981). To the best of the author's knowledge, Delbaen (1986) was the first to apply the martingale pricing theory to equity-linked contracts.

[5] In the single premium case the guaranteed amount could be given by a guaranteed rate of return on the deposit made by the customer initially. This the case in for example Persson and Aase (1997) and Grosen and Jørgensen (1997).

[6] In this survey mortality risk is ignored and the payout date is therefore fixed. If mortality risk is considered, the payout date is either the maturity date of the contract or the time of death of the customer as discussed earlier.

d: Amount deposited each period in the periodic premium case and deemed to be invested in the reference portfolio.

$c(t, T, S, K)$: Date t value of a European call option on the reference portfolio with an exercise price of K and maturity date T.

$\pi(t, T, S, K)$: Date t value of a European put option on the reference portfolio with an exercise price of K and maturity date T.

2.1.1 Asset dynamics

The dynamics of one unit of the reference portfolio is assumed to be of the following form under the equivalent martingale measure, Q,

$$dS(t) = S(t)[r(t)dt + \sigma dW(t)], \quad S(0) = s, \tag{2.1.1}$$

where σ is a constant.[7]

The bank account evolves according to

$$dB(t) = r(t)B(t)dt, \tag{2.1.2}$$

where $r(t)$ might be stochastic.

The date t value of a zero coupon bond maturing at date T is given by

$$P(t, T) = E_t^Q[e^{-\int_t^T r(s)ds}], \tag{2.1.3}$$

where $E_t^Q[\cdot]$ denotes the conditional expectation under Q, at date t. With deterministic interest rates, (2.1.3) is equivalent to $P(t, T) = \frac{B(t)}{B(T)}$.

Let $\sigma_P(t, T)$ denote the instantaneous volatility at date t of a zero coupon bond expiring at date T. Assume that $\sigma_P(t, T)$ is a deterministic function of t and T. Then the dynamics of the zero coupon bond under Q is given by

$$dP(t, T) = P(t, T)[r(t)dt + \sigma_P(t, T)dZ(t)],$$

where $Z(\cdot)$ is a standard Brownian motion under Q, which is correlated with $W(\cdot)$ with correlations coefficient ρ, $\rho \in [0, 1]$.

2.1.2 Payout to the customer

Assume that the payout is due at a known date, τ. The payout in its general form is the maximum of the date τ value of customer's account and the guaranteed amount at

[7] All of the derivations that follow also hold when the volatility rate of the risky asset is a deterministic function of time.

date τ.

$$Y(\tau) = \max\bigl(X(\tau), G(\tau)\bigr), \tag{2.1.4}$$

which can be rewritten as,

$$Y(\tau) = G(\tau) + \max\bigl(X(\tau)) - G(\tau), 0\bigr) \tag{2.1.5}$$

$$Y(\tau) = X(\tau) + \max\bigl(G(\tau) - X(\tau), 0\bigr). \tag{2.1.6}$$

So the payout is given as either *i*) the guaranteed amount plus the payout from a call option on the customer's account[8] with an exercise price equal to the guaranteed amount and maturity date τ, or as *ii*) the value of the customer's account plus the payout from a put option on the customer's account with the same exercise price and maturity date as the call option.

If the company actually invests the deposits (single premium or periodic) it receives from the customer in the reference portfolio, it will be sure to have $X(\tau)$ at the date τ. Hence, the value of the guarantee is equal to the value of a put option on the customer's account, see (2.1.6). The valuation of this particular type of contract therefore basically amounts to valuing an option on a reference portfolio. In the above set-up this is a fairly simple task.

Remark 2.1.1. One thing to note is that the no-arbitrage evaluation used builds on the assumption that the payout from any contingent claim including the insurance contract above can be replicated. It follows that if the issuing company receives a payment for the contract equal to the market value of the payout in (2.1.4) and hedges the option involved (basically the hedging strategy known from Black-Scholes should be used), it will have zero profits. □

2.1.3 Valuation in the single premium case

An amount equal to D is deposited by the customer initially. D is the starting level of the customer's account, i.e. $D = X(0)$. The amount is invested in the reference portfolio, which means that the customer's portfolio or account evolves as $m = \frac{D}{S(0)}$ units of the reference portfolio.[9] From (2.1.5) and (2.1.6) it follows that the date 0

[8] The deposits made by the customer are linked to the reference portfolio and therefore follow the same dynamics as the reference portfolio. This does not mean that the deposits actually have to be placed in the reference portfolio, but merely that the return on the customer's account is determined by the return on one unit of the reference portfolio.

[9] Note here that the value of the customer's account is a market value as opposed to other models considered later in the survey. In particular, a model that works with the book value of the customer's account is presented in chapter 3.

value of a single premium contract is given by,

$$\begin{aligned} V_0\big(Y(\tau)\big) &= V_0\big(G(\tau)\big) + V_0\big(\max(mS(\tau) - G(\tau), 0)\big) \\ &= G(\tau)P(0,\tau) + mc(0,\tau,S,G(\tau)/m) \qquad (2.1.7) \end{aligned}$$

or

$$\begin{aligned} V_0\big(Y(\tau)\big) &= V_0\big(mS(\tau)\big) + V_0\big(\max(G(\tau) - mS(\tau), 0)\big) \\ &= mS(0) + m\pi(0,\tau,S,G(\tau)/m) \qquad (2.1.8) \\ &= D + m\pi(0,\tau,S,G(\tau)/m), \qquad (2.1.9) \end{aligned}$$

where $V_t(\cdot)$ is the date t market value operator. In (2.1.8), the property that discounted prices are Q-martingales is used, i.e. that $V_u(S(t)) = S(u)$ for $u \leq t$.

The discounted price process for one unit of the reference portfolio, $\big(\frac{S(t)}{P(t,\tau)}\big)$, is a martingale under the τ-forward measure, thus

$$d\Big(\frac{S(t)}{P(t,\tau)}\Big) = \phi(t,\tau)\frac{S(t)}{P(t,\tau)}dW^{Q^\tau},$$

where W^{Q^τ} is a standard Brownian motion under Q^τ and $\phi(\cdot,\cdot)$ is given by

$$\phi^2(t,\tau) = \sigma^2 + 2\rho\sigma\sigma_P(t,\tau) + \sigma_P^2(t,\tau).$$

Therefore the discounted price process is log-normally distributed according to

$$\ln\Big(\frac{S(t)}{P(t,\tau)}\Big) \sim N\Big(\ln\frac{S(u)}{P(u,\tau)} - \frac{1}{2}\int_u^t \phi^2(s,\tau)ds, \int_u^t \phi^2(s,\tau)ds\Big), \quad \forall \quad u \leq t.$$

The call option value from (2.1.7) is therefore equal to[10]

$$\begin{aligned} c(t,\tau,S,G(\tau)/m) &= P(t,\tau)E^{Q^\tau}\big[\max(S(\tau) - G(\tau)/m, 0)\big] \qquad (2.1.10) \\ &= S(t)N(d_1(t,\tau)) - \frac{G(\tau)}{m}P(t,\tau)N(d_2(t,\tau)) \qquad (2.1.11) \end{aligned}$$

where

$$\begin{aligned} d_1(t,\tau) &= \frac{\ln\big(\frac{S(t)m}{P(t,\tau)G(\tau)}\big) + \frac{1}{2}\int_t^\tau \phi^2(s,\tau)ds}{\int_t^\tau \phi^2(s,\tau)ds}, \qquad (2.1.12) \\ d_2(t,\tau) &= d_1(t,\tau) - \int_t^\tau \phi^2(s,\tau)ds, \qquad (2.1.13) \end{aligned}$$

[10]Follows from a change of numeraire argument, see for example Geman, El Karoui, and Rochet (1995), and the properties of a log-normally distributed variable.

and $N(x)$ denotes the standard normal distribution evaluated in x. The put option value follows from the put-call parity, i.e.

$$\pi(t,\tau,S,G(\tau)/m) = \frac{G(\tau)}{m}P(t,\tau)N(-d_2(t,\tau)) - S(t)N(-d_1(t,\tau)). \tag{2.1.14}$$

m times this put option value can be interpreted as the up-front premium that must be paid for the guarantee. The total initial single premium is given by

$$\text{Total single premium} = \mathrm{D} + \mathrm{m}\pi(0,\tau,\mathrm{S},\mathrm{G}(\tau)/\mathrm{m}). \tag{2.1.15}$$

where $\pi(\cdot,\cdot,\cdot,\cdot)$ is given by (2.1.14).

Remark 2.1.2. In a standard Black-Scholes-Merton set-up[11] with a constant interest rate equal to r, the formula for the call option in (2.1.10) simplifies into

$$c(t,\tau,S,G(\tau)/m) = S(t)N(d_1(t,\tau)) - \frac{G(\tau)}{m}e^{-r(\tau-t)}N(d_2(t,\tau))$$

with

$$d_1(t,\tau) = \frac{\ln\left(\frac{S(t)m}{G(\tau)}\right) + (r+\frac{1}{2}\sigma^2)(\tau-t)}{\sigma\sqrt{\tau-t}}, \tag{2.1.16}$$

$$d_2(t,\tau) = d_1(t,\tau) - \sigma\sqrt{\tau-t}. \tag{2.1.17}$$

This set-up is used in Brennan and Schwartz (1976), Boyle and Schwartz (1977), and Brennan and Schwartz (1979). The latter paper is an extension of Brennan and Schwartz (1976) to consider the effect of transaction costs. In particular, they investigate different approximations to the Black-Scholes type hedging strategies found in Brennan and Schwartz (1976). All three papers consider endowment policies. Thus, in order to have the correct value of the contract, the value found above must be "weighted" by the probability distribution for the payout date. With the assumption of independence between mortality and financial risks this is fairly easy. □

Remark 2.1.3. Persson and Aase (1994) consider equity-linked pure endowments and term insurance contracts with a guarantee. They determine premiums for the contracts using pricing techniques known from financial theory. Both the single premium case and a case where the customer pays a premium rate are presented. Moreover, the authors derive a partial differential equation (PDE) that the premium reserve for the contract must follow and relate this PDE to the well-known Thiele differential equation[12] heavily

[11] Black and Scholes (1973) and Merton (1973b).

[12] See Gerber (1997) for an introduction to the application of the Thiele differential equation in life insurance.

used in actuarial theory. Finally, hedging strategies that the issuing company can use to hedge financial risks are also discussed. □

Models with constant interest rates

For the case with constant interest rates various extensions to the model presented above have been made. For example, Bacinello and Ortu (1993a) consider single and periodic premium endowment contracts with guaranteed benefits that are functions of the total premium paid for the contracts—so-called endogenous guarantees.[13] With a known payout date τ, the total premium for a single premium contract, i.e. the date 0 value of the contract, U, follows from (2.1.9):

$$U = D + \pi(0, \tau, S, G(\tau, U)). \tag{2.1.18}$$

The dependence of the guaranteed benefit on the premium is illustrated by the additional argument in $G(\cdot, \cdot)$. For the periodic premium case an equivalent expression arises. Under the assumption of constant interest rates, the authors put forward sufficient conditions for existence of a solution, U, to the fix point problem[14] (2.1.18). In particular, they analyze the case where the guaranteed amount is given by $G(\tau, U) = Ue^{r_g \tau}$, that is, where the customer is guaranteed an average rate of return of r_g on the total payments made for the contract. They also look at the case where the guaranteed amount equals the actuarial price of an endowment paying U units of account at the payout date.

Ekern and Persson (1996) calculate single premiums for various types of equity-linked contracts that are extensions of the one considered in section 2.1.2. In particular, they consider pure equity-linked contracts, equity-linked with a guarantee, equity-linked with a cap on benefits, and equity-linked with both a guarantee and a cap. For each of the different contract types, they consider cases with one or two reference portfolios or funds.[15] Finally, they also consider the case where "the link" is determined at a certain date prior to maturity. An example is a contract where at date t prior to the payout date[16], given that the customer is alive, the customer chooses whether to receive the value of the reference fund or the guaranteed amount at the payout date.

A completely different issue is the subject of Grosen and Jørgensen (1997). In this paper the issue of early surrender of the contract is analyzed. Early surrender is the situation where the customer wishes to terminate or "buy back" the insurance contract

[13] Nielsen and Sandmann (1995) extend the analysis of the periodic premium case. See Remark 2.1.5.

[14] It can be shown that the function on the right-hand side of (2.1.18) is a contraction and therefore by Banach's Fixed Point Theorem there exists a unique fix point.

[15] If S is the value of the reference in the one fund case, then this is replaced by $\max(S^1, S^2)$ in the two fund case, where S^1, S^2 denote the values of reference fund one and two, respectively.

[16] The payout date is either determined by the time of death or the maturity of the contract.

before the maturity date of the contract. They work in a setting as above with a constant interest rate and where mortality risk is completely ignored.[17] That is, the contract has a fixed maturity date, T. The possible cost that the customer most often must pay in order to surrender is disregarded.[18] The guaranteed amount at any date t prior to the maturity date is given by initial deposit, D, accumulated at a guaranteed rate of return, r_g, that is, $G(t) = De^{r_g t}$. Assuming that the customer receives the amount of his account at the date he surrenders, the value of a contract allowing for early surrender can be determined by valuing American type of options instead of European type contracts as above. The difference between the value of the American and the European type of contract is then the value of this so-called *surrender option*, which the customer has and should pay for. As would be expected, the value of the surrender option can be quite substantial. In practice of course, the fee that must be paid in the case of early termination provides an incentive not to exercise early, and the value of the surrender option can be much lower.

Models with stochastic interest rates

The model presented above is based on Bacinello and Ortu (1994) who extend the analysis of Brennan and Schwartz (1976) to include stochastic interest rates in the form of a Vasicek term structure of interest rates. A Vasicek term structure of interest rates is a one-factor term structure model with the short term interest rate as the explaining factor. In the model the short interest rate follows an Ornstein-Uhlenbeck process of the form[19]

$$dr(t) = \kappa(\theta - r(t))dt + \sigma_r dZ^Q(t), \quad \text{under} \quad Q,$$

where θ, κ, and σ_r are constants. This implies that the volatility of the zero coupon bond expiring at date τ is given by

$$\sigma_P(t, \tau) = -\frac{\sigma_r}{\kappa}(1 - e^{-\kappa(\tau - t)}).$$

As in Brennan and Schwartz (1976), Bacinello and Ortu (1994) consider both the case of a single premium contract and a contract where premiums are paid periodically. In the periodic premium case they use the martingale pricing technology as opposed to the

[17] Note that mortality risk cannot be implemented in the usual fashion in this model. Complications arise because the exercise or surrender strategy is determined by the customer, who does not know when he is going to die and cannot apply the law of large numbers as is possible for the insurance company. Therefore the hedging arguments underlying the no-arbitrage valuation do not work in the usual sense.

[18] Typically, a certain percentage is deducted from the customer's account when the customer surrenders early.

[19] See Vasicek (1977).

PDE-approach. Their work is therefore an extension of Delbaen (1986) to stochastic interest rates.

In another paper Bacinello and Ortu (1993b) analyze single premium endowment contracts using two different reference portfolios. First, the case of a reference portfolio consisting purely of equities is considered. More specifically, the asset dynamics is as in (2.1.1). Secondly, a case where the reference portfolio consists only of fixed-income securities is analyzed. Stochastic interest rates are modeled using Vasicek (1977). However, stochastic interest rates in the form of Cox, Ingersoll, and Ross (1985) are also considered in connection with the fixed-income reference portfolio. In all cases closed form solutions are available since only single premium contracts are valuated. The case with a fixed-income reference portfolio is also considered in Bacinello and Ortu (1996). Persson and Aase (1997) also consider an economy where interest rates are stochastic in the form of a Vasicek model. Only bonds and a bank account, however, are traded. They look at single premium contracts with a payout structure as above, but where the reference portfolio is merely the bank account. Moreover, the guaranteed amount arises from a minimum guaranteed rate of return on the customer's account or portfolio over the life of the contract as mentioned in footnote 5. Furthermore, the case of a periodic guaranteed rate of return is discussed. However, only the two-period case is analyzed.

A general framework allowing for stochastic interest rates in the form of a general Heath-Jarrow-Morton framework[20] is proposed in Miltersen and Persson (1999). More specifically, the authors consider contracts with a fixed payout date that have either the bank account or an equity portfolio as the reference portfolio. They allow for a slightly more general dynamics for the equity reference portfolio than the one in (2.1.1). In particular, the volatility is allowed to be a deterministic function of time as opposed to being constant in the above set-up. The guaranteed amount stems from a guaranteed rate of return on average. Special cases of the Heath-Jarrow-Morton model, namely the Vasicek and the Cox-Ingersoll-Ross term structures of interest rates, are considered. The subject of periodic guarantees is also discussed—again in a two-period setting.

2.1.4 Valuation in the periodic premium case

Premiums are typically paid periodically, mostly once a year, and paid in advance. The customer pays an annual deposit, d, at the beginning of each year. As in the single premium case, one may think of these deposits as being invested in the reference portfolio.[21]

[20] See Heath, Jarrow, and Morton (1992).

[21] Boyle and Schwartz (1977) formulate the problem using a constant rate of payment, d. That is, over the period $[t, T]$ the customer pays $\int_t^T d\, ds = d(T-t)$. The payments are again assumed to be invested in the reference portfolio. The options involved can now be interpreted as options on an asset evolving like the reference portfolio. The difference is that the reference portfolio now pays a negative dividend. The options are valued using standard techniques, and the results are very similar

The present value of the deposits made by the customer over the course of time (still assuming τ is the known payout date) is given by

$$V_0(\text{deposits}) = d\sum_{t=0}^{\tau-1} P(0,t).$$

At the beginning of each year t, $d/S(t)$ units of the reference portfolio are added to the customer's account. Hence, at date τ the customer's account consists of $m(\tau) = \sum_{t=0}^{\tau-1} \frac{d}{S(t)}$ units of the reference portfolio. The value of the customer's portfolio at the payment date is therefore given by, $X(\tau) = d\sum_{t=0}^{\tau-1} \frac{S(\tau)}{S(t)}$. The value of the customer's contract can therefore be written as

$$\begin{aligned} V_0\big(Y(\tau)\big) &= V_0\big(G(\tau)\big) + V_0\Big(\max(d\sum_{t=0}^{\tau-1}\frac{S(\tau)}{S(t)} - G(\tau), 0)\Big) \\ &= G(\tau)P(0,\tau) + E_0^Q\Big[e^{-\int_0^\tau r(s)ds}\max(d\sum_{t=0}^{\tau-1}\frac{S(\tau)}{S(t)} - G(\tau), 0)\Big] \qquad (2.1.19) \end{aligned}$$

and

$$\begin{aligned} V_0\big(Y(\tau)\big) &= V_0\Big(d\sum_{t=0}^{\tau-1}\frac{S(\tau)}{S(t)}\Big) + V_0\Big(G(\tau) - \max(d\sum_{t=0}^{\tau-1}\frac{S(\tau)}{S(t)}, 0)\Big) \\ &= d\sum_{t=0}^{\tau-1} P(0,t) + E_0^Q\Big[e^{-\int_0^\tau r(s)ds}\max(G(\tau) - d\sum_{t=0}^{\tau-1}\frac{S(\tau)}{S(t)}, 0)\Big]. \qquad (2.1.20) \end{aligned}$$

In (2.1.20) it is used that the date 0 value of the customer's reference portfolio must equal the present value of the deposits made to the reference in order to preclude arbitrage. The expectations in (2.1.19) and (2.1.20) can be evaluated by Monte-Carlo simulation. Delbaen (1986) was the first to apply the martingale methodology to the problem. Realizing that with constant interest rates, the reference portfolio follows a geometric Brownian motion: it follows that at date t, $S(t) = S(0)e^{(r-\frac{1}{2}\sigma)t+\sigma W(t)}$ and therefore $\sum_{t=0}^{\tau-1}\frac{S(\tau)}{S(t)} = \sum_{t=0}^{\tau-1} e^{(r-\frac{1}{2}\sigma^2)(\tau-t)+\sigma(W(\tau)-W(t))}$, which is easily simulated. For the case of Vasicek interest rates Bacinello and Ortu (1994) show that the expectation can be rewritten as a function of two variables that are jointly normally distributed. Therefore simulation is also relatively straightforward in this case.

Remark 2.1.4. The case with periodic premiums is also considered in Brennan and Schwartz (1976), Boyle and Schwartz (1977), and Brennan and Schwartz (1979). They, however, calculate the value of the option element by solving the partial differential equation which the price of the option must satisfy. □

to Brennan and Schwartz (1976).

Remark 2.1.5. In extension of the work of Bacinello and Ortu (1993b), Nielsen and Sandmann (1995) consider the case with endogenous guarantees. They use a set-up with stochastic interest rates and consider only the periodic premium case. The guaranteed amount at the payout date is a function of the periodic premium paid by the customer. The difference from Bacinello and Ortu (1993b) is that not all of the premium payment is invested in the reference portfolio, only a certain fraction is invested. Let K denote the periodic premium, then the guaranteed amount at payout date τ is given as $G(\tau, K)$, whereas the payout from the call option is equal to $\max(aK\sum_{t=0}^{\tau-1}\frac{S(\tau)}{S(t)} - G(\tau), 0)$, where $a \in [0, 1]$ is the fraction of the premium that is invested in the reference portfolio. Sufficient conditions for the existence of a unique solution to the pricing problem are found. More specifically, the functional form of the function $G(\cdot, \cdot)$ must be such that $\frac{G(\cdot,K)}{K}$ is bijective. The case with a constant guaranteed amount satisfies the condition, whereas a guaranteed amount equal to only a fraction of the periodic premium never yields a solution to the fair pricing problem. □

Remark 2.1.6. Milevsky and Posner (2001) consider a variety of contracts with guaranteed minimum death benefits. In particular, they consider a set-up similar to the above, however, the way the premium for the option element is collected is different. They deduct a certain percentage from the customer's part of the reference portfolio. That is, the drift of the reference portfolio is adjusted so as to incorporate the premium. The authors compute the percentage fee that makes the contract fair or, in other words, that equates the expected present value of the fees to the expected present value of the option. Mortality risk is incorporated using a simple exponential as well as the so-called *Gompertz* distribution for the time of death.[22] The focus in the paper is on the comparison between the theoretically fair percentage fee or risk charges for the option and available data for the so-called Mortality & Expense risk charge reported by Morningstar Inc. A fairly detailed description of different kinds of contracts offered on the Canadian market can be found in the introduction to the paper. □

2.1.5 Valuation of periodic guarantees

Some equity-linked contracts are issued with guarantees, which instead of guaranteeing a certain sum at the payout date (maturity guarantee type), guarantee a fixed rate of return over specific periods of time. In particular, contracts are often offered with an annual rate of return guarantee. This case is considered in the following. Let r_g denote

[22] That is, using an assumption of $\mu_{x+t} = \textit{constant}$ for all t and $\mu_{x+t} = Bc^{x+t}$, B constant, as the forces of mortality, respectively. See Gerber (1997) page 18. The Gompertz force of mortality can be rewritten as $\mu_{x+t} = \frac{1}{b}e^{\frac{(x+t)-m}{b}}$, with m and b constants related to B and c through $\ln c = \frac{1}{b}$ and $B = e^{-m\ln c}\ln c$. This is the form used by Milevsky and Posner (2001).

the rate of return guarantee on a specific contract. Only single premium contracts are investigated.

Payout to the customer

Consider again a fixed payout date, τ. Each year t, i.e. over the period $[t-1,t)$, the customer is guaranteed a minimum rate of return equal to r_g on his account. The payout at date τ with an initial deposit of D is given by

$$Y(\tau) = D\prod_{t=1}^{\tau}\max(\frac{S(t)}{S(t-1)}, e^{r_g}) \tag{2.1.21}$$

$$\tag{2.1.22}$$

With the assumed dynamics for the reference portfolio, equation (2.1.1), the continuously compounded annual returns on the reference portfolio are i.i.d. normal. More specifically, let δ_t denote the continuously compounded rate of return on the reference portfolio over the period $[t-1,t)$—then (2.1.21) can be rewritten as

$$Y(\tau) = D\prod_{t=1}^{\tau}\max(e^{\delta_t}, e^{r_g}) = D\prod_{t=1}^{\tau}e^{\max(\delta_t, r_g)} \tag{2.1.23}$$

The difference from the earlier payout is that the payout at date τ now depends on how the reference portfolio performed up until this date, that is, payout is now path dependent. The value is still given by the usual expectation:

$$V_0(Y(\tau)) = E_0^Q[e^{-\int_0^\tau r(s)ds}Y(\tau)].$$

The contract can always be evaluated using Monte-Carlo simulation. Closed-form solutions are available. They, however, involve evaluation of a multi-dimensional standard normal distribution. In particular, the dimension equals the number of years the contract lasts when the guarantees are annual.

Remark 2.1.7. Within a Black-Scholes set-up Hipp (1996) analyzes several different payout structures related to equity-linked polices. Examples include an annual minimum rate of return guarantee similar to the case above, but also more complex structures. In relation to hedging, the Delta and Vega of the option elements in the payout structure are calculated.[23] □

Remark 2.1.8. Persson and Aase (1997) consider a two period case where the reference portfolio is the bank account, and interest rates are stochastic in the form of a Vasicek

[23] The Delta and Vega of a portfolio of derivatives are measures of the sensitivity of the position with respect to the underlying asset and the volatility of the underlying asset, respectively. See for example Hull (1997).

term structure of interest rates. Miltersen and Persson (1999) can been seen as an extension of Persson and Aase (1997) as they find valuation formulas for the two-period case in a Heath-Jarrow-Morton framework. They consider both an equity reference (as above) and the case with the bank account as the reference portfolio. Finally, Lindset (2001) extends the results in Miltersen and Persson (1999) to the multi-period case. □

Chapter 3

Participating policies

The analysis of participating policies is relatively new within financial economics. Whereas an investigation of equity-linked policies with a guarantee dates back to Brennan and Schwartz (1976), it seems that the focus on participating policies did not attract much attention until the mid-nineties. Briys and de Varenne (1994) were among the first to turn to the analysis of participating policies. They considered a contract with an average guaranteed rate of return over the life of the contract and with bonus which is distributed according to a very simple rule. The payout from the policy that they investigate is simply a portfolio of options on the company's investment portfolio. Several extensions of the type of model considered in Briys and de Varenne (1994) have since been done. Within a framework similar to Briys and de Varenne interesting aspects can be captured, however, if one is interested in contracts with an annual minimum rate of return guarantee and perhaps a more complex bonus distribution scheme, a different framework is typically called for. This chapter will start with a description of a model similar to Briys and de Varenne and discuss work related to this type of model. These models are referred to as being concerned with *average rate of return guarantees.* The contracts are also known as *maturity guarantees.* The word maturity guarantee reflects that it is the return over the entire holding period which is guaranteed. Later, models that consider annual guarantees with bonus are discussed. In particular, the presentation of this area will be based on Miltersen and Persson (2000), Grosen and Jørgensen (2000b) and Hansen and Miltersen (1999). These models are referred to as dealing with *annual guarantees.*

Common for all the models discussed is that a slightly different way of pricing is used. In the equity-linked case, the single or periodic premium paid by the customer was the sum of a deposit and a payment for guarantee, i.e. the option in the payout. The payment for the option or guarantee is therefore in the form of an up-front premium. Most of the models concerning participating contracts presented here do not use up-front premiums as payment for the option elements. Instead the terms of the contracts are initially set in such a way that the contract is said to be *fair*, where fair means that the market value of payout from a contract must equal the market value of the payments for the contract.[1] Basically, all it means is that the company must have zero expected profit.[2]

3.1 Average rate of return guarantees

The model

Assumptions:

1. The life insurance market and the financial market are competitive.
2. The financial market is frictionless, complete, and free of arbitrage, i.e. a unique equivalent martingale measure Q exists.
3. Several risky assets are traded on the financial market.
4. A bank account and zero-coupon bonds of all maturities are traded in the economy.
5. The life insurance company is formed by equity holders and policy holders, each party invests money initially. The group of equity holders and the group of policy holders are both assumed to be homogeneous, and therefore one can think of the company as being composed of one equity holder and one policy holder (customer).

Notation:

T**:** Maturity date of the contract.[3]

$r(t)$**:** Short risk free interest rate at date t.

$P(t,T)$**:** Date t price of a zero coupon bond expiring at date T.

L_0**:** Deposit made by the customer initially.

E_0**:** Deposit made by the equity holder initially.

A_0**:** Total capital inflow to the company initially, and hence the initial asset value.

[1] In the equity-linked case the contracts are fair given that the total premiums are set equal to the deposit and the up front premium for the option.

[2] Once again it is implicitly assumed that the company implements the correct hedging strategy which follows from option theory.

[3] Briys and de Varenne (1994) interpret T as the date an inspection of the company is made. That is, at date T the values of the assets and liabilities of the company in respect to its contracts is assessed.

α: Fraction of total deposits, i.e. A_0 that the customer contributes with. That is, $L_0 = \alpha A_0$ and $E_0 = (1-\alpha)A_0$.

According to the above, a simplified balance sheet of the company at date 0 is given by

Assets	Liabilities
A_0	$L_0 = \alpha A_0$ $E_0 = (1-\alpha)A_0$
A_0	A_0

3.1.1 Asset dynamics

The company invests the initial deposits in a certain portfolio. The asset value of the company is assumed to evolve according to the following stochastic differential equation under Q,

$$dA(t) = A(t)[r(t)dt + \sigma_A dW(t)] \tag{3.1.1}$$

$$A(0) = A_0, \tag{3.1.2}$$

where σ_A is a constant.

The dynamics of the price of a zero coupon bond with maturity T is given by

$$dP(t,T) = P(t,T)[r(t)dt + \sigma_P(t,T)dZ(t)],$$

where the two standard Brownian motions W and Z are correlated with correlation coefficient $\rho \in [0,1]$. $\sigma_P(t,T)$ is assumed to be a deterministic function of t and T.

3.1.2 Payout to the customer

The customer is "guaranteed"[4] an average minimum rate of return, r_g, on his deposit, i.e. the sum $L_0 e^{r_g T}$ at date T, plus possibly an additional bonus. The customer's bonus is a fraction, $\delta \in (0,1)$, of the possible surplus that might be generated. Surplus arises when the value of the customer's part of the assets,[5] is above the guaranteed amount. It is important to note that the company might default on the claim (guarantee), that is, the funds available in the company at date T might not be large enough to cover the guaranteed amount. In this case the customer receives the funds that are there,

[4]The word *guaranteed* is in quotation marks since in the model the company might actually default on the obligation toward the customer, and hence the customer is not guaranteed to receive the minimum rate of return.

[5]His part of the asset amounts to a fraction, α, of the asset value at any date since this is the share he contributed with initially.

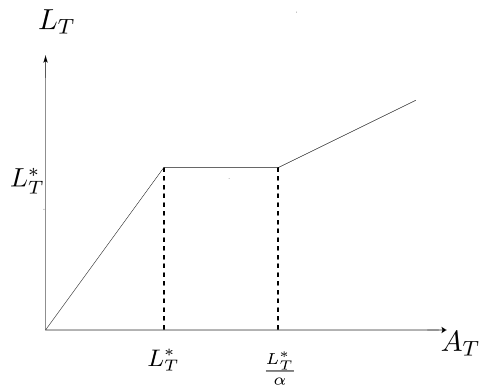

Figure 3.1: Payout to the customer at date T, where $L_T^* = L_0 e^{r_g T}$.

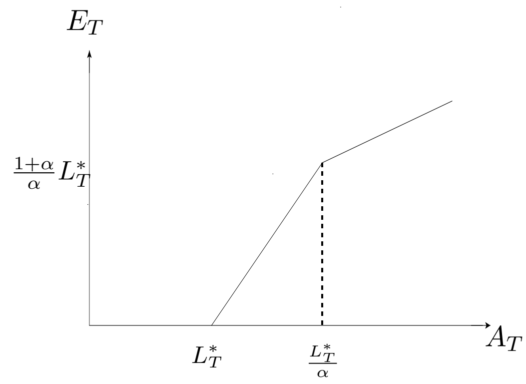

Figure 3.2: Payout to the equity holder at date T, where $L_T^* = L_0 e^{r_g T}$.

and the equity holders walk away with nothing. The payout to the customer at date T is denoted L_T. L_T can be written as a portfolio of bonds and options, in particular

$$\begin{aligned} L_T &= L_0 e^{r_g T} - \max(L_0 e^{r_g T} - A(T), 0) + \delta \max(\alpha A(T) - L_0 e^{r_g T}, 0) \\ &= L_0 e^{r_g T} - \max(L_0 e^{r_g T} - A(T), 0) + \delta\alpha \max(A(T) - \frac{L_0 e^{r_g T}}{\alpha}, 0). \end{aligned} \tag{3.1.3}$$

So the payout to the customer is equivalent to the payout from a portfolio consisting of: $L_0 e^{r_g T}$ zero coupon bonds with maturity T, a short put option on the assets with an exercise price equal to the guaranteed amount and maturity T, and $\delta\alpha$ call options on the assets with an exercise price of $L_0 e^{r_g T}/\alpha$ also with maturity T. The put option illustrates the limited liability of the equity holder. That is, he does not have to make up for the difference between the guaranteed amount and the funds available if it should happen that the funds cannot cover the guaranteed amount. The payout at date T as a function of the asset value is illustrated graphically in figure 3.1.

3.1.3 Payout to the equity holder

The equity holder's payout at date T is as usual the residual claim between the asset value and the liabilities. Let E_T denote the equity holder's payout, then

$$E_T = A(T) - L_T = \max(A(T) - L_0 e^{r_g T}, 0) - \delta\alpha \max(A(T) - \frac{L_0 e^{r_g T}}{\alpha}, 0).$$

It follows that the equity holder's payout is the same as the payout from a long call option on the assets with an exercise price of $L_0 e^{r_g T}$ and maturity T and $\delta\alpha$ short call options on the assets with an exercise price equal to $L_0 e^{r_g T}/\alpha$ and maturity T. The payout as a function of the asset value is depicted in figure 3.2.

Remark 3.1.1. Observe that if the guarantee is binding (in the sense that there is no put option in the payout to the customer) and the company's investment portfolio is known to the customer and fixed throughout the life time of the contract, then for $\delta = 1$ the payout is equal to the payout of an equity-linked contract with a guarantee where α is the number of units of the reference portfolio that the customer "receives" for his deposit, i.e. m in the previous chapter. □

3.1.4 Valuation of the claims

Given the payout structures in (3.1.3) and (3.1.3) and the dynamics for the assets and zero coupon bonds, closed form solutions are available for the value of the claims of the customer and equity holder, respectively. Let L_T^* denote the guaranteed amount, i.e. $L_T^* = L_0 e^{r_g T}$, and let $c(t, T, A, K)$ and $\pi(t, T, A, K)$ denote the date t values of a call and put option on the assets with an exercise price of K and maturity T, respectively. The values of the customer's and equity holder's claims at date $t \leq T$ are given by

Customer:

$$V_t(L_t) = L_T^* P(t,T) - \pi(t,T,A,L_T^*) + \delta\alpha \; c(t,T,A,L_T^*/\alpha). \tag{3.1.4}$$

Equity holder:

$$V_t(E_T) = c(t,T,A,L_T^*) - \delta\alpha \; c(t,T,A,L_T^*/\alpha). \tag{3.1.5}$$

Using the same techniques as in section 2.1.3, closed form solutions for the options involved can be found. The results are:

$$c(t,T,A,L_T^*) = A(t)N(d_1(t,T)) - L_T^* P(t,T)N(d_2(t,T)), \tag{3.1.6}$$

$$c(t,T,A,L_T^*/\alpha) = A(t)N(d_3(t,T)) - \frac{L_T^*}{\alpha} P(t,T)N(d_4(t,T)), \tag{3.1.7}$$

$$\pi(t,T,A,L_T^*) = L_T^* P(t,T)N(-d_2(t,T)) - A(t)N(-d_1(t,T)), \tag{3.1.8}$$

where

$$d_1(t,T) = \frac{\ln\left(\frac{A(t)}{P(t,T)L_T^*}\right) + \frac{1}{2}\int_t^T \phi^2(s,T)ds}{\int_t^T \phi^2(s,T)ds}, \tag{3.1.9}$$

$$d_2(t,T) = d_1(t,T) - \int_t^T \phi^2(s,T)ds, \tag{3.1.10}$$

$$d_3(t,T) = \frac{\ln\left(\frac{\alpha A(t)}{P(t,T)L_T^*}\right) + \frac{1}{2}\int_t^T \phi^2(s,T)ds}{\int_t^T \phi^2(s,T)ds}, \tag{3.1.11}$$

$$d_4(t,T) = d_3(t,T) - \int_t^T \phi^2(s,T)ds, \tag{3.1.12}$$

and

$$\phi^2(t,T) = \sigma_A^2 + 2\rho\sigma_A\sigma_P(t,T) + \sigma_P^2(t,T). \tag{3.1.13}$$

The date t values of the customer's and the equity holder's contracts are hence given by the following expressions:

$$V_t(L_T) = A(t)\big[N(-d_1(t,T)) + \delta\alpha N(d_3(t,T))\big] + L_T^* P(t,T)\big[N(d_2(t,T)) - \delta N(d_4(t,T))\big] \tag{3.1.14}$$

$$V_t(E_T) = A(t)\big[N(d_1(t,T)) - \delta\alpha N(d_3(t,T))\big] - L_T^* P(t,T)\big[N(d_2(t,T)) - \delta N(d_4(t,T))\big]. \tag{3.1.15}$$

3.1.5 Fair contracts

The term fair is used about a contract that in a competitive market generates zero expected profits for the issuing company. One can think of a fair contract as a contract with a premium that equals the no-arbitrage value of the payout the contract gives rise to. Given the description of the payout above, a fair contract in this set-up must satisfy the two following conditions,

Customer:

$$V_0(L_T) = L_0 = \alpha A_0 \tag{3.1.16}$$

Equity holder:

$$V_0(E_T) = E_0 = (1-\alpha)A_0. \tag{3.1.17}$$

The problem is now to determine parameters of the contract, i.e. α, δ, r_g, such that (3.1.16) and (3.1.17) are fulfilled. Note that since the total asset value, $A(T)$, is divided between the claim holder at the maturity date, it follows that if (3.1.16) is satisfied then

(3.1.17) is automatically also satisfied and vice versa. Hence, when searching for fair parameter constellations, it is sufficient to consider either of (3.1.16) and (3.1.17). Since the average minimum rate of return guarantee, r_g, is typically given exogenously, the focus in this presentation will be on determining the fair level of the share of surplus to the customer, that is, the fair level of δ.

Given the asset dynamics in (3.1.1) and a specification for the zero coupon bond volatility, a closed form solution exist for the fair δ. The fair level of δ is found by substituting (3.1.14) into (3.1.16), using that $L_T^* = \alpha A_0 e^{r_g T}$, and solving for δ. This yields

$$\begin{aligned} \alpha A_0 &= A_0\big[N(-d_1(0,T)) + \delta\alpha N(d_3(0,T))\big] + L_T^* P(0,T)\big[N(d_2(0,T)) - \delta N(d_4(0,T))\big] \\ \Leftrightarrow \quad \delta &= \frac{\alpha\big[1 - e^{r_g T} P(0,T) N(d_2(0,T)\big] - N(-d_1(0,T))}{\alpha\big[N(d_3(0,T)) - e^{r_g T} P(0,T) N(d_4(0,T))\big]}. \end{aligned} \tag{3.1.18}$$

Note that if one is interested in finding the fair rate of return guarantee, r_g, given δ, α, etc. numerical methods must be applied since no closed form solution can be found for the fair rate of return guarantee.

Remark 3.1.2. Briys and de Varenne (1994) suggested the above set-up for analyzing participating policies. In their 1994 paper they use a Heath-Jarrow-Morton model for the term structure of interest rates. More specifically, they assume a constant volatility of the instantaneous forward rates, which implies that they use the term structure of interest rates suggested by Ho and Lee (1986). With this model for the bond prices they find the values of the two different claims and investigate the behaviour of the fair conditions. The specific choice of model for the term structure of interest rates yields a zero-coupon bond price volatility equal to $\sigma_P(t,T) = \sigma_f(T-t)$, where σ_f is the constant volatility of the forward rate.[6] The values of the claims of the customer and the equity holder are therefore given by (3.1.14) and (3.1.15), respectively, with $\int_t^T \phi^2(s,T)ds = \sigma^2(T-t) + \rho\sigma\sigma_f(T-t)^2 + \frac{1}{3}\sigma_f^2(T-t)^3$ in (3.1.9)–(3.1.12).

In Briys and de Varenne (1997) the authors use a specification of the forward rate volatility that implies the use of a Vasicek term structure of interest rates. The focus in this paper is on issues related to risk assessment of the claims issued. In particular, the durations of the different claims are calculated and discussed. □

Grosen and Jørgensen (2000a) extend[7] the above framework to the case where the customer and the equity holder only receive the payout, given by (3.1.3) and (3.1.3), at

[6] That is, the dynamics of the instantaneous forward rate with maturity date T is given by $df(t,T) = drift\ term + \sigma_f dZ(t)$.

[7] They assume a constant interest rate so in this sense the analysis is simplified compared to Briys and de Varenne (1994).

maturity if the company has not been closed down by some regulatory authority prior to maturity. If the company is closed down early, the assets are divided between the two claim holders according to a certain scheme known at initiation. The company is closed down at date $\tau < T$ if $A(\tau) \leq \lambda L_0 e^{r_g \tau}$, where λ is a constant. The right-hand side of the inequality is called the regulatory boundary. In the case of a close-down at date τ, the customer receives the following payout at this date:

$$\Theta_L(\tau) = \begin{cases} L_0 e^{r_g \tau} & \text{if} \quad \lambda \geq 1 \\ \lambda L_0 e^{r_g \tau} & \text{if} \quad \lambda < 1 \end{cases}$$

The equity holder receives some payout (the residual claim as usual) in the case of a close-down at date τ only when $\lambda \geq 1$. One can interpret the situation of $\lambda \geq 1$ as the case where the regulatory authority closes down the company if it does not hold a certain "buffer", i.e. has asset value somewhat higher than the amount guaranteed to the customer on the inspection date. The case of $\lambda < 1$ is then a situation where the regulatory authority only closes down the company if its asset value is lower than the guaranteed amount. Given the described set-up, Grosen and Jørgensen are able to find closed form expressions for the market values of the claims and to investigate how to set for instance δ such that the arrangement is fair. One might, however, argue that it would be more appropriate if the regulatory boundary was given by the present value[8] of the guarantee to the customer, $L_0 e^{r_g T} e^{-r(T-\tau)}$ and active intervention by the authorities was demanded as opposed to "passively" closing the company down. An active intervention could be to demand that the company reallocates everything to the risk free asset at date τ if it hits the boundary, $L_0 e^{r_g T} e^{-r(T-\tau)}$, at date τ. This situation is considered in Hansen and Hansen (2000) which is discussed next.

In Hansen and Hansen (2000) the above analysis is extended to the case where the asset value is given through dynamic portfolio optimization. That is, the company can change the asset composition over time. The company can, however, only choose between a risk free security earning, a constant rate of interest, and a risky security (or the mean variance efficient tangency portfolio) with a price evolving as a geometric Brownian motion. The conditions for fair contracts are determined and compared to the case where the company's portfolio is static.[9] In the Briys and de Varenne model the guarantee is not really a guarantee since the company can default on it. In Hansen and Hansen (2000) the case with a binding guarantee is also analyzed, both without and with dynamic portfolio choice. The company can be sure to be able to satisfy the guarantee with certainty if it only makes investment decisions concerning the so-called *free reserves*. This basically amounts to making sure that the asset value is above the

[8] Assuming a constant risk free interest rate of r.

[9] Here static means that the assets evolve as the risky asset, i.e. as a geometric Brownian motion.

present value of the guaranteed amount at any time and if the asset value ever reaches the present value of the guaranteed amount, the company must instantly reallocate all wealth to the risk free security. In this way the company is certain to have the guaranteed amount at the maturity date. The strategies are highly dependent on the objective of the company. The results in Hansen and Hansen (2000) are based on a situation where the company maximizes expected utility for the customer. The paper is included as part III.

Remark 3.1.3. Jensen and Sørensen (2001) consider portfolio choice decisions in a pension fund. In particular, they investigate the welfare loss of a customer who is forced into a contract with an average minimum rate of return. That is, they compare the solutions to the customer's portfolio choice problem without and with the constraint that the rate of return guarantee imposes. They solve the problem with the martingale method[10] in a set-up with stochastic interest rates in the form of a Vasicek model. Moreover, they consider effects of pooling customers with different risk aversions together. □

3.2 Annual guarantees

It is somewhat difficult to set up a general framework for considering participating policies with an annual rate of return guarantee. Instead, a few specific examples are therefore summarized below. As previously, it is assumed that there exists an equivalent martingale measure Q and so on.

The model

Consider a set-up with a life insurance company that has issued a specific type of interest rate guarantee to a customer. The maturity date is fixed and mortality issues are ignored. The customer makes an initial deposit with the company, and in return he receives an account with the company starting at that particular value. At maturity he receives the book value of the account and perhaps some bonus. Whether there is terminal bonus depends on whether the contract is offered without or with a bonus reserve, and if the bonus is positive. In the case where the bonus is negative the customer only receives the book value of his account. The company can collect payment for the bonus option element in different ways. Common for the cases considered is the fact that the contracts must be fair. The focus will, however, be on the case where the company has a specific account where payment for the option element is collected.

[10] See for example Cox and Huang (1989).

Notation:

T**:** Payout date of the contract.

$A(\cdot)$**:** The customer's account.

$B(\cdot)$**:** The bonus account or bonus reserve.

$C(\cdot)$**:** The company's account.

$X(\cdot)$**:** The customer's part of the company's investment portfolio. The investment portfolio is, however, modeled as a fixed reference portfolio.[11]

r**:** The risk free rate of return which is assumed to be constant.

r_g**:** Annual minimum rate of return guarantee. Continuously compounded.

r_P**:** Annual rate of return on the customer's account.

X**:** Initial deposit made by the customer. This is the initial book value of his account. The amount is assumed to be placed in the reference portfolio,[12] i.e. $X(0) = X$.

A simplified balance sheet for the company towards one customer can be represented as

Assets	Liabilities
X	A
	C
	B
X	X

The customer's account starts with an initial value of X while the bonus reserve and the company's account both start with an initial value of zero, i.e. $B(0) = C(0) = 0$.

3.2.1 Asset dynamics

The value of the customer's part of the reference portfolio is modeled as a geometric Brownian motion. Hence, the annual returns under the equivalent martingale measure,

[11] This assumption about the company's investment portfolio is made in order to keep things simple and focus on the liability side of the company's balance sheet. Moreover, linking the payout to the customer to a fixed reference eliminates any incentive the company might have to manipulate the investment portfolio. For instance, manipulation by changing the volatility of the portfolio in such a way as to minimize the payout to the customer.

[12] The deposit does not actually have to be placed in the reference portfolio, however, since the return on the customer's account is linked to this portfolio, it is safe to assume that the deposit is actually placed in the reference.

Q, are *i.i.d.* normal. In particular, let $\delta(t)$ denote the rate of return on the reference portfolio in year t, that is, over the interval $[t-1,t]$, then

$$\delta(t) = r - \frac{1}{2}\sigma^2 + \sigma(W(t) - W(t-1)), \tag{3.2.1}$$

where W is a Brownian motion under Q, and σ is the volatility of the reference portfolio, which is assumed to be constant.

3.2.2 Payout

At the maturity of the contract the customer receives the book value of his account plus the terminal value of the bonus reserve if it is positive. That is, he receives

$$A(T) + B^+(T), \tag{3.2.2}$$

where $B^+ = \max(B, 0)$.
The company receives the book value of the C account at maturity. However, it must cover a possible negative terminal bonus reserve, and hence it receives

$$C(T) - B^-(T), \tag{3.2.3}$$

where $B^- = \max(-B, 0)$.

The way the different accounts, A, B, and C, accumulate over time depends on the specifics of the contract and hence differs from model to model. Some examples are discussed in subsection 3.2.4.

3.2.3 Fair contracts

As in section 3.1.5 the value of the reference portfolio at date T is divided between the two parties, that is, the bookkeeping condition, $X(T) = A(T) + B(T) + C(T)$, is satisfied. Applying the market value operator on both sides yields the following condition, which must be fulfilled initially:

$$V_0\big(X(T)\big) = V_0\big(A(T) + B^+(T) + C(T) - B^-(T)\big), \tag{3.2.4}$$

where $V_0(X(T)) = X(0) = X$. Here it is used that the bonus reserve can be divided into its positive and negative parts, i.e. $B = B^+ - B^-$, that the reference portfolio is a traded asset, and finally that the deposit into the reference is initially equal to the customer's deposit of X.

Also as previously, (3.2.4) is satisfied when the contract is fair and the market value of payout to the customer and company, respectively, equal their initial deposits. For the company this amounts to the condition of zero expected profits, i.e. $V_0(C(T) - B^-(T)) = 0$, and thus fair contracts are simply characterized by

$$V_0(A(T) + B(T)^+) = X. \tag{3.2.5}$$

3.2.4 Examples of account dynamics

The model by Miltersen and Persson (2000) yields one example of how the different accounts might evolve. Other examples are given in Grosen and Jørgensen (2000b) and Hansen and Miltersen (1999).

In Miltersen and Persson (2000) the annual rate of return to the customer in year t, $r_P(t)$, is given by

$$r_p(t) = r_g + \hat{\alpha}\max(\delta(t) - r_g, 0), \tag{3.2.6}$$

where $\hat{\alpha} \in [0, 1]$ is the share of excess return that the customer receives. The company collects payment for the option element by receiving an annual return in year t determined by book value of the customer's account in the beginning of the year. In particular, the company's account is given by the book value of the customer's account accumulated at a rate of return equal to a fraction β of the excess rate of return (if there is any). Finally, the bonus reserve is determined residually. Written in mathematical terms the accounts evolve according to the following:

$$\begin{aligned} A(t) &= A(t-1)e^{r_P(t)} = Xe^{\sum_{i=1}^t r_P(i)} \\ C(t) &= C(t-1) + A(t-1)\big(e^{\beta(\delta(t)-r_g)^+} - 1\big) = \sum_{i=1}^t \big(e^{\beta(\delta(i)-r_g)^+} - 1\big)A(i-1) \\ B(t) &= Xe^{\sum_{i=1}^t \delta(i)} - A(t) - C(t), \end{aligned} \tag{3.2.7}$$

using that the initial balance of the company's account is zero.

Remark 3.2.1. Miltersen and Persson (2000) consider both a contract without and with a bonus account. That is, both a case where the customer receives the book value of his account at the maturity of the contract and a case where he receives the book value of his account plus possibly some terminal bonus as addressed above. The authors actually formulate the model with a minimum rate of return guarantee which can vary from year to year. Since the minimum rate of return guarantee is most often constant, only this case is presented in the present survey. □

In practice, a bonus reserve is typically used to smooth the returns on the customer's account. In an attempt to model this feature a specific bonus distribution mechanism is suggested in Grosen and Jørgensen (2000b). This way of distributing returns to the customer, i.e. determining the policy rate, $r_P(\cdot)$, is also used in Hansen and Miltersen (1999). In their model, one way of collecting payment for the option element is for the company to deduct a certain percentage of the customer's account each year. The bonus distribution mechanism works in the following way: when the bonus reserve reaches a certain target size (determined as a fraction, γ, of the sum of the customer's and the company's accounts),[13] a fraction, α, of the excess bonus is distributed to the customer's and the company's accounts.[14] This return can be reformulated into a rate of return.[15] Since the contract is issued with an annual rate of return guarantee, the annual rate of return that arises is equal to the maximum of the minimum rate of return guarantee and the rate arising from an excess bonus. Disregarding the percentage fee charged for the option element, the annual rate of return on the customer's and the company's accounts in year t is given by

$$r_P(t) = \max\Big\{ r_g, \ln\Big(1 + \alpha\big(\frac{B(t-1)}{(A+C)(t-1)} - \gamma\big)\Big)\Big\}.$$

Observe that $r_P(t)$ is known at the beginning of year t. This was not the case in the previous model, that is, in (3.2.6) $\delta(t)$ is not known in the beginning of year t.

Let $\xi \in [0,1]$ denote the percentage fee charged by the company. The development of the different accounts according to the model of Hansen and Miltersen (1999), which is presented in part IV, is then given by

$$A(t) = A(t-1)e^{r_P(t)-\xi}, \tag{3.2.9}$$

$$C(t) = C(t-1)e^{r_P(t)} + A(t-1)e^{r_P(t)}(1-e^{-\xi}), \tag{3.2.10}$$

$$B(t) = B(t-1) + \underbrace{X(t) - X(t-1)}_{\text{return on the assets}} - A(t) + A(t-1) - C(t) + C(t-1) \tag{3.2.11}$$

for $t \in \{1, \ldots, T\}$. Note that the term $A(t-1)e^{r_P(t)}(1-e^{-\xi})$ in (3.2.10) is merely the annual fee the customer pays.[16]

Remark 3.2.2. Hansen and Miltersen (1999) investigate how to set the terms of a contract such that the contract is fair. They also try to analyze possible redistribution

[13] That is, the target level of the bonus reserve is given by $\gamma(A+C)(\cdot)$.

[14] Hence the sum of the accounts evolves as

$$\begin{aligned}(A+C)(t) &= (A+C)(t-1) + \alpha\big(B(t-1) - \gamma(A+C)(t-1)\big)\\ &= (A+C)(t-1)\Big(1 + \alpha\big(\frac{B(t-1)}{(A+C)(t-1)} - \gamma\big)\Big)\\ &= (A+C)(t-1)e^{\ln\left(1+\alpha\left(\frac{B(t-1)}{(A+C)(t-1)}-\gamma\right)\right)}.\end{aligned} \tag{3.2.8}$$

[15] Grosen and Jørgensen (2000b) use annual compounding whereas Hansen and Miltersen (1999) use a formulation with continuous compounding.

[16] Recall, that for small ξ, $1 - e^{\xi} \approx \xi$.

effects arising from the sharing of a bonus reserve by a heterogenous group of customers. The paper is included in part IV. A few minor extensions are considered in an appendix. □

Remark 3.2.3. Mertens (2000) analyzes German life insurance contracts using a framework similar to the one presented in (3.2.9)–(3.2.11). Given the German rules she needs, however, an extra account on the liability side. More specifically, the customer's account is split into two accounts. One which accumulates at the minimum rate of return guarantee and another that collects distributed bonus. □

Remark 3.2.4. Grosen and Jørgensen (2000b) consider a contract where the customer only receives the book value of his account at the payout date. In their set-up, however, they do not operate with a C account, which means that they need an alternative way to collect premium so that the contract is fair. In their model the deposit made by the customer is placed in a reference portfolio as above. The customer's account, however, does not start with an initial value equal to the deposit but a value which can be smaller than or larger than the deposit. Whether the starting value of the customer's account is smaller than or larger than the initial deposit depends on whether the initial bonus reserve connected to the contract is positive or negative initially.[17] The authors find the value of the deposit such that there is no arbitrage. Given the starting level of the customer's account and other parameters, the deposit is determined as the arbitrage free value of the contract. There is no one to receive a possible positive terminal bonus and since there is no C account, there is no one to cover a possible negative bonus reserve either. A discussion of what happens with this terminal bonus would be nice. This is of course connected to the question of where the positive starting bonus reserve comes from. Because of the way the contract is constructed, it is, however, the case that the market value of the terminal bonus reserve is zero.[18] Despite the problem just mentioned, the paper does provide a significant contribution to the literature because of the way the distribution of bonus, i.e. the policy rate, to the customer is modeled. □

[17] For instance, if the bonus reserve is positive initially, the deposit is larger than the starting value, and the customer is paying for having a bonus reserve that is positive initially.

[18] At maturity, $X(T) = A(T)+B(T)$. The initial deposit, X, is placed in the reference portfolio, thus $X = V_0(X(T)) = V_0(A(T) + B(T))$. Since X is determined as the no-arbitrage value of the contract, i.e. $X = V_0(A(T)))$, this yields that $V_0(B(T)) = 0$.

Part III

Portfolio choice and Fair Pricing in Life Insurance Companies

Portfolio Choice and Fair Pricing in Life Insurance Companies

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Abstract

We investigate the implications of optimal portfolio choice on *fair* pricing of specific insurance contracts. More specifically we consider *participating* policies. We motivate that the manager should optimize utility of final payout for the policy holders. The payout of a participating policy is highly non-linear in wealth, implying that the optimization problem is non-trivial. Still, we find closed form solutions. We consider both the case where default is avoided for sure and the case where default is allowed. Simultaneously with the optimal portfolio choice we find fair contract specifications in the sense that the market values of future claims to policy and equity holders, respectively, are equal to their initial deposits. We find that the introduction of a dynamic optimal portfolio changes the fair contract specifications as well as the comparative statics considerably compared to the case with a fixed underlying portfolio. The utility loss, measured by a *certainty equivalent wealth*, of not introducing portfolio choice can have an arbitrary sign since the fair contract specifications change by the introduction of optimal portfolio choice. Our findings illustrate that dynamic portfolio choice should not be ignored when analyzing fair life insurance contracts.

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4.1 Introduction

Billions of dollars are invested in life and pension insurance contracts around the world today. These contracts typically have a payout which consists of various option elements plus some kind of guarantee. In particular, an increasing amount of money is invested in policies where the payout is linked to an actively managed portfolio. The contract specifications are adjusted in a way such that the contracts are fair. A fair contract is a contract that at initiation has a market value equal to the present value of the deposits or premiums paid by the corresponding contract holder. In the analysis of such contracts a fixed reference portfolio is assumed in order to facilitate for instance the use of standard Black-Scholes pricing techniques. This assumption might, however, seriously affect the fair specifications, etc. of the contracts. A study of the implications of portfolio choice is therefore needed. As far as we know no one has previously included dynamic portfolio choice in the analysis of fair contracts.

In this paper we model and solve an insurance company's optimal portfolio and fair pricing problem simultaneously. Our contribution is three-fold. First, we solve and analyze the non-trivial portfolio choice problem of the company. The problem is non-trivial since the payout is tied to a basket of options which implies that the objective function is not concave in wealth. Second, we find and perform comparative statics on the fair contract specifications both in the case with a fixed reference portfolio and in the case with an optimal chosen dynamic portfolio as the underlying asset. Finally, we investigate which of the two types of contracts a risk-averse agent would prefer. We find that the inclusion of optimal portfolio choice changes the fair contract specifications considerably. In fact the specifications can change in such a way that a risk averse policy holder will prefer a fair contract based on a fixed reference portfolio.

We study a company that offers so-called participating policies. We think of a participating policy (in its most simple form) as consisting of two elements—a fixed guaranteed payment and a call option which gives the policy holder a fraction of the surplus where surplus is defined as the policy holders' share of total wealth less the guaranteed amount. A participating policy based on a fixed reference portfolio is equivalent to an equity-linked policy with guarantee. A company might not always be able to fulfill its obligations towards the policy holder. That is, it might default on the claim in the sense that the policy holder receives an amount which is smaller than the minimum guaranteed amount. The participating policy, hence, includes an additional option—a put option. In practice monitoring by regulators could imply that the companies are forced to invest in a way such that they can satisfy the guarantee for sure. First we analyze the case with the possibility of default on the guarantee and then we analyze the case where the guarantee has to be satisfied for sure. In both cases closed form solutions for the optimal wealth level and the portfolio strategies are available with dynamic portfolio choice but we have to use simulations to obtain the fair contract specifications. However, closed form solutions for the fair contract specifications can be obtained when a fixed reference portfolio is used.

We assume that the insurance company is operating in a competitive insurance market. The company consists of two types of claims holders and it is managed by a third part, the manager. One group of claims holders is risk averse policy holders who buy contracts from the company. The policy holders are assumed to be unable to invest in the financial market on their own account. Another group is risk averse equity holders who are assumed to be large investors able to trade in the financial market on their own. There might be several reasons why the policy holders consider to invest in the company. Firstly, life insurance contracts, as the name suggests, include some kind of insurance in the case of death. This mortality risk is non-hedgeable for the single policy holder, but by the Law of Large Numbers the company can diversify that risk away by pooling a large group of homogeneous policy holders. Secondly, the policy holders are often retail investors who cannot invest on their own in risky assets without bearing, for example, large transactions costs, see Brennan (1993). We do not model any of these imperfections explicitly. We merely assume that at date zero the policy holders make a decision to invest in the contracts. In order to attract the policy holders, the manager promises to manage the portfolio to which returns are linked so as to maximize the policy holders' expected utility of their payout from the company. Ex-ante the equity holders are willing to participate since the contracts are valued at the (competitive) market price. As soon as the policy holders have entered into the contract, the equity holders, however, have incentives to make the manager choose a strategy that maximizes their expected utility instead of the policy holders'. We assume that there is some regulatory power[20] or simply that the manager's compensation scheme is such that this possibility is eliminated. Therefore the manager maintains the strategy of maximizing the policy holders' expected utility at all times.[21]

The call option element of a participating policy is due to the actuarial practice of basing the pricing of a life insurance contract on a set of conservative estimates of interest rates, mortality rates, etc.—the so-called *first order* or *technical basis*. The premiums are set such that the present value of the guaranteed payments, i.e. the bond part, is equal to the present value of the premiums under this first order basis. As time evolves, the true values of the different variables, i.e. *the second order basis*, are known. Usually, the true values are favorable compared to the first order basis and the policy holders will, hence, receive an additional payout stream.[22] In case of

[20] We do not model this explicitly.

[21] The policy holders recognize the manager's incentive to deviate as soon as they have entered and hence if there are no actions to prevent the manager from deviating, for instance a specific compensation scheme, the policy holders do not enter to begin with.

[22] We say usually since the first order basis normally includes a safety margin. At least this was the

an unfavorable development in the variables (compared to the first order basis), the company cannot charge the policy holder an extra premium and thus the policy holder in fact has a call option with the guaranteed payments as the strike level. The company can receive payment for the option indirectly by setting the fraction of surplus that the policy holders receive in such a way that the market value[23] of the future payoffs is equal to the initial deposit (i.e. a fair contract specification). This is exactly the way we adjust the contracts to be fair.

The problem of finding fair prices, i.e. fair conditions, of different types of life insurance and pension contracts, is not new. Several authors have analyzed the problem with different types of contracts. To mention a few, Briys and de Varenne (1994), Miltersen and Persson (2000), and Grosen and Jørgensen (2000b). The papers concerned with fair pricing of life insurance or pension contracts (at least the ones known to the authors of this paper) all assume that the contract payoffs are linked to the return on a exogenously given fixed reference portfolio. While this might be a reasonable assumption when considering equity-linked policies, cf. Brennan and Schwartz (1976), this is not the case with participating (with-profit) policies. The participating policies are offered with a payoff that is linked to the return on a dynamic portfolio which is managed by the issuing company. Therefore it is important to consider the implications of portfolio choice when analyzing fair contracts. This paper does exactly that.

Our point of departure is the set-up of Briys and de Varenne (1994) and therefore we provide a brief introduction to their model. Briys and de Varenne (1994) use the insight that the payout of the participating policy equals the payout of standard contingent claims. They do not treat the guarantee as a binding guarantee. Instead they assume that if the wealth at the date of maturity is below the guaranteed amount, the policy holders receive the remaining wealth[24] and equity holders receive nothing. By imposing the restriction that the wealth is invested in a fixed reference portfolio, closed form solutions are obtained for the market values. The bonus scheme (given as a call option payout at the date of maturity) used is quite simple compared to schemes used in practice. However, it has the most important characteristics and it is possible to obtain closed form solutions to the portfolio choice problems given this scheme. A more realistic bonus distribution could for instance be linked to the surplus each year. Observe, that this would change the maturity guarantee (which is what Briys and de Varenne (1994) look at) to a yearly interest rate guarantee. Yearly minimum rate of return guarantees are analyzed by e.g. Grosen and Jørgensen (2000b), Miltersen and Persson (2000), Hansen and Miltersen (1999), and Grosen and Jørgensen (2000a).

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idea, but the historically low interest rates have actually diminished the safety margin, see for instance Grosen and Jørgensen (2000b).

[23]The market value is defined as the value that precludes arbitrage opportunities. One can also think of it as the competitive price, i.e. the value of the contract that generates zero expected profit.

[24]Wealth is greater than or equal to zero with their specification of the wealth accumulation.

Merton (1971) considered the problem of optimizing utility of consumption and final wealth. He found that the optimal investment strategy for a CRRA investor facing a constant investment opportunity set is to keep a constant fraction of wealth in the mean-variance efficient tangency portfolio and the rest in the risk free bank account, i.e. a two fund separation. If the payout in our setup was equal to the final wealth and the fixed reference portfolio was chosen to be equal to the optimal combination between the risk free asset and the static mean-variance efficient tangency portfolio, there would not be any differences between the fixed and the dynamic portfolio case. The payouts to the policy holders are, however, of the form of a fixed payment plus a portfolio of put and call options. The portfolio problem is therefore non-trivial. Carpenter (2000) investigates the portfolio choice of a fund manager who is equipped with a number of call options on the fund that he manages. Besides the payout from the option, the manager also receives a fixed compensation. The options are in-the-money when the fund performs better than a stochastic benchmark. As a special case Carpenter (2000) considers the case where the benchmark is constant. Hence, she considers a payout structure which is exactly like the payout from the insurance contract that we consider when there is a binding guarantee. In this case the contract pays a fixed guaranteed amount plus a call option payout. We can therefore use the results from Carpenter (2000) to characterize the optimal portfolio. The fraction of wealth invested in the risky assets consists of the solution known from Merton (1971) plus an additional term that adjusts for the special option feature. The optimal final wealth will either be equal to zero or it will be so high that the call options end in-the-money. In the case where default on the guarantee might occur we extend the results in Carpenter (2000) by allowing a put option element in the payout. This changes the portfolio choice problem significantly.[25]

The rest of the paper is organized as follows. The model is presented in section 4.2. In section 4.3 we set up the dynamic portfolio choice problem that must be solved when the company is allowed to default on the guarantee. The fair conditions are discussed in section 4.4 along with a discussion of how to measure the policy holders' preferences for the different types of fair contracts. Some numerical results are presented in section 4.5. In section 4.6 we discuss the case where the company must satisfy the guarantee with certainty. Finally, we conclude in section 4.7.

[25] Cuoco and Kaniel (2001) study the effect that performance fees which are given to portfolio managers has on equilibrium prices of traded assets. As part of their analysis they solve a portfolio choice problem similar to the one we solve in this paper. The solutions to the portfolio choice problems have, however, been derived independently of each other.

4.2 The model

We consider an economy with two traded assets—a risk free bank account with price process B and a risky asset with price process S. We assume that this financial market is frictionless and can be represented by a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$. The time horizon is fixed at T. $\{\mathcal{F}_t\}_{t\in[0,T]}$ is the filtration generated by a one-dimensional Brownian motion, W, which represents the financial uncertainty in the economy.

We assume that there is a constant risk free interest rate denoted by r. The dynamics of the bank account is hence given as

$$B_t = e^{rt} \quad \Rightarrow \quad dB_t = rB_t dt, \quad B_0 = 1. \tag{4.2.1}$$

The value of one unit of the risky asset is assumed to follow a geometric Brownian motion,

$$dS_t = \mu S_t dt + \sigma S_t dW_t. \tag{4.2.2}$$

In the economy life-insurance companies are assumed to exist. The companies offer various types of contracts to homogeneous private investors. In the model we focus on a specific contract which will be specified below. The insurance industry is characterized by perfect competition.

The financial market is dynamically complete from the insurance company's perspective. The claim holders, i.e. the policy holders and the equity holders, delegate the management of the company to a manager. The equity holders also face a complete market,[26] whereas the policy holders face an incomplete market. As mentioned in the introduction we assume that the policy holders buy contracts from the company and are unable to invest in any other assets.

We concentrate on analyzing single premium contracts. At date 0 the company receives premiums for the contracts offered. Moreover, equity holders invest an amount in the company at date 0. The total wealth in the company at date 0, A_0, is the sum of the premiums and the investment made by the equity holders.

A simplified version of the company's balance sheet at date 0 can be represented as:

Assets	Liabilities
A_0	$L_0 = \alpha A_0$
	$E_0 = (1-\alpha)A_0$
A_0	A_0

[26] One can think of the equity holders as either large corporations which have infinite life horizons or as large investors who have already bought life insurance (pension) elsewhere. In either case they are not faced with undiversifiable mortality risk or high trading/information costs.

where $\alpha \in (0, 1)$ is the fraction of the asset value that the policy holders contribute with. The equity holders contribute with the fraction $(1 - \alpha)$. At each date $t \geq 0$ the value of the assets, A_t, is equal to the sum of the value of the policy holders' position, L_t, and the value of the equity holders' position, E_t. We will operate with only a single policy holder and a single equity holder since the group of policy holders and the group of equity holders are both assumed to be homogeneous. The company invests the total wealth in the financial market.

We consider the effects of introducing portfolio choice into the model of Briys and de Varenne (1994). In the case with a fixed reference portfolio we use the risky asset as the reference portfolio.

The assets evolve according to the usual wealth equation with dynamic portfolio choice. If we let Π_t denote the dollar *amount* held in the risky security at date t, then

$$dA_t = (rA_t + \Pi_t(\mu - r))dt + \Pi_t \sigma dW_t$$

under the physical (real-world) probability measure, P.

4.2.1 The contract payout

We interpret the payout date T as the date when the policy holder retires. The policy holder receives a payout, L_T, at date T. The payout depends on the value of the assets at date T, i.e. A_T. The payout can be written as follows:

$$L_T = \begin{cases} A_T & \text{if } A_T \leq L_0 e^{r_g T} \\ L_0 e^{r_g T} & \text{if } L_0 e^{r_g T} \leq A_T \leq \frac{1}{\alpha} L_0 e^{r_g T} \\ L_0 e^{r_g T} + \delta(\alpha A_T - L_0 e^{r_g T}) & \text{if } \frac{1}{\alpha} L_0 e^{r_g T} \leq A_T \end{cases} \tag{4.2.3}$$

where r_g is the minimum rate of return guarantee and δ is the fraction of surplus that the policy holder receives. This fraction is often called the participation coefficient. Observe that the contract is in fact a maturity guarantee since it only pays out at date T, and the policy holder is therefore only guaranteed a return of r_g on average over the life time of the contract, $[0, T]$, and not every year. The payout can equivalently be stated as

$$L_T = L_0 e^{r_g T} - (L_0 e^{r_g T} - A_T)^+ + \delta(\alpha A_T - L_0 e^{r_g T})^+. \tag{4.2.4}$$

Remember that αA_T is exactly the fraction of the asset value at date T that the policy holder has contributed to (through his initial deposit). The policy holder's payout is composed of three elements: a risk free component and two option elements. The first option arises from the fact that if there is not enough wealth to cover the guarantee, the policy holder only gets the asset value. The second option element is what might be called the bonus option, since it pays out when the policy holder's share of wealth is above the guaranteed amount.

The equity holder has the residual claim and his payout at date T is given by

$$E_T = (A_T - L_0 e^{r_g T})^+ - \delta(\alpha A_T - L_0 e^{r_g T})^+.$$

Notice that the equity holder cannot lose more than the initial deposit. The equity holders cannot be forced to cover the difference between the asset value and the guaranteed amount should the asset value not be large enough. The first option part, $(A_T - L_0 e^{r_g T})^+$, illustrates that the equity holder has limited liability.

4.3 Dynamic portfolio choice

When the payout of the insurance contract is linked to a portfolio controlled by the company, we must simultaneously determine the optimal investment strategy and the value of the contract payout. We therefore set up the company's optimization problem for given contract specifications. In section 4.4 we analyze how to set the terms of the contract to sustain a fair contract in the sense that the market value a date 0 of the policy holder's contract is equal to L_0. Hence, in the end we solve for the optimal portfolio strategy and the fair contract specification simultaneously.

We assume that the portfolio choice problem of the company is delegated to the manager. The manager is assumed to be fully informed of the policy holder's preferences and as mentioned earlier his objective is to maximize the policy holder's expected utility of payout.

Let U be the policy holder's utility function, which is assumed to be of the form

$$U(x) = \frac{x^{1-\gamma}}{1-\gamma} \quad \forall\, x \geq 0, \quad 0 < \gamma, \gamma \neq 1,$$

i.e. a power utility function. Hence, U belongs to the class of CRRA utility functions and the relative risk aversion coefficient is given by γ. The log investor is obtained by letting γ approach 1.

The following optimization problem must be solved:

$$\sup_{\Pi} E\big[U\big(L_T\big)\big] = \sup_{\Pi} E\Big[U\Big(L_0 e^{r_g T} - (L_0 e^{r_g T} - A_T)^+ + \delta\alpha(A_T - \frac{L_0 e^{r_g T}}{\alpha})^+\Big)\Big] \quad (4.3.1)$$

$$\text{s.t.} \quad dA_t = (rA_t + \Pi_t(\mu - r))dt + \Pi_t \sigma dW_t \quad \text{and} \quad A_T \geq 0. \quad (4.3.2)$$

Here Π is the amount invested in the risky asset and $E[\cdot]$ denotes the expectation under the real-world probability measure, P.

The objective function, $E[U(\cdot)]$, in (4.3.1) is not concave in the underlying state variable, i.e. in A. The maximization problem can, however, be solved using the same methodology as in Carpenter (2000), i.e. applying the martingale approach combined with a concavified objective function. In figure 4.1 we have shown graphically how to concavify the utility function. Observe that there exists a point, $\hat{a}$, at which the chord from the point $(L_0e^{r_gT}, U(L_0e^{r_gT}))$ is tangent to $U(\cdot)$. The function that equals the original utility function on $[0, L_0e^{r_gT}]$ and on $[\hat{a}, \infty)$ and takes on values on the line from $(L_0e^{r_gT}, U(L_0e^{r_gT}))$ to $(\hat{a}, U(\hat{a}))$ for values of the terminal wealth in $[L_0e^{r_gT}, \hat{a}]$ is clearly concave in the terminal wealth. The function is, however, not differentiable in the point $L_0e^{r_gT}$. We define the subdifferential of the convavified utility function and solve the problem using methods similar to Carpenter (2000). The interested reader is referred to section A of the appendix, where we elaborate on the procedure. The solution to the concavified problem is, in fact, the same as the solution to our original problem since the optimal terminal wealth never takes on values where the original and the concavified utility functions differ. The manager is never interested in a terminal wealth in $[L_0e^{r_gT}, \frac{1}{\alpha}L_0e^{r_gT}]$ because such a level of A_T would not increase the utility but still cost something. The manager would also never want a terminal wealth in $(\frac{1}{\alpha}L_0e^{r_gT}, \hat{a})$ for a set of states (that occur with positive probability) since this level of terminal wealth could be dominated by a strategy that yields $A_T = L_0e^{r_gT}$ for part of the set of states and $A_T = \hat{a}$ on the other part of the set.

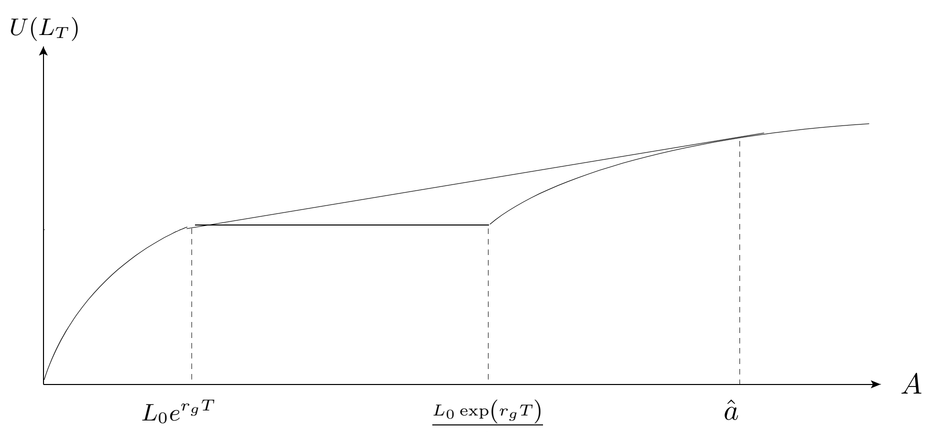

Figure 4.1: Illustration of the concavification of the utility function.

The martingale approach is a technique to transform a dynamic optimization problem into a static one, cf. Cox and Huang (1989), to solve for the optimal level of wealth.[27] The optimal trading strategy can be obtained from the optimal wealth. An application of Itô's formula to the expression for the optimal wealth gives a stochastic differential equation that the optimal wealth must obey. The wealth also satisfies (4.2) and since an Itô process has a unique representation, the diffusion terms of the two stochastic differential equations must be equal. The optimal trading strategy follows by equating the diffusion terms. Formulating the optimization problem given in (4.3.1) using the martingale approach yields the following problem to be solved:

$$\sup_{A_T} E[U(L_T)] \quad \text{s.t.} \quad E[\xi_T A_T] \leq A_0 \quad \text{and} \quad A_T \geq 0, \tag{4.3.3}$$

where ξ is the stochastic discount factor or state price deflator defined by,

$$\xi_t = e^{-(r+\frac{1}{2}\theta^2)t-\theta W_t} \tag{4.3.4}$$

and θ is the market price of risk, i.e. $\theta = \frac{\mu - r}{\sigma}$. The solution to the portfolio choice problem formulated in (4.3.3) is summarized in proposition 4.3.1. The derivations are placed in the appendix.

Proposition 4.3.1. *Let a denote the level of the state variable, i.e. the assets, and define the following function by the original utility function,*

$$u(a) = U\big(L_0 e^{r_g T} - (L_0 e^{r_g T} - a)^+ + \delta\alpha(a - \frac{1}{\alpha} L_0 e^{r_g T})^+\big), \quad a \geq 0. \tag{4.3.5}$$

(i) The optimal level of the assets is given by

$$\begin{aligned} A_T = {} & I(\lambda\xi_T) 1_{\{\lambda\xi_T \geq u'(L_0 \exp(r_g T))\}} + L_0 e^{r_g T} 1_{\{u'(\hat{a}) \leq \lambda\xi_T < u'(L_0 \exp(r_g T))\}} \\ & + \Big[\frac{I(\frac{\lambda\xi_T}{\delta\alpha}) - L_0 e^{r_g T}}{\delta\alpha} + \frac{1}{\alpha} L_0 e^{r_g T}\Big] 1_{\{\lambda\xi_T < u'(\hat{a})\}} \end{aligned} \tag{4.3.6}$$

where $I(\cdot)$ is the inverse of the original marginal utility function and λ is the Lagrange multiplier arising from the budget constraint, i.e. the solution to

$$E[\xi_T A_T] = A_0. \tag{4.3.7}$$

[27] And of course the optimal rate of consumption if the agent cares about intermediate consumption.

(ii) The optimal value of the assets at date $t < T$ is given by

$$
\begin{aligned}
A_t = e^{-r(T-t)}\Big[&\hat{a}N(d_{1,t}) + (\hat{a} - \frac{1}{\alpha}L_0 e^{r_g T} + \frac{L_0 e^{r_g T}}{\delta\alpha})\Big(\big(N(d_{2,t}) \\
&+ (\delta\alpha)^{\frac{\gamma-1}{\gamma}} N(d_{4,t})\big)\frac{N'(d_{1,t})}{N'(d_{2,t})} - N(d_{1,t})\Big) \\
&+ L_0 e^{r_g T}\big(N(-d_{1,t}) - N(d_{3,t} + \theta\sqrt{T-t})\big)\Big] \qquad (4.3.8)
\end{aligned}
$$

where

$$
\begin{aligned}
d_{1,t} &= \frac{\log\big(\frac{u'(\hat{a})}{\lambda\xi_t}\big) + (r - \frac{1}{2}\theta^2)(T-t)}{\theta\sqrt{T-t}} \\
d_{2,t} &= d_{1,t} + \frac{\theta\sqrt{T-t}}{\gamma} \\
d_{3,t} &= \frac{-\log\big(\frac{u'(L_0 \exp(r_g T))}{\lambda\xi_t}\big) - (r + \frac{1}{2}\theta^2)(T-t)}{\theta\sqrt{T-t}} \\
d_{4,t} &= d_{3,t} - \frac{1-\gamma}{\gamma}\theta\sqrt{T-t}.
\end{aligned}
$$

(iii) The optimal portfolio strategy, Π_t, is given by

$$
\begin{aligned}
\Pi_t = \frac{\mu - r}{\sigma^2}\Big\{&\frac{A_t}{\gamma} + e^{-r(T-t)}\Big[\frac{\hat{a}N'(d_{1,t})}{\theta\sqrt{T-t}} - \frac{\frac{1}{\alpha}L_0 e^{r_g T} - \frac{L_0 \exp(r_g T)}{\delta\alpha}}{\gamma}N(d_{1,t}) \qquad (4.3.9) \\
&- \frac{(\hat{a} - \frac{1}{\alpha}L_0 e^{r_g T} + \frac{L_0 \exp(r_g T)}{\delta\alpha})(\delta\alpha)^{\frac{\gamma-1}{\gamma}}}{\theta\sqrt{T-t}} N'(d_{4,t})\frac{N'(d_{1,t})}{N'(d_{2,t})} \\
&- L_0 e^{r_g T}\Big(\frac{N(-d_{1,t})}{\gamma} + \frac{N'(-d_{1,t})}{\theta\sqrt{T-t}} + \frac{N(d_{3,t} + \theta\sqrt{T-t})}{\gamma} + \frac{N'(d_{3,t} + \theta\sqrt{T-t})}{\theta\sqrt{T-t}}\Big)\Big]\Big\}.
\end{aligned}
$$

The optimal terminal wealth in (4.3.6) depends on the realization of the state price deflator at maturity, i.e. ξ_T. The states where payout is relatively expensive are represented by a high value of the state price deflator. The manager chooses to let the company default on the claim in the expensive states. This corresponds to the first term in (4.3.6). In the cheap states the manager invests in such a way that the policy holder's bonus option ends in-the-money. This is represented by the third term in (4.3.6). Finally, the second term in (4.3.6) corresponds to intermediate states where the manager invests in such a way that the payout to the policy holder is exactly equal to the guaranteed amount.

The optimal value of the assets at date t given by (4.3.8) follows from (4.3.6) and the fact that $A_t = E_t[\frac{\xi_T}{\xi_t}A_T]$. The calculation of the expectation can be found in section B of the appendix. The derivation of the optimal trading strategy given in (4.3.9) is done using (4.3.8), see section C of the appendix. Observe that the optimal portfolio strategy consists of the solution to the standard Merton problem, e.g. Merton (1971), and several correction terms arising from the non-linear payout structure.

4.4 Fair contracts and certainty equivalent of wealth

In section 4.3 we characterized the manager's portfolio choice problem given the specific contract terms (i.e. δ, r_g, etc.) of the insurance contract. In practice the company sets the terms of the contract offered. We want to analyze how the contract parameters should be set in the two cases discussed so far, that is, using a fixed or dynamic reference portfolio. The parameters should be set to prevent the company from having arbitrage opportunities. Contracts that preclude arbitrage opportunities are what we call *fair* contracts.

As mentioned, the company faces a complete financial market. It is therefore able to determine the market value of the policy holder's contract as the present value of the payoff, L_T, under the unique equivalent martingale measure, Q, cf. Harrison and Kreps (1979) and Harrison and Pliska (1981).

We focus on determining the level of the participation coefficient, δ, that makes the contract fair. That is, we search for δ, such that premiums are equal to the present value of payoff, i.e.

$$L_0 = \alpha A_0 = E^Q[e^{-rT} L_T]. \tag{4.4.1}$$

Equivalently, we could have solved for δ such that the present value of the equity holder's payout is equal to their initial deposit,

$$E_0 = (1-\alpha)A_0 = E^Q[e^{-rT} E_T]. \tag{4.4.2}$$

Rewriting equation (4.4.1) in terms of the physical probability measure yields

$$L_0 = E[e^{-rT}\frac{dQ}{dP}L_T] = E[\xi_T L_T] \tag{4.4.3}$$

where L_T is given by (4.2.4).

Using a fixed reference portfolio

In the case where the payout is linked to a fixed reference portfolio with a constant volatility, σ, the value of the policy holder's contract is found using the Black-Scholes formula, cf. Black and Scholes (1973), and the put-call parity.

Recall that the date T value of the policy holder's contract, L_T, is given as the guaranteed amount minus the payout of a put option on A with strike $L_0 e^{r_g T}$ plus the payout of $\delta\alpha$ call options on A with strike $\frac{1}{\alpha}L_0 e^{r_g T}$, see (4.2.4). Both options have maturity T. Therefore, the date 0 value, $V_0(L_T)$, is given as

$$V_0(L_T) = L_0 e^{r_g T} e^{-rT} - \big(L_0 e^{r_g T} e^{-rT} N(-d_2') - A_0 N(-d_1')\big) + \delta\alpha\big(A_0 N(d_1) - \frac{1}{\alpha} L_0 e^{r_g T} e^{-rT} N(d_2)\big), \tag{4.4.4}$$

where

$$d_1 = \frac{\log\left(\frac{\alpha A_0}{L_0 \exp(r_g T)}\right) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T},$$
$$d_1' = d_1 - \frac{\log\alpha}{\sigma\sqrt{T}}, \quad d_2' = d_1' - \sigma\sqrt{T}.$$

Equation (4.4.4) shows that the solution to $L_0 = V_0(L_T)$ yields a fair level of δ equal to

$$\delta = \frac{L_0 - L_0 e^{r_g T} e^{-rT} + L_0 e^{r_g T} e^{-rT} N(-d_2') - A_0 N(-d_1')}{\alpha\big(A_0 N(d_1) - \frac{1}{\alpha} L_0 e^{r_g T} e^{-rT} N(d_2)\big)}. \tag{4.4.5}$$

Note that δ is independent of the initial wealth level, A_0, since $L_0 = \alpha A_0$.

Using a dynamic portfolio

For the dynamic case, we can find the optimal terminal value of the assets using equation (4.3.6) for a given δ. Hence, we can solve for the date 0 value, V_0, of the policy holder's contract by simulation. Simultaneously with the simulation procedure we implement a search algorithm that searches for the participation coefficient, δ, that solves $V_0 = L_0$. In other words, we solve the optimization problem (4.3.3) with the additional constraint that the contract must be fair. That is, we solve

$$V_0 = E[\xi_T L_T] = L_0 \tag{4.4.6}$$

where the payout, L_T, is given by (4.2.4) with A_T equal to the expression for the optimal terminal wealth, i.e. (4.3.6).

Measuring certainty equivalent wealth

We want to measure the policy holder's preferences for the different contracts. In particular, we are interested in comparing fair contracts with payout based on a fixed reference portfolio to fair contracts where the payout is linked to a portfolio that the manager controls. For this purpose we introduce a variable, CE, which measures the amount of wealth that must be added initially, that is, added to A_0, in order for the policy holder to be indifferent between the contracts.

We introduce some notation: Let $E\big[U_{fix}(A_0, r_g, r, \sigma)\big]$ denote the policy holder's expected utility of final payout, L_T, in the case of a fixed reference portfolio, given the fair level of δ, the initial wealth in the company, A_0, the minimum rate of return guarantee, r_g, the risk free interest rate, r, and the volatility of the risky asset, σ. Equivalently, let $E\big[U_{dyn}(A_0, r_g, r, \sigma)\big]$ denote the policy holder's expected utility of final payout with dynamic portfolio. We have left δ out from the notation since the levels of the fair δs are independent of the initial wealth and hence does not change when we search for the certainty equivalent. Note that the fair δs in the dynamic and the static portfolio cases will typically differ.

We find the values of CE such that the following equation holds:

$$E\big[U_{fix}(A_0 + CE, r_g, r, \sigma)\big] = E\big[U_{dyn}(A_0, r_g, r, \mu, \sigma)\big] \tag{4.4.7}$$

CE is the amount that must be added to the overall initial wealth, A_0, such that the policy holder is indifferent between a fair contract based on a fixed reference portfolio and an equivalent fair contract based on a dynamically chosen portfolio. The certainty equivalent wealth for the policy holder, given that we keep the capital structure, i.e. α, fixed, is therefore α times CE. This is exactly the additional amount that the holder of a fair contract with payouts that are linked to a fixed reference portfolio must deposit initially in order to be indifferent between this contract and a fair contract with a payout which is linked to a portfolio that changes dynamically. The analysis below is based on the values of CE, i.e. on what might be called the total certainty equivalent instead of the policy holder's certainty equivalent, since it is merely a matter of scaling.

4.5 Results

As our benchmark we will use the following parameter values:

$$\begin{aligned} A_0 &= 100, & \mu &= 0.10, \\ \alpha &= 0.90, & \sigma &= 0.20, \\ r &= 0.05, & \gamma &= 1.25, \\ r_g &= 0.025, & T &= 10. \end{aligned}$$

The optimal portfolio choice in the standard problem with no options, solved by Merton (1971), is to place a fraction of wealth equal to $\frac{1}{\gamma}\frac{\mu - r}{\sigma^2}$ in the risky asset (the mean-variance tangency portfolio if there is more than one risky asset). With our choice of parameter values the constant equals one. Recall that in the static portfolio case all wealth is placed in the risky asset. If there were no options in the payout, the solution to the dynamic and the static problems would therefore be the same for the base case parameter values.

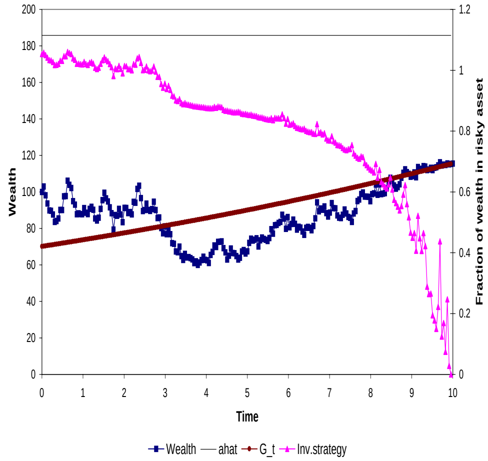

Figure 4.2: Example of the optimal wealth and the corresponding optimal trading strategy for the base case parameter values.

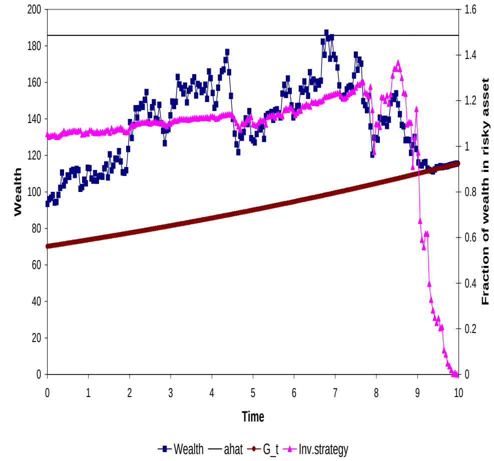

Figure 4.3: Example of the optimal wealth and the corresponding optimal trading strategy for the base case parameter values.

In the case with dynamic portfolio choice, the initial value of the contract is found by Monte-Carlo simulation. In the simulation procedures we use 100,000 simulation paths and anti-thetic variance reduction. We use Newton-Raphson to find the value of the concavification point, $\hat{a}$, and a simple bisection procedure to determine the value of the Lagrange multiplier, λ, and the participation coefficient, δ. Note, that when searching for the fair level of δ we must in each iteration determine a new $\hat{a}$ and a new λ.

4.5.1 The trading strategies

Figures 4.2–4.5 show four different paths of the optimal wealth and the corresponding optimal trading strategy. G_t denotes the date t value of the guaranteed amount, i.e. $G_t = L_0 e^{r_g T} e^{-r(T-t)}$. In most cases the solution will deviate from the Merton solution—at least close to maturity. Whether this happens or not depends on the value of the assets compared to the exercise price of the put option and the concavification point, $\hat{a}$. If the asset value is so low that the put option is (or is close to being) in-the-money, then the manager allocates more to the risk free asset, i.e. lowers the fraction of wealth in the risky asset (or equivalently the volatility) in order to lower the value of the put. This is what happens in parts of the paths in figures 4.2 and 4.3. Figure 4.4 illustrates an extreme case where the asset value is between the guaranteed amount and $\hat{a}$ close to maturity. Since a level of wealth in this area is not optimal, the manager

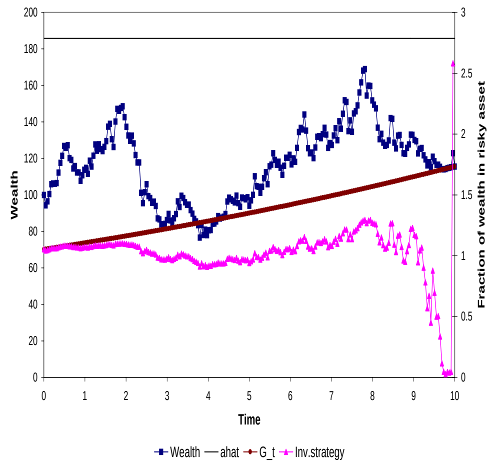

Figure 4.4: Example of the optimal wealth and the corresponding optimal trading strategy for the base case parameter values.

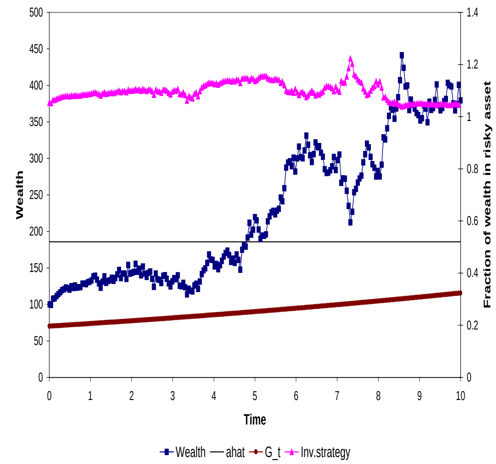

Figure 4.5: Example of the optimal wealth and the corresponding optimal trading strategy for the base case parameter values.

picks a very risky position at this point in time. Finally, figure 4.5 shows a case where the manager will stay close to the Merton solution towards maturity since the asset value is extremely high. When the asset value is very high, the call options are deeply in-the-money. The manager therefore treats the problem as the case where payout is linear in wealth. Figure 4.5 illustrates a case where the call option actually ends in-the-money. That is, the asset value is above the concavification point at maturity.

4.5.2 Fair deltas

We have calculated the fair δs as a function of the minimum rate of return guarantee for different choices of the volatility of the risky asset, σ, the interest rate, r, and the risk aversion, γ. We show the fair δs with a dynamic portfolio and with a fixed reference portfolio in the same graphs to illustrate the differences between using a dynamic and a static portfolio.

Changing σ: In figure 4.6 we see that the fair δ is decreasing in the minimum rate of return guarantee in both the case with a fixed reference portfolio and the case with a dynamically changing portfolio. The fair δ is decreasing in r_g because when r_g increases, the present value of the guaranteed amount increases more than the values of the option elements decrease.[28]

[28]The call (put) option value decreases (increases) with r_g since an increase in r_g is equivalent to an increase in the exercise price.

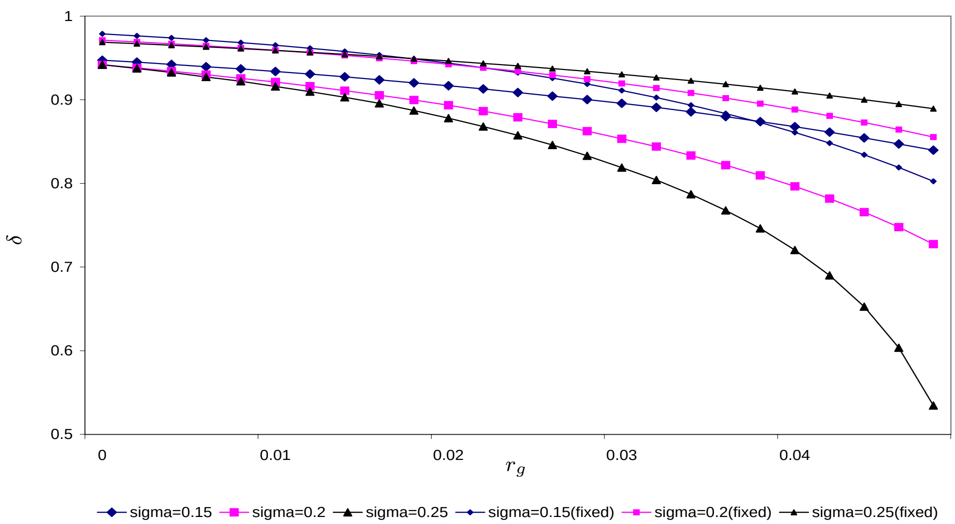

Figure 4.6: Values of the fair δs for different minimum rate of return guarantees and volatilities. Static and dynamic portfolio.

A significant difference between the static portfolio and the dynamic portfolio case is seen for relatively high levels of r_g when the volatility parameter, σ, changes. With a fixed reference portfolio, the fair δ is higher, the higher σ is. The opposite is true for the dynamic portfolio case, i.e. here δ decreases with σ. In the static portfolio case an increase in σ implies an increase in the volatility on the underlying asset. Both the value of the put option and the value of the call options increase with the volatility of the underlying. In fact the put option value increases more with σ than the call options. Since the put option enters as a short position, the overall value of the contract decreases with σ and the policy holder must be given more call options, i.e. a higher δ.

In the dynamic portfolio case the level of the volatility of the portfolio does not necessarily increase with σ since the manager can change the portfolio composition so as to decrease volatility. This is in fact what happens when σ is increased.[29] The put and the call option values are therefore lower, the higher σ is. The decrease in the value of the put option is larger than the decrease in the call option value. The number of call options, δ, must therefore be decreased as σ increases.

[29]We have decomposed the value of the policy holder's position into the present value of the guarantee, the put option and the call options. The decomposition shows that the values of the options decreases with σ, which indicates that the volatility of the underlying portfolio decreases with σ. The decomposition results are available by the authors upon request.

We note that dynamic portfolio choice does not necessarily imply a lower number of call options, i.e. lower δ, than does a fixed reference portfolio. The opposite is for instance the case in figure 4.6 for $\sigma = 0.15$ and a high minimum rate of return guarantee, r_g.

Changing r: In figure 4.7 we change the risk free interest rate, r, instead of the volatility of the risky asset. We see that δ increases with r in both the case with a fixed reference portfolio and with a dynamic portfolio. Increasing r implies that the value of the guaranteed amount decreases. With a fixed reference portfolio an increase in r implies furthermore that the values of the short put and the long call options increase. The increase in the value of the options is, however, not high enough to offset the decrease in the value of the guaranteed amount. A higher interest rate therefore implies that the fair δ is lower. When dynamic portfolio choice is introduced, we have, with our choices of parameter values, that the value of the put option is reduced when r increases, whereas the value of the call options is more or less unaffected by an increase in r.[30] In total, an increase in r decreases the value of the contract and a higher δ is needed in order to establish a fair contract.

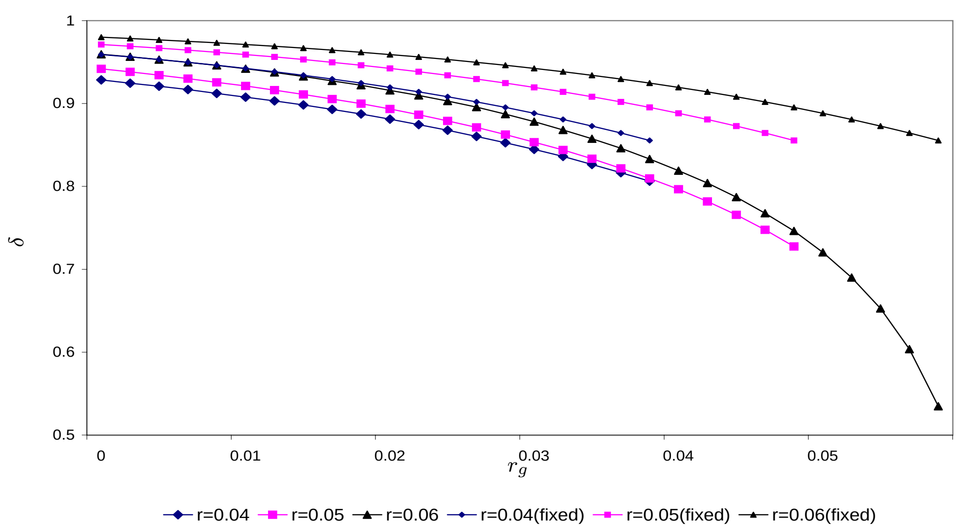

Figure 4.7: Values of the fair δs for different minimum rate of return guarantees and levels of the risk free interest rate. Static and dynamic portfolio.

Changing γ: The fair δ is of course independent of the risk aversion parameter, γ,

[30] Again we have used results from a decomposition of the value of the policy holder's position.

when we have a fixed reference portfolio, and therefore only one curve is shown in figure 4.8 for the fixed reference portfolio case.

Consider a case with dynamic portfolio choice and with a relatively high minimum rate of return guarantee in figure 4.8. In this case, δ increases when the policy holder becomes less risk averse. A less risk averse agent will hold a more risky portfolio. The value of the call options increases and the value of the shorted put option decreases with the riskiness of the portfolio. The decrease in the value of the shorted put is higher than the increase in the value of the call options, which implies an increase in the value of the contract and hence that δ decreases with γ. A policy holder who is not very risk averse wants a relatively risky portfolio in order to increase the possibility of receiving bonus, well aware that this also increases the default risk.

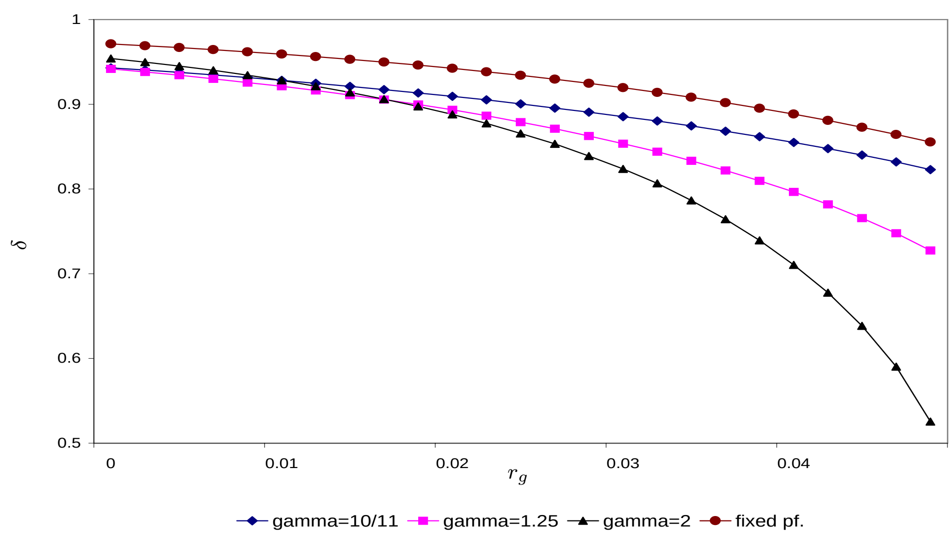

Figure 4.8: Values of the fair δs for different minimum rate of return guarantees and levels of the risk free aversion parameter, γ. Static and dynamic portfolio.

4.5.3 Certainty equivalent wealth

Consider now the certainty equivalents which we defined in section 4.4. In figures 4.9, 4.10, and 4.11 we show the certainty equivalents, CE, for the cases discussed in the analysis of the fair fraction of surplus, δ. That is, for different choices of σ, r, and γ, respectively. The results show whether a policy holder prefers a fair contract based on a dynamic portfolio or a fair contract based on a fixed reference portfolio. Recall from the previous section that the fair fraction of surplus, δ, is typically lower with

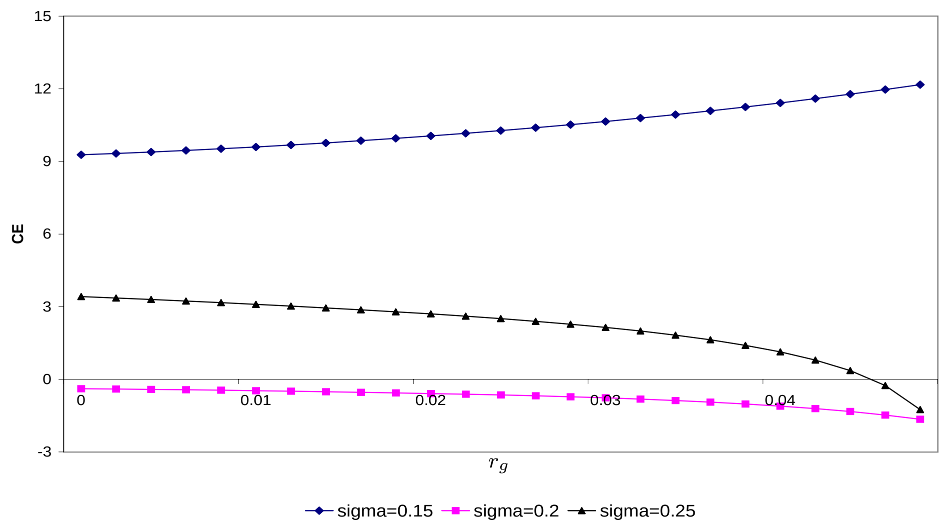

Figure 4.9: Values of the *certainty equivalent* for different r_g's and volatilities without a binding guarantee.

dynamic portfolio choice than with a fixed reference portfolio. In some sense the policy holder pays for the flexibility of dynamic portfolio choice by accepting a lower fraction of surplus.

In figures 4.9–4.11 we see that the certainty equivalents are often negative. At first glance this might seem counterintuitive since the fixed reference portfolio is in the investment opportunity set of the dynamic problem. The reason why the fixed reference portfolio is sometimes preferred is that the 'value' of the higher fair fraction of surplus, δ, offered on a contract with fixed reference, outweighs the 'value' of the flexibility of a dynamic portfolio. If one, however, compares two contracts with the same specifications, i.e. the same δs and so on, a contract based on a dynamic portfolio is of course always preferred to a contract based on a fixed reference portfolio. More specifically, we see in figures 4.9–4.11 that the smallest certainty equivalents correspond to the base case. Recall that with the base case parameters the Merton constant, $\frac{1}{\gamma}\frac{\mu-r}{\sigma^2}$, equals one. With a fixed reference portfolio, 100 percent of wealth is invested in the risky asset as in the Merton case (with the base case parameters), see Merton (1971). The Merton constant indicates the desired fraction of wealth to invest in the risky asset even with the payout structure given here. When we change either σ, r, or γ, the Merton constant changes and therefore portfolio choice becomes more valuable to the policy holder. However, since the fair δ with a dynamic portfolio is typically lower than

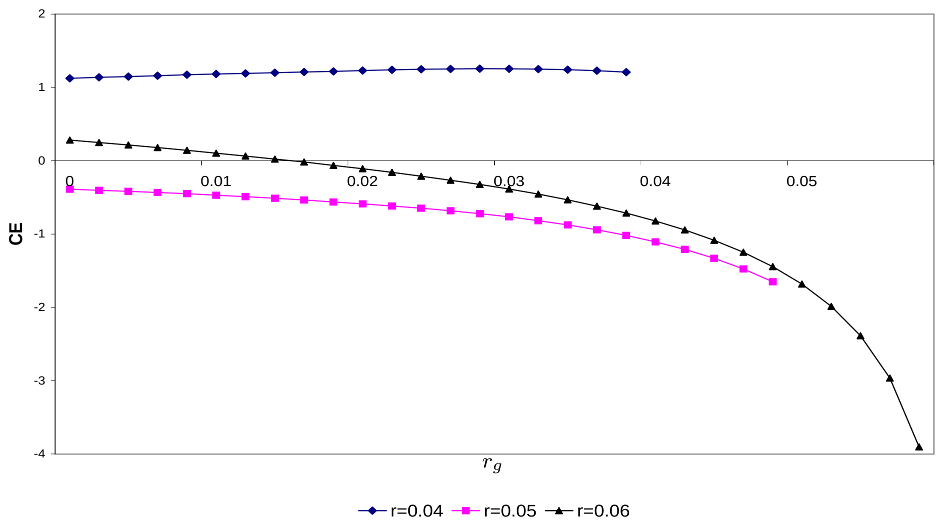

Figure 4.10: Values of the *certainty equivalent* for different r_g's and levels of the risk free interest rate, r, without a binding guarantee.

with a fixed reference portfolio, the policy holder might still prefer a contract based on a fixed reference. Hence, the notion of fair contracts makes it possible to have a policy holder who prefers a (fair) contract based on a fixed reference portfolio to a contract based on a dynamic portfolio.

4.6 The case of a 'true' guarantee

In practice, regulations might force the companies to invest in such a way that they can satisfy the guarantee for sure. In this section we give a brief presentation of the results when the guarantee is binding.

In the case of a binding guarantee, the fixed reference portfolio is a combination of the risk free and the risky asset because money must be placed in the risk free asset initially in order to cover the guarantee for sure. Therefore the fixed reference portfolio in the case with a binding guarantee is actually less risky than in the previous analysis where the guarantee was not binding.

When the policy holder receives the guaranteed amount for sure at date T, the put option disappears from the payout. Hence, the payouts to the policy holder and equity

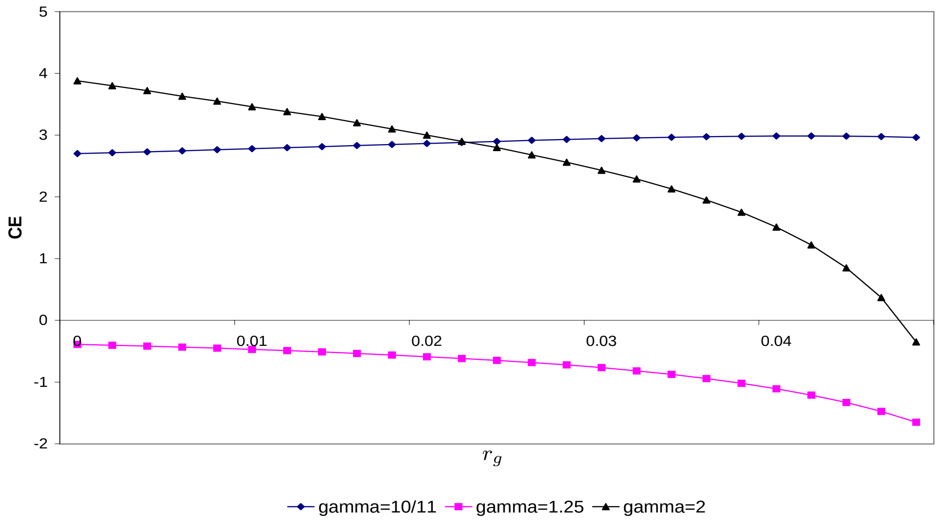

Figure 4.11: Values of the *certainty equivalent* for different r_g's and γs without a binding guarantee.

holder are given as

$$L_T = L_0 e^{r_g T} + \delta(\alpha A_T - L_0 e^{r_g T})^+ \tag{4.6.1}$$

and

$$E_T = A_T - L_0 e^{r_g T} - \delta(\alpha A_T - L_0 e^{r_g T})^+. \tag{4.6.2}$$

Let G_t denote the present value of the future guaranteed benefits at date t, i.e. $G_t = L_0 e^{r_g T} e^{-r(T-t)}$. For technical reasons we introduce a state variable F defined as

$$F_t = A_t - G_t \quad \forall\ 0 \le t \le T. \tag{4.6.3}$$

With a fixed reference, the company is assumed to place an amount, G_0, in the risk free asset at date zero in order to satisfy the guarantee for sure. The remaining wealth, F_0, is placed in the risky asset initially. This portfolio is the fixed reference portfolio to which the payout is linked.[31] With a dynamically changing portfolio, F is the amount that the company can invest freely. We call F_t the *free reserves* at date t.

[31]Recall, that in the non-binding case, it is the total wealth, i.e. A_0, which is placed in the risky asset initially.

Note that with dynamic portfolio choice, the company need not actually hold G_t in the risk free asset at date t in order to satisfy the guarantee at date T. However, the total asset value is required to be above G_t at any date t, and if it reaches G_t, everything is instantly reallocated to the risk free asset in order to be able to satisfy the guarantee for sure. Using (4.2) the dynamics of F is given as

$$\begin{aligned} dF_t &= dA_t - dG_t \\ &= (rA_t + \Pi_t(\mu - r))dt + \Pi_t \sigma dW_t - rG_t dt \\ &= (rF_t + \Pi_t(\mu - r))dt + \Pi_t \sigma dW_t. \end{aligned} \tag{4.6.4}$$

We can rewrite the contract payouts, i.e. L_T and E_T, in terms of F_T. We have that

$$\begin{aligned} L_T &= L_0 e^{r_g T} + \delta(\alpha A_T - L_0 e^{r_g T})^+ \\ &= L_0 e^{r_g T} + \delta\alpha(A_T - L_0 e^{r_g T} - \frac{1-\alpha}{\alpha} L_0 e^{r_g T})^+ \\ &= L_0 e^{r_g T} + \delta\alpha(F_T - \frac{1-\alpha}{\alpha} L_0 e^{r_g T})^+ \end{aligned} \tag{4.6.5}$$

and equivalently for the equity holder

$$E_T = F_T - \delta\alpha(F_T - \frac{1-\alpha}{\alpha} L_0 e^{r_g T})^+. \tag{4.6.6}$$

Dynamic portfolio choice

The optimization problem that must be solved when the guarantee is binding is given as

$$\sup_{F_T} E[U(L_T)] \quad \text{s.t.} \quad E[\xi_T F_T] \le F_0 \quad \text{and} \quad F_T \ge 0. \tag{4.6.7}$$

This optimization problem is exactly mathematically equivalent to the problem analyzed by Carpenter (2000). Hence, the solution for the optimal level of the free reserves as well as the characterization of the optimal portfolio strategy is already known. The concavification of the objective function is illustrated in figure 4.12.[32] Let $\hat{f}$ denote the concavification point.

The solution to the optimization problem follows directly from Carpenter (2000) and can be formulated in a proposition equivalent to proposition 4.3.1.

[32] There exists a point, $\hat{f}$, at which the chord from $(0, U(L_0 e^{r_g T}))$ is tangent to $U(\cdot)$. The function, which takes on values on this line for values of the free reserves in $[0, \hat{f}]$ and values of $U(\cdot)$ for values of the free reserves in $[\hat{f}, \infty]$, is concave in the free reserves.

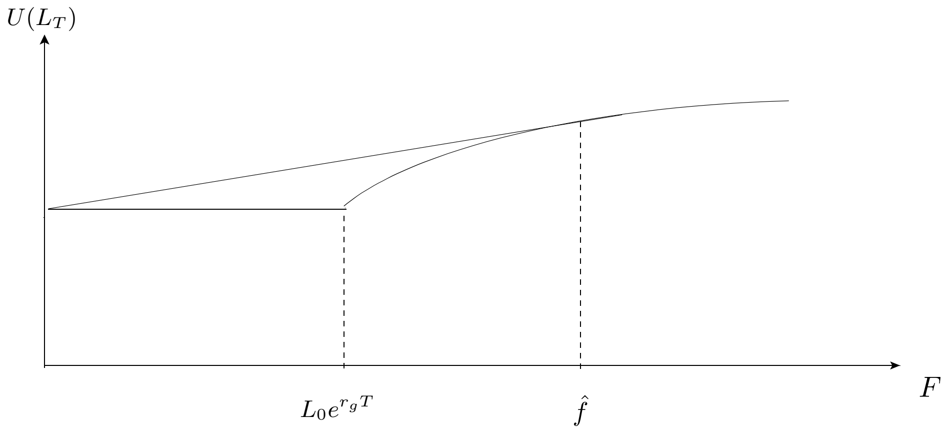

Figure 4.12: Illustration of the concavification of the utility function when the guarantee is binding.

Proposition 4.6.1 (As in Carpenter (2000)). *Let f denote the level of the free reserves, and define u_F by*

$$u_F(f) = U\big(L_0e^{r_gT} + \delta\alpha(f - \frac{1-\alpha}{\alpha}L_0e^{r_gT})^+\big), \quad f \geq 0. \tag{4.6.8}$$

$$\tag{4.6.9}$$

(i). The optimal value of the reserves, F_T, is given as

$$F_T = \Big[\frac{I(\frac{\lambda\xi_T}{\delta\alpha}) - L_0e^{r_gT}}{\delta\alpha} + \frac{1-\alpha}{\alpha}L_0e^{r_gT}\Big]1_{\{\lambda\xi_T < u_F'(\hat{f})\}} \tag{4.6.10}$$

where λ is the Lagrange multiplier that solves $E[\xi_T F_T] = F_0$.

(ii). The optimal level of the reserves at date t, F_t, is given by

$$F_t = e^{-r(T-t)}\Big[\hat{f}N(d_{1,t}^F) + (\hat{f} - \frac{1-\alpha}{\alpha}L_0e^{r_gT} + \frac{L_0e^{r_gT}}{\delta\alpha})\big(N(d_{2,t}^F)\frac{N'(d_{1,t}^F)}{N'(d_{2,t}^F)} - N(d_{1,t}^F)\big)\Big], \qquad t \in [0,T). \tag{4.6.11}$$

(iii). The optimal portfolio strategy, Π_t*, is given by*

$$\Pi_t = \frac{\mu - r}{\sigma^2}\Big\{\frac{F_t}{\gamma} + e^{-r(T-t)}\Big[\frac{\hat{f}N'(d^F_{1,t})}{\theta\sqrt{T-t}} - \frac{\frac{1-\alpha}{\alpha}L_0 e^{r_g T} - \frac{L_0\exp(r_g T)}{\delta\alpha}}{\gamma}N(d^F_{1,t})\Big]\Big\}, \tag{4.6.12}$$

where $d^F_{1,t}$ *and* $d^F_{2,t}$ *are equal to*

$$d^F_{1,t} = \frac{\log\big(\frac{u'_F(\hat{f})}{\lambda\xi_t}\big) + (r - \frac{1}{2}\theta^2)(T-t)}{\theta\sqrt{T-t}}$$

and

$$d^F_{2,t} = d^F_{1,t} + \frac{\theta\sqrt{T-t}}{\gamma}.$$

A discussion of the properties of *(i)–(iii)* can be found in Carpenter (2000).

Observe that the optimal level of the free reserves at maturity given by (4.6.10) consists of only one term and it corresponds to the third term in the expression for the optimal terminal wealth in (4.3.6). This simpler expression for the optimal free reserves at maturity leads to expressions for the optimal free reserves prior to maturity and the optimal trading strategy that are simpler than for the case where there is no binding guarantee.

Fair delta with a fixed reference portfolio

From (4.6.5) we see that the date T payout equals the guaranteed amount plus the payout from $\delta\alpha$ call options on the free reserves with an exercise price equal to $\frac{1-\alpha}{\alpha}L_0 e^{r_g T}$. Hence, the date 0 value, $V_0(L_T)$, when the underlying portfolio is fixed, is given as

$$V_0(L_T) = L_0 e^{r_g T}e^{-rT} + \delta\alpha\big(F_0 N(e_1) - \frac{1-\alpha}{\alpha}L_0 e^{r_g T}e^{-rT}N(e_2)\big),$$

where

$$e_1 = \frac{\log\big(\frac{\alpha F_0}{(1-\alpha)L_0\exp(r_g T)}\big) + (r + \frac{1}{2}\sigma)^2 T}{\sigma\sqrt{T}} \quad \text{and} \quad e_2 = e_1 - \sigma\sqrt{T}.$$

Analogously to the case without a binding guarantee, the fair δ is found as

$$\delta = \frac{L_0 - L_0 e^{r_g T}e^{-rT}}{\alpha\big(F_0 N(e_1) - \frac{1-\alpha}{\alpha}L_0 e^{r_g T}e^{-rT}N(e_2)\big)}. \tag{4.6.13}$$

Observe that δ is again independent of the wealth level, A_0.

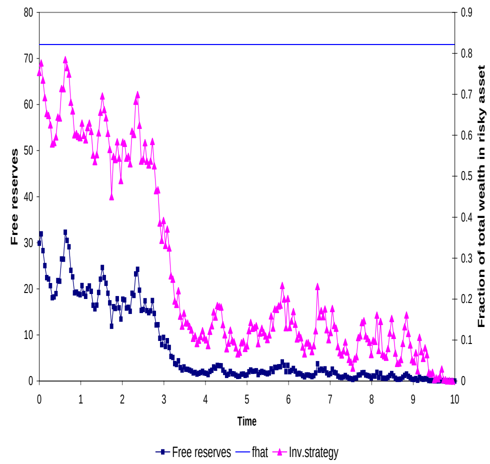

Figure 4.13: Example of the optimal free reserves and the corresponding investment strategy.

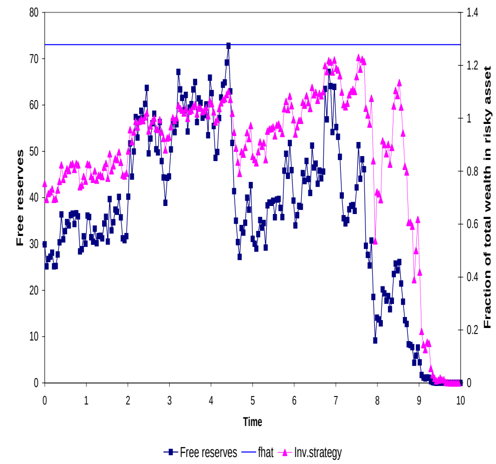

Figure 4.14: Example of the optimal free reserves and the corresponding investment strategy.

Numerical results

Figures 4.13–4.16 show the solution for the case with a binding guarantee, that is, the optimal free reserves and the corresponding fraction of *total wealth* placed in the risky asset, respectively. The paths in the figures are based on the same state price deflator as in figures 4.2–4.5. In general, the level of the free reserves and the investment strategy change in the same direction, that is, when the free reserves rise, the investment strategy becomes more risky and vice versa. This was not the case when the guarantee was not binding. In particular, consider figure 4.13 where the free reserves approach zero quite early. In order to be able to satisfy the guarantee for sure, the manager lowers the fraction of wealth in the risky asset substantially. This was not necessary with the non-binding guarantee, see figure 4.2, where wealth can move below the present value of the guaranteed amount. Hence, the manager is not inclined to decrease the riskiness of the portfolio as much. In figures 4.13–4.15 the free reserves and the fraction of total wealth in the risky asset approach zero toward maturity. However, the fraction of the *free reserves* invested in the risky asset approaches infinity towards maturity as shown by Carpenter (2000) in proposition 1. As in the case without a binding guarantee, figure 4.16 illustrates a case where the call option element ends in-the-money and the fraction of wealth in the risky asset remains relatively constant toward maturity. In particular, the constant fraction of wealth invested in the risky asset converges to the

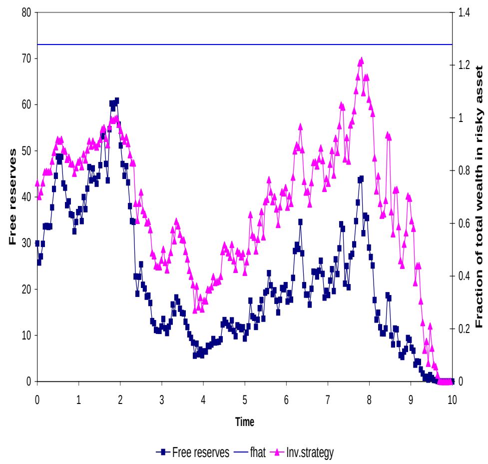

Figure 4.15: Example of the optimal free reserves and the corresponding investment strategy.

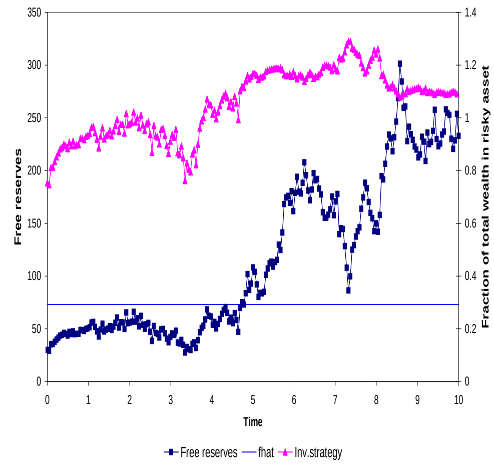

Figure 4.16: Example of the optimal free reserves and the corresponding investment strategy.

Merton constant $\frac{1}{\gamma}\frac{\mu-r}{\sigma^2}$, as the free reserves approach infinity according to proposition 1 in Carpenter (2000).

As in section 4.5, we have found the fair level of δ and the certainty equivalents for various choices of the parameter values, i.e. different σs, rs, and γs. The results for the fair δs are shown in figures 4.17, 4.19, and 4.21, respectively, and the corresponding certainty equivalents are shown in figures 4.18, 4.20, and 4.22, respectively.

We see that the fair δs are decreasing with the minimum rate of return guarantee, r_g, in all figures—just as they were in the case without a binding guarantee. However, with a binding guarantee, δ decreases towards zero as the minimum rate of return guarantee moves closer to the level of the risk free interest rate. This happens because the present value of the guaranteed amount approaches the initial deposit, L_0, as r_g approaches r and the call option value is positive.

The certainty equivalents are positive in all of the cases we have investigated. Moreover, they converge to zero as the minimum rate of return guarantee approaches the risk free interest rate. This happens because δ converges to zero as r_g approaches r, both when the underlying portfolio is a fixed reference and when it is a dynamic portfolio. Hence, the payout approaches the guaranteed amount in both cases and there is nothing to be gained from dynamic portfolio choice. With a dynamic portfolio a change in σ does not have a large impact on the level of the fair δ as can been seen in figure 4.17. The manager changes the volatility of the underlying portfolio in such a way that the value of the call option does not change considerably. We observe that with a fixed

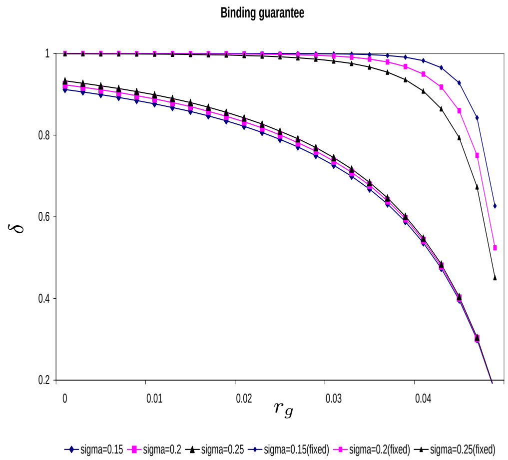

Figure 4.17: Fair δs for different minimum rate of return guarantees and the level of volatility, σ. Static and dynamic portfolio with a binding guarantee.

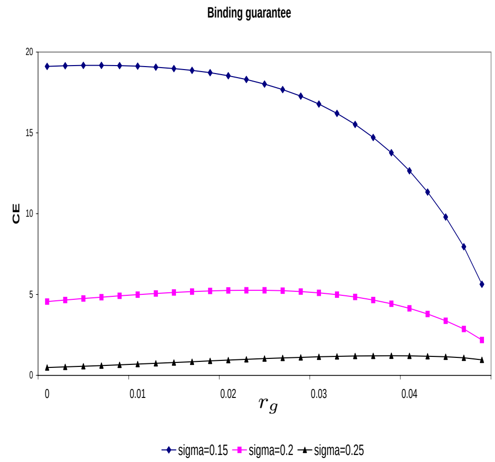

Figure 4.18: Certainty equivalent wealth for different minimum rate of return guarantees and levels of volatility, σ, with a binding guarantee.

reference portfolio the fair δ is decreasing with σ for all levels of the minimum rate of return guarantee, r_g. Recall that with a non-binding guarantee, this was only the case for low minimum rate of return guarantees. For the high levels of r_g the shorted put option present in the non-binding case dominates and pulls the results in the opposite direction.

When the guarantee must be satisfied for sure, a higher risk free interest rate, r, results in a higher fair δ, both with a static and a dynamic portfolio, as seen in figure 4.19. A higher risk free interest rate implies that it is cheaper for the insurance company to provide the guarantee and the policy holder therefore does not have to pay as much for the guarantee. In other words, he does not need to give up as much of the potential bonus, i.e. he is given a higher δ. The impact on the fair δ of a change in r is larger with a binding guarantee than in the case without, see figure 4.7—simply because there is no longer a short put option in the payout.

In figure 4.21 we see that δ increases with γ. A relatively risk averse policy holder wants a portfolio with relatively less risk. This portfolio choice results in a decrease in the value of the bonus options. In order to compensate for this, he is given higher fraction of the bonus, i.e. a higher δ. The opposite was true with a non-binding guarantee—again because of the short put option.

In general, the certainty equivalents are higher, the higher the Merton constant, $\frac{1}{\gamma}\frac{\mu-r}{\sigma^2}$, is. A decrease in either of the parameters, σ, r, or γ will increase the Merton constant. A higher Merton constant implies that the manager chooses a more risky

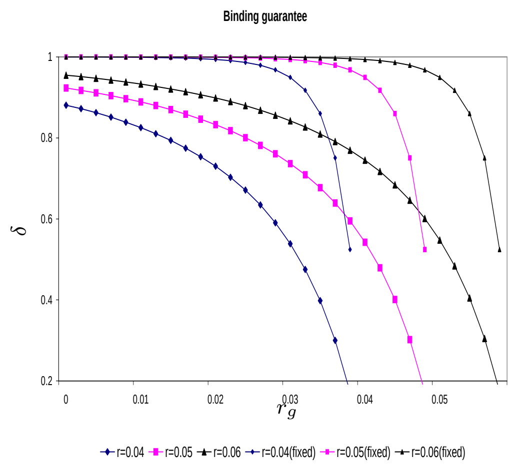

Figure 4.19: Fair δs for different minimum rate of return guarantees and levels of the risk free interest rate. Static and dynamic portfolio with a binding guarantee.

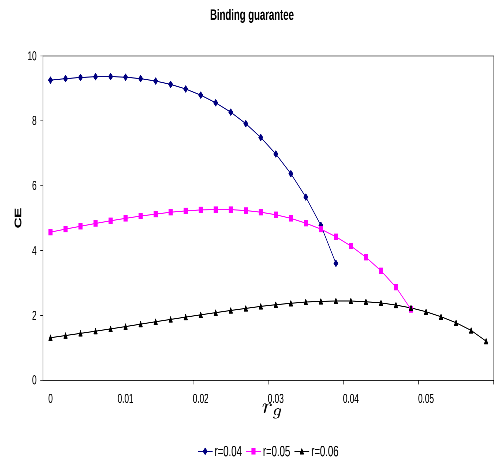

Figure 4.20: Certainty equivalent wealth for different minimum rate of return guarantees and levels of the risk free interest rate with a binding guarantee.

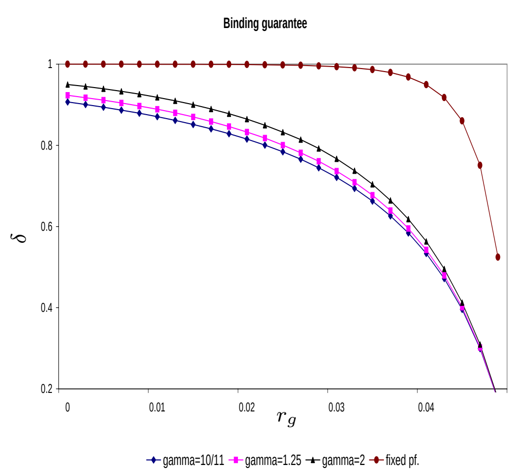

Figure 4.21: Fair δs for different minimum rate of return guarantees and levels of the risk free aversion parameter, γ. Static and dynamic portfolio with a binding guarantee.

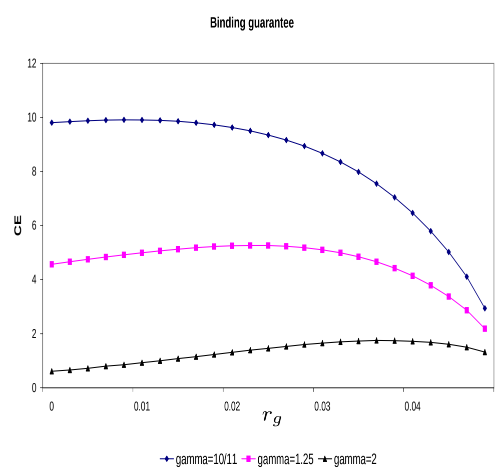

Figure 4.22: Certainty equivalent wealth for different minimum rate of return guarantees and levels of the risk free aversion parameter, γ, with a binding guarantee.

portfolio. The benefits of being able to choose a portfolio which is riskier than the fixed reference outweighs the loss implied by the lower fair δ caused by introducing a dynamic portfolio.

4.7 Conclusion

In this paper we have analyzed the portfolio choice problem of an insurance company which offers contracts that are issued on fair conditions. Our contribution is three-fold. First of all we have solved the non-trivial portfolio choice problem that the manager in a life insurance company is faced with. Secondly, we have found the fair terms of contracts offered by the insurance company. Finally, we have compared fair contracts with payouts based on a dynamic portfolio to fair contracts with payouts based on a fixed reference portfolio.

The analysis of fair contract specifications have so far been based on an assumption of returns being linked to a fixed verifiable reference portfolio. That is, the impact of portfolio choice on the fair contracts specifications has been ignored until now. In practice, most of the contracts that are offered by life insurance companies are linked to an actively managed portfolio. Therefore, if a company calculates fair terms of such contracts under the assumption of a fixed reference portfolio, they might incur significant losses.

We have analyzed both a case where the insurance company might default on the guarantee that it has offered to its policy holders, and a case where the company makes sure that it can always satisfy the guarantee. No matter if one believes that a guarantee is non-binding or binding, one cannot neglect the effects of portfolio choice. Both the level of the fair share of surplus and the comparative statics of the fair contracts change considerably by the introduction of portfolio choice. The fair share of surplus is typically lower with a dynamic portfolio than with a fixed reference portfolio. With a non-binding guarantee the opposite situation can, however, occur because of the short put option in the payout structure.

We find that a policy holder sometimes prefers the higher fair fraction of surplus, δ, combined with a fixed reference portfolio to a lower fair δ and a dynamic portfolio. When the guarantee must be satisfied for sure, we find the policy holder is always better off with a fair contract that builds on dynamic portfolio choice compared to a fair contract based on a fixed reference portfolio.

Our analysis is based on the assumptions of a complete market, a constant risk free interest rate, and a risky asset that follows a geometric Brownian motion. It is our belief that the main conclusion, i.e. that portfolio choice matters a great deal when analyzing fair contracts, would hold in a more general setting.

Appendix

A Concavification and solution method

We show how to concavify the utility function in the problem without the additional constraint that the guarantee must be satisfied for sure. In this case there exists a point, $\hat{a}$, at which the chord from the point $(L_0 e^{r_g T}, U(L_0 e^{r_g T}))$ is tangent to $U(\cdot)$. Now let a denote the level of the state variable, i.e. the assets. We can then introduce the following function defined by the original utility function,

$$u(a) = U\big(L_0 e^{r_g T} - (L_0 e^{r_g T} - a)^+ + \delta\alpha(a - \frac{1}{\alpha} L_0 e^{r_g T})^+\big), \quad a \geq 0. \tag{A.1}$$

Equivalent to Lemma 1 in Carpenter (2000) we know that there exists a unique $\hat{a}$ such that

$$\frac{u(\hat{a}) - u(L_0 e^{r_g T})}{\hat{a} - L_0 e^{r_g T}} = u'(\hat{a}).$$

We can now define the concavified object function, $\tilde{u}$, as

$$\tilde{u}(a) = \begin{cases} u(a) & \text{if } 0 \leq a < L_0 e^{r_g T} \\ u(L_0 e^{r_g T}) + u'(\hat{a})(a - L_0 e^{r_g T}) & \text{if } L_0 e^{r_g T} \leq a < \hat{a} \\ u(a) & \text{if } \hat{a} \leq a. \end{cases} \tag{A.2}$$

Observe that the function is not differentiable at the points 0 and $L_0 e^{r_g T}$. We can, however, define the subdifferential, $\tilde{u}'$, by

$$\tilde{u}'(a) = \begin{cases} u'(a) & \text{if } 0 \leq a < L_0 e^{r_g T} \\ [u'(\hat{a}); u'(L_0 e^{r_g T})) & \text{if } a = L_0 e^{r_g T} \\ u'(\hat{a}) & \text{if } L_0 e^{r_g T} \leq a < \hat{a} \\ u'(a) & \text{if } \hat{a} \leq a. \end{cases} \tag{A.3}$$

The inverse of the marginal of the concavified object function, $\tilde{u}'$, is therefore given as

$$\begin{aligned} i(y) = {} & I(y) 1_{\{y \geq u'(L_0 \exp(r_g T))\}} + L_0 e^{r_g T} 1_{\{u'(\hat{a}) \leq y < u'(L_0 \exp(r_g T))\}} \\ & + \big[\frac{I(\frac{y}{\delta\alpha}) - L_0 e^{r_g T}}{\delta\alpha} + \frac{1}{\alpha} L_0 e^{r_g T}\big] 1_{\{y < u'(\hat{a})\}} \end{aligned}$$

where $I(\cdot)$ is the inverse of the original marginal utility function.

The optimal level of the assets is given by

$$A_T = i(\lambda \xi_T), \tag{A.4}$$

where λ is the solution to

$$E[\xi_T i(\lambda \xi_T)] = A_0. \tag{A.5}$$

Furthermore, the optimal value of the assets at date t is given by

$$A_t = E_t[\frac{\xi_T}{\xi_t} i(\lambda \xi_T)]. \tag{A.6}$$

B The optimal level of wealth

We need to calculate the conditional expectation in (A.6). We use the notation $K = L_0 e^{r_g T}$, $B = \frac{L_0 \exp(r_g T)}{\alpha}$ and a for the value of the assets.

From section A we have that

$$E_t\big[\frac{\xi_T}{\xi_t} i(\lambda\xi_T)\big] = \underbrace{E_t\big[\frac{\xi_T}{\xi_t} I(\lambda\xi_T) 1_{\{\lambda\xi_T \geq u'(K)\}}\big]}_{(1)} + K \underbrace{E_t\big[\frac{\xi_T}{\xi_t} 1_{\{u'(\hat{a}) \leq \lambda\xi_T < u'(K)\}}\big]}_{(2)}$$
$$+ \underbrace{E_t\big[\frac{\xi_T}{\xi_t}\big[\frac{I(\frac{\lambda\xi_T}{\delta\alpha}) - K}{\delta\alpha} + B\big] 1_{\{\lambda\xi_T < u'(\hat{a})\}}\big]}_{(3)} \tag{B.1}$$

We evaluate the three expectations separately. The third expectation is of the type that needs to be evaluated when there is a binding guarantee.

We show how to calculate the general expectation, $E_t[\xi_T^k \mathbf{1}_{\{a \leq \xi_T \leq b\}}]$, where $k \in \{1, \frac{\gamma-1}{\gamma}\}$ and a, $b \in \mathcal{R}_+$, since the calculations of the other expectations needed are very similar. In the following x denotes a normally distributed variable with mean zero and a variance of one.

Using the definition of the state price deflator in (4.3.4) we get

$$\begin{aligned} E_t[\xi_T^k \mathbf{1}_{\{a \leq \xi_T \leq b\}}] &= \int_{-\infty}^{\infty} \xi_t^k e^{-k(r+\frac{1}{2}\theta^2)(T-t) - k\theta\sqrt{T-t}x} \mathbf{1}_{\{a \leq \xi_t \exp\left(-(r+\frac{1}{2}\theta^2)(T-t) - \theta\sqrt{T-t}x\right) \leq b\}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ &= \int_c^d \xi_t^k e^{-k(r+\frac{1}{2}\theta^2)(T-t) - k\theta\sqrt{T-t}x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ &= \xi_t^k e^{-k(r+\frac{1}{2}\theta^2)(T-t) + k^2\frac{1}{2}\theta^2(T-t)} \left[N(d + k\theta\sqrt{T-t}) - N(c + k\theta\sqrt{T-t})\right] \end{aligned} \tag{B.2}$$

where

$$d = \frac{-\log(\frac{a}{\xi_t}) - (r + \frac{1}{2}\theta^2)(T-t)}{\theta\sqrt{T-t}}$$

and

$$c = \frac{-\log(\frac{b}{\xi_t}) - (r + \frac{1}{2}\theta^2)(T-t)}{\theta\sqrt{T-t}}$$

and the last equality in (B.2) follows by completing the square in the usual fashion.

Ad. (1) in equation (B.1):

$$
\begin{aligned}
&E_t\big[\frac{\xi_T}{\xi_t} I(\lambda\xi_T) 1_{\{\lambda\xi_T \geq u'(K)\}}\big] = \frac{1}{\xi_t} E_t\big[\xi_T (\lambda\xi_T)^{-\frac{1}{\gamma}} 1_{\{\xi_T \geq \frac{u'(K)}{\lambda}\}}\big] = \frac{1}{\xi_t}\lambda^{-\frac{1}{\gamma}} E_t\big[(\xi_T)^{\frac{\gamma-1}{\gamma}} 1_{\{\xi_T \geq \frac{u'(K)}{\lambda}\}}\big] \\
&\quad = \lambda^{-\frac{1}{\gamma}} \xi_t^{-1} \xi_t^{\frac{\gamma-1}{\gamma}} e^{\frac{1-\gamma}{\gamma}(r+\frac{1}{2}\theta^2)(T-t)+\frac{1}{2}\frac{(1-\gamma)^2}{\gamma^2}\theta^2(T-t)} \big[N\big(\frac{-\log\big(\frac{u'(K)}{\lambda\xi_t}\big) - (r+\frac{1}{2}\theta^2)(T-t) - \frac{1-\gamma}{\gamma}\theta^2(T-t)}{\theta\sqrt{T-t}}\big)\big] \\
&= \lambda^{-\frac{1}{\gamma}} \xi_t^{-1} \xi_t^{\frac{\gamma-1}{\gamma}} e^{\frac{1-\gamma}{\gamma}(r+\frac{1}{2}\theta^2)(T-t)+\frac{1}{2}\frac{(1-\gamma)^2}{\gamma^2}\theta^2(T-t)} N(d_{4,t}).
\end{aligned}
\tag{B.3}
$$

Ad. (2) in equation (B.1): This expectation follows directly from (B.2) with $k = 1$, $a = \frac{u'(\hat{a})}{\lambda}$, and $b = \frac{u'(K)}{\lambda}$.

Ad. (3) in equation (B.1): Let $d_{1,t}$, $d_{2,t}$, $d_{3,t}$, and $d_{4,t}$ be defined as in proposition 4.3.1 in section 4.3. Then

$$
\begin{aligned}
&E_t\big[\frac{\xi_T}{\xi_t}\big[\frac{I(\frac{\lambda\xi_T}{\delta\alpha}) - K}{\delta\alpha} + B\big] 1_{\{\lambda\xi_T < u'(\hat{a})\}}\big] \\
&= \xi_t^{-1} E_t\big[\xi_T \frac{I(\frac{\lambda\xi_T}{\delta\alpha})}{\delta\alpha} 1_{\{\lambda\xi_T < u'(\hat{a})\}}\big] - \big(\frac{K}{\delta\alpha} - B\big)\xi_t^{-1} E_t\big[\xi_T 1_{\{\lambda\xi_T < u'(\hat{a})\}}\big] \\
&= \xi_t^{-1}(\delta\alpha)^{\frac{1-\gamma}{\gamma}} E_t\big[\xi_T (\lambda\xi_T)^{\frac{-1}{\gamma}} 1_{\{\lambda\xi_T < u'(\hat{a})\}}\big] - \big(\frac{K}{\delta\alpha} - B\big)\xi_t^{-1} E\big[\xi_T 1_{\{\lambda\xi_T < u'(\hat{a})\}}\big] \\
&= \xi_t^{-1}(\delta\alpha)^{\frac{1-\gamma}{\gamma}} \lambda^{-\frac{1}{\gamma}} \xi_t^{-\frac{1-\gamma}{\gamma}} e^{-\frac{1-\gamma}{\gamma}(r+\frac{1}{2}\theta^2)(T-t)+\frac{1}{2}\frac{(1-\gamma)^2}{\gamma^2}\theta^2(T-t)} \Big[\\
&\quad N\Big(\frac{\log\big(\frac{u'(\hat{a})}{\lambda\xi_t}\big) + (r+\frac{1}{2}\theta^2)(T-t) + \frac{1-\gamma}{\gamma}\theta^2(T-t)}{\theta\sqrt{T-t}}\Big)\Big] \\
&\quad - \big(\frac{K}{\delta\alpha} - B\big) e^{-r(T-t)} N\Big(\frac{\log\big(\frac{u'(\hat{a})}{\lambda\xi_t}\big) + (r+\frac{1}{2}\theta^2)(T-t) - \theta^2(T-t)}{\theta\sqrt{T-t}}\Big) \\
&= \lambda^{-\frac{1}{\gamma}} \xi_t^{-\frac{1}{\gamma}} e^{\frac{1-\gamma}{\gamma}(r+\frac{1}{2}\theta^2)(T-t)+\frac{1}{2}\frac{(1-\gamma)^2}{\gamma^2}\theta^2(T-t)} (\delta\alpha)^{\frac{1-\gamma}{\gamma}} N(d_{2,t}) - \big(\frac{K}{\delta\alpha} - B\big) e^{-r(T-t)} N(d_{1,t}),
\end{aligned}
\tag{B.4}
$$

since

$$\frac{N'(d_{1,t})}{N'(d_{2,t})} = \lambda^{-\frac{1}{\gamma}} \xi_t^{-\frac{1}{\gamma}} u'(\hat{a})^{\frac{1}{\gamma}} e^{\frac{1}{\gamma}r(T-t)+\frac{1-\gamma}{\gamma^2}\frac{1}{2}\theta^2(T-t)}$$

and

$$u'(\hat{a}) = \delta\alpha(K + \delta\alpha(\hat{a} - B))^{-\gamma}.$$

Equations (B.3) and (B.4) can be rewritten as

$$e^{-r(T-t)}\big[(\hat{a} - B + \frac{K}{\delta\alpha})(\delta\alpha)^{-\frac{1-\gamma}{\gamma}} N(d_{4,t}) \frac{N'(d_{1,t})}{N'(d_{2,t})}\big]$$

and

$$e^{-r(T-t)}\big[\hat{a}N(d_{1,t}) + (\hat{a} - B + \frac{K}{\delta\alpha})(N(d_{2,t})\frac{N'(d_{1,t})}{N'(d_{2,t})} - N(d_{1,t}))\big],$$

respectively.

Collecting terms, we find

$$A_t = e^{-r(T-t)}\Big[\hat{a}N(d_{1,t}) + (\hat{a} - B + \frac{K}{\delta\alpha})\Big(\big(N(d_{2,t}) + (\delta\alpha)^{-\frac{1-\gamma}{\gamma}}N(d_{4,t})\big)\frac{N'(d_{1,t})}{N'(d_{2,t})} - N(d_{1,t})\Big) \tag{B.5}$$

$$+ K(N(-d_{1,t}) - N(d_{3,t} + \theta\sqrt{T-t}))\Big].$$

C The optimal portfolio strategy

We want to find the strategy, Π_t, that generates the optimal level of wealth, A_t. From (A.6) we know that the wealth can be regarded as a function, h, of time, t, and the state price deflator, ξ_t, that is,

$$A_t = h(t, \xi_t), \quad \text{where} \quad d\xi_t = -r\xi_t dt - \theta\xi_t dW_t. \tag{C.1}$$

Hence, by Itô's formula we have that

$$dA_t = \big(\frac{\partial h}{\partial t} - r\xi_t \frac{\partial h}{\partial \xi} + \frac{1}{2}\frac{\partial^2 h}{\partial \xi^2}\theta^2\xi_t^2\big)dt - \frac{\partial h}{\partial \xi}\theta\xi_t dW_t \tag{C.2}$$

We also know that the dynamics of the wealth is given by the stochastic differential equation in (4.3.1), i.e. as

$$dA_t = (rA_t + \Pi_t(\mu - r))dt + \Pi_t \sigma dW_t. \tag{C.3}$$

Since an Itô process has a unique representation, we know that the diffusion terms in (C.2) and (C.3) must be equal. This yields the following equation for the optimal portfolio strategy,

$$-\frac{\partial h}{\partial \xi}\theta\xi_t = \Pi_t\sigma \quad \Leftrightarrow \quad \Pi_t = -\frac{\partial h}{\partial \xi}\frac{\theta}{\sigma}\xi_t = -\frac{\partial A_t}{\partial \xi}\frac{\mu - r}{\sigma^2}\xi_t. \tag{C.4}$$

The last equality sign follows from the facts that $A_t = h(t, \xi_t)$ and $\theta = \frac{\mu - r}{\sigma}$.

All we need to do is therefore to differentiate the expression for A_t that we found in (B.5) with respect to ξ and rearrange. This yields the portfolio strategy Π_t,

$$\Pi_t = \frac{\mu - r}{\sigma^2}\Big\{\frac{A_t}{\gamma} + e^{-r(T-t)}\Big[\frac{\hat{a}N'(d_{1,t})}{\theta\sqrt{T-t}} - \frac{\frac{1}{\alpha}L_0 e^{r_g T} - \frac{L_0 \exp(r_g T)}{\delta\alpha}}{\gamma}N(d_{1,t}) \tag{C.5}$$

$$- \frac{(\hat{a} - \frac{1}{\alpha}L_0 e^{r_g T} + \frac{L_0 \exp(r_g T)}{\delta\alpha})(\delta\alpha)^{-\frac{1-\gamma}{\gamma}}}{\theta\sqrt{T-t}}N'(d_{4,t})\frac{N'(d_{1,t})}{N'(d_{2,t})} \tag{C.6}$$

$$- L_0 e^{r_g T}\Big(\frac{N(-d_{1,t})}{\gamma} + \frac{N'(-d_{1,t})}{\theta\sqrt{T-t}} + \frac{N(d_{3,t} + \theta\sqrt{T-t})}{\gamma} + \frac{N'(d_{3,t} + \theta\sqrt{T-t})}{\theta\sqrt{T-t}}\Big)\Big]\Big\}, \tag{C.7}$$

where we have used that $K = L_0 e^{r_g T}$ and $B = \frac{L_0 \exp(r_g T)}{\alpha}$.

Part IV

Minimum Rate of Return Guarantees: The Danish Case

Minimum Rate of Return Guarantees: The Danish Case

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Abstract

We analyze minimum rate of return guarantees for life-insurance (investment) contracts and pension plans with a smooth surplus distribution mechanism. We specifically model the smoothing mechanism used by most Danish life-insurance companies and pension funds. The annual distribution of bonus will be based on this smoothing mechanism after taking the minimum rate of return guarantee into account. In addition, based on the contribution method, the customer will receive a final (non-negative) undistributed surplus when the contract matures.

We consider two different methods that the company can use to collect payment for issuing these minimum rate of return guarantee contracts: the *direct* method where the company gets a fixed (percentage) fee of the customer's savings each year, e.g. 0.5% in Denmark, and the *indirect* method where the company gets a share of the distributed surplus. In both cases we analyze how to set the terms of the contract in order to have a fair contract between an individual customer and the company.

Having analyzed the one-customer case, we turn to analyzing the case with two customers. We consider the consequences of pooling the undistributed surplus over two inhomogeneous customers. This implies setting up different mechanisms for distributing final bonus (undistributed surplus) between the customers.

5.1 Introduction

The historically low interest rates and correspondingly low expected returns on portfolios of risky securities have left Danish pension funds and life-insurance companies in a situation where it is difficult for them to find investment opportunities with a return distribution enabling them to meet the guaranteed minimum rate of return they have promised their customers in the past. Standard life-insurance practice is to set fairly conservative terms (i.e. premia, annual return, etc.) of a life-insurance (or investment) contract based on estimates of the future development of the financial market (and other types of risk including mortality risk) when the contract is initiated and to compensate the customer through a surplus distribution mechanism as time evolves and the true development of the financial market is gradually revealed. This distributed surplus is normally termed *bonus*, cf. e.g. Norberg (1999). In some contracts the surplus is accumulated over the life of the contract and not distributed until the maturity of the contract, in other contracts the surplus is gradually distributed over time. This practice of setting fairly conservative terms initially and compensating the customer with bonus payments as the contract matures is also adapted by most Danish pension funds. Since the terms of the contract cannot be altered by the company during the life of the contract in a way unfavorable for the customer, the initial conservative terms of the contract is de facto a minimum rate of return guarantee issued by the company to the customers. In financial terms the company has issued an option to its customers. In principle, the company could have set the initial terms of the contracts extremely conservatively such that the issued option is so far out of the money that it is valueless for all practical purposes. However, competition among these companies has forced them to set less conservative terms for the contracts and hence made the option valuable (i.e. less out of the money). Furthermore, the latest development in the financial market has even driven these minimum rate of return guarantees into the money. In the present paper we will try to model the way Danish life-insurance investment contracts and pension plans are (or can be) designed to collect this option premium. Moreover, we will try, for fairly realistic parameter values, to find the terms of the contracts such that the company gets a fair option premium for the issued minimum rate of return guarantee.

In the paper we consider a hybrid of the models by Grosen and Jørgensen (2000b) and Miltersen and Persson (2000).[33] Grosen and Jørgensen (2000b) price minimum rate of return guarantees with profits where the surplus (i.e. profit) is distributed to the customer gradually, based on a so-called *smoothing mechanism*. This way of distributing

[33]The idea of merging these two models was inspired by the discussions of Henrik Ramlau-Hansen and Paul Brüniche-Olsen at the conference *Financial Markets in the Nordic Countries*, in Århus, January 14–15, 1999, where early version both papers were presented.

surplus gives the customer a rate of return on his stake which does not fluctuate very much. They look at a European contract that at maturity gives the customer the amount which has accumulated on his account through both the guaranteed minimum rate of return and the profit paid out during the life of the contract. However, the customer does not receive the undistributed surplus when the contract matures. Grosen and Jørgensen (2000b) compare the European contract with an equivalent American contract, i.e. a contract on the same terms except that it also includes a surrender option. They operate with two accounts on the liability side of the insurance company's balance sheet: the customer's account and the bonus reserve (or buffer). On the asset side a given reference portfolio is specified. The return on the reference portfolio (positive or negative) is credited to the bonus reserve. It is the amount of the bonus reserve that determines (via the smoothing mechanism) the amount the customer receives as bonus payment: when the amount of the bonus reserve is at least a certain fraction of the value of the customer's account, the company distributes some of the bonus reserve. At maturity the company keeps the remaining amount (positive or negative) in the bonus reserve. Hence, this is a way for the company to collect payment for issuing the guarantee, since in some states of nature it collects a payment and in others it must cover the deficit.

Miltersen and Persson (2000) take a somewhat different approach. They also consider minimum rate of return guarantees with a surplus distribution mechanism. Their way of distributing surplus is, however, different from that of Grosen and Jørgensen (2000b). Miltersen and Persson (2000) simply distribute a fraction of the annual excess return (if positive) to the customer. Moreover, they consider contracts which, besides the amount in the customer's account, also pay out the amount of the bonus reserve (if bonus is positive) at maturity. Hence, the obligation of the company in this type of contract is to issue a (European) call option on the bonus reserve with an exercise price of zero. Opposed to Grosen and Jørgensen (2000b) where the company in some states of nature collects payment for issuing the guarantee by keeping the remaining bonus reserve, Miltersen and Persson (2000) must have another way for the company to collect this payment. Therefore, Miltersen and Persson (2000) work with a third account on the liability side—the account whereto payments to the company for issuing the call option (on the bonus reserve) is made. On the asset side, they also specify a given reference portfolio which is used to determine the annually distributed bonus. They price the contract indirectly by finding the terms of the contract (e.g. guaranteed minimum rate of return, cf. Miltersen and Persson (2000)) such that the present value (using an equivalent martingale measure) of the total net payments to the company from the customer up to the date of expiration is zero (i.e. such that the contract is *fair*).

The model in this paper makes use of the surplus distribution mechanism from Grosen and Jørgensen (2000b) since this type of smoothing mechanism is often used in practice. Our model considers a contract of the type where the customer receives a specified annual minimum rate of return, some of the bonus reserve during the life of the contract, and the amount on the bonus reserve (if positive) at maturity (here date T). If the bonus reserve is negative at date T, then the company covers the deficit as in Miltersen and Persson (2000). This means that the company has issued a series of options on the annual returns. These are covered by the bonus reserve, hence, the company, de facto, has only issued an option on the final bonus reserve.

We work with three accounts on the liability side of the balance sheet: the customer's account, the bonus reserve, and the company's account. On the asset side we have the value of the customer's investment which the company administers.

Observe that when the bonus payments are linked to the company's own investment portfolio, the company has incentives to lower the volatility of the investment portfolio in order to decrease the risk of the final value of the bonus reserve becoming negative. The customer would recognize this incentive and therefore value the option using the volatility that gives the lowest possible price, that is, using a volatility of zero. A volatility of zero would degenerate our model, and more importantly, it is not what we observe in practice. The reason we do not observe a volatility of zero is, according to our beliefs, due to competition among the companies. To model competition among insurance companies is, however, outside the scope of this paper. Instead we assume that the surplus distribution is linked to a certain verifiable reference portfolio with a given volatility. This eliminates the company's incentive to manipulate the investment portfolio.

At initiation of the contract an amount, X, is paid by the customer to the company. This is the amount in the customer's account at date zero. It is invested by the insurance company for the duration of the contract. Besides investing X in the reference portfolio, the company has the opportunity to set up a hedge portfolio that completely eliminates any financial risk the company faces as a result of the contract issued to the customer.

Firstly, we investigate the one-customer case and characterize fair contracts between the customer and the company. In this case the customer will always have an initial bonus reserve of zero when he enters. At the maturity of the contract the customer will receive the remaining (positive) undistributed surplus. Secondly, we investigate a situation with two customers. The two customers can differ with respect to minimum rate of return guarantees, entry dates, and exit dates. We propose different mechanisms for distributing the final bonus between the customers depending on how the customers differ.

The paper is organized as follows. Section 5.2 describes the modeling framework.

This includes describing the surplus distribution mechanism and examples of possible payment schemes the company can use to collect payment for the contract it has set up with the customer. The condition that characterizes a *fair* contract is given in section 5.3, and the results follow in section 5.4. Finally, in section 5.5 we investigate the situation where two customers have one common bonus reserve and compare this to the situation with individual bonus reserves. Section 5.6 concludes. In the main body of the text we ignore issues such as stochastic interest rates and mortality risk. We have, however, included a small investigation of these issues in the appendices A and B.

5.2 The model

In the case with only one customer we use a brutal simplification of the company's balance sheet. We only include the accounts relevant for determining the customer's contract. That is, we exclude the company's hedge activities from the balance sheet. Hence, the balance sheet can be represented graphically as

Assets	Liabilities
X	A
	B
	C
X	X

A short description of the different accounts is provided below:

Account A: This is the customer's main account. The initial deposit is credited to this account. In any year the total amount on this account earns the guaranteed minimum rate of return, g, (specified in the contract) and possibly some bonus as interest. Bonus is distributed when the so-called *buffer ratio* is above a certain level γ.[34] The buffer ratio is determined by the bonus reserve, B, in a manner equivalent to Grosen and Jørgensen (2000b).[35] The contribution from the bonus reserve to the customer's account is also known as distributed surplus.

Account C: This is the account where the company collects the payment for issuing and guaranteeing the contract. In the case of a negative amount in the bonus reserve at the maturity of the contract, the deficit is covered by the company. We consider two different ways of collecting payments, both determined so that the contract is fair.[36]

[34]The size of γ is typically around 10% in Denmark, according to regulated solvency rules.

[35]The buffer ratio is defined as the bonus reserve over the sum of the accounts A and C since $A+C$ play the role of the so-called policy account in the Grosen-Jørgensen model (the account on the liability side other than the bonus reserve). That is, the buffer ratio is equal to $\frac{B}{A+C}$.

[36]By fair we simply mean that the present value of the company's net payments from the customer up to date T is equal to 0, i.e. $V_0(C(T) - B^-(T)) = 0$, where $C(T)$ is the total amount on account C at date T, $B^-(T)$ is the potential deficit in the bonus reserve, and $V_0(\cdot)$ denotes the date zero market value operator.

These two methods will be termed the *indirect* and the *direct* method. More about this in subsection 5.2.1.

Account B: This is the bonus reserve (or undistributed surplus) for the individual customer. It is determined residually in the sense that the annual return (positive or negative) of the customer's investment in the reference portfolio is first distributed to this account. Then the required return to account A and the payments to account C are subtracted from this account. This should become clearer when looking at the model in mathematical terms.

Account X: The account X keeps track of the value of the investment that the company has made in the reference portfolio on behalf of the customer. We assume that the change in this value can be described by a geometric Brownian motion, that is, the value of the investment at date t, $X(t)$, is given by

$$dX(t) = rX(t)dt + \sigma X(t)dW(t), \quad X(0) = X,$$

under the equivalent martingale measure, Q. r is the instantaneous riskless interest rate, which is assumed to be constant,[37] σ is the volatility of the reference portfolio, and W is a Brownian motion under Q.

We model X under the equivalent martingale measure, Q, since for valuation purposes we would like to use the traditional Harrison-Kreps/Harrison-Pliska approach, cf. Harrison and Kreps (1979) and Harrison and Pliska (1981). This approach relies on assumptions of no-arbitrage and complete markets. Hence, it is possible for the company to dynamically hedge the contract (issued to the customer) completely. That is, one can think of the company's entire investment portfolio, Y, as composed of the investment on behalf of the customer, X, in the reference portfolio and a hedge portfolio, H.

Note that we have implicitly assumed that there are no dividend payments[38] on the assets included in the reference portfolio since the drift is equal to the short term interest rate.

The given specification of the value of the reference portfolio implies that the customer's investment in the reference portfolio has a random continuously compounded annual rate of return equal to

$$\delta(t) = \ln\left(\frac{X(t)}{X(t-1)}\right) = (r - \frac{1}{2}\sigma^2) + \sigma\big(W(t) - W(t-1)\big), \tag{5.2.1}$$

[37] In appendix A we briefly extend the model to include a stochastic term structure of interest rates in the form of a Vasicek model, cf. Vasicek (1977).

[38] An equivalent interpretation is that potential dividends on the assets included in the reference portfolio are immediately reinvested into the portfolio.

i.e. $\delta(t) \sim N(r - \frac{1}{2}\sigma^2, \sigma^2)$. Note, moreover, that the returns in different years are stochastically independent because increments of the Wiener process are independent.

5.2.1 Distributing to the accounts

Our method for distributing bonus is somewhat complicated. Firstly, we model the development of the sum of the customer's and the company's accounts. The model incorporates possible ways of collecting payments (for the contract) from the customer to the company.

We distribute the guaranteed minimum return (determined by the minimum rate of return guarantee, g) and possibly an extra amount depending primarily on the size of the bonus reserve to the sum of the two accounts $(A + C)$. More specifically, the accounts together receive either the guaranteed minimum return, $(e^g - 1)(A+C)(t-1)$, at date t or a certain fraction, $\alpha + \rho$, where $\alpha + \rho \in [0, 1]$, of the excess bonus reserve (bonus above the optimal buffer level, $\gamma(A + C)$), whichever amount is the greater. Hence, the sum of accounts A and C is compounded (continuously) at the following annual rate[39]

$$\max\Big\{g, \ln\Big(1 + (\alpha + \rho)\Big(\frac{B(t-1)}{(A+C)(t-1)} - \gamma\Big)\Big)\Big\}. \tag{5.2.2}$$

In order to see that our method for distributing to the sum of accounts A and C in fact distributes a certain fraction, $\alpha + \rho$, of the excess bonus reserve in the case where the return is greater than the guaranteed minimum return, consider the following: the desired level of the bonus reserve at date t is $\gamma(A + C)(t)$, $t \leq T$. If $g < \ln\Big(1 + (\alpha + \rho)\Big(\frac{B(t-1)}{(A+C)(t-1)} - \gamma\Big)\Big)$ then the sum of accounts A and C develops as[40]

$$\begin{aligned}(A + C)(t) &= (A + C)(t-1)e^{\ln\left(1+(\alpha+\rho)\left(\frac{B(t-1)}{(A+C)(t-1)}-\gamma\right)\right)} \\ &= (A + C)(t-1)\Big(1 + (\alpha + \rho)\Big(\frac{B(t-1)}{(A+C)(t-1)} - \gamma\Big)\Big) \\ &= (A + C)(t-1) + (\alpha + \rho)\big(B(t-1) - \gamma(A + C)(t-1)\big).\end{aligned} \tag{5.2.3}$$

Since $\gamma(A + C)(t-1)$ is the targeted level of the bonus reserve and $B(t-1)$ is the actual level, the difference, $B(t-1) - \gamma(A + C)(t-1)$, is the excess bonus mentioned above.[41]

[39] Grosen and Jørgensen (2000b) use an annually compounded rate of $\max\{g_a, r_a\}$, where $r_a = (\alpha+\rho)\big(\frac{B(t-1)}{(A+C)(t-1)} - \gamma\big)$. This is equivalent to the continuously compounded rate from expression (5.2.2), since $1 + r_a = e^{r_c} \Leftrightarrow r_c = \ln(1 + r_a)$. Here a and c denote annual and continuous compounding, respectively.

[40] This idea is borrowed from Grosen and Jørgensen (2000b).

[41] Note that equation (5.2.3) only makes sense if $\Big(1 + (\alpha + \rho)\big(\frac{B(t-1)}{(A+C)(t-1)} - \gamma\big)\Big) > 0$; otherwise $A + C$ can change sign from date $t-1$ to date t and we may start chasing a negative optimal buffer level. Fortunately, this will never happen when we have the minimum rate of return guarantee, g, since this prevents us from ever emptying the sum of the accounts A and C totally. That is, whenever

After having determined the method for distributing the return from the buffer account to the sum of the accounts A and C, we will now specify the method for distributing between the two accounts. The development of the amount in the customer's account is modeled similarly to the sum of the accounts A and C. Account A receives the guaranteed minimum return or a fraction of the excess bonus reserve, whichever is the greater. This fraction, however, is smaller than for $A+C$. Only the fraction α is distributed to the account A. Moreover, a (percentage) fee, ξ, is subtracted from the customer's rate of return. This fee, which will be referred to as the rate of payment fee, is introduced as a method for collecting payment for the contract. Hence, the rate of return on account A is

$$\max\Big\{g, \ln\Big(1+\alpha\Big(\frac{B(t-1)}{(A+C)(t-1)}-\gamma\Big)\Big)\Big\}-\xi.$$

That is, we have modeled two ways that the company can collect payment (into the account C) for issuing the guarantee. Either the contract can be specified with a positive ξ or a positive ρ. When the contract is specified with a positive ξ (and $\rho=0$), we say that the company uses the *direct* method for collecting payment for the contract, whereas when the contract is specified with a positive ρ (and $\xi=0$), we say that the company uses the *indirect* method for collecting payment for the contract. Of course, the *indirect* and the *direct* methods can be combined, setting $\rho>0$ and $\xi>0$ at the same time. However, analyzing the effects of this is not the purpose of the present paper. We use the term *direct* in the case where ξ is positive since in this case the payment is collected directly from account A. In particular, a certain fraction, ξ, of the amount in the customer's account is paid to the company.

Observe that payment for the contract is made over time. No up-front premium is paid.

A simple subtraction of A from the value of $A+C$ gives us the amount in the company's account, that is, the amount the company collects for issuing the contract.

Let us summarize: the development in $A+C$, A, and C from year to year can be written as

$$(A+C)(t)=(A+C)(t-1)e^{\max\left\{g,\ln\left(1+(\alpha+\rho)\left(\frac{B(t-1)}{(A+C)(t-1)}-\gamma\right)\right)\right\}},\quad \alpha,\rho\in[0,1],\alpha+\rho\in[0,1], \tag{5.2.4}$$

$$A(t)=A(t-1)e^{\max\left\{g,\ln\left(1+\alpha\left(\frac{B(t-1)}{(A+C)(t-1)}-\gamma\right)\right)\right\}-\xi},\qquad \xi\in[0,1], \tag{5.2.5}$$

$\Big(1+(\alpha+\rho)\big(\frac{B(t-1)}{(A+C)(t-1)}-\gamma\big)\Big)\le 0$, we know that expression (5.2.2) will be equal to g even though the second term in the max is not well-defined. We have taken that into account in our computer implementation of the model.

and

$$C(t) = (A + C)(t) - A(t), \tag{5.2.6}$$

where $t \in \{1, \ldots, T\}$ and T is the maturity date. Moreover, note that if both ξ and ρ are zero, then the company does not collect any premium and hence contracts cannot be fair.

As mentioned, account B is determined residually. It starts off at zero value, (the individual bonus reserve is always zero at the date of entry, since the customer has not built any reserve yet). At the end of each year, the return on the customer's investment in the reference portfolio is added to account B while the amount going to $A + C$ (according to equation (5.2.4)) is withdrawn. That is, everything is first put into account B and then the amounts are distributed to A and C according to equations (5.2.4)–(5.2.6). We can write

$$B(t) = B(t-1) + \underbrace{X(t) - X(t-1)}_{\text{return on the assets}} - (A + C)(t) + (A + C)(t-1). \tag{5.2.7}$$

According to equation (5.2.1) we have that the value of the customer's investment portfolio can be given recursively as

$$X(t) = X(t-1)e^{\delta(t)}. \tag{5.2.8}$$

The initial values of the different accounts are

$$\begin{aligned} X(0) &= X, \\ A(0) &= X, \\ B(0) &= 0, \end{aligned}$$

and

$$C(0) = 0.$$

This allows us to rewrite equations (5.2.4), (5.2.5), and (5.2.8) as

$$\begin{aligned} (A + C)(t) &= (A + C)(0)e^{\sum_{i=1}^{t} \max\left\{g, \ln\left(1+(\alpha+\rho)\left(\frac{B(i-1)}{(A+C)(i-1)}-\gamma\right)\right)\right\}} \\ &= Xe^{\sum_{i=1}^{t} \max\left\{g, \ln\left(1+(\alpha+\rho)\left(\frac{B(i-1)}{(A+C)(i-1)}-\gamma\right)\right)\right\}}, \end{aligned} \tag{5.2.9}$$

$$A(t) = Xe^{\sum_{i=1}^{t} \max\left\{g, \ln\left(1+\alpha\left(\frac{B(i-1)}{(A+C)(i-1)}-\gamma\right)\right)\right\}} e^{-t\xi}, \tag{5.2.10}$$

and

$$X(t) = Xe^{\sum_{i=1}^{t}\delta(i)}. \tag{5.2.11}$$

Recall that in the case where the bonus reserve is negative at date T, a transfer of $B^-(T)$ takes place from account C to B where $B^-(T)$ denotes the amount of the potential deficit. That is, we have that the value of the company's account at date T is, $C(T) - B^-(T)$.

5.3 Pricing the contract fairly

We will use the same method of pricing the contract as Miltersen and Persson (2000). We assume that the insurance market is characterized by perfect competition. This competitiveness forces abnormal profits to be zero. Since the company's profits are collected in the account C, abnormal profit equal to zero is equivalent to the present value of future (total) profits being zero, i.e. $V_0\big(C(T)-B^-(T)\big) = 0$. Here $V_t(\cdot)$ denotes the date t market value operator, i.e.

$$V_t\big(Z(T)\big) = e^{-r(T-t)}E_t^Q\big[Z(T)\big],$$

where $E_t^Q[\cdot]$ denotes the conditional expectation under the equivalent martingale measure, Q, given the information at date t and $Z(T)$ is a (stochastic) payoff at date T.

The present value of the future profits must be zero, otherwise, if e.g.

$$V_0\big(C(T) - B^-(T)\big) > 0$$

then another company could offer a contract with better terms for the customer and still have $V_0\big(C(T) - B^-(T)\big) > 0$. This mechanism of the market will eventually drive $V_0\big(C(T) - B^-(T)\big)$ to zero.

At any date t, $t \leq T$, we have that

$$X(t) = A(t) + B(t) + C(t),$$

since the usual accounting principle has to apply (i.e. sum of assets equals sum of liabilities). Writing the account B as the difference between its positive and negative part, $B = B^+ - B^-$, we get

$$X(t) = A(t) + B^+(t) - B^-(t) + C(t).$$

For $t = T$ we have $X(T) = A(T) + B^+(T) + C(T) - B^-(T)$. The use of the market value operator and equation (5.2.11) yields

$$V_0\big(Xe^{\sum_{i=1}^T \delta(i)}\big) = V_0\big(A(T) + B^+(T)\big) + V_0\big(C(T) - B^-(T)\big),$$

which implies by the competitive market argument that we have

$$V_0\big(Xe^{\sum_{i=1}^T \delta(i)}\big) = V_0\big(A(T) + B^+(T)\big). \tag{5.3.1}$$

Since the date zero market value of investing X in the reference portfolio has to equal X in order to preclude arbitrage,[42] we end up with the following requirement for a fair contract

$$X = V_0\big(A(T) + B^+(T)\big) \quad \Leftrightarrow \quad 1 = V_0\Big(\frac{A(T)}{X}\Big) + V_0\Big(\frac{B^+(T)}{X}\Big). \tag{5.3.2}$$

This final condition determines the relation between:

(i) the annual minimum rate of return guarantee, g, the fraction of the bonus reserve distributed to the customer, α, and the *indirect* payment fee, ρ, for the contracts offered by an insurance company using the *indirect* method, and

(ii) the annual minimum rate of return guarantee, g, the fraction of the bonus reserve distributed to the customer, α, and the rate of (*direct*) payment fee, ξ, for the contracts offered by an insurance company using the *direct* method.

Note that we assume that the company is able to invest in a portfolio, Y, which completely replicates the payoff, $A(T) + B^+(T)$, to the customer at the maturity of the contract. That is, the value of this portfolio can be expressed as

$$Y(t) = V_t\big(A(T) + B^+(T)\big).$$

Since the hedge portfolio, H, can be expressed as $H(t) = Y(t) - X(t)$ we have

$$H(0) = Y(0) - X = V_0\big(A(T) + B^+(T)\big) - X = 0,$$

where the last equality follows from Equation (5.3.2). Thus, it is costless for the company to set up the hedge.

The theoretically correct representation (in accounting sense) of the company's balance sheet (in the single customer case) should, besides the customer's investment in the reference portfolio, include the hedge portfolio on the asset side. Moreover, the liability side of the balance sheet should simply consist of the current market value of the future obligations to the customer. The market value of the future obligations can

[42] The reference portfolio is a traded asset.

be expressed as

$$L(t) = V_t\big(A(T) + B^+(T)\big).$$

Hence, at any given date, t, the bookkeeping condition is fulfilled as the following (true and fair) balance sheet shows:

Assets	Liabilities
$X(t)$ $H(t)$	$L(t)$
$Y(t)$	$Y(t)$

5.4 Results

We use numerical methods to find the terms of the contract so that condition (5.3.2) is fulfilled. Bonus is distributed when the buffer ratio is above 10% (i.e. $\gamma = 0.1$).

The payout from the contract at maturity is $A(T) + B^+(T)$. It is determined using Monte Carlo simulation. Specifically, we simulate the amount in the different accounts A, B, C, and X, thereby finding the value of $A(T) + B^+(T)$. We use the assumption of a constant interest rate, r, to discount the payoff back to date zero, i.e. we find

$$V_0\big(A(T) + B^+(T)\big) = e^{-rT} E^Q\big[A(T) + B^+(T)\big].$$

This simulation procedure gives the value of the contract issued to the customer for a specific combination of the parameters g, α, ρ, and ξ. We have to search for combinations of parameter values which fulfill requirement (5.3.2). This is done through the use of a modified Newton-Raphson algorithm. In order to simplify requirement (5.3.2) we assume that $X = 1$. This assumption implies that we are searching for parameter values such that

$$V_0\big(A(T) + B^+(T)\big) = 1. \tag{5.4.1}$$

We analyze a few different cases. Firstly, we look for values of g (using Newton-Raphson on g) such that requirement (5.4.1) is satisfied for different combinations of α and ξ, and α and ρ.

With a choice of $r = (1-0.26)5\% = 3.7\%$[43] and $\sigma = 10\%$ we find the values of g for different values of α and ξ (i.e. using the *direct* method, $\rho = 0$). The values are given in table 5.1. As an illustration, consider a company that wants to offer a contract with an α equal to 20% and a rate of payment fee of 0.75%. What guaranteed minimum rate of return can the company offer in this case? That is, what value of g makes the contract fair? According to our calculations the contract with the features mentioned

[43] Approximately the present (after-tax) short interest rate in Denmark.

	α (%)										
ξ	0	10	20	30	40	50	60	70	80	90	100
0.25	0.15	0.18	0.22	–0.04	–0.09	–0.26	–0.36	–0.62	–0.90	–1.01	–1.18
0.50	1.45	1.46	1.54	1.42	1.39	1.26	1.22	1.14	0.96	0.88	0.73
0.75	2.31	2.28	2.37	2.34	2.28	2.23	2.20	2.10	1.99	1.92	1.81
1.00	2.95	2.96	2.99	2.99	2.96	2.92	2.90	2.83	2.78	2.71	2.64
1.25	3.54	3.50	3.54	3.57	3.54	3.51	3.45	3.43	3.37	3.29	3.27
1.50	3.99	3.98	4.04	4.07	4.02	4.02	3.98	3.95	3.89	3.85	3.81
1.75	4.42	4.46	4.47	4.48	4.48	4.46	4.41	4.40	4.38	4.33	4.29
2.00	4.87	4.85	4.88	4.88	4.88	4.86	4.84	4.80	4.79	4.75	4.71
2.25	5.25	5.23	5.26	5.27	5.27	5.25	5.23	5.21	5.18	5.16	5.14
2.50	5.60	5.61	5.62	5.64	5.61	5.62	5.60	5.57	5.54	5.53	5.52

Table 5.1: Values of g in percent (%) for different choices of ξ and α in percent (%); $\sigma = 10\%$, $T = 10$, $\rho = 0$, and $r = 3.7\%$.

	α (%)										
ρ (%)	0	10	20	30	40	50	60	70	80	90	100
10	1.32	1.74	1.63	1.34	1.10	0.82	0.57	0.24	–0.10	–0.34	—
20	2.04	2.32	2.26	2.11	1.88	1.68	1.44	1.25	1.05	—	—
30	2.39	2.60	2.57	2.45	2.27	2.11	1.93	1.72	—	—	—
40	2.63	2.78	2.77	2.66	2.52	2.35	2.20	—	—	—	—
50	2.78	2.90	2.89	2.80	2.68	2.55	—	—	—	—	—
60	2.90	2.99	2.99	2.91	2.81	—	—	—	—	—	—
70	2.98	3.05	3.06	2.99	—	—	—	—	—	—	—
80	3.05	3.11	3.11	—	—	—	—	—	—	—	—
90	3.11	3.15	—	—	—	—	—	—	—	—	—
100	3.16	—	—	—	—	—	—	—	—	—	—

Table 5.2: Values of g in percent (%) for different choices of ρ and α in percent (%); $\sigma = 10\%$, $T = 10$, $\xi = 0$, and $r = 3.7\%$.

is fair when the company offers a minimum rate of return guarantee, g, of 2.37%.[44]

In table 5.2 values of g for different choices of α and ρ are given (i.e. using the *indirect* method). Remember that we have assumed that the company cannot distribute more than 100% of the bonus reserve to accounts A and C, that is, $\alpha + \rho \leq 100\%$. Therefore only half of the table is full. Looking through the column with $\alpha = 20\%$ in table 5.2, we see that a minimum rate of return guarantee near the one offered for $\xi = 0.75\%$ in the case of *direct* payment method (i.e. $g = 2.37\%$) can be given for a ρ between 20% and 30%.

In order to find the magnitude of ξ or ρ that the company should claim for a

[44] We have similar results for the extensions to mortality risk (table B.1) and the extension to stochastic interest rates that we consider in the appendices A and B. We note, however, that for the case of stochastic interest rates the results are only similar for a correlation coefficient of –0.5 between the returns on the reference portfolio and the short term interest rates (table A.3).

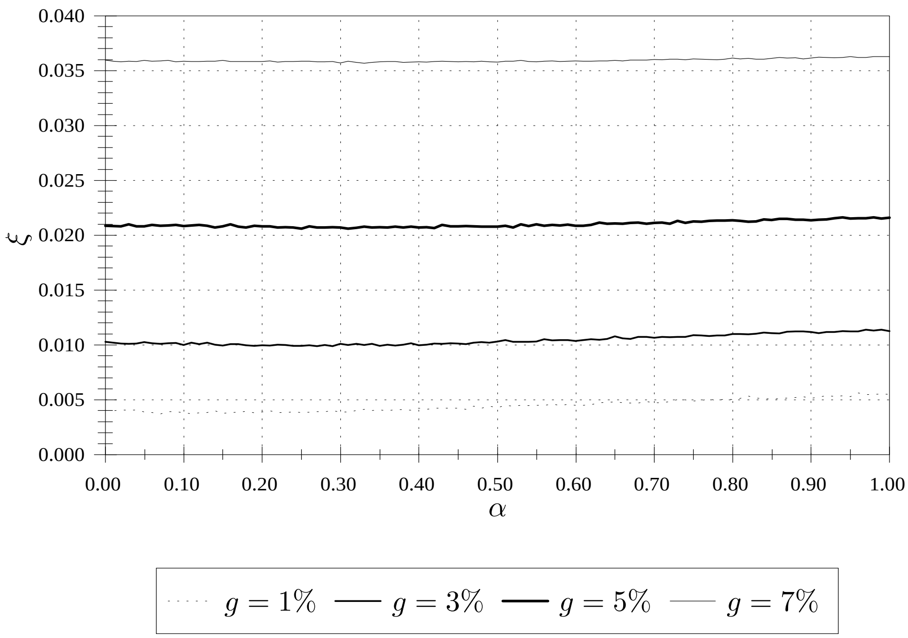

Figure 5.1: Corresponding values of α and ξ for four different values of g; $T = 10$, $\rho = 0$, $\sigma = 10\%$, and $r = 3.7\%$.

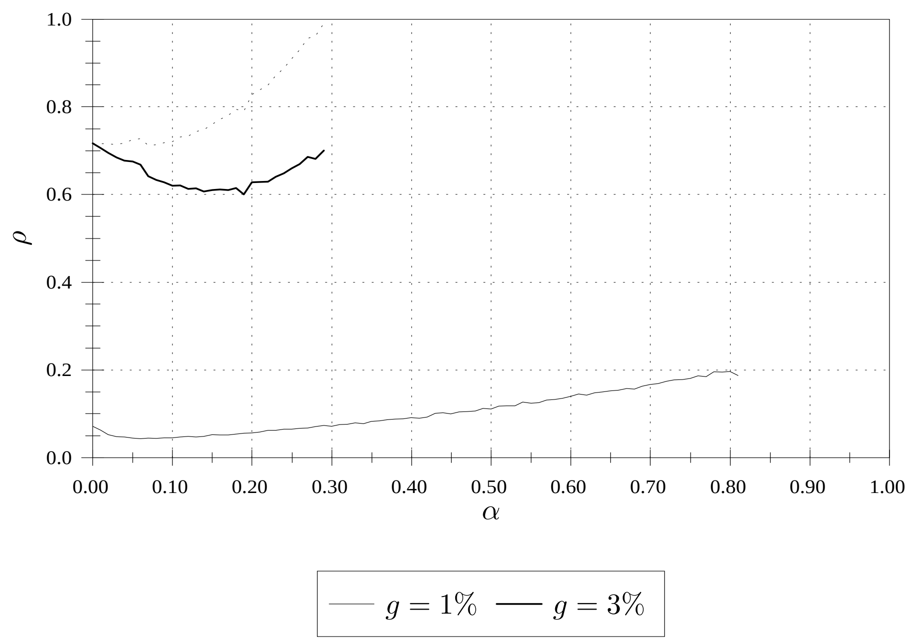

Figure 5.2: Corresponding values of α and ρ for two different values of g; $T = 10$, $\xi = 0$, $\sigma = 10\%$, and $r = 3.7\%$.

contract with a guaranteed minimum rate of return as high as the ones that exist in Denmark today (i.e. 3% and 5%) we have found combinations of fair αs and ξs (ρs) for g fixed at these values. For given αs the search algorithm has found values of ξ and ρ, respectively, so that the contract is fair. The results are depicted in figures 5.1 and 5.2, respectively.

We observe, in figure 5.1, that for a minimum rate of return guarantee, g, of 3%, ξ is about 1% regardless of the size of α. Similarly in the case of a minimum rate of return guarantee of 5%: here ξ is around 2.1% for the different values of α. The higher the minimum rate of return guarantee is, the more the company has to claim (ξ is higher) to be able to honour the contract. We observe that α does not have any significant influence on the size of the rate of payment fee, ξ, necessary to retain a fair contract. We explain this by the way the payment scheme is constructed: the customer pays a certain fraction of the amount of his account each year. A higher α results in a larger amount in account A, however, the fee for the contract is calculated on the basis of this larger value, and therefore, as it turns out, ξ is more or less independent of the size of α. This is a convenient feature since the company does not have to worry about fine tuning the size of α. A closer look at figure 5.1, however, indicates that the rate of payment fee required by the company might be increasing marginally in α for small values of the minimum rate of return guarantee, g. To further justify this claim we have also depicted the curves for $g = 1\%$ and $g = 7\%$. Notice how the curve for $g = 7\%$ is literally horizontal and as g gets smaller, the curves increase more and more in α—even though this effect is only marginal. An explanation for this is that for small values of the minimum rate of return guarantee, g, there is a greater possibility of distributing more than g, and hence the size of the bonus reserve will be more sensitive to the chosen α. The larger the value of α, the more of the excess bonus is distributed and the larger the probability of ending up with a negative bonus reserve. Therefore a larger rate of payment fee, ξ, is needed as α increases.

In figure 5.2 we are only able to depict corresponding values of α and ρ for a minimum rate of return guarantee of 3% and a limited range of αs. The reason is that for a minimum rate of return guarantee of 5% there is no way, even by setting $\rho = 1$, that the company can collect payment enough for the contract when they use the *indirect* method (see table 5.2). This is also the case for $g = 3\%$ and $\alpha \geq 30\%$. That is, for α greater than 30%, even the highest possible ρ (i.e. $\rho = 1 - \alpha$) does not provide the company with enough payments to make the contract fair (see table 5.2). In figure 5.2 we have also depicted the curve for $g = 1\%$, here we have fair contracts for α as high as 80%.

Firstly consider the curve for $g = 3\%$. Above the curve we have depicted the sum $\alpha + \rho$ (the dotted curve). We see that this curve is non-decreasing in α; for $\alpha \in [0, 10\%]$

the curve is more or less constant and for $\alpha \in [10\%, 30\%]$ it is strictly increasing. The curve for combinations of α and ρ that corresponds to $g = 3\%$ can also be divided into two parts, one for each of the intervals of α just mentioned. This curve is decreasing in α in the first interval and non-decreasing in α in the second. We interpret the curve in the following way: for small values of α (i.e. $\alpha \in [0, 10\%]$) there is a high probability that the customer's account only grows at the minimum rate of return guarantee, g. Hence, the company receives almost all the surplus distributed to accounts A and C. However, the contract has to be fair and since there is no distribution of extra funds to the customer, the conditions between the customer and the company are almost the same for all $\alpha \in [0, 10\%]$, hence $\alpha + \rho$ must be more or less constant. This explains why the curve for ρ is decreasing for small αs. For higher αs the customer starts getting more than the minimum rate of return guarantee and, at the same time, the probability that the bonus reserve will be negative is increased, hence the company must have a higher payment for the contract, i.e. a higher ρ.

For $g = 1\%$ we have a more traditional picture. That is, the higher α is, the higher is the share of the distributed bonus, ρ, that must be distributed to the company for the contract to remain fair.

Note that, compared to the *direct* payment method, the *indirect* payment method does not have the same convenient feature that the rate of payment fee is more or less independent of α.

Because of the limitations in the contract design and therefore also the available menu of fair contracts when the *indirect* method of payment is used, we will not investigate further into this payment method. Moreover, the *direct* payment method is more in line with the way real-life Danish contracts are designed.

All the tables and curves considered so far are for contracts with a maturity of 10 years. To see the influence of maturity on the guaranteed minimum rate of return, we have depicted values of g for varying T's in figure 5.3. We have drawn the curves for five different values of α. These curves are derived using the *direct* method of payment with $\xi = 0.5\%$[45], $\sigma = 10\%$, and $r = 3.7\%$. There are two features of figure 5.3 which should be emphasized:

(i) For $\alpha \neq 0$ and T fixed we have that g decreases as α rises. This effect can be explained in the following way: as α increases, more and more of the bonus is distributed and this increases the probability of a negative terminal bonus reserve (i.e. a higher option value). Therefore, in order for the contract to be fair, a lower minimum rate of return guarantee will be offered.

(ii) The minimum rate of return guarantee, g, rises as T increases for a fixed value of α. There are two effects explaining why g increases in T. The first effect follows

[45] Most insurance companies in Denmark collect a 0.5% fee for administrative costs, etc.

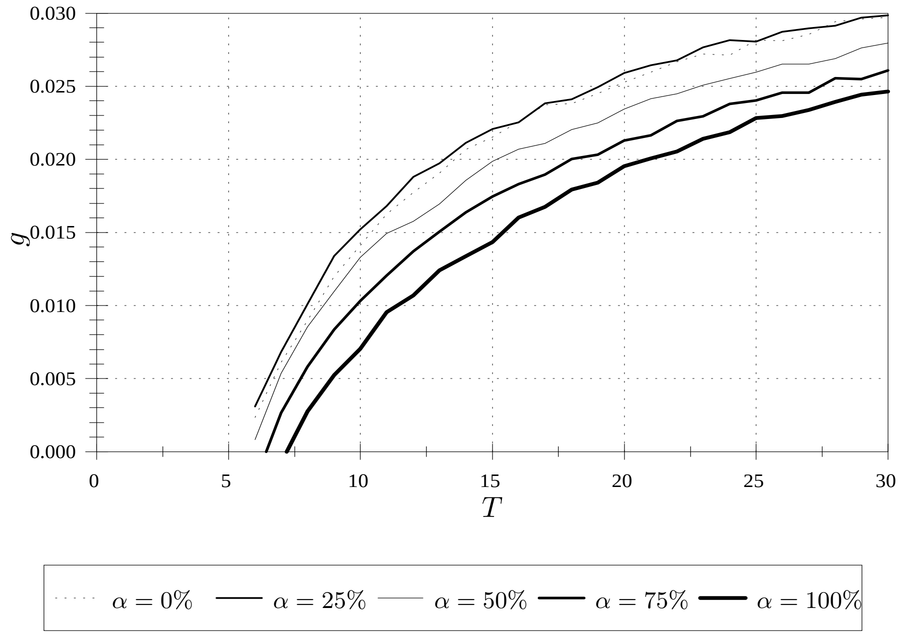

Figure 5.3: Corresponding values of T and g for five different values of α; $\xi = 0.5\%$, $\rho = 0$, $\sigma = 10\%$, and $r = 3.7\%$.

from equations (5.2.9) and (5.2.10), since we have, for $\rho = 0$,

$$A(T) = Xe^{\sum_{i=1}^{T} \max\{g, \ln(1+\alpha(\frac{B(i-1)}{(A+C)(i-1)} - \gamma))\}} e^{-T\xi}$$

and

$$C(T) = Xe^{\sum_{i=1}^{T} \max\{g, \ln(1+\alpha(\frac{B(i-1)}{(A+C)(i-1)} - \gamma))\}} \left(1 - e^{-T\xi}\right).$$

That is, as T increases, a larger share of the amount distributed to the accounts A and C is distributed to account C and, hence, a higher minimum rate of return guarantee can be offered. The second effect is that the bonus reserve increases with time since the targeted buffer size increases with the sum of the accounts A and C, and, hence, the probability that the bonus reserve will end up being negative at the maturity of the contract decreases with the maturity of the contract.

Finally, observe that for $\alpha = 0$ and $\alpha = 25\%$, the minimum rate of return guarantee of 3%, which has been offered until recently in Denmark, is consistent with a fair contract when the maturity of the contract is around thirty years.

5.5 Pooled bonus reserve: The two-customer case

Most insurance companies and pension funds today do not keep track of the individual customers' bonus reserves. Normal practice for the companies is to place the different customers' bonus reserves in one pool. From this pooled bonus reserve the company then distributes bonus to the customers. The group of customers is, however, not homogeneous, e.g. the customers have different minimum rate of return guarantees and/or different maturities. One of the questions that currently raises great debate is whether this practice causes a redistribution of bonus from one group of customers (with similar contracts) to another group of customers (with similar contracts). In Denmark the big issue is whether the group of customers with a 3% guarantee (new contracts) is treated unfairly compared to the group of customers with a 5% guarantee (old contracts). In order to analyze the question in a simple setting, we consider the case with only two customers, customer one and customer two. We compare the situation where the customers have individual bonus reserves with the case where there is only one pooled bonus reserve, and bonus is distributed to the customers using some ad hoc criteria. The way bonus is distributed is known by the customer at the date he enters into the contract. Whether it is possible for a company to alter the way of distributing bonus during the life of the contracts, is a question that we will leave to qualified people to answer. Here we will assume that the company cannot change the bonus distribution mechanism.

We consider five different scenarios. The first scenario considered is the base case where the customers are identical, that is, they have the same minimum rate of return guarantee and so on. In scenario two we look at the isolated effect of the customers having different minimum rate of return guarantees. In the third scenario we analyze the isolated effect of different maturities. More specifically, we consider the case where the two customers engage in a contract at the same date but the contracts have different maturities. The effect of different maturities is also investigated in scenario four. However, this time the two customers enter into a contract at different dates but their contracts expire at the same date. The last scenario considers the combined effect of different minimum rate of return guarantees and different maturities, in particular it combines scenario two and four. Each scenario consists of two different parts:

Part (a) In this part we consider the question of how the bonus is redistributed as a result of pooling. The rate of payment fee, ξ, that the company gets as *direct* payment for issuing the guarantee is calculated individually for the two customers in such a way that their contracts are fair with individual bonus reserves. That is, for individual rate of payment fees, ξ_1 and ξ_2, which make the individual contracts fair, we compare the value of the customer's contract in the case of an individual bonus reserve to the case with a pooled bonus reserve and consider who benefits from the use of a pooled bonus reserve instead of individual bonus reserves.

The values of ξ_1 and ξ_2 that make the contracts fair (using individual bonus reserves, i.e. using the method from section 5.3) are the ξs which solve

$$X = V_0(A_1 + B_1^+) \tag{5.5.1}$$

and

$$X = V_0(A_2 + B_2^+), \tag{5.5.2}$$

where the subscripts on the accounts A and B refer to the two customers. Recall that we have set $X = 1$ so the value of each contract equals one (at the date of entry).

Part (b) In this part we consider the question of how bonus is redistributed when we use a common rate of payment fee, ξ, for all customers. That is, for the value of ξ that makes the sum of the contracts fair, we again compare the values of the contracts in the pooled bonus case to the individual bonus case. The common ξ is found as the ξ which solves

$$2X = V_0(A_1 + A_2 + B^+), \tag{5.5.3}$$

where B represents the pooled bonus reserve.

The method for distributing terminal bonus, and hence the values of the contracts in the case of a pooled bonus reserve, depends on which scenario we are in. The method for distributing the terminal bonus reserve depends mainly on the individual customer's date of entry and exit. The general principle for distributing the terminal bonus reserve is to distribute bonus according to the fraction of the assets that each customer has contributed to. This fraction depends on the dates of entry and exit of the customers. A detailed description is provided below for each scenario separately. In each of the scenarios below α is equal to 25% and each customer deposits $X = 1$ with the company at the date of entry.

Scenario One: In this case the two customers, one and two, have the same minimum rate of return guarantee, $g_1 = g_2 = 3\%$, and they enter into and exit the contracts at the same date. Maturity is 10 years. In the case of a pooled bonus reserve, bonus is distributed equally among the two customers since they have contributed equally to the assets, that is, we distribute half of the terminal bonus to each of the two customers. Since the terms of the two contracts are the same and the customers receive equal shares of the terminal bonus, the values of the contracts based on pooled bonus should be equal to the values found using individual bonus reserves.

Scenario Two: In this scenario we change the value of the minimum rate of return guarantee so that customer one has $g_1 = 5\%$ and customer two has $g_2 = 3\%$. All other parameters remain unchanged. Since the entry and exit dates are the same for customers one and two, they receive an equal fraction of the terminal bonus reserve (as in scenario one). That is, we find the values of the contracts using a pooled bonus reserve as $V_0(A_1 + \frac{1}{2}B^+)$ and $V_0(A_2 + \frac{1}{2}B^+)$.

Scenario Three: In this scenario we go back to the case where the customers have the same minimum rate of return guarantee, $g_1 = g_2 = 3\%$. Moreover, the customers enter at the same date. However, their contracts have different maturities. We set the maturity for customer one equal to 20 years, i.e. $T_1 = 20$, and the maturity for customer two equal to 10 years, $T_2 = 10$. Since they enter at the same date and both pay X at the beginning, this amount will have grown equally for both at the date customer two exits. That is, at date $T_2 = 10$ they have helped build the same fraction of the bonus reserve in the case of the pooled bonus reserve. More specifically one half each. Note, however, that because customer two leaves the company before customer one, we have to adjust the bonus account (and the asset side) at date $T_2 = 10$ when customer two exits. Customer two receives half of the bonus reserve (if positive) at this date. The rest is kept in the bonus reserve, and whatever amount (if positive) there is in the bonus reserve at date $T_1 = 20$ goes to customer one. That is, the contract values for customers one and two, respectively, are calculated as

customer one:

$$V_0(A_1(T_1) + B^+(T_1)), \tag{5.5.4}$$

and customer two:

$$V_0(A_2(T_2) + \frac{1}{2}B^+(T_2)). \tag{5.5.5}$$

At date $T_2 = 10$ we adjust the asset side in order to maintain the bookkeeping equality (assets = liabilities). This is done in the simulations by withdrawing customer two's amount, $A_2(T_2) + \frac{1}{2}B^+(T_2)$, from the account on the asset side. The amount on the asset side at date T_2, when customer two has exited, is therefore $2Xe^{\sum_{i=1}^{T_2}\delta(i)} - A_2(T_2) - \frac{1}{2}B^+(T_2)$.

Scenario Four: In this scenario customer one enters at date zero and exits at date 20, i.e. $T_1 = 20$. However, customer two enters 10 years later than customer one, that is, at date $T = 10$. His contract has a maturity of 10 years, $T_2 = 10$, hence he exits at the same date as customer one. All other parameters remain the same.

At the date of entry of customer two, date $T = 10$, customer one has already built a bonus reserve, $B(T)$. In the case where the final pooled bonus reserve is positive, we distribute bonus to the two customers in such a way that customer one in principle receives all the bonus built up to date $T = 10$ (discounted forward at the rate at which the reference portfolio grows). The bonus built in the period from date $T = 10$ to date $T_1 = 20$ is distributed among the two customers relative to their fractions of the total amount the company has invested on their behalf in the reference portfolio at date $T = 10$ when customer two enters. The two customers' portfolio weights at date $T = 10$ are given by β and $1 - \beta$, respectively, where β is defined as

$$\beta = \frac{Xe^{\sum_{i=1}^{T}\delta(i)}}{X + Xe^{\sum_{i=1}^{T}\delta(i)}} = \frac{e^{\sum_{i=1}^{T}\delta(i)}}{1 + e^{\sum_{i=1}^{T}\delta(i)}}.$$

The fraction, ϵ, of the total bonus reserve at date $T_1 = 20$ that originates from the period up to date $T = 10$, is given by

$$\epsilon = \frac{B(T)e^{\sum_{i=T+1}^{T_1}\delta(i)}}{B(T_1)}.$$

Note that we have discounted $B(T)$ forward to date T_1.

Using these variables, the (pooled) terminal bonus at date $T_1 = 20$ is distributed according to the following:

customer one:

$$\big(\min\{\epsilon + (1-\epsilon)\beta, 1\}\big)^+ B^+(T_1),$$

and customer two:

$$\big(\min\{(1-\epsilon)(1-\beta), 1\}\big)^+ B^+(T_1).$$

We have used the $\min\{\cdot,\cdot\}$ operator in combination with the $(\cdot)^+$ operator in order to make sure that the company does not distribute more than the total terminal bonus. This rule[46] for distributing final bonus favorizes customer one slightly. This happens because customer one receives a final bonus reflecting all the bonus built up to the date when customer two enters. Moreover, from the date when customer two enters, customer one also receives annual returns based on the total bonus (i.e. including the (positive) bonus built before customer two entered). Hence, in some sense customer one receives part of the bonus twice.

Scenario Five: We make one change from scenario four. We look at the case where the customers have different minimum rate of return guarantees. In particular, $g_1 = 5\%$

[46] We have not been able to find a rule that is more fair and at the same time simple and tractable.

Scenario 1a	$(\xi_1, \xi_2) = (0.0099, 0.0099)$	
	Individual bonus	Pooled bonus
Customer One	1.0008	1.0012
Customer Two	0.9992	1.0012
Sum	2.0000	2.0024

Table 5.3: Individual ξs, $g_1 = g_2 = 3\%$, and $T_1 = T_2 = 10$.

Scenario 1b	$\xi = 0.0099$	
	Individual bonus	Pooled bonus
Customer One	1.0004	1.0008
Customer Two	0.9992	1.0008
Sum	1.9995	2.0015

Table 5.4: Common ξ, $g_1 = g_2 = 3\%$, and $T_1 = T_2 = 10$.

and $g_2 = 3\%$. That is, the customer who enters first is offered a higher minimum rate of return guarantee. The way of distributing the pooled bonus reserve is the same as in scenario four. The only difference compared to scenario four is that the amount of the pooled bonus reserve evolves differently.

5.5.1 Results

In tables 5.3–5.12 the results of the simulations are given. Looking at the tables we have to consider the following issues in particular:

- Does a redistribution of bonus take place, and who benefits (or is worse off) in the case of a redistribution?
- Are the contracts fair? Together as a whole as well as individually.
- The use of individual ξs versus the use of one common ξ.

With respect to the last question, it is important to note that in scenarios one, two, and three the two contracts as a whole is fair if the sum of their values is equal to two (the amount deposited at date zero), whereas in scenarios four and five the fair value is only 1.6907. The reason is that in scenarios four and five customer two does not enter until date $T = 10$. Since the fair value is a 'date zero' value, we have to discount customer two's deposit (of 1) back to date zero. The discounting is done at the risk free rate, $r = 3.7\%$ which we have used in all the simulations, yielding a present value of $1e^{-0.05(1-0.26)10} = 0.6907$.

Firstly, consider tables 5.3 and 5.4. They should in theory be identical because the two customers have identical contracts with respect to the minimum rate of return

Scenario 2a	$(\xi_1, \xi_2) = (0.0207, 0.0099)$	
	Individual bonus	Pooled bonus
Customer One	0.9997	1.0288
Customer Two	0.9996	0.9602
Sum	1.9993	1.9889

Table 5.5: Individual ξs, $g_1 = 5\%$, $g_2 = 3\%$, and $T_1 = T_2 = 10$.

Scenario 2b	$\xi = 0.0151$	
	Individual bonus	Pooled bonus
Customer One	1.0545	1.0817
Customer Two	0.9550	0.9154
Sum	2.0095	1.9971

Table 5.6: Common ξ, $g_1 = 5\%$, $g_2 = 3\%$, and $T_1 = T_2 = 10$.

guarantee, entry date, and exit date. Hence, all values in the two tables should be either one or two. Because of simulation errors, however, there are small deviations from these values. We use tables 5.3 and 5.4 as indicators of how well the simulations perform. Since these deviations are so small (within 0.15% deviations), the simulation procedure seems to be working quite well.

Table 5.5 shows the effect of pooled bonus if the two customers have different minimum rate of return guarantees. In this table the contracts have different rate of payment fees, ξ, determined so that the contracts are individually fair. Thus, $\xi_1 > \xi_2$ reflects that customer one has a higher minimum rate of return guarantee than customer two. Even though the customers are charged different rates of payment fees, we observe a significant redistribution of bonus from customer two to customer one when the bonus accounts are pooled. That is, the use of a pooled bonus account negatively affects the customer with the lowest minimum rate of return guarantee (as we would expect, since customer one always receives at least as much as customer two from the bonus account in each period and the final bonus is shared equally at maturity). Moreover, notice that the company also benefits from the use of pooled bonus (to a lesser extent), since the payout from the option that the company has issued to customer one is less when customer two is also contributing to the bonus reserve than when bonus is individual.

In table 5.6 we use a common rate of payment fee, ξ, determined such that the sum of the contracts is fair. In this case we see an even further redistribution from customer two to customer one. Most of this redistribution arises from the use of a common rate of payment fee as we see by comparing the contract values for the

Scenario 3a	$(\xi_1, \xi_2) = (0.0065, 0.0101)$	
	Individual bonus	Pooled bonus
Customer One	1.0005	0.9860
Customer Two	0.9987	0.9993
Sum	1.9993	1.9853

Table 5.7: Individual ξs, $g_1 = g_2 = 3\%$, $T_1 = 20$, and $T_2 = 10$.

Scenario 3b	$\xi = 0.0072$	
	Individual bonus	Pooled bonus
Customer One	0.9856	0.9736
Customer Two	1.0254	1.0254
Sum	2.0110	1.9990

Table 5.8: Common ξ, $g_1 = g_2 = 3\%$, $T_1 = 20$, and $T_2 = 10$.

individual and common rate of payment fee both based on individual bonus accounts.

We consider the effect of different exit dates in tables 5.7 and 5.8. When the rates of payment fees are determined individually (table 5.7), we do not observe a very profound effect of pooled bonus. There is only a small redistribution to the company from the customer holding the long maturity contract. The value of the short maturity contract is unaffected by the introduction of pooled bonus. The two customers contribute equally to the bonus reserve until date T_2, where customer two exits. If, at this date, the bonus reserve is negative, customer one carries the whole load, whereas if the bonus reserve is positive, customer two leaves with half of the bonus reserve. This means that the company, de facto, has transferred its liabilities with respect to customer two to customer one.

When we change to a common rate of payment fee (table 5.8), we find the same kind of redistribution to the company from the customer with the long maturity contract. However, in this case this customer is also negatively affected by a redistribution to the customer with the short maturity contract following from the higher (common) rate of payment fee compared to his individually determined rate of payment fee.

Tables 5.9 and 5.10 illustrate the effect of different entry dates. In table 5.9 we see a redistribution to the company from customer one as well as customer two. However, customer one, who enters first, loses more by pooling than customer two, who enters when the first customer is halfway through his contract. We explain this by the following three effects: (i) The bonus that customer one has built by the time the second customer enters is positive (on average under the equivalent martingale measure) when customer one's minimum rate of return guarantee is 3%. Customer two benefits by

Scenario 4a	$(\xi_1, \xi_2) = (0.0065, 0.0099)$	
	Individual bonus	Pooled bonus
Customer One	0.9991	0.9876
Customer Two	0.6914	0.6871
Sum	1.6905	1.6747

Table 5.9: Individual ξs, $g_1 = g_2 = 3\%$, entry date (1)=0, $T_1 = 20$, entry date (2)=10, and $T_2 = 10$.

Scenario 4b	$\xi = 0.0070$	
	Individual bonus	Pooled bonus
Customer One	0.9892	0.9825
Customer Two	0.7091	0.7067
Sum	1.6983	1.6892

Table 5.10: Common ξ, $g_1 = g_2 = 3\%$, entry date (1)=0, $T_1 = 20$, entry date (2)=10, and $T_2 = 10$.

the 'participation' in the bonus already built by customer one. That is, he receives excess return (above the minimum rate of return guarantee) much faster than had the bonus been zero when he entered. This 'sharing' of bonus will of course hurt customer one. However, the second effect (ii) pulls in the opposite direction: the way we have modeled the sharing of the final bonus favorizes customer one slightly (and therefore hurts customer two). Finally (iii), the company benefits since the probability (under the equivalent martingale measure) of having to cover negative bonus for either customer is smaller. In total the effects are such that both customers are worse off and the company better off with pooling, c.f. table 5.9.

As usual we observe (table 5.10) a redistribution from the customer with the low individual rate of payment fee to the customer with the high individual rate of payment fee when we introduce a common rate of payment fee.

Scenario 5a	$(\xi_1, \xi_2) = (0.0173, 0.0101)$	
	Individual bonus	Pooled bonus
Customer One	1.0012	1.0106
Customer Two	0.6902	0.6446
Sum	1.6914	1.6553

Table 5.11: Individual ξs, $g_1 = 5\%$, $g_2 = 3\%$, entry date (1)=0, $T_1 = 20$, entry date (2)=10, and $T_2 = 10$.

Scenario 5b	$\xi = 0.0142$	
	Individual bonus	Pooled bonus
Customer One	1.0619	1.0711
Customer Two	0.6662	0.6210
Sum	1.7280	1.6921

Table 5.12: Common ξ, $g_1 = 5\%$, $g_2 = 3\%$, entry date (1)=0, $T_1 = 20$, entry date (2)=10, and $T_2 = 10$.

Finally, we consider the combined effect of different minimum rate of return guarantees and different entry dates in tables 5.11 and 5.12. This case illustrates the situation (at least as we see it) in Denmark today. In table 5.11 there is a redistribution from the customer with the short maturity contract to both the other customer and the company—the larger part goes to the company. The net redistribution stems from two separate effects working in the same direction. The first effect arises from customer one having a higher minimum rate of return guarantee than customer two as in scenario two. The second effect is due to the different entry dates of the customers as in scenario four. However, in this case (with customer one's high minimum rate of return guarantee, i.e. 5%) the bonus reserve at date $T = 10$ is negative (on average) whereas it was positive in scenario four. Therefore the direction of the redistribution between the customers is the other way around (i.e. from customer two to customer one). Furthermore, the way the rule for sharing final bonus is modeled favorizes customer one, thereby leaving customer two even worse off.

We observe the same kind of redistribution in table 5.12, where we use a common rate of payment fee, as we observed in table 5.11. Moreover, we see the usual redistribution as a result of switching from the individual rate of payment fee to a common rate of payment fee. That is, a redistribution from customer two to customer one. In this combined case the effect is quite profound.

In addition, we have recalculated the different scenarios using different volatilities, σ. However, this does not alter the redistribution effects significantly, and therefore we have not reported the result of these calculations.

There are no major changes in the redistribution effects when we introduce stochastic interest rates in the form of a Vasicek model, c.f. appendix A. However, a small difference in scenario four is worth nothing. With stochastic interest rates customer two is indifferent (or slightly) better off in the case of pooling.

We have also tried to extend the model to include mortality risk. More specifically, we consider a single-premium (with a deposit of 1) contract which pays out 0.5 in the case of death before maturity and the usual amount, $A + B^+$, if the customer survives until maturity. None of our findings are altered significantly by the introduction of mortality risk. The results are reported in appendix B.

5.6 Concluding remarks

In this paper we have set up a model which we think is fairly close to the institutional setup that prevails within the life-insurance and pension-fund industry in Denmark. The model prices contracts with minimum rate of return guarantees using the principle of fair valuation. The minimum rate of return guarantees that we consider are equipped with an option on the final bonus reserve. We use a smooth bonus distribution mechanism in order to even out the annual returns on the customers' accounts. Since we have used the principle of fair valuation to find the terms of the contracts, there is no need for an up-front premium. The customers simply pay for the guarantees by paying an annual fee. Of course they also have to provide an initial deposit when entering into the contracts.

We have looked at two different ways in which the company can collect payment for issuing the contracts. The *direct* method, where the company collects payment by charging a rate of payment fee (i.e. a certain fraction of the amount in the customer's account), and the *indirect* method, where the company receives a fraction of the excess bonus. We have found that the *direct* method allows for a greater variety of contract specifications, that is, different minimum rate of return guarantees and αs. In particular, the rate of payment fee is more or less independent of α. The *direct* method is, in addition, much easier for the company to implement.

The current market practice in Denmark is to charge a rate of payment fee of 0.5% and to offer a minimum rate of return guarantee of 3%.[47] We have shown that under the current market conditions (i.e. an (after-tax) short term interest rate of 3.7% and a volatility of 10% on the reference portfolio) the offered contracts are fair if their maturity is thirty years (see figure 5.3). This illustrates, according to our model, that the companies have charged a correct premium for the minimum rate of return guarantees issued.[48] The companies in Denmark today claim that with the current low interest rate level it is difficult to construct investment portfolios which yield a return distribution sufficient to cover the issued guarantees, hence they indirectly claim that the contracts are not fair (indeed favorable to the customers). Therefore they wish to lower the minimum rate of return on the already established contracts. This is, however, regulated by legislation and it is therefore a decision to be made by the politicians.

[47] Up until spring 1999 companies have offered contracts with minimum rate of return guarantees of 3%. However, during 1999 some companies lowered their minimum rate of return guarantee offered to new customers, after having experienced difficulties finding investment opportunities with returns high enough to cover the guarantee. In June 1999 the Danish authorities lowered the maximum allowed minimum rate of return guarantee to 2%.

[48] Note that with mortality risk (as it is modeled in appendix B) a similar thirty year contract is slightly favorable for the company.

Since the current contracts are fair according to our model, reasons for the inadequate investment opportunities must be found outside our model. This could be related to e.g. incomplete markets, in the form of lacking trading (or hedging) opportunities and/or transactions costs.

Moreover, our model has shown that the practice of pooling the inhomogeneous customers' bonus reserves makes the company better off leaving at least one of the customers worse off. This weakens the companies' claim even further, since a thirty year contract with a minimum rate of return guarantee of 3% and a 0.5% rate of payment fee must be a favorable contract for the companies if the customers are entering or leaving at different dates.

Lately in Denmark there has been a lot of discussion about whether old customers with a 5% minimum rate of return guarantee 'cheat' new(er) customers with a minimum rate of return guarantee of 3%. In our model we have shown that this is, in fact, the case. More precisely, figures from scenario five, c.f. table 5.12, show a redistribution in the area of 10% of the initial deposit from new(er) customers to old customers. This last observation indicates that the companies should keep track of each customer's bonus reserve separately and not pool them, implying that they should also calculate an individual rate of payment fee for each customer.

It is important to remember that the findings of our model are limited by the Black-Scholes/Merton assumptions of log-normally distributed asset returns and complete markets with either a constant interest rate or a Vasicek term structure of interest rate model. In addition, we have only analyzed the case of one initial deposit by the customer.[49] Interesting extensions of our model would be to consider different hedging aspects in markets with some degree of incompleteness and the incentive issues of premature surrender of contracts.

[49]New results indicate that introducing annual payments for minimum rate of return guarantee contracts does not alter the redistribution effects in the two-customer case except in scenario four. In this scenario customer two is better off with pooling contrary to the case with one initial deposit. The ad hoc distribution rules for terminal bonus might be driving the result since the rules are slightly more complicated when annual payments are introduced.

A Stochastic interest rates

In order to study effects of stochastic interest rates we introduce a term structure of interest rates as modeled by Vasicek (1977). Using the same dynamics for the value of the reference portfolio as earlier, we have the following dynamics of the short interest rates and the return on the reference portfolio

$$dr(t) = \kappa\big(\theta - r(t)\big)dt + \sigma_r dW^1(t) \tag{A.1}$$

and

$$d\delta(t) = \big(r(t) - \frac{1}{2}\sigma_\delta^2\big)dt + \sigma_\delta \rho dW^1(t) + \sigma_\delta\sqrt{1-\rho^2}dW^2(t), \tag{A.2}$$

where W^1 and W^2 are two uncorrelated Brownian motions and ρ is the correlation coefficient between the interest rate and the return. κ, θ, σ_r, and σ_δ are constants. We use $\kappa = 0.30723$, $\theta = 3.7\%$, $\sigma_r = 2.258\%$, and $\sigma_\delta = 10\%$.[50]

For discounting purposes we need to keep track of the variable, β, defined by

$$\beta(t) = \int_0^t r(u)du.$$

Note that with stochastic interest rates the date zero value of a deposit of one unit at date t is given by the date zero value of the zero-coupon bond with maturity date t. Since we are using a Vasicek term structure of interest rates model, we have a closed form solution for the zero-coupon bond price. With an initial interest rate of 3.7% and the parameter values above the zero-coupon bond price with a maturity of ten years is 0.7009.

In order to do the simulations we need the simultaneous distribution of $r(t)$, $\delta(t)$, and $\beta(t)$ conditional on the information at date s. Tedious calculations give

$$\left\{\begin{array}{c|c} r(t) & r(s) \\ \delta(t) & \delta(s) \\ \beta(t) & \beta(s) \end{array}\right\} \sim N\left(\left\{\begin{array}{c} E_s r(t) \\ E_s\delta(t) \\ E_s\beta(t) \end{array}\right\}, \left\{\begin{array}{lll} \mathrm{var}_s\, r(t) & \mathrm{Cov}_s\big(r(t),\delta(t)\big) & \mathrm{Cov}_s\big(r(t),\beta(t)\big) \\ \mathrm{Cov}_s\big(\delta(t),r(t)\big) & \mathrm{var}_s\,\delta(t) & \mathrm{Cov}_s\big(\delta(t),\beta(t)\big) \\ \mathrm{Cov}_s\big(\beta(t),r(t)\big) & \mathrm{Cov}_s\big(\beta(t),\delta(t)\big) & \mathrm{var}_s\,\beta(t) \end{array}\right\}\right), \tag{A.3}$$

where the means, variances, and covariances (conditional on the date s information)

[50] κ and σ_r are obtained from Jørgensen, Miltersen, and Sørensen (1996). θ is set at the same level as our initial short term interest rate.

are calculated as

$$
\begin{aligned}
E_s r(t) &= \big(r(s)-\theta\big)e^{-\kappa(t-s)}+\theta,\\
E_s \delta(t) &= \delta(s)+\frac{1}{\kappa}\big(1-e^{-\kappa(t-s)}\big)\big(r(s)-\theta\big)+\big(\theta-\frac{1}{2}\sigma_\delta^2\big)(t-s),\\
E_s \beta(t) &= \beta(s)+\frac{1}{\kappa}\big(1-e^{-\kappa(t-s)}\big)\big(r(s)-\theta\big)+\theta(t-s),
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{var}_s r(t) &= \frac{\sigma_r^2}{2\kappa}\big(1-e^{-2\kappa(t-s)}\big),\\
\operatorname{var}_s \delta(t) &= \big(\frac{\sigma_r^2}{\kappa^2}+\sigma_\delta^2+\frac{2\sigma_r\sigma_\delta\rho}{\kappa}\big)(t-s)-\frac{2\sigma_r}{\kappa^2}\big(\frac{\sigma_r}{\kappa}+\sigma_\delta\rho\big)\big(1-e^{-\kappa(t-s)}\big)\\
&\quad+\frac{\sigma_r^2}{2\kappa^3}\big(1-e^{-2\kappa(t-s)}\big),\\
\operatorname{var}_s \beta(t) &= \frac{\sigma_r^2}{\kappa^2}\Big((t-s)-\frac{2}{\kappa}\big(1-e^{-\kappa(t-s)}\big)+\frac{1}{2\kappa}\big(1-e^{-2\kappa(t-s)}\big)\Big),\\
\operatorname{Cov}_s\big(r_t,\delta_t\big) &= \frac{\sigma_r^2}{\kappa^2}\big(1-e^{-\kappa(t-s)}\big)-\frac{\sigma_r^2}{2\kappa^2}\big(1-e^{-2\kappa(t-s)}\big)+\frac{\sigma_r\sigma_\delta\rho}{\kappa}\big(1-e^{-\kappa(t-s)}\big),\\
\operatorname{Cov}_s\big(r(t),\beta(t)\big) &= \frac{\sigma_r^2}{\kappa}\Big(\frac{1}{\kappa}\big(1-e^{-\kappa(t-s)}\big)-\frac{1}{2\kappa}\big(1-e^{-2\kappa(t-s)}\big)\Big),\\
\operatorname{Cov}_s\big(\delta(t),\beta(t)\big) &= \frac{\sigma_r^2}{\kappa^2}\Big((t-s)-\frac{2}{\kappa}\big(1-e^{-\kappa(t-s)}\big)+\frac{1}{2\kappa}\big(1-e^{-2\kappa(t-s)}\big)\Big)\\
&\quad+\frac{\sigma_r\sigma_\delta\rho}{\kappa}\Big((t-s)-\frac{1}{\kappa}\big(1-e^{-\kappa(t-s)}\big)\Big).
\end{aligned}
$$

A.1 Results

	α (%)										
ξ (%)	0	10	20	30	40	50	60	70	80	90	100
0.0025	—	—	—	—	—	—	—	—	—	—	—
0.50	0.668	0.736	0.750	0.612	0.600	0.471	0.333	0.257	0.101	-0.014	-0.174
0.75	1.595	1.679	1.708	1.692	1.592	1.539	1.458	1.336	1.330	1.193	1.126
1.00	2.327	2.347	2.395	2.379	2.337	2.291	2.231	2.196	2.119	2.044	1.998
1.25	2.954	2.956	3.054	3.041	2.986	2.955	2.883	2.857	2.818	2.726	2.657
1.50	3.515	3.518	3.567	3.544	3.516	3.517	3.477	3.422	3.383	3.344	3.303
1.75	3.956	3.978	3.976	4.018	4.029	4.009	3.968	3.920	3.909	3.870	3.844
2.00	4.390	4.422	4.458	4.463	4.457	4.440	4.410	4.386	4.331	4.295	4.300
2.25	4.828	4.812	4.862	4.868	4.878	4.837	4.827	4.785	4.773	4.738	4.727
2.50	5.191	5.216	5.223	5.221	5.250	5230	5.233	5.194	5.183	5.136	5.125

Table A.1: Values of g in percent (%) for different choices of ξ and α in percent (%); Stochastic interest rates, $\sigma_\delta = 10\%$, $\sigma_r = 2.258\%$, $\kappa = 0.30723$, $T = 10$, $\rho = 0$, and $\theta = 3.7\%$.

Results for a 10 year contract and three different choices of correlation coefficients are given in tables A.1–A.3 for the one customer case. We observe that the fair minimum rate of return guarantees offered increase as the correlation coefficient changes from

	α (%)										
ξ (%)	0	10	20	30	40	50	60	70	80	90	100
0.25	—	—	—	—	—	—	—	—	—	—	—
0.50	—	—	—	—	—	—	—	—	—	—	—
0.75	0.691	0.840	0.840	0.861	0.815	0.691	0.626	0.570	0.446	0.329	0.266
1.00	1.565	1.661	1.720	1.705	1.656	1.629	1.526	1.514	1.411	1.362	1.246
1.25	2.192	2.292	2.381	2.348	2.340	2.327	2.300	2.237	2.160	2.119	2.066
1.50	2.805	2.952	2.996	2.984	2.985	2.920	2.904	2.864	2.806	2.725	2.757
1.75	3.354	3.461	3.499	3.518	3.515	3.495	3.428	3.422	3.381	3.358	3.303
2.00	3.848	3.864	3.950	4.010	4.011	3.983	3.933	3.941	3.901	3.885	3.812
2.25	4.344	4.351	4.427	4.425	4.435	4.413	4.417	4.364	4.323	4.346	4.302
2.50	4.754	4.814	4.827	4.853	4.853	4.852	4.810	4.809	4.780	4.773	4.718

Table A.2: Values of g in percent (%) for different choices of ξ and α in percent (%); Stochastic interest rates, $\sigma_\delta = 10\%$, $\sigma_r = 2.258\%$, $\kappa = 0.30723$, $T = 10$, $\rho = 0.5$, and $\theta = 3.7\%$.

	α (%)										
ξ (%)	0	10	20	30	40	50	60	70	80	90	100
0.25	0.583	0.701	0.602	0.535	0.221	0.138	-0.103	-0.294	-0.472	-0.722	-0.938
0.50	1.766	1.826	1.812	1.758	1.701	1.560	1.507	1.345	1.220	1.168	0.976
0.75	2.573	2.577	2.584	2.560	2.493	2.436	2.359	2.250	2.192	2.127	2.028
1.00	3.168	3.178	3.198	3.214	3.175	3.101	3.050	2.964	2.921	2.885	2.785
1.25	3.676	3.714	3.727	3.710	3.671	3.622	3.617	3.562	3.518	3.451	3.409
1.50	4.144	4.136	4.169	4.144	4.150	4.122	4.094	4.046	4.003	3.977	3.932
1.75	4.581	4.545	4.565	4.587	4.544	4.547	4.513	4.478	4.440	4.399	4.367
2.00	4.947	4.950	4.938	4.942	4.940	4.939	4.886	4.874	4.835	4.815	4.776
2.25	5.304	5.281	5.306	5.310	5.284	5.294	5.263	5.257	5.220	5.199	5.167
2.50	5.634	5.630	5.644	5.626	5.636	5.622	5.602	5.588	5575	5.547	5.526

Table A.3: Values of g in percent (%) for different choices of ξ and α in percent (%); Stochastic interest rates, $\sigma_\delta = 10\%$, $\sigma_r = 2.258\%$, $\kappa = 0.30723$, $T = 10$, $\rho = -0.5$, and $\theta = 3.7\%$.

positive to zero and to negative, c.f. e.g. tables A.1–A.3. More specifically, in the case of positive and zero correlation coefficient the minimum rate of return guarantees are lower than the corresponding guarantees with constant interest rates, c.f. table 5.1, whereas they are similar for the case of a negative correlation coefficient. A positive correlation between the returns and the interest rates means that whenever the interest rate increases (decreases) so does the return on the portfolio. Hence, there is a greater variability in the contract payoff and the customer has to pay for the larger variation through a higher rate of payment fee or equivalently a lower minimum rate of return guarantee. In other words, since the customer's contract includes an option on the final bonus (determined by the return process, δ), an increase in the volatility of the underlying return process raises the option premium, i.e. the rate of payment fee for a given guarantee.

Hence, with this argument a correlation coefficient of zero should give the same rate of payment fees/minimum rate of return guarantees as in the case with a constant interest rate. However, we observe higher rate of payment fees/lower minimum rate of return guarantees indicating a 'hidden' positive correlation between the interest rate and the return on the reference portfolio. This 'hidden' correlation follows from the way the interest rate enters into the drift of the return (i.e. drift term $= r_t - \frac{1}{2}\sigma_\delta^2$). Therefore, the overall correlations are, in general, higher than the correlations illustrated by the correlation coefficients, ρ. In fact, the case with $\rho = -0.5$ gives almost identical rate of payment fees/minimum rate of return guarantees as the case with constant interest rates.

Scenario 1a	$\rho = 0$		$\rho = 0.5$		$\rho = -0.5$	
(ξ_1, ξ_2)	$(0.0124, 0.0124)$		$(0.0151, 0.0151)$		$(0.0092, 0.0092)$	
	Ind. bonus	Pooled	Ind. bonus	Pooled	Ind. bonus	Pooled
Customer One	0.9991	1.0000	0.9999	1.0006	0.9992	0.9996
Customer Two	0.9989	1.0000	0.9987	1.0006	0.9997	0.9996
Sum	1.9980	2.0001	1.9986	2.0013	1.9989	1.9991

Table A.4: Individual ξs with stochastic interest rates, $g_1 = g_2 = 0.03$, and $T_1 = T_2 = 10$.

The results for the two customer case are given in tables A.4-A.13. In scenarios one to three the effects are the same as in the constant interest rate case. That is, in scenario two the customer with the high minimum rate of return guarantee (customer one) and the company both benefit from pooling while the customer with the low minimum rate of return guarantee (customer two) is hurt by pooling. Recall that the higher the correlation is the higher is the variation or volatility on the return. Customer one has a higher minimum rate of return guarantee than customer two and he therefore receives some of customer two's bonus. This can be interpreted as if customer two gives customer one an asset with uncertain payments with values that may rise or fall from period to period just as a stock and hence more volatility is bad for customer one (who has the asset) and good for customer two (who is short the asset). This

Scenario 1b	$\rho = 0$		$\rho = 0.5$		$\rho = -0.5$	
ξ	0.0123		0.0150		0.0091	
	Ind. bonus	Pooled	Ind. bonus	Pooled	Ind. bonus	Pooled
Customer One	1.0006	1.0012	0.9995	1.0001	1.0006	1.0014
Customer Two	1.0008	1.0012	1.0005	1.0001	1.0008	1.0014
Sum	2.0014	2.0025	2.0000	2.0001	2.0015	2.0028

Table A.5: Common ξ with stochastic interest rates, $g_1 = g_2 = 0.03$, and $T_1 = T_2 = 10$.

Scenario 2a	$\rho = 0$		$\rho = 0.5$		$\rho = -0.5$	
(ξ_1, ξ_2)	$(0.0235, 0.0123)$		$(0.0261, 0.0150)$		$(0.0204, 0.0091)$	
	Ind. bonus	Pooled	Ind. bonus	Pooled	Ind. bonus	Pooled
Customer One	1.0001	1.0250	0.9996	1.0237	1.0004	1.0266
Customer Two	1.0006	0.9620	0.9996	0.9641	1.0005	0.9587
Sum	2.0007	1.9870	1.9993	1.9878	2.0009	1.9853

Table A.6: Individual ξs with stochastic interest rates, $g_1 = 0.05$, $g_2 = 0.03$, and $T_1 = T_2 = 10$.

Scenario 2b	$\rho = 0$		$\rho = 0.5$		$\rho = -0.5$	
ξ	0.0177		0.0204		0.0144	
	Ind. bonus	Pooled	Ind. bonus	Pooled	Ind. bonus	Pooled
Customer One	1.0553	1.0824	1.0540	1.0795	1.0582	1.0862
Customer Two	0.9549	0.9185	0.9564	0.9215	0.9542	0.9140
Sum	2.0102	2.0008	2.0104	2.0010	2.0124	2.0003

Table A.7: Common ξ with stochastic interest rates, $g_1 = 0.05$, $g_2 = 0.03$, and $T_1 = T_2 = 10$.

explains why for higher correlation customer one is relatively worse off and customer two is relatively better off. The company also benefits from customer two because of customer two's contribution to the common bonus reserve. This can be interpreted as the company receiving an asset from customer two (with less value than the asset to customer one) and the relative redistribution effects are similar to those for customer one, i.e. the company is (marginally) worse off the higher the variability or correlation is. The redistribution effects for scenario 3 are more or less the same with and without stochastic interest rates, that is, the company benefits on account of customer one. The pooled bonus reserve might be negative when customer two exits the contract and this negative bonus has to be covered by customer one. This means that customer one has in fact written an option on the bonus reserve (with maturity at customer two's exit date) to the company. This option is more expensive for high volatility and

Scenario 3a	$\rho = 0$		$\rho = 0.5$		$\rho = -0.5$	
(ξ_1, ξ_2)	$(0.0090, 0.0124)$		$(0.0112, 0.0151)$		$(0.0063, 0.0092)$	
	Ind. bonus	Pooled	Ind. bonus	Pooled	Ind. bonus	Pooled
Customer One	0.9968	0.9832	0.9973	0.9808	0.9984	0.9863
Customer Two	0.9994	0.9976	0.9996	0.9978	0.9994	0.9977
Sum	1.9962	1.9809	1.9968	1.9786	1.9978	1.9840

Table A.8: Individual ξs with stochastic interest rates, $g_1 = g_2 = 0.03$, $T_1 = 20$ and $T_2 = 10$.

Scenario 3b	$\rho = 0$		$\rho = 0.5$		$\rho = -0.5$	
ξ	0.0095		0.0117		0.0067	
	Ind. bonus	Pooled	Ind. bonus	Pooled	Ind. bonus	Pooled
Customer One	0.9904	0.9782	0.9892	0.9752	0.9920	0.9806
Customer Two	1.0260	1.0269	1.0295	1.0304	1.0211	1.0223
Sum	2.0165	2.0051	2.0187	2.0056	2.0131	2.0028

Table A.9: Common ξ with stochastic interest rates, $g_1 = g_2 = 0.03$, $T_1 = 20$ and $T_2 = 10$.

therefore customer one is relatively worse off and the company relatively better off the higher correlation coefficient, ρ, is. The main differences between the stochastic and

Scenario 4a	$\rho = 0$		$\rho = 0.5$		$\rho = -0.5$	
(ξ_1, ξ_2)	$(0.0090, 0.0122)$		$(0.0112, 0.0150)$		$(0.0063, 0.0089)$	
	Ind. bonus	Pooled	Ind. bonus	Pooled	Ind. bonus	Pooled
Customer One	1.0007	0.9912	1.0011	0.9897	1.0008	0.9921
Customer Two	0.7027	0.7048	0.7022	0.7030	0.7026	0.7063
Sum	1.7034	1.6959	1.7033	1.6927	1.7034	1.6984

Table A.10: Individual ξs with stochastic interest rates, $g_1 = g_2 = 0.03$, entry date (1) =0, $T_1 = 20$, entry date (2) = 10, and $T_2 = 10$.

the constant interest rate cases are seen in scenario 4. The customer who enters the contract first (customer one) is more or less unaffected by stochastic interest rates, meaning, he is still left worse off by pooling. Customer two, (the one who enters last) however, is now indifferent or even slightly better off by pooling as opposed to being worse off in the constant interest rate case. This of course means that the company does not benefit as much by pooling when there are stochastic interest rates.

Recall the three effects that played a role in the redistribution in scenario four in the case with constant interest rates. First, (i) customer two gains from participating in customer one's bonus (and hence customer one is worse off), secondly (ii) customer one is slightly favorized by the sharing rule for the terminal bonus, and last (iii) the

Scenario 4b	$\rho = 0$		$\rho = 0.5$		$\rho = -0.5$	
ξ	0.0096		0.0118		0.0068	
	Ind. bonus	Pooled	Ind. bonus	Pooled	Ind. bonus	Pooled
Customer One	0.9882	0.9812	0.9869	0.9799	0.9905	0.9825
Customer Two	0.7180	0.7209	0.7210	0.7230	0.7151	0.7196
Sum	1.7061	1.7022	1.7079	1.7029	1.7056	1.7022

Table A.11: Common ξ with stochastic interest rates, $g_1 = 0.05$, $g_2 = 0.03$, entry date (1) =0, $T_1 = 20$, entry date (2) = 10, and $T_2 = 10$.

Scenario 5a	$\rho = 0$		$\rho = 0.5$		$\rho = -0.5$	
(ξ_1, ξ_2)	$(0.0203, 0.0124)$		$(0.0223, 0.0150)$		$(0.0180, 0.0092)$	
	Ind. bonus	Pooled	Ind. bonus	Pooled	Ind. bonus	Pooled
Customer One	1.0009	1.0091	1.0008	1.0068	1.0004	1.0102
Customer Two	0.7007	0.6640	0.7009	0.6658	0.7006	0.6616
Sum	1.7016	1.6731	1.7017	1.6726	1.7010	1.6718

Table A.12: Individual ξs with stochastic interest rates, $g_1 = 0.05$, $g_2 = 0.03$, entry date (1) =0, $T_1 = 20$, entry date (2) = 10, and $T_2 = 10$.

Scenario 5b	$\rho = 0$		$\rho = 0.5$		$\rho = -0.5$	
ξ	0.0173		0.0194		0.0148	
	Ind. bonus	Pooled	Ind. bonus	Pooled	Ind. bonus	Pooled
Customer One	1.0577	1.0663	1.0548	1.0603	1.0640	1.0734
Customer Two	0.6721	0.6347	0.6760	0.6393	0.6667	0.6280
Sum	1.7298	1.7010	1.7308	1.6996	1.7307	1.7014

Table A.13: Common ξ with stochastic interest rates, $g_1 = 0.05$, $g_2 = 0.03$, entry date (1) =0, $T_1 = 20$, entry date (2) = 10, and $T_2 = 10$.

company benefits from an improvement in the probability of having to cover a negative bonus for either customer. This effect we will call the diversification effect.

When we observe a bonus account above the average level, it must be because the past returns have been high. Hence, in this case it is more likely that the stochastic short term interest rate is also high. This, moreover, implies that the future returns are more likely to be large. Therefore, if we observe a bonus account above the average level at any given date, the likelihood of a terminal bonus account above the average level is higher with stochastic interest rates than with constant interest rates and vice versa for a bonus account below the average level. Thus, it is less likely that one of the customers' terminal bonus account is small when that of the other customer is large. Hence, it is also less likely that the two customers' accounts have opposite sign. Since the company stands to gain by pooling exactly when the terminal bonus accounts are of opposite sign (effect (*iii*)), the company gains less when interest rates are stochastic.

The higher likelihood of observing a bonus account above average at a given date simultaneously with higher future returns after this date also explains the higher redistribution from customer one to customer two (effect (*i*)) with stochastic interest rates. The reason is that a significant redistribution requires both a bonus account above average when customer two enters and high returns after this date. In scenario five, which is a combination of scenario two and scenario four, the redistribution effects are similar to those for the constant interest rates for the three different correlation coefficients. Therefore, we will not comment further on these effects.

B Mortality risk

In this section we study the isolated effects of mortality risk. That is, we go back to the assumption of constant interest rates. We assume that mortality risk can be diversified away by the law of large numbers. That is, the company has a large pool of homogeneous customers which die independently of each other. Each customer has a known probability distribution for date of death.[51] Hence, the company can use this probability distribution for the individual customer to determine the fraction of the pool of customers who die in each period. Moreover, we assume that mortality risk is orthogonal to financial risk.

In particular, we look at a contract that in the case of death (of the customer) before maturity pays out a fixed amount, Y, at the end of the year death occurs or pays out the usual amount, $A + B^+$, if the customer survives until maturity. In the case with one initial deposit of 1 the fixed amount, Y, is set equal to 0.5 for a 10 year contract. The contract we consider is hence a combination of a common life insurance (i.e. a term insurance) and a pension contract (a pure endowment)—this contract is also known as an endowment insurance.[52]

Based on the G82 foundation[53] the probability that an x year old customer dies before reaching age $x + n$ is[54]

$$_{n}q_x = 1 - {}_{n}p_x = 1 - e^{-An - \frac{B}{\ln c}(c^{x+n} - c^x)},$$

where A, B, and c are constants which are estimated on the basis of past observations of deaths in a population. We use, $A = 0.0005075787$, $B = 0.000039342435$, $c = 1.10291509$, and $x = 30$ years. The constants are obtained from Delbaen (1986).

Every year some customers die and the amounts that must be paid to them are subtracted from the account held by the pool of customers, i.e. each year, say t, the probability of death times the death sum, $_{t+1+x}q_{t+x}Y$, is subtracted from the A account. This amount is also subtracted from the reference portfolio, the X account, since the company must liquidate some of the portfolio in order to pay the customer (his heirs) the death sum. Thus, we take into account that out of a large pool of customers only a certain fraction will survive until maturity of their contracts and thereby receive $A + B^+$.

[51]The distribution is estimated from historical data and is commonly parameterized by Makeham's formula.

[52]See Black and Skipper (1994).

[53]The G82 foundation is the set of 'rules' or principles that all life insurance companies in Denmark must follow. It lays out valuation principles and what types of contracts may be offered.

[54]The formula arises from the Makeham formula.

	α (%)										
ξ (%)	0	10	20	30	40	50	60	70	80	90	100
0.25	0.73	0.75	0.77	0.56	0.52	0.37	0.28	0.07	-0.15	-0.24	-0.39
0.50	1.78	1.79	1.85	1.74	1.71	1.60	1.55	1.49	1.32	1.25	1.11
0.75	2.56	2.52	2.59	2.57	2.51	2.46	2.43	2.34	2.24	2.18	2.07
1.00	3.14	3.15	3.18	3.17	3.15	3.11	3.09	3.02	2.97	2.91	2.84
1.25	3.70	3.66	3.70	3.73	3.70	3.67	3.61	3.59	3.54	3.46	3.44
1.50	4.14	4.13	4.18	4.21	4.16	4.16	4.12	4.09	4.03	4.00	3.95
1.75	4.55	4.59	4.60	4.60	4.60	4.58	4.54	4.53	4.51	4.46	4.43
2.00	4.99	4.97	4.99	5.00	4.99	4.98	4.96	4.92	4.91	4.87	4.83
2.25	5.36	5.34	5.37	5.38	5.37	5.35	5.34	5.32	5.29	5.27	5.25
2.50	5.70	5.71	5.72	5.74	5.72	5.72	5.70	5.67	5.65	5.63	5.62

Table B.1: Values of g in percent (%) for different choices of ξ and α in percent (%); Mortality risk included, fixed death sum of $\frac{1}{2}$ and $T = 10$.

Scenario 1a	$(\xi_1, \xi_2) = (0.0099, 0.0099)$	
	Individual bonus	Pooled bonus
Customer One	0.9997	1.0003
Customer Two	1.0001	1.0003
Sum	1.9998	2.0006

Table B.2: Individual ξs with mortality risk, death sum equal to 0.5, $g_1 = g_2 = 3\%$, and $T_1 = T_2 = 10$.

B.1 Results

In table B.1 we give the fair values of the minimum rate of return guarantee, g, for different combinations of rate of payment fee, ξ, and α for this contract. We observe that the minimum rate of return guarantee is slightly higher, in general, when we include mortality risk, (c.f. table 5.1). Intuitively, the reason for this is that funds are withdrawn from the accounts in the event of death which implies that the amount the company must guarantee is smaller. Hence, a fair contract has a higher minimum rate of return guarantee, *ceteris paribus*. The way the minimum rate of return guarantee changes as function of ξ and/or α is, however, similar to the case with no mortality risk.

The results for the two-customer case (or equivalently two different groups of homogenous customers) are given in tables B.2-B.11. All the results are identical to the case without mortality risk.

Scenario 1b	$\xi = 0.0099$	
	Individual bonus	Pooled bonus
Customer One	0.9993	1.0002
Customer Two	0.9998	1.0002
Sum	1.9992	2.0003

Table B.3: Common ξ with mortality risk, death sum equal to 0.5, $g_1 = g_2 = 3\%$, and $T_1 = T_2 = 10$.

Scenario 2a	$(\xi_1, \xi_2) = (0.0207, 0.0099)$	
	Individual bonus	Pooled bonus
Customer One	1.0004	1.0290
Customer Two	1.0013	0.9611
Sum	2.0018	1.9901

Table B.4: Individual ξs with mortality risk, death sum equal to 0.5, $g_1 = 5\%$, $g_2 = 3\%$, and $T_1 = T_2 = 10$.

Scenario 2b	$\xi = 0.0150$	
	Individual bonus	Pooled bonus
Customer One	1.0551	1.0832
Customer Two	0.9566	0.9174
Sum	2.0117	2.0006

Table B.5: Common ξ with mortality risk, death sum equal to 0.5, $g_1 = 5\%$, $g_2 = 3\%$, and $T_1 = T_2 = 10$.

Scenario 3a	$(\xi_1, \xi_2) = (0.0064, 0.0100)$	
	Individual bonus	Pooled bonus
Customer One	1.0026	0.9874
Customer Two	0.9993	1.0000
Sum	2.0019	1.9874

Table B.6: Individual ξs with mortality risk, death sum equal to 0.5, $g_1 = g_2 = 3\%$, $T_1 = 20$, and $T_2 = 10$.

Scenario 3b	$\xi = 0.0072$	
	Individual bonus	Pooled bonus
Customer One	0.9859	0.9746
Customer Two	1.0239	1.0244
Sum	2.0098	1.9990

Table B.7: Common ξ with mortality risk, death sum equal to 0.5, $g_1 = g_2 = 3\%$, $T_1 = 20$, and $T_2 = 10$.

Scenario 4a	$(\xi_1, \xi_2) = (0.0065, 0.0099)$	
	Individual bonus	Pooled bonus
Customer One	0.9991	0.9876
Customer Two	0.6914	0.6871
Sum	1.6905	1.6747

Table B.8: Individual ξs with mortality risk, death sum equal to 0.5, $g_1 = g_2 = 3\%$, entry date (1)=0, $T_1 = 20$, entry date (2)=10, and $T_2 = 10$.

Scenario 4b	$\xi = 0.0070$	
	Individual bonus	Pooled bonus
Customer One	0.9893	0.9826
Customer Two	0.7091	0.7066
Sum	1.6984	1.6892

Table B.9: Common ξ with mortality risk, death sum equal to 0.5, $g_1 = g_2 = 3\%$, entry date (1)=0, $T_1 = 20$, entry date (2)=10, and $T_2 = 10$.

Scenario 5a	$(\xi_1, \xi_2) = (0.0174, 0.0099)$	
	Individual bonus	Pooled bonus
Customer One	1.0003	1.0079
Customer Two	0.6901	0.6449
Sum	1.6904	1.6528

Table B.10: Individual ξs with mortality risk, death sum equal to 0.5, $g_1 = 5\%$, $g_2 = 3\%$, entry date (1)=0, $T_1 = 20$, entry date (2)=10, and $T_2 = 10$.

Scenario 5b	$\xi = 0.0142$	
	Individual bonus	Pooled bonus
Customer One	1.0617	1.0709
Customer Two	0.6663	0.6212
Sum	1.7281	1.6921

Table B.11: Common ξ with mortality risk, death sum equal to 0.5, $g_1 = 5\%$, $g_2 = 3\%$, entry date (1)=0, $T_1 = 20$, entry date (2)=10, and $T_2 = 10$.

C Supplements to Hansen and Miltersen

C.1 Stochastic interest rates

In section 5.1–5.6 and appendix A and B, Hansen and Miltersen (1999) was presented. Stochastic interest rates were introduced into the model in the form of a Vasicek term structure of interest rates. This section contains the calculations of the simultaneous conditional distribution of the interest rates and the returns used in Hansen and Miltersen (1999).

Recall that the dynamics of the return on the the benchmark portfolio is given as

$$d\delta(t) = (r - \frac{1}{2}\sigma_\delta^2)dt + \sigma_\delta dW^\delta(t) \tag{C.1}$$

under the equivalent martingale measure, Q. The interest rate, r, and the volatility, σ_δ, are constant. We assume a Vasicek term structure of interest rates, that is, the dynamics of the short interest rate under Q is given as

$$dr(t) = \kappa(\theta - r(t))dt + \sigma_r dW^r(t), \tag{C.2}$$

where κ, θ, and σ_r are constants. The two Brownian motions W^δ and W^r are correlated with correlation coefficient ρ. Replacing the interest rate, r, in (C.1) with $r(t)$ and rewriting (C.2) and (C.1) in terms of two uncorrelated Brownian motions under Q, W^1, and W^2 yields

$$dr(t) = \kappa(\theta - r(t))dt + \sigma_r dW^1(t) \tag{C.3}$$

$$d\delta(t) = (r(t) - \frac{1}{2}\sigma_\delta^2)dt + \sigma_\delta \rho dW^1(t) + \sigma_\delta\sqrt{1-\rho^2}\, dW^2(t). \tag{C.4}$$

We need to find the conditional simultaneous distribution of $r(t)$ and $\delta(t)$ given the information at time $s \leq t$. The reason we need the conditional distribution and not just the distribution is that the distribution of the annual returns on the benchmark portfolio in year t depends on the interest rate at date $t-1$ (i.e. for the period $[t-1, t]$). Moreover, we need to keep track of the interest rates over the life time of the contract we are pricing (denoted T) since we need to discount future payoffs using $e^{-\int_0^T r(u)du}$ and not e^{-rT} as we did when the interest rate was constant. This means that we also need the distribution $\int_0^T r(u)du$ given the information at any given date. The future payoff is the amount on the customer's account, $A(T)$, plus the bonus reserve if it is positive, i.e. $B^+(T)$.

We denote $\int_0^t r(u)du$ by $\beta(t)$, that is,

$$\beta(t) = \int_0^t r(u)du.$$

All in all we have to find the simultaneous distribution of $r(t)$, $\delta(t)$, and $\beta(t)$ conditional on the information available at date $s \leq t$ (i.e. $\mathcal{F}_s$, the filtration generated by the Brownian motions, W^1 and W^2).

The dynamics of β is given through the use of Ito's Lemma:

$$d\beta(t) = r(t)dt.$$

Therefore we can write the dynamics of the vector $(r(t), \delta(t), \beta(t))'$ (where ' denotes the transpose) as

$$d\begin{bmatrix} r(t) \\ \delta(t) \\ \beta(t) \end{bmatrix} = \left(\begin{bmatrix} -\kappa & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} r(t) \\ \delta(t) \\ \beta(t) \end{bmatrix} + \begin{bmatrix} \kappa\theta \\ -\frac{1}{2}\sigma_\delta^2 \\ 0 \end{bmatrix} \right) dt + \begin{bmatrix} -\sigma_r & 0 \\ \sigma_\delta \rho & \sigma_\delta \sqrt{1-\rho^2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} dW^1(t) \\ dW^2(t) \end{bmatrix}. \tag{C.5}$$

Denote the vectors $(r(t), \delta(t), \beta(t))'$ and $(W^1(t), W^2(t))'$ by $p(t)$ and $W(t)$, respectively. From (C.5) we see that the dynamics of p is of the form

$$dp(t) = \big(a(t)p(t) + m(t)\big)dt + b(t)dW(t), \tag{C.6}$$

where $a(t)$ and $b(t)$ are 3×3 and 3×2 matrices, respectively, and $m(t)$ is a 3-dimensional vector. We write $a(t)$, $b(t)$, and $m(t)$ to illustrate that we could also allow for time dependence, i.e. the transformation results also hold for dynamics with time-varying parameters instead of the constants in (C.5).

In order to be able to simulate without taking small time steps we need to find a transformation, f, of p that will give a system of dynamics where the drift and volatility terms only depend on time and not on p itself.

Let $Y(t) = f(t, p)$, $f : \mathcal{R}^3 \times \mathcal{R} \to \mathcal{R}^3$ denote the transformation of $p(t)$. Ito's Lemma yields

$$dY(t) = f_t dt + f_p(a(t)p(t) + m(t))dt + f_p b(t) dW(t) + \frac{1}{2} f_{pp} b(t) b'(t) dt \tag{C.7}$$

$$= \underbrace{\Big[f(t) + f_p(a(t)p(t) + m(t)) + \frac{1}{2} f_{pp} b(t) b'(t)\Big]}_{(*)} dt + \underbrace{f_p b(t)}_{(**)} dW(t), \tag{C.8}$$

where f_t, f_p, and f_{pp} denote the partial derivatives of f.

Observe that since $f(\cdot) \in \mathcal{R}^3$, the time derivative f_t will also belong to $\mathcal{R}^3$ while the derivative w.r.t. p is a 3×3 matrix. As we will see below, we do not need to be concerned with the term involving the second order partial derivative of f w.r.t. p, since the transformation we are looking for must have a second order partial derivative equal to zero.

Recall that we are interested in a transformation such that $(*)$ and $(**)$ in (C.8) only depend on time t.

Ad. $(**)$**:**

We want to have that $f_p(t,p)b(t) = G(t)$, where $G(\cdot)$ is some function that only depends on t. This, however, means that f has to fulfill the following condition:

$$f_p(t,p) = g(t) \tag{C.9}$$

where g again is a function that only depends on time. (C.9) gives us that the second derivative of f with respect to p is zero, i.e. $f_{pp}(t,p) = 0$ and that f is of the form

$$f(t,p) = g(t)p(t) \quad \text{(ignoring the integration constant)}.$$

Hence, the derivative with respect to time is

$$f_t(t,p) = \frac{dg}{dt}p. \tag{C.10}$$

Ad. $(*)$**:**

Let H be a function that only depends on t, then (C.9)-(C.10) gives that f must also satisfy the condition

$$\begin{aligned} f_t(t,p) + f_p(t,p)(a(t)p(t) + m(t)) + \frac{1}{2}f_{pp}(t,p)b(t)b'(t) &= H(t) \\ \Leftrightarrow \qquad \frac{dg}{dt}p(t) + g(t)a(t)p(t) + g(t)m(t) &= H(t). \end{aligned}$$

Putting the term $g(t)m(t)$ (which only depends on t) over to the right-hand side yields the condition

$$(\frac{dg}{dt} + a(t)g(t))p(t) = h(t),$$

where h is a function that only depends on t. This condition must be satisfied for every $p(t)$, and since there is no $p(t)$ term on the right-hand side, we must have that

$$\frac{dg}{dt} + a(t)g(t) = 0 \quad \Leftrightarrow \quad \frac{dg}{dt} = -a(t)g(t).$$

Therefore we have that g is on the form $g(t) = ce^{-\int_0^t a(s)ds}$, where $a(t)$ is the matrix which is multiplied by $p(t) = (r(t), \delta(t), \beta(t))'$ in (C.5), i.e. here $a(s) = a$ for all s, and c is an integration constant.[55] Recall that $g(\cdot)$ is a 3×3 matrix. We now have the

[55] $\frac{dg}{dt} = -a(t)g(t) \Leftrightarrow \int_0^t \frac{dg}{g} = -\int_0^t a(s)ds + K \Leftrightarrow \ln|g| = -\int_0^t a(s)ds + K \Leftrightarrow g(t) = e^{-\int_0^t a(s)ds}c$, where $c = e^K$ for $g > 0$, $c = -e^K$ for $g < 0$, and $c = 0$ corresponds to the solution $g = 0$.

functional form of the transformation, f:

$$f(t,p) = g(t)p(t) = c\ e^{-\begin{bmatrix} -\kappa & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} t} p(t) = c\ e^{\begin{bmatrix} \kappa t & 0 & 0 \\ -t & 0 & 0 \\ -t & 0 & 0 \end{bmatrix}} p(t). \tag{C.11}$$

The exponential series, $e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!}$, $X \in \mathcal{R}$, can be generalized to work for matrices. The calculations for e^{-at} are given below, where I is the 3×3 identity matrix.

$$e^{\begin{bmatrix} \kappa t & 0 & 0 \\ -t & 0 & 0 \\ -t & 0 & 0 \end{bmatrix}} = I + \begin{bmatrix} \kappa t & 0 & 0 \\ -t & 0 & 0 \\ -t & 0 & 0 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} \kappa^2 t^2 & 0 & 0 \\ -\kappa t^2 & 0 & 0 \\ -\kappa t^2 & 0 & 0 \end{bmatrix} + \frac{1}{3!}\begin{bmatrix} \kappa^3 t^3 & 0 & 0 \\ -\kappa^2 t^3 & 0 & 0 \\ -\kappa^2 t^3 & 0 & 0 \end{bmatrix} \tag{C.12}$$

$$+\frac{1}{4!}\begin{bmatrix} \kappa^4 t^4 & 0 & 0 \\ -\kappa^3 t^4 & 0 & 0 \\ -\kappa^3 t^4 & 0 & 0 \end{bmatrix} + \cdots \tag{C.13}$$

$$= I + \sum_{i=1}^{\infty} \frac{1}{i!}\begin{bmatrix} (\kappa t)^i & 0 & 0 \\ -\kappa^{i-1} t^i & 0 & 0 \\ -\kappa^{i-1} t^i & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 + \sum_{i=1}^{\infty} \frac{1}{i!}(\kappa t)^i & 0 & 0 \\ -\frac{1}{\kappa}\sum_{i=1}^{\infty} \frac{1}{i!}(\kappa t)^i & 1 & 0 \\ -\frac{1}{\kappa}\sum_{i=1}^{\infty} \frac{1}{i!}(\kappa t)^i & 0 & 1 \end{bmatrix} \tag{C.14}$$

$$= \begin{bmatrix} \sum_{i=0}^{\infty} \frac{1}{i!}(\kappa t)^i & 0 & 0 \\ \frac{1}{\kappa} - \frac{1}{\kappa}\sum_{i=0}^{\infty} \frac{1}{i!}(\kappa t)^i & 1 & 0 \\ \frac{1}{\kappa} - \frac{1}{\kappa}\sum_{i=0}^{\infty} \frac{1}{i!}(\kappa t)^i & 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{\kappa t} & 0 & 0 \\ \frac{1}{\kappa}(1 - e^{\kappa t}) & 1 & 0 \\ \frac{1}{\kappa}(1 - e^{\kappa t}) & 0 & 1 \end{bmatrix} \tag{C.15}$$

Hence, the transformation of p is given as

$$f(t,p) = \begin{bmatrix} ce^{\kappa t} & 0 & 0 \\ \frac{1}{\kappa}c(1 - e^{\kappa t}) & c & 0 \\ \frac{1}{\kappa}c(1 - e^{\kappa t}) & 0 & c \end{bmatrix} \begin{bmatrix} r(t) \\ \delta(t) \\ \beta(t) \end{bmatrix} = \begin{bmatrix} z(t) \\ y(t) \\ x(t) \end{bmatrix}$$

where we have denoted the three coordinates of f by $z(t)$, $y(t)$, and $x(t)$, respectively. Without loss of generality we let $c = 1$. We then have

$$z(t) = e^{\kappa t} r(t) \tag{C.16}$$

$$y(t) = \frac{1}{\kappa}(1 - e^{\kappa t}) r(t) + \delta(t) \tag{C.17}$$

$$x(t) = \frac{1}{\kappa}(1 - e^{\kappa t}) r(t) + \beta(t). \tag{C.18}$$

In order to find the simultaneous distribution of $r(t)$, $\delta(t)$, and $\beta(t)$ conditional on the information at a given date, we first find the dynamics of $z(t)$, $y(t)$, and $x(t)$.

Given the information at time $s < t$ the dynamics of $z(t)$ is

$$\begin{aligned} dz(t) &= e^{\kappa t} dr(t) + \kappa e^{\kappa t} r(t) dt \\ &= \left(e^{\kappa t}\kappa(\theta - r(t)) + \kappa e^{\kappa t} r(t)\right) dt + e^{\kappa t}\sigma_r dW(t)^1 \\ &= \kappa\theta e^{\kappa t} dt + e^{\kappa t}\sigma_r dW(t)^1. \end{aligned}$$

Hence given the information at time $s < t$

$$
\begin{aligned}
z(t) &= z(s) + \kappa\theta \int_s^t e^{\kappa v} dv + \sigma_r \int_s^t e^{\kappa v} dW^1(v) \\
&= z(s) + \theta(e^{\kappa t} - e^{\kappa s}) + \sigma_r \int_s^t e^{\kappa v} dW^1(v).
\end{aligned}
$$

Using Itô on $y(t)$ yields the following dynamics:

$$
\begin{aligned}
dy(t) &= -e^{\kappa t} r(t)dt + \frac{1}{\kappa}(1 - e^{\kappa t})dr(t) + d\delta(t) \\
&= -e^{\kappa t} r(t)dt + \big(\frac{1}{\kappa}(1 - e^{\kappa t})\kappa(\theta - r(t)) + r(t) - \frac{1}{2}\sigma_\delta^2\big)dt + \frac{1}{\kappa}(1 - e^{\kappa t})\sigma_r dW(t)^1 \\
&\quad + \sigma_\delta \rho dW^2(t) + \sigma_\delta \sqrt{1 - \rho^2}\ dW^2(t) \\
&= \big((1 - e^{\kappa t})\theta - \frac{1}{2}\sigma_\delta^2\big)dt + \big(\frac{1}{\kappa}(1 - e^{\kappa t})\sigma_r + \sigma_\delta \rho\big)dW^1(t) + \sigma_\delta \sqrt{1 - \rho^2}\ dW^2(t)
\end{aligned}
$$

and on integral form we have

$$
\begin{aligned}
y(t) &= y(s) + \theta \int_s^t (1 - e^{\kappa v})dv - \frac{1}{2}\sigma_\delta^2(t - s) + \int_s^t \big(\frac{\sigma_r}{\kappa}(1 - e^{\kappa v}) + \sigma_\delta \rho\big)dW^1(v) \\
&\quad + \sigma_\delta \sqrt{1 - \rho^2}(W^2(t) - W^2(s)) \\
&= y(s) + \theta(t - s) - \frac{\theta}{\kappa}(e^{\kappa t} - e^{\kappa s}) - \frac{1}{2}\sigma_\delta^2(t - s) + \int_s^t \big(\frac{\sigma_r}{\kappa}(1 - e^{\kappa v}) + \sigma_\delta \rho\big)dW^1(v) \\
&\quad + \sigma_\delta \sqrt{1 - \rho^2}(W^2(t) - W^2(s)) \\
&= y(s) + (\theta - \frac{1}{2}\sigma_\delta^2)(t - s) - \frac{\theta}{\kappa}(e^{\kappa t} - e^{\kappa s}) + \int_s^t \big(\frac{\sigma_r}{\kappa}(1 - e^{\kappa v}) + \sigma_\delta \rho\big)dW^1(v) \\
&\quad + \sigma_\delta \sqrt{1 - \rho^2}(W^2(t) - W^2(s)).
\end{aligned}
$$

For $x(t)$, the dynamics and the integral form are found as

$$
\begin{aligned}
dx(t) &= -e^{\kappa t} r(t)dt + \frac{1}{\kappa}(1 - e^{\kappa t})dr(t) + d\beta(t) \\
&= \big(-e^{\kappa t} r(t) + \frac{1}{\kappa}(1 - e^{\kappa t})\kappa(\theta - r(t)) + r(t)\big)dt + \frac{1}{\kappa}(1 - e^{\kappa t})\sigma_r dW^1(t) \\
&= (1 - e^{\kappa t})\theta dt + \frac{1}{\kappa}(1 - e^{\kappa t})\sigma_r dW^1(t)
\end{aligned}
$$

and

$$
\begin{aligned}
x(t) &= x(s) + \theta \int_s^t (1 - e^{\kappa v})dv + \frac{\sigma_r}{\kappa} \int_s^t (1 - e^{\kappa v})dW^1(v) \\
&= x(s) + \theta(t - s) - \frac{\theta}{\kappa}(e^{\kappa t} - e^{\kappa s}) + \frac{\sigma_r}{\kappa} \int_s^t (1 - e^{\kappa v})dW^1(v).
\end{aligned}
$$

Rearranging (C.16)-(C.18) yields

$$r(t) = e^{-\kappa t} z(t). \tag{C.19}$$

$$\delta(t) = y(t) - \frac{1}{\kappa}(1 - e^{\kappa t})e^{-\kappa t} z(t) = y(t) + \frac{1}{\kappa}(1 - e^{-\kappa t})z(t). \tag{C.20}$$

$$\beta(t) = x(t) - \frac{1}{\kappa}(1 - e^{\kappa t})e^{-\kappa t} z(t) = x(t) + \frac{1}{\kappa}(1 - e^{-\kappa t})z(t). \tag{C.21}$$

Observe, that $(1 - e^{-\kappa t})z(t)$ can be rewritten as

$$\begin{aligned}
(1 - e^{-\kappa t})z(t) &= (1 - e^{-\kappa t})\big(z(s) + \theta(e^{\kappa t} - e^{\kappa s}) + \sigma_r \int_s^t e^{\kappa v} dW^1(v)\big) \\
&= (1 - e^{-\kappa t})e^{\kappa s} r(s) + \theta(1 - e^{-\kappa t})(e^{\kappa t} - e^{\kappa s}) + \sigma_r (1 - e^{-\kappa t}) \int_s^t e^{\kappa v} dW^1(v) \\
&= (e^{\kappa s} - e^{-\kappa(t-s)})r(s) + \theta(e^{\kappa t} - e^{\kappa s} - 1 + e^{-\kappa(t-s)}) \\
&\quad + \sigma_r \int_s^t (e^{\kappa v} - e^{-\kappa(t-v)}) dW^1(v).
\end{aligned}$$

Using this expression for $(1 - e^{-\kappa t})z(t)$ and (C.20), we can find an expression for $\delta(t)$:

$$\begin{aligned}
\delta(t) &= y(t) + \frac{1}{\kappa}(1 - e^{-\kappa t})z(t) \\
&= y(s) + (\theta - \frac{1}{2}\sigma_\delta^2)(t - s) - \frac{\theta}{\kappa}(e^{\kappa t} - e^{\kappa s}) + \int_s^t \big(\frac{\sigma_r}{\kappa}(1 - e^{\kappa v}) + \sigma_\delta \rho\big) dW^1(v) \\
&\quad + \sigma_\delta \sqrt{1 - \rho^2}(W^2(t) - W^2(s)) + \frac{1}{\kappa}(e^{\kappa s} - e^{-\kappa(t-s)})r(s) + \frac{\theta}{\kappa}(e^{\kappa t} - e^{\kappa s} - 1 + e^{-\kappa(t-s)}) \\
&\quad + \frac{\sigma_r}{\kappa} \int_s^t (e^{\kappa v} - e^{-\kappa(t-v)}) dW^1(v) \\
&= \frac{1}{\kappa}(1 - e^{\kappa s})r(s) + \delta(s) + \frac{1}{\kappa}(e^{\kappa s} - e^{-\kappa(t-s)})r(s) + (\theta - \frac{1}{2}\sigma_\delta^2)(t - s) - \frac{\theta}{\kappa}(e^{\kappa t} - e^{\kappa s}) \\
&\quad + \frac{\theta}{\kappa}(e^{\kappa t} - e^{\kappa s} - 1 + e^{-\kappa(t-s)}) + \int_s^t \big(\frac{\sigma_r}{\kappa}(1 - e^{\kappa v} + e^{\kappa v} - e^{-\kappa(t-v)}) + \sigma_\delta \rho\big) dW^1(v) \\
&\quad + \sigma_\delta \sqrt{1 - \rho^2}(W^2(t) - W^2(s)) \\
&= \delta(s) + \frac{1}{\kappa}(1 - e^{-\kappa(t-s)})r(s) + (\theta - \frac{1}{2}\sigma_\delta^2)(t - s) - \frac{\theta}{\kappa}(1 - e^{-\kappa(t-s)}) \\
&\quad + \int_s^t \big(\frac{\sigma_r}{\kappa}(1 - e^{\kappa-(t-v)} + e^{\kappa v}) + \sigma_\delta \rho\big) dW^1(v) + \sigma_\delta \sqrt{1 - \rho^2}(W^2(t) - W^2(s)).
\end{aligned}$$

Analogously we can find an expression for $\beta(t)$:

$$\begin{aligned}
\beta(t) &= x(t) + \frac{1}{\kappa}(1 - e^{-\kappa t})z(t) \\
&= x(s) + \theta(t-s) - \frac{\theta}{\kappa}(e^{\kappa t} - e^{\kappa s}) + \frac{\sigma_r}{\kappa}\int_s^t (1 - e^{\kappa v})dW^1(v) + \frac{1}{\kappa}(e^{\kappa s} - e^{-\kappa(t-s)})r(s) \\
&\quad + \frac{\theta}{\kappa}(e^{\kappa t} - e^{\kappa s} - 1 + e^{-\kappa(t-s)}) + \frac{\sigma_r}{\kappa}\int_s^t (e^{\kappa v} - e^{-\kappa(t-v)})dW^1(v) \\
&= \frac{1}{\kappa}(1 - e^{\kappa s})r(s) + \beta(s) + \theta(t-s) - \frac{\theta}{\kappa}(e^{\kappa t} - e^{\kappa s}) + \int_s^t \frac{\sigma_r}{\kappa}(1 - e^{\kappa v})dW^1(v) \\
&\quad + \frac{1}{\kappa}(e^{\kappa s} - e^{-\kappa(t-s)})r(s) + \frac{\theta}{\kappa}(e^{\kappa t} - e^{\kappa s} - 1 + e^{-\kappa(t-s)}) + \frac{\sigma_r}{\kappa}\int_s^t (1 - e^{-\kappa(t-v)})dW^1(v) \\
&= \beta(s) + \frac{1}{\kappa}(1 - e^{\kappa(t-s)})(r(s) - \theta) + \theta(t-s) + \frac{\sigma_r}{\kappa}\int_s^t (1 - e^{-\kappa(t-v)})dW^1(v).
\end{aligned}$$

The conditional means (conditional on the information given at date s) of $r(t)$, $\delta(t)$, and $\beta(t)$ are therefore

$$E_s r(t) = (r(s) - \theta)e^{-\kappa(t-s)} + \theta, \tag{C.22}$$

$$E_s\delta(t) = \delta(s) + \frac{1}{\kappa}(1 - e^{-\kappa(t-s)})(r(s) - \theta) + (\theta - \frac{1}{2}\sigma_\delta^2)(t-s) \tag{C.23}$$

and

$$E_s\beta(t) = \beta(s) + \frac{1}{\kappa}(1 - e^{\kappa(t-s)})(r(s) - \theta) + \theta(t-s). \tag{C.24}$$

The conditional variances of $r(t)$, $\delta(t)$, and $\beta(t)$ are calculated using the Itô isometry and that the Brownian motions, W^1 and W^2, are uncorrelated:

$$\begin{aligned}
\text{var}_s\ r(t) &= \sigma_r^2 \int_s^t e^{-2\kappa(t-v)}dv = \sigma_r^2 e^{-2\kappa t}\int_s^t e^{2\kappa v}dv = \frac{\sigma_r^2}{2\kappa}e^{-2\kappa t}(e^{2\kappa t} - e^{2\kappa s}) \\
&= \frac{\sigma_r^2}{2\kappa}(1 - e^{-2\kappa(t-s)}). \\
\text{var}_s\,\delta(t) &= E[(\int \ldots dW^1(v))^2] + E[(\int \ldots dW^2(v))^2] + 2Cov(\ldots dW^1, \ldots dW^2) \\
&= \int_s^t \big[\frac{\sigma_r^2}{\kappa^2}(1 - 2e^{-\kappa(t-v)} + e^{-2\kappa(t-v)}) + \sigma_\delta^2\rho^2 + \frac{2\sigma_r\sigma_\delta\rho}{\kappa}(1 - e^{-\kappa(t-v)})\big]dv \\
&\quad + \sigma_\delta^2(1-\rho^2)(t-s) \\
&= \frac{\sigma_r^2}{\kappa^2}(t-s) - \frac{2\sigma_r^2}{\kappa^3}e^{-\kappa t}(e^{\kappa t} - e^{\kappa s}) + \frac{\sigma_r^2}{2\kappa^3}e^{-2\kappa t}(e^{2\kappa t} - e^{2\kappa s}) + \sigma_\delta^2\rho^2(t-s) \\
&\quad + \frac{2\sigma_r\sigma_\delta\rho}{\kappa}(t-s) - \frac{2\sigma_r\sigma_\delta\rho}{\kappa^2}e^{-\kappa t}(e^{\kappa t} - e^{\kappa s}) + \sigma_\delta^2(1-\rho^2)(t-s) \\
&= \big(\frac{\sigma_r^2}{\kappa^2} + \sigma_\delta^2 + \frac{2\sigma_r\sigma_\delta\rho}{\kappa}\big)(t-s) - \big(\frac{2\sigma_r^2}{\kappa^3} + \frac{2\sigma_r\sigma_\delta\rho}{\kappa^2}\big)(1 - e^{-\kappa(t-s)}) \\
&\quad + \frac{\sigma_r^2}{2\kappa^3}(1 - e^{-2\kappa(t-s)})
\end{aligned}$$

$$
\begin{aligned}
&= \big(\frac{\sigma_r^2}{\kappa^2} + \sigma_\delta^2 + \frac{2\sigma_r\sigma_\delta\rho}{\kappa}\big)(t-s) - \frac{2\sigma_r}{\kappa^2}\big(\frac{\sigma_r}{\kappa} + \sigma_\delta\rho\big)(1-e^{-\kappa(t-s)}) \\
&\quad + \frac{\sigma_r^2}{2\kappa^3}(1-e^{-2\kappa(t-s)}). \\
\mathrm{var}_s\,\beta(t) &= \frac{\sigma_r^2}{\kappa^2}\int_s^t (1-2e^{-\kappa(t-v)} + e^{-2\kappa(t-v)})dv = \frac{\sigma_r^2}{\kappa^2}\big[(t-s) - \frac{2}{\kappa}e^{-\kappa t}(e^{\kappa t}-e^{\kappa s}) \\
&\quad + \frac{1}{2\kappa}e^{-2\kappa t}(e^{2\kappa t}-e^{2\kappa s})\big] \\
&= \frac{\sigma_r^2}{\kappa^2}\big[(t-s) - \frac{2}{\kappa}(1-e^{-\kappa(t-s)}) + \frac{1}{2\kappa}(1-e^{-2\kappa(t-s)})\big].
\end{aligned}
$$

The conditional covariances are

$$
\begin{aligned}
\mathrm{Cov}_s(r(t),\delta(t)) &= \sigma_r\int_s^t e^{-\kappa(t-v)}\big[\frac{\sigma_r}{\kappa}(1-e^{-\kappa(t-v)}) + \sigma_\delta\rho\big]dv \\
&= \frac{\sigma_r^2}{\kappa}\int_s^t e^{-\kappa(t-v)}dv - \frac{\sigma_r^2}{\kappa}\int_s^t e^{-2\kappa(t-v)}dv + \sigma_r\sigma_\delta\rho\int_s^t e^{-\kappa(t-v)}dv \\
&= \frac{\sigma_r^2}{\kappa^2}e^{-\kappa t}(e^{\kappa t}-e^{\kappa s}) - \frac{\sigma_r^2}{2\kappa}e^{-2\kappa t}(e^{2\kappa t}-e^{2\kappa s}) + \frac{\sigma_r\sigma_\delta\rho}{\kappa}e^{-\kappa t}(e^{\kappa t}-e^{\kappa s}) \\
&= \frac{\sigma_r^2}{\kappa^2}(1-e^{-\kappa(t-s)}) - \frac{\sigma_r^2}{2\kappa}(1-e^{-2\kappa(t-s)}) + \frac{\sigma_r\sigma_\delta\rho}{\kappa}(1-e^{-\kappa(t-s)}) \\
\mathrm{Cov}_s(r(t),\beta(t)) &= \frac{\sigma_r^2}{\kappa}\int_s^t e^{-\kappa(t-v)}(1-e^{-\kappa(t-v)})dv \\
&= \frac{\sigma_r^2}{\kappa}\big[\frac{1}{\kappa}(1-e^{-\kappa(t-s)}) - \frac{1}{2\kappa}(1-e^{-2\kappa(t-s)})\big] \\
\mathrm{Cov}_s(\delta(t),\beta(t)) &= \frac{\sigma_r}{\kappa}\int_s^t (1-e^{-\kappa(t-v)})\big[\frac{\sigma_r}{\kappa}(1-e^{-\kappa(t-v)}) + \sigma_\delta\rho\big]dv \\
&= \frac{\sigma_r^2}{\kappa^2}\int_s^t (1-2e^{-\kappa(t-v)} + e^{-2\kappa(t-v)} + \frac{\sigma_r\sigma_\delta\rho}{\kappa}(1-e^{-\kappa(t-v)}))dv \\
&= \frac{\sigma_r^2}{\kappa^2}\big[(t-s) - \frac{2}{\kappa}(1-e^{-\kappa(t-s)}) + \frac{1}{2\kappa}(1-e^{-2\kappa(t-s)})\big] + \frac{\sigma_r\sigma_\delta\rho}{\kappa}\big[(t-s) \\
&\quad - \frac{1}{\kappa}(1-e^{-\kappa(t-s)})\big].
\end{aligned}
$$

Hence, we have

$$
\begin{bmatrix} r(t) \\ \delta(t) \\ \beta(t) \end{bmatrix} \sim N\left(\begin{bmatrix} E_s r(t) \\ E_s\delta(t) \\ E_s\beta(t) \end{bmatrix}, \begin{bmatrix} \mathrm{var}_s\ r(t) & \mathrm{Cov}_s(r(t),\delta(t)) & \mathrm{Cov}_s(r(t),\beta(t)) \\ \mathrm{Cov}_s(\delta(t),r(t)) & \mathrm{var}_s\ \delta(t) & \mathrm{Cov}_s(\delta(t),\beta(t)) \\ \mathrm{Cov}_s(\beta(t),r(t)) & \mathrm{Cov}_s(\beta(t),\delta(t)) & \mathrm{var}_s\ \beta(t) \end{bmatrix}\right). \tag{C.25}
$$

C.2 Annual deposits

In this section we consider minor extensions of the model analyzed in Hansen and Miltersen (1999), section 5.1-5.6 and appendix A and B. We consider the case where the customer pays annual premiums/deposits for his contract with the company as opposed to one initial deposit—the case analyzed in Hansen and Miltersen (1999). Moreover, we briefly consider the effect of introducing mortality risk into the model when there are annual deposits.

Annual deposits without mortality risk

We investigate the case where the customer pays the same premium each year. The annual payment is without loss of generality normalized to 1 unit. We are interested in several things. First, in the one-customer case, what rate of return guarantee is fair for given values of the share of excess bonus distributed, α, and the rate of payment fee, ξ? Second, for the two-customer case, we want to see if the introduction of annual payments has any influence on the redistribution effects that we observed in the case with only one initial deposit.

In order to get numerical results we only have to make minor changes to the program which was used in Hansen and Miltersen (1999). The changes are outlined below.

- Each year, from the time the customer enters and until he exits we add 1 unit to the X account (the reference portfolio) and 1 unit to the customer's account, A (as before the customer receives at least his deposit accumulated at the guaranteed interest rate).

- When we search for fair combinations of the parameters, α, ξ, and guaranteed minimum rate of return, g, we use the present value of the annual payments as the value that the contract must have to be fair. In the case with only one deposit we searched for parameter values that made the contract's value equal to the initial deposit.

  We have that for $|a| < 1$: $\sum_{i=0}^{\infty} a^i = \frac{1}{1-a}$. Let $n \in \mathcal{N}$ then

$$\sum_{i=0}^{n-1} a^i = \sum_{i=0}^{\infty} a^i - \sum_{i=0}^{\infty} a^{i+n} = \frac{1}{1-a} - a^n \sum_{i=0}^{\infty} a^i \tag{C.26}$$

$$= \frac{1}{1-a} - a^n \frac{1}{1-a} = \frac{1}{1-a}(1 - a^n). \tag{C.27}$$

  In the beginning of each year the customer deposits 1 unit. With $a = e^{-r}$ we find that the present value of a series of 1 unit deposits over T years is $\frac{1-\exp(-rT)}{1-\exp(-r)}$.

- In the case where one customer enters later than the other one we must reformulate the "weighting" scheme or bonus distribution mechanism. Let $\tilde{t}$ denote the entry date of the second customer,[56] and let $\tilde{T} > \tilde{t}$ denote the first date when either of the customers exits. We distribute according to the following:

When the second customer enters at date $\tilde{t}$, customer one's deposits have grown to $X_{\tilde{t}}$ units (just after the annual deposit for year $\tilde{t}$ is made). Discounting the amount forward to date $\tilde{T}$ at the rate of return on the reference portfolio yields that customer one's deposits are worth $X_{\tilde{t}}e^{(\sum_{i=\tilde{t}+1}^{\tilde{T}}\delta_i)}$ at date $\tilde{T}$. From date $\tilde{t}$ and until one of the customers leaves the company they contribute equally (i.e. same annual deposits) to the bonus, hence they should share the amount of bonus built between the two dates equally. Remember that $X_{\tilde{T}}$ is the total value of the asset side at date $\tilde{T}$, i.e. stemming from payments from both customer one and customer two. We define the fraction, β, of the bonus available at date $\tilde{T}$ that should go to customer one as follows:

$$\beta = \frac{X_{\tilde{t}}e^{(\sum_{i=t+1}^{\tilde{T}}\delta(i))} + \frac{1}{2}(X_{\tilde{T}} - X_{\tilde{t}}e^{(\sum_{i=\tilde{t}+1}^{\tilde{T}}\delta(i))})}{X_{\tilde{T}}} \tag{C.28}$$

$$= \frac{1}{2} + \frac{X_{\tilde{t}}e^{(\sum_{i=\tilde{t}+1}^{\tilde{T}}\delta(i))}}{2X_{\tilde{T}}}. \tag{C.29}$$

Observe that this definition of β conforms with the one used in Hansen and Miltersen (1999).[57]

Results

The one customer case. In order to compare with the single premium case we find the fair minimum rate of return guarantees for different combinations of the share of excess bonus distributed, α, and rate of payment fee, ξ, and maturity of 10 years, see table C.1.

We have the usual picture: for a fixed α, the minimum rate of return guarantee increases with the rate of payment fee. This is quite intuitive and needs not to be commented upon. Comparing with the single-premium case we see that with annual deposits the minimum rate of return guarantees which are offered are lower. For example, with a rate of payment fee of 0.5% and an α equal to 20% the company can offer a guarantee of 0.9% with annual deposits, whereas it could offer 1.54% in the single

[56] The first customer enters at date zero.

[57] In Hansen and Miltersen (1999), $\beta = \frac{X\exp\left(\sum_{i=1}^{\tilde{t}}\delta(i)\right)}{X+X\exp\left(\sum_{i=1}^{\tilde{t}}\delta(i)\right)}$, which does not change if we extrapolate to date $\tilde{T}$ analogously to the above definition, i.e. multiply by $e^{\sum_{i=\tilde{t}+1}^{\tilde{T}}\delta(i)}$ in the numerator and denominator. The numerator, as in (C.28), resembles the amount that the first customer has built by the time the second customer enters, and the denominator is the total value of the assets built by the customers.

	α (%)										
ξ (%)	0	10	20	30	40	50	60	70	80	90	100
0.25	—	—	—	—	—	—	—	—	—	—	—
0.50	0.95	1.01	0.90	0.88	0.81	0.76	0.57	0.41	0.36	0.24	0.06
0.75	1.87	1.93	1.87	1.86	1.80	1.77	1.76	1.60	1.58	1.47	1.41
1.00	2.60	2.64	2.64	2.61	2.59	2.57	2.51	2.44	2.40	2.31	2.27
1.25	3.21	3.20	3.22	3.21	3.23	3.20	3.17	3.13	3.09	3.04	3.01
1.50	3.73	3.75	3.76	3.77	3.73	3.71	3.71	3.69	3.63	3.63	3.57
1.75	4.20	4.24	4.26	4.22	4.24	4.21	4.22	4.18	4.16	4.12	4.08
2.00	4.67	4.65	4.65	4.68	4.66	4.65	4.65	4.65	4.60	4.59	4.57
2.25	5.06	5.06	5.07	5.07	5.07	5.06	5.05	5.05	5.02	5.00	5.00
2.50	5.45	5.45	5.46	5.45	5.45	5.45	5.44	5.43	5.42	5.40	5.38

Table C.1: Values of g in percent (%) for different choices of ξ and α in percent (%) with annual payments; $\sigma = 10\%$, $T = 10$, $\rho = 0$, and $r = (1 - 0.26)5\%$.

premium case. The reason for this is that a contract with annual deposits is a collection of single-premium contracts. Implying that having a 10 year contract with annual deposits is the same as having a 10 year (starting now), a 9 year (beginning in one year), . . . , and a 1 year (starting in nine years) single-premium contract with the same deposit, i.e. X units initially. We know that the company can offer a higher minimum rate of return guarantee for a longer maturity, c.f. Hansen and Miltersen (1999). The duration of an annual deposit contract with a certain maturity, T, is shorter than T, which is the maturity or duration of the single-premium contract to which we are making a comparison. Hence, the minimum rate of return guarantee offered on an annual deposit contract must be lower than the one offered on a single-premium contract of the same maturity for the contract to be fair.

We see, that the fair minimum rate of return guarantee, g, does not change much (decreases only slightly) as the share of excess bonus distributed, α, increases. Or equivalently, that the fair rate of payment fee is quite insensitive to changes in α for a fixed minimum rate of return guarantee. This is the same as in the single-premium case, see figure 5.1 in section 5.4. The way the payouts and the collection of premia are constructed yields that the rate of payment fee is relatively unaffected by changes in α and this does not change when we allow for annual deposits.

The two customer case. We run the simulations for the two-customer case with the adjustments mentioned earlier. Looking at the tables with the results for the annual premium case, one must keep in mind that the present value of the deposits are now different. In scenarios 1–3 the value of the discounted annual deposits for a 10 year contract is $1\frac{1-\exp(-r10)}{1-\exp(-r)} = 8.5141\ (8.4971, 8.5311)$, where r is the risk free interest rate which we set equal to 0.037 as in Hansen and Miltersen (1999). For a 20 year contract the value is $14.3951\ (14.3663, 14.4239)$. In scenarios 4 and 5 the value of the contract for

Sc.1a	$(\xi_1, \xi_2) = (0.0115, 0.0115)$	
	Individual bonus	Pooled bonus
Customer One	8.5136	8.5170
Customer Two	8.5086	8.5170
Sum	17.0222	17.0340

Table C.2: Ind. ξs, $g_1 = g_2 = 3\%$, and $T_1 = T_2 = 10$ - annual deposits.

Sc.1b	$\xi = 0.0115$	
	Individual bonus	Pooled bonus
Customer One	8.5118	8.5139
Customer Two	8.5073	8.5139
Sum	17.0191	17.0277

Table C.3: Common ξ, $g_1 = g_2 = 3\%$, and $T_1 = T_2 = 10$ - annual deposits.

the customer who enters late must equal $1\frac{1-\exp(-r10)}{1-\exp(-r)}e^{-r10} = 5.8810\ (5.8692, 5.8928)$. The intervals given in parentheses are the present values minus/plus 0.2 percent. The simulation procedure will generate some deviation from the present value and therefore we do not interpret a change from the present value (going from the individual bonus case to the pooled bonus case) as a redistribution until the value falls outside the intervals given above.

The results show that in scenarios two, three, and five (tables C.4–C.7 and C.10–C.11) the redistributions are in the same directions as with a single premium. That is, in scenarios two and five, customer one and the company both benefit on account of customer two, and in scenario three customer one is worse off and customer two is indifferent, leaving the company better off by pooling. Since the redistributions are the same as with single premium, we will not go through them here but refer to Hansen and Miltersen (1999), i.e. sections 5.1–5.6 and appendix A and B.

In scenario 4 we observe a deviation from the results in the single premium case, see tables C.8 and C.9. In the single premium case both customer one and customer two were worse off by pooling, now only customer one is worse off while customer two is better off. Recall the three effects discussed in Hansen and Miltersen (1999): (*i*)

Sc.2a	$(\xi_1, \xi_2) = (0.0221, 0.0114)$	
	Individual bonus	Pooled bonus
Customer One	8.5073	8.6804
Customer Two	8.5203	8.2984
Sum	17.0276	16.9787

Table C.4: Ind. ξs, $g_1 = 5\%$, $g_2 = 3\%$, and $T_1 = T_2 = 10$ - annual deposits.

Sc.2b	$\xi = 0.0163$	
	Individual bonus	Pooled bonus
Customer One	8.7893	8.9561
Customer Two	8.3066	8.0879
Sum	17.0959	17.0441

Table C.5: Common ξ, $g_1 = 5\%$, $g_2 = 3\%$, and $T_1 = T_2 = 10$.

Sc.3a	$(\xi_1, \xi_2) = (0.0075, 0.0116)$	
	Individual bonus	Pooled bonus
Customer One	14.4256	14.2202
Customer Two	8.5048	8.5085
Sum	22.9305	22.7288

Table C.6: Ind. ξs, $g_1 = g_2 = 3\%$, $T_1 = 20$, and $T_2 = 10$ - annual deposits.

Sc.3b	$\xi = 0.0075$	
	Individual bonus	Pooled bonus
Customer One	14.3874	14.2199
Customer Two	8.6925	8.6975
Sum	23.0799	22.9174

Table C.7: Common ξ, $g_1 = g_2 = 3\%$, $T_1 = 20$, and $T_2 = 10$ - annual deposits.

Sc.4a	$(\xi_1, \xi_2) = (0.0076, 0.0116)$	
	Individual bonus	Pooled bonus
Customer One	14.3940	14.2754
Customer Two	5.8825	5.9500
Sum	20.2766	20.2253

Table C.8: Ind. ξs, $g_1 = g_2 = 3\%$, entry date (1)=0, $T_1 = 20$, entry date (2)=10, and $T_2 = 10$ - annual deposits.

Sc.4b	$\xi = 0.0079$	
	Individual bonus	Pooled bonus
Customer One	14.3276	14.2377
Customer Two	5.9895	6.0677
Sum	20.3170	20.3054

Table C.9: Common ξ, $g_1 = g_2 = 3\%$, entry date (1)=0, $T_1 = 20$, entry date (2)=10, and $T_2 = 10$ - annual deposits.

Sc.5a	$(\xi_1, \xi_2) = (0.0182, 0.0116)$	
	Individual bonus	Pooled bonus
Customer One	14.4069	14.4604
Customer Two	5.8771	5.6771
Sum	20.2840	20.1375

Table C.10: Ind. ξs, $g_1 = 5\%$, $g_2 = 3\%$, entry date (1)=0, $T_1 = 20$, entry date (2)=10, and $T_2 = 10$ - annual deposits.

Sc.5b	$\xi = 0.0164$	
	Individual bonus	Pooled bonus
Customer One	14.6728	14.7347
Customer Two	5.7358	5.5290
Sum	20.4086	20.2638

Table C.11: Common ξ, $g_1 = 5\%$, $g_2 = 3\%$, entry date (1)=0, $T_1 = 20$, entry date (2)=10, and $T_2 = 10$ - annual deposits.

customer two gains from participating in customer one's bonus (and leaves customer one worse off), (*ii*) customer one is slightly favorized by the sharing rule for the terminal bonus (and hence customer two is slightly worse off), and (*iii*) the company benefits from an improvement in the probability of having to cover a negative bonus for either customer. We have changed the sharing rule to accommodate the annual deposits. Even though the sharing rule should be equivalent to the one used in the single-premium case, it seems to be the case that effect (*ii*) is no longer present. The result is that both customer two and the company benefit by pooling on account of customer one.

Introducing mortality risk

In this section we consider the pricing of a certain contract with annual deposits when we take mortality risk into consideration. As in Hansen and Miltersen (1999) it is assumed that mortality risk can be diversified away by the Law of Large Numbers, i.e. that the company has a large pool of homogeneous customers and that the probability distribution for the time of death is the same for each customer. It is then possible to use this probability distribution for the entire pool since it is reasonable to work under the assumption that the customers' deaths are independent events. Moreover, it is assumed that mortality risk is orthogonal to financial risk.

In particular, we look at a contract that in the case of death (of the insured/the customer) before maturity pays out whatever has accumulated on his or her own account (A) by the time of death. We assume that the payment is made at the end of the year in which death occurs and after interests have been ascribed to the accounts. If the the customer lives until maturity of the contract, he or she will receive the "usual" amount, i.e. the sum of the A account and the bonus account (if positive), at this date.

The contract is therefore slightly different from the one considered in Hansen and Miltersen (1999) since the payout from the term insurance part (A) is now varying over time as opposed to being constant in Hansen and Miltersen (1999). We introduce mortality risk in the form of *Makeham's formula*, which is what is commonly used in practice, see e.g. Cederbye and Pedersen (1997). This is the same way mortality risk is handled in Hansen and Miltersen (1999).

Using the Makeham formula the probability that a x year old lives to be $x+n$ years old is

$$ {}_n p_x = e^{-An-\frac{B}{\ln c}(c^{x+n}-c^x)}, $$

where A, B, and c are constants which are estimated on the basis of past observations of death in a population. They differ slightly for males and females, mostly because females tend to live longer than males. We use, $A = 0.0005075787$, $B = 0.000039342435$, $c = 1.10291509$, and $x = 30$ years. The constants are obtained from Delbaen (1986).

Pricing the contract

We have to make some adjustments to the way we calculate the present value of deposits, i.e. the value that the value of the contract must match. In the case with annual deposits this was simple. when adding mortality risk, we have to adjust for the fact that people die and that from the time of death they will not make further deposits. Therefore the present value of deposits will be a sum of the discounted annual payments each multiplied by a correction factor equal to the probability that the customer makes the deposit, i.e. that he is alive at the particular date. The present value, PV, of the deposits made on a contract with maturity in T years, is hence found as

$$ PV = X \sum_{i=0}^{T-1} e^{-ri} Prob(\text{person age } x \text{ will live until age } x+i) \tag{C.30} $$

$$ = X \sum_{i=0}^{T-1} e^{-ri} {}_i p_x = X \sum_{i=0}^{T-1} e^{-ri} [e^{-Ai-\frac{B}{\ln c} c^x (c^i - 1)}]. \tag{C.31} $$

$$ \tag{C.32} $$

The present values for 10 and 20 year contracts are given below for the different scenarios. We allow a margin of error of ± 0.2 percent as before. The allowed interval for the fair values are given below as well.[58]

[58] Recall that as long as the values for the pooled bonus case with individual rate of payment fees lie within the given intervals, we cannot determine whether there are in fact bonus redistributions.

	α (%)										
ξ (%)	0	10	20	30	40	50	60	70	80	90	100
0.25	0.92	0.90	0.90	0.73	0.67	0.58	0.50	0.28	0.10	0.02	-0.12
0.50	1.81	1.81	1.86	1.75	1.71	1.63	1.59	1.56	1.39	1.35	1.23
0.75	2.57	2.54	2.56	2.55	2.50	2.46	2.42	2.35	2.27	2.22	2.12
1.00	3.13	3.14	3.14	3.14	3.11	3.11	3.08	3.04	2.99	2.94	2.89
1.25	3.69	3.66	3.68	3.69	3.69	3.63	3.60	3.59	3.57	3.51	3.50
1.50	4.14	4.12	4.17	4.19	4.14	4.15	4.11	4.11	4.07	4.04	4.00
1.75	4.57	4.60	4.58	4.58	4.60	4.57	4.56	4.55	4.55	4.50	4.48
2.00	5.00	4.98	4.99	5.00	4.99	4.99	4.98	4.96	4.96	4.92	4.90
2.25	5.38	5.37	5.38	5.39	5.38	5.38	5.37	5.36	5.34	5.33	5.31
2.50	5.73	5.75	5.74	5.76	5.74	5.74	5.73	5.70	5.71	5.70	5.70

Table C.12: Values of g in percent (%) for different choices of ξ and α in percent (%) with annual payments and mortality risk; $\sigma = 10\%$, $T = 10$, and $r = (1 - 0.26)5\%$.

Scenarios 1 and 2	PV	$\mp$ 0.2 percent
Customer One – 10yr	8.4595	$(8.4426, 8.4764)$
Customer Two – 10yr	8.4595	$(8.4426, 8.4764)$
Total	16.9190	$(16.9528, 16.8852)$

Scenario 3	PV	$\mp$ 0.2 percent
Customer One – 20yr	14.1594	$(14.1028, 14.1877)$
Customer Two – 10yr	8.4595	$(8.4426, 8.4764)$
Total	22.6190	$(22.5737, 22.6642)$

Scenarios 4 and 5	PV	$\mp$ 0.2 percent
Customer One – 20yr	14.1594	$(14.1028, 14.1877)$
Customer Two – 10yr	5.8433	$(5.8316, 5.8433)$
Total	20.0027	$(19.9627, 20.0428)$

Results

One customer case

In table C.12, fair minimum rate of return guarantees are given for different combinations of ξ and α for a contract of the first mentioned type with maturity of 10 years and including mortality risk.

The general picture with mortality risk is that the company can offer a higher minimum rate of return guarantee for all combinations of rate of payment fee, ξ, and α than without mortality risk. In order to understand why this is so, we must remember that the contract considered is different with and without mortality risk. When mortality risk is included, the customer will receive the exact same payout as without mortality risk only if he survives until maturity of the contract, T. This only happens with a certain probability as mentioned earlier. If the customer dies before T, he only receives the amount on his account, A, and the undistributed bonus that he has built at the time of death stays with the company. In order for the contract to be fair, the company must hence offer a higher rate of return guarantee to compensate for retaining undistributed bonus if the customer dies.

Two customer case

In tables C.13-C.22 the results for the two customer case with annual deposits and mortality risk are given. The results are completely equivalent to the case without mortality risk. Since there are no differences we will not comment further on the results except to say that as expected implementing mortality risk does not seem to add anything to the analysis.

Sc.1a	$(\xi_1, \xi_2) = (0.0128, 0.0128)$	
	Individual bonus	Pooled bonus
Customer One	8.4590	8.4624
Customer Two	8.4540	8.4624
Sum	16.9130	16.9248

Table C.13: Ind. ξs, $g_1 = g_2 = 3\%$, and $T_1 = T_2 = 10$ - annual deposits and mortality risk.

Sc.1b	$\xi = 0.0127$	
	Individual bonus	Pooled bonus
Customer One	8.4572	8.4593
Customer Two	8.4528	8.4593
Sum	16.9100	16.9185

Table C.14: Common ξ, $g_1 = g_2 = 3\%$, and $T_1 = T_2 = 10$ - annual deposits and mortality risk.

Sc.2a	$(\xi_1, \xi_2) = (0.0233, 0.0127)$	
	Individual bonus	Pooled bonus
Customer One	8.4528	8.6248
Customer Two	8.4657	8.2451
Sum	16.9184	16.8699

Table C.15: Ind. ξs, $g_1 = 5\%$, $g_2 = 3\%$, and $T_1 = T_2 = 10$ - annual deposits and mortality risk.

Sc.2b	$\xi = 0.0175$	
	Individual bonus	Pooled bonus
Customer One	8.7297	8.8956
Customer Two	8.2565	8.0392
Sum	16.9862	16.9347

Table C.16: Common ξ, $g_1 = 5\%$, $g_2 = 3\%$, and $T_1 = T_2 = 10$ - annual deposits and mortality risk.

Sc.3a	$(\xi_1, \xi_2) = (0.0091, 0.0129)$	
	Individual bonus	Pooled bonus
Customer One	14.1894	13.9882
Customer Two	8.4503	8.4540
Sum	22.6397	22.4422

Table C.17: Ind. ξs, $g_1 = g_2 = 3\%$, $T_1 = 20$, and $T_2 = 10$ - annual deposits and mortality risk.

Sc.3b	$\xi = 0.0090$	
	Individual bonus	Pooled bonus
Customer One	14.1639	14.0000
Customer Two	8.6221	8.6271
Sum	22.7861	22.6271

Table C.18: Common ξ, $g_1 = g_2 = 3\%$, $T_1 = 20$, and $T_2 = 10$ - annual deposits and mortality risk.

Sc.4a	$(\xi_1, \xi_2) = (0.0092, 0.0128)$	
	Individual bonus	Pooled bonus
Customer One	14.1583	14.0545
Customer Two	5.8448	5.8992
Sum	20.0031	19.9537

Table C.19: Ind. ξs, $g_1 = g_2 = 3\%$, entry date (1)=0, $T_1 = 20$, entry date (2)=10, and $T_2 = 10$ - annual deposits and mortality risk.

Sc.4b	$\xi = 0.0095$	
	Individual bonus	Pooled bonus
Customer One	14.1023	14.0264
Customer Two	5.9402	6.0051
Sum	20.0425	20.0314

Table C.20: Common ξ, $g_1 = g_2 = 3\%$, entry date (1)=0, $T_1 = 20$, entry date (2)=10, and $T_2 = 10$ - annual deposits and mortality risk.

Sc.5a	$(\xi_1, \xi_2) = (0.0197, 0.0129)$	
	Individual bonus	Pooled bonus
Customer One	14.1711	14.2311
Customer Two	5.8395	5.6328
Sum	20.0105	19.8639

Table C.21: Ind. ξs, $g_1 = 5\%$, $g_2 = 3\%$, entry date (1)=0, $T_1 = 20$, entry date (2)=10, and $T_2 = 10$ - annual deposits and mortality risk.

Sc.5b	$\xi = 0.0179$	
	Individual bonus	Pooled bonus
Customer One	14.4406	14.5088
Customer Two	5.6952	5.4817
Sum	20.1358	19.9905

Table C.22: Common ξ, $g_1 = 5\%$, $g_2 = 3\%$, entry date (1)=0, $T_1 = 20$, entry date (2)=10, and $T_2 = 10$ - annual deposits and mortality risk.

Part V

Competition among Life Insurance Companies

Competition among Life Insurance Companies: The driving force of high policy rates?

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Abstract

We analyze the effect competition has on the decisions of life insurance companies. In particular, we are interested in the companies' choices of policy rates and investment strategies given that they have issued contracts with a minimum rate of return guarantee. Our modeling framework is a one-period Cournot model of duopoly. We find policy rates and investment strategies that sustain a Nash equilibrium. We compare the results to the cooperative solution, that is, the case where the companies operate as a monopoly company and share the profits. Our model illustrates how competition between companies drives companies to offer relatively high policy rates, in particular rates above the risk free rate of return.

6.1 Introduction

Many contracts offered by life insurance companies and pension funds are offered with a minimum rate of return guarantee. The minimum rate of return guarantee is lower than the risk free rate of return when the contract is issued. The guarantee therefore provides a 'floor' on the future payout to the customers. Besides the guaranteed minimum payout, the customers are typically entitled to profits, i.e. bonus, that might be generated by their contract as a result of changes in financial and demographic conditions.[59] In the end of each year life insurance companies typically announce the rate of return they will give their customers in the year to come. This rate of return—the *policy rate*—is a promised rate of return. One can think of the rate of return as including some expected bonus, in the sense that it is higher than the minimum rate of return guarantee. The companies must be able to give the customers the minimum rate of return guarantee. The policy rate, however, does not have to be fulfilled with certainty and it might actually not be possible for the company to honor the promise. For instance, changes in the financial market could influence the value of the company's investment portfolio so that funds are simply not large enough to give the customers a return equal to the policy rate. The companies could simply offer the minimum rate of return guarantee as the policy rate and then later distribute bonus arising from the contract to the customer. This would be a way in which they can be certain not to promise the customers too much. However, competition among the life insurance companies seems to drive the policy rate up well above the minimum rate of return guarantee. In this paper we provide a model that explains this feature.

Other authors have analyzed contracts that provide a minimum rate of return guarantee and possibly some bonus, see for example Brennan and Schwartz (1976), Briys and de Varenne (1994), Grosen and Jørgensen (2000b), Miltersen and Persson (1998), and Hansen and Miltersen (1999). The first two papers consider maturity guarantees, whereas the others deal with annual guarantees. In Briys and de Varenne (1994) policy holders receive a minimum rate of return on average over the life of the contract and a fraction of possible surplus. Surplus arises if the company's investment portfolio performs well and the value of the customer's part of the investment portfolio is higher than the guaranteed minimum amount. In the model, however, the guarantee is not binding in the sense that the company can default on the claim it has sold to the customer. Hansen and Hansen (2000) investigate the model of Briys and de Varenne (1994)

[59] Bonus arises from the difference in the so-called first order and second order basis. The terms of an insurance contract are set initially according to a first order basis, which is a set of assumptions about the future values of demographic and financial variables. Typically, constant intensities, e.g. mortality rates and constant interest rates, are assumed. The companies try to set these conservatively so that as time evolves and the true values of mortality and the financial variables become known, i.e. the second order basis is known, the companies typically generate a surplus which is known as bonus once it is distributed to the customers, see Norberg (1999).

where the guarantee must be satisfied for sure and, more importantly, they extend the framework to the case of a dynamic investment portfolio instead of a static portfolio. Grosen and Jørgensen (2000b) consider a contract offered by a pension fund[60] with an annual minimum rate of return guarantee where the policy rate is determined each year by the previous year's level of a bonus reserve compared to the sum of equity and the customer's account. If the bonus reserve in the company reaches a certain size, some of the bonus is distributed according to a specific mechanism. Common for the papers is the assumption of an insurance market which is perfectly competitive and, hence, that the terms of the contract should be set so that there is no expected profit (also sometimes referred to as 'fair'). Instead of perfect competition we consider a one-period Cournot model of duopoly,[61] hence, only two companies operate in the life insurance market, and no entry to or exit from the insurance market is possible. We provide a model which allows us to study how competition among life insurance companies influences the companies' choices of policy rates and investment strategies. We compare the companies' equilibrium strategies in a model of duopoly with the outcome from the case where the companies cooperate and operate as one company and share the profits.

The paper is organized as follows: we present the model in section 6.2. Section 6.3 contains a description of how to solve for the equilibrium, i.e. the optimal choices of policy rates and investment strategies, while general as well as numerical results for the cooperative and the competitive case are presented in sections 6.4 and 6.5, respectively. In section 6.6 a slightly altered model is presented which yields results that are more in accordance with empirical facts. Finally, some concluding remarks are given in section 6.7. Most calculations and proofs are delegated to an appendix.

6.2 The model

There are two life insurance companies in the market. The companies are competing for the deposits of a large group of customers. Each company offers a contract with a specific payout structure that depends on the announced policy rates, the minimum rate of return guarantee, and the company's investment strategy. We return to the specifics of the payouts later. Should the customer die before payout to his contract is made, his heirs inherit the contract.[62] The companies each have a number of risk averse equity

[60] A distinction between life insurance companies and pension funds is made since the shareholders in a pension fund are typically the customers and this is not the case in life insurance companies.

[61] For an introduction to the Cournot model of duopoly, consult a standard textbook on game theory in economics, c.f. Gibbons (1992).

[62] This assumption could also be used in a multi-period model. In such a model one could also consider incorporating mortality risk using a Law of Large Numbers argument. That is, it is assumed that the company has a large enough number of similar customers (i.e. same age, etc.) to diversify mortality risk and hence calculate expected benefits and premiums based on a deterministic distribution of the customers' times of death.

holders[63] who invest an amount in the company initially. This initial capital or equity can differ between the companies and hence they might differ in size. We restrict the companies from short sales since this is typically the case for life and pension insurance companies. Furthermore, the companies are not allowed to invest more than the so-called free reserves in risky assets.[64] We assume that there is a frictionless competitive financial market with several risky assets and one risk free asset. The equity holders and the insurance companies are able to trade in this market, whereas the customers are restricted from doing so. The life insurance companies are price takers on the financial market in the sense that they are not large enough to influence security prices. Instead of modeling the dynamics of all the risky assets, we simply model the portfolio of risky assets that the life insurance companies invest in as a single risky asset. Finally, we assume that the stocks of the life insurance companies are traded on the competitive financial market so that the objective of each life insurance company is to maximize the value of its equity.

The number of customers in each of the two life insurance companies depends on how the companies set their policy rates. The company with the highest policy rate will have the highest number of customers.[65]

The companies compete for the customers since the companies receives a certain premium (a percentage of the customers' deposits) from the customers, and they can use this to generate additional future profits. We normalize the number of customers in the economy, and the total number of units of account that the companies are competing for to one.[66] Recall that the customers are assumed not to be able to trade in the financial market themselves, for instance due to large transactions costs.[67] They are, moreover, forced into a life insurance contract of the type the companies offer and are only allowed to choose between the two companies.[68]

The insurance company must make sure that it is able to fulfill its obligations toward the customers, at least with respect to the minimum rate of return guarantee. Therefore we assume that the company initially places at least an amount equal to the present value of the future (minimum) obligations in the risk free asset. In other words, the company initially decides how to invest only the so-called *free reserves*. The

[63] The risk aversion of the equity holders need not be the same.

[64] Otherwise we cannot make sure that the minimum rate of return guarantee given to the customers is fulfilled with certainty.

[65] In order to keep things tractable we have only allowed the number of customers that go to a certain company to depend on the difference in policy rates. It is not possible to allow the number of customers to vary with the investment strategy, for instance, since the company cannot commit to a strategy ex ante that can be verified by the customers. The investment decision will be made when the company knows the other parameters in their decision problem.

[66] One can think of it as each customer depositing one unit of account upon entering into a company.

[67] A similar argument is used in Brennan (1993) in the discussion of the existence of life insurance companies as financial intermediaries.

[68] This could for instance be the case if customers want employment and they can choose between two similar jobs with different mandatory pension and life insurance plans.

free reserves are equal to the total asset value less the value of the company's future obligations with respect to the minimum rate of return guarantee.

6.2.1 The financial market

We assume that the companies can invest in a risk free asset—the bank account—with date t price of one unit equal to $B(t)$ and in a risky asset[69] with a date t unit price $S(t)$.

We assume that the risk free rate of return is constant. The risk free interest rate is denoted by r. The date t value of the bank account, i.e. the date t value of one unit deposited in the bank account at date 0, is then given by

$$B(t) = e^{rt}.$$

We assume there exists a unique risk neutral probability measure, Q.[70] The price of the risky asset is assumed to follow a geometric Brownian motion under Q, that is, the continuously compounded returns on the risky asset are normally distributed. Let μ denote the expected rate of return on the risky asset under Q, and σ denote the volatility of the risky asset under Q. Since the market is perfect, i.e. frictionless, complete, and free of arbitrage, we have that $\mu = r$.[71]

The dynamics of the risky asset is given by

$$S(t) = S(u)e^{(\mu - \frac{1}{2}\sigma^2)(t-u)+\sigma(W(t)-W(u))}, \quad \text{for} \quad u \leq t, \tag{6.2.1}$$

where W is a standard Brownian motion under the risk neutral probability measure, Q. We introduce some notation that is used throughout the paper.

6.2.2 Notation

r_g: Periodic minimum rate of return guarantee. We assume that $r_g < r$.

r: Risk free rate of return.

μ: Expected rate of return on the risky asset under the risk neutral probability measure.

σ: Volatility rate of the risky asset.

η: Premia charged for the contract. A certain percentage of the initial deposits made by the customers. $\eta \geq 0$.

[69] Recall that this risky asset is actually a portfolio of risky assets.
[70] That is, the financial market is free of arbitrage and complete.
[71] The reason for operating with this slightly more general notation will become clearer later on.

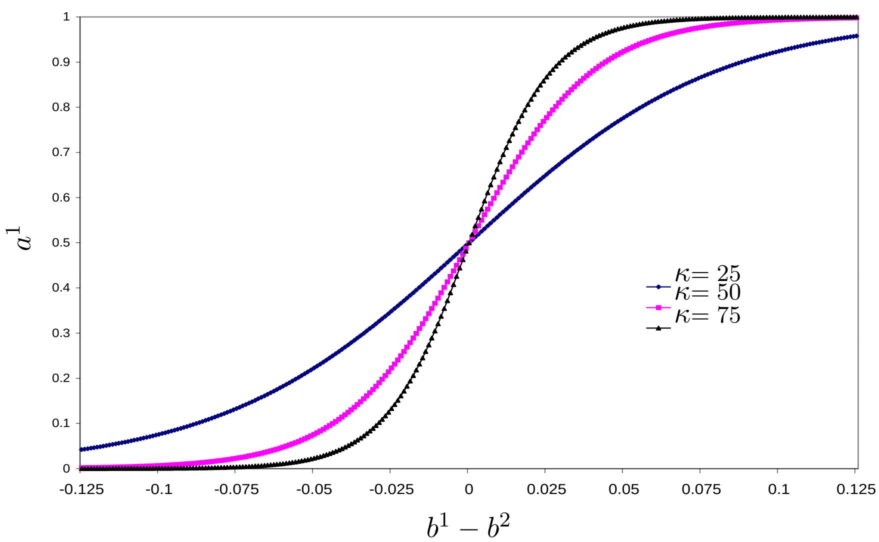

Figure 6.1: The number of customers in company 1 as a function of the difference in policy rates, $b^1 - b^2$.

E_0^k**:** Initial capital deposited in company k, $k = 1, 2$, by the equity holders before customers have entered into a contract.

b^k**:** *Policy rate* or announced promised rate of return in company k, $k = 1, 2$, for one period. Note that $b^k \geq r_g$ and can be thought of as a rate that incorporates some expected bonus.

a^k**:** The number of customers in company k, $k = 1, 2$. We normalize the total number of customers to one, hence $a^1 + a^2 = 1$. The initial deposits made by the group of customers choosing company k is equal to a^k. The number of customers is determined by the difference in the policy rates offered by the two companies. We assume that the number of customers in company 1, a^1, is given by

$$a^1 = \frac{1}{1 + e^{-\kappa(b^1 - b^2)}}, \quad \kappa \quad \text{a constant,}$$

and analogously for company 2. The constant, κ, controls how sensitive the customers are to the difference in policy rates, i.e. a large κ implies that even a small difference causes a large difference in the number of customers that the companies receive. Figure 6.1 shows a^1 as a function of $b^1 - b^2$ for different choices of κ. Observe that the choice of function satisfies $a^1 + a^2 = 1$. Moreover, for a fixed b^2, we have

$$a^1 \to 1 \ \ (\text{and } a^2 \to 0) \quad \text{as} \quad b^1 \to \infty \quad \text{and} \quad a^1 \to \frac{1}{2} \ \ (\text{and } a^2 \to \frac{1}{2}) \quad \text{as} \quad b^1 \to b^2,$$

and analogously for a^2 for a fixed b^1. The function exhibits the basic features that we want, namely that the higher policy rate a company promises (given the other company's policy rate), the more customers it will attract and that the companies share the number of customers equally if they set the same policy rate.

δ: Fraction of 'extra' bonus that goes to a company, where extra bonus is bonus besides that which is already included in the policy rate.

F_0^k: The free reserves for company k, $k = 1, 2$, defined by $F_0^k = E_0^k + a^k - a^k(1-\eta)e^{(r_g - r)}$. That is, initial equity plus cash flow from customers minus the amount that must be invested in the risk free asset initially to cover the guarantee for sure. Note that the free reserve can never be negative since $\eta \geq 0$ and $r > r_g$.

π^k: Fraction of the free reserves of company k, $k = 1, 2$, that are placed in the risky asset initially. The fraction is determined initially and cannot be altered during the life of the contract. π^k is restricted to the set $[0, 1]$, i.e. the life insurance companies cannot short sell the risky asset and they must fulfill the guarantee with certainty.

6.2.3 Timing

The timing of the game is as follows:

At date zero,

- The company announces the policy rate, b, for the next period.
- The (potential) customers observe the policy rates offered and decide which company to turn to on the basis of the difference in policy rates. The company that announces the highest policy rate receives the largest inflow of money since more customers enter into a contract with this company.
- The company observes the customers' decisions, i.e. the capital inflow and, hence, premium payments, and decides on an investment strategy, that is, the company determines its π.

At date one,

- The payouts to the company (equity holders) and the customers are determined.

The company's objective is to maximize the value of its equity.

6.2.4 Payout

The customers in company k, $k = 1, 2$, pay a total of a^k units initially. After the premium payments are deducted, the residual amount is guaranteed a minimum rate of return of r_g. If the company's investments perform well, the customers receive a rate of return equal to the promised policy rate, b^k, and possibly some extra bonus. The extra bonus arises when the company's total asset value rises above a certain level (see below). If the company's investments do not perform well, that is, not good enough to honor the policy rate, then the customers receive whatever asset value there is in the company.[72] Hence, the company has limited liability.

Let A^k denote the date one value of company k's assets then,

$$\begin{aligned} A^k &= \pi^k F_0^k e^{r-\frac{1}{2}\sigma^2+\sigma W} + (1-\pi^k)F_0^k e^r + a^k(1-\eta)e^{r_g} && (6.2.2) \\ &= \pi^k F_0^k S(1) + (1-\pi^k)F_0^k e^r + a^k(1-\eta)e^{r_g}. && (6.2.3) \end{aligned}$$

That is, the asset value at date one is equal to the date one value of the free reserves (the first two terms) and the date one value of the position taken in the risk free asset to cover the minimum rate of return guarantee (the last term). In mathematical terms, we have that payout to the customers in company k is given by

$$\min\left\{A^k, a^k(1-\eta)e^{b^k}\right\} + \text{extra bonus}.$$

Since the value of the free reserves cannot fall below zero, the guarantee is always fulfilled. The second term is called extra bonus since it is extra in the sense that some bonus is already included in the policy rate, b^k.[73] More about the extra bonus part below.

The sum of the customers' and the company's payouts must equal the total asset value at date one. The so-called extra bonus arises when company k's asset value rises above a certain level. This level is given by $e^{b^k}(\eta a^k + E_0^k)$—the total initial equity (premia paid by the customers plus initial capital) accumulated at the policy rate. The extra bonus is divided between the company and the customers according to their initial capital. That is, a fraction, $\delta = \frac{\eta a^k + E_0^k}{a^k + E_0^k}$, of the extra bonus goes to the company and the rest, $(1-\delta)$, goes to the customers.

[72] Observe that asset value is always larger than the minimum guaranteed amount to the customer.

[73] In a multi-period framework this extra bonus can be thought of as undistributed surplus which is collected over the life time of the contract and distributed to the customers by the end of the contract or perhaps during the life of the contract.

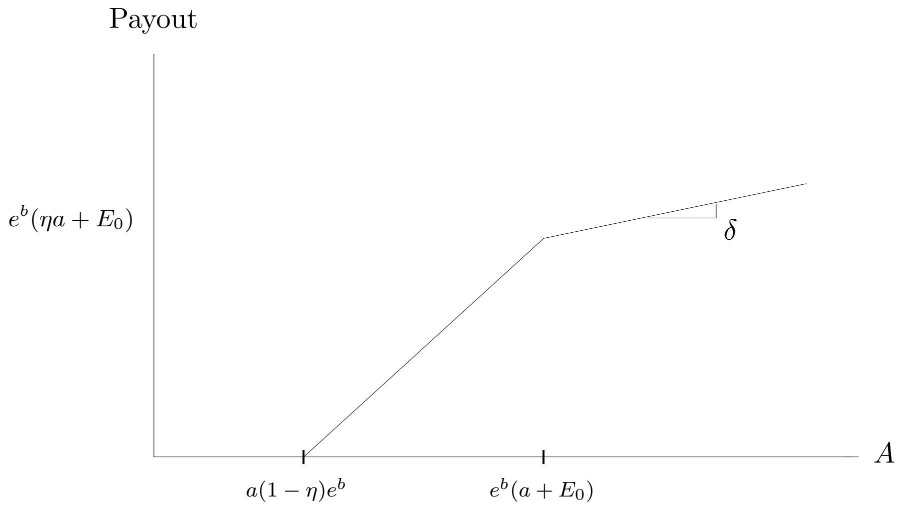

Figure 6.2: Payout to the company's equity holders as a function of asset value.

The payout at date one to the stock holders of company k is given by

$$\begin{aligned} &\min\Big\{\max\Big\{A^k - a^k(1-\eta)e^{b^k}, 0\Big\}, e^{b^k}(\eta a^k + E_0^k)\Big\} \\ + \quad &\delta \max\Big\{A^k - a^k(1-\eta)e^{b^k} - e^{b^k}(\eta a^k + E_0^k), 0\Big\}. \end{aligned} \tag{6.2.4}$$

By the dynamics of the risky asset we have that the asset value at date one is always greater than or equal to zero. Using this and that $a^k(1-\eta)e^{b^k} + e^{b^k}(\eta a^k + E_0^k) = e^{b^k}(a^k + E_0^k)$ we see that payout in (6.2.4) is equal to the payout from a portfolio consisting of call options on the asset value, A^k. In particular,

$$C\big(A^k, a^k(1-\eta)e^{b^k}\big) - (1-\delta)C\big(A^k, e^{b^k}(a^k + E_0^k)\big), \tag{6.2.5}$$

where $C(A, Z)$ denotes the payout from a call option on A with exercise price Z and time to maturity equal to one period. Figure 6.2 shows the payout to the equity holders of the company as a function of asset value.

The level, $Y = e^{b^k}(\eta a^k + E_0^k)$, above which any further profits that the company makes are shared with the customers, is chosen to have this particular form because companies are usually not allowed to provide a rate of return on equity above the rate of return on the policies issued (possibly plus a fixed percentage for instance 2 percent).[74]

[74] The 'extra bonus' part is not going to contradict this since for reasonable parameter constellations the equity holders' fraction δ is much smaller than the customers' fraction of the extra bonus, $1-\delta$.

6.3 Optimization and equilibrium

The objective of the insurance companies is to maximize their equity holders' expected utility. Since the companies operate in a perfect competitive and complete capital market, this is accomplished by maximizing the value of their company's shares.[75] Hence, the objective of company k is to maximize the value of the equity with respect to its choice variables, π^k and b^k, given the policy rate of the other company. The value of a company's equity is given as the expected discounted payouts to the equity holders, i.e. the company, where discounting is done with the risk free interest rate and the expectation is with respect to the risk neutral measure, c.f. Harrison and Kreps (1979) and Harrison and Pliska (1981).

The life insurance companies are operating in a duopoly and hence there is not perfect competition in the life insurance business. The terms of the contracts are therefore typically not fair. In fact, since there are only two companies on the market and no companies are allowed to enter into or exit the insurance market, the date zero value of the equity of an existing company is larger than or equal to the initial deposits made by the equity holders.

We consider the two following optimization problems:[76] company 1 solves

$$\begin{aligned}\sup_{\pi^1,b^1} \ & E^Q\Big[e^{-r}\min\Big\{\max\Big\{A^1 - a^1(1-\eta)e^{b^1}, 0\Big\}, e^{b^1}(\eta a^1 + E_0^1)\Big\} \\ & + \quad e^{-r}\delta\max\Big\{(A^1 - a^1(1-\eta)e^{b^1} - e^{b^1}(\eta a^1 + E_0^1), 0\Big\}\Big]\end{aligned} \tag{6.3.1}$$

for a given b^2. Recall that the difference $b^1 - b^2$ determines the number of customers in company 1, i.e. determines a^1.

Analogously, company 2 solves,

$$\begin{aligned}\sup_{\pi^2,b^2} \ & E^Q\Big[e^{-r}\min\Big\{\max\Big\{A^2 - a^2(1-\eta)e^{b^2}, 0\Big\}, e^{b^2}(\eta a^2 + E_0^2)\Big\} \\ & + \quad e^{-r}\delta\max\Big\{A^2 - a^2(1-\eta)e^{b^2} - e^{b^2}(\eta a^2 + E_0^2), 0\Big\}\Big]\end{aligned} \tag{6.3.2}$$

for a given b^1.

We are able to calculate the expectations in (6.3.1) and (6.3.2). The calculations are placed in section A of the appendix.

Let V^k denote the expectation for company k, $k = 1, 2$. We then find that

$$V^k = f1_{(X \geq \exp(b^k)(\eta a^k + E_0^k))} + g1_{(0 \leq X < \exp(b^k)(\eta a^k + E_0^k))} + h1_{(X<0)},$$

[75] See for example Copeland and Weston (1988) pp. 124–125.
[76] Recall that $A^k = \pi^k F_0^k S(1) + (1-\pi^k)F_0^k + a^k(1-\eta)e^{r_g}$, $k = 1, 2$.

where $1_{(\dots)}$ is the indicator function and

$$
\begin{aligned}
f &= e^{-r}\Big((1-\delta)e^{b^k}(\eta a^k + E_0^k) + \delta(\pi^k F_0^k e^{\mu} + X)\Big),\\
g &= e^{-r}\big((1-\delta)(\pi^k F_0^k e^{\mu} N(l-\sigma) + XN(l) + e^{b^k}(\eta a^k + E_0^k)(1-N(l))) + \delta(\pi^k F_0^k e^{\mu} + X)\big),\\
h &= e^{-r}\big(\pi^k F_0^k e^{\mu}(N(l-\sigma) - N(d-\sigma)) + X(N(l) - N(d)) + (1-\delta)e^{b^k}(\eta a^k + E_0^k)(1-N(l))\\
&\quad + \delta\pi^k F_0^k e^{\mu}(1 - N(l-\sigma)) + \delta X(1-N(l))\big),\\
X &= (1-\pi^k)F_0^k e^r - a^k(1-\eta)(e^{b^k} - e^{r_g}),\\
F_0^k &= E_0^k + a^k - a^k(1-\eta)e^{r_g - r},\\
d &= \frac{1}{\sigma}\Big\{\ln\Big(\frac{-X}{\pi^k F_0^k}\Big) - (\mu - \frac{1}{2}\sigma^2)\Big\},\\
l &= \frac{1}{\sigma}\Big\{\ln\Big(\frac{e^{b^k}(\eta a^k + E_0^k) - X}{\pi^k F_0^k}\Big) - (\mu - \frac{1}{2}\sigma^2)\Big\}.
\end{aligned}
$$

6.3.1 Equilibrium

We solve for Nash Equilibria. That is, we are searching for policy rates b^1 and b^2 that solve (6.3.1) and (6.3.2) simultaneously or in other words, b^1 must be company 1's best response[77] to company 2 choosing b^2, while b^2 must be company 2's best response to b^1. By the nature of a Nash Equilibrium there can be several equilibria. The numerical results of our model suggest, however, that there exists at most one equilibrium for a given set of parameters.

Symmetric equilibrium

In the case where the two companies are identical, in the sense that they have equal initial equity, $E_0^1 = E_0^2$, and hence are of equal size, we know that the equilibrium is a symmetric equilibrium where the two companies choose the same policy rates and investment strategies, i.e. $b^1 = b^2$ and $\pi^1 = \pi^2$.

The cooperative case

We want to investigate the effects of competition. We therefore need to consider what happens when there is no competition for customers, that is, when the companies cooperate and share the profits. In this situation their joint company has monopoly power and therefore gets all the customers in the economy, i.e. one. The problem is solved by solving for the policy rate and investment strategy that maximize the value of the equity of the monopoly company. This problem is of course much simpler since one does not have to consider the decisions of another company. We solve the maximization

[77] Note that the best response incorporates the optimal choice of investment strategy, π^1.

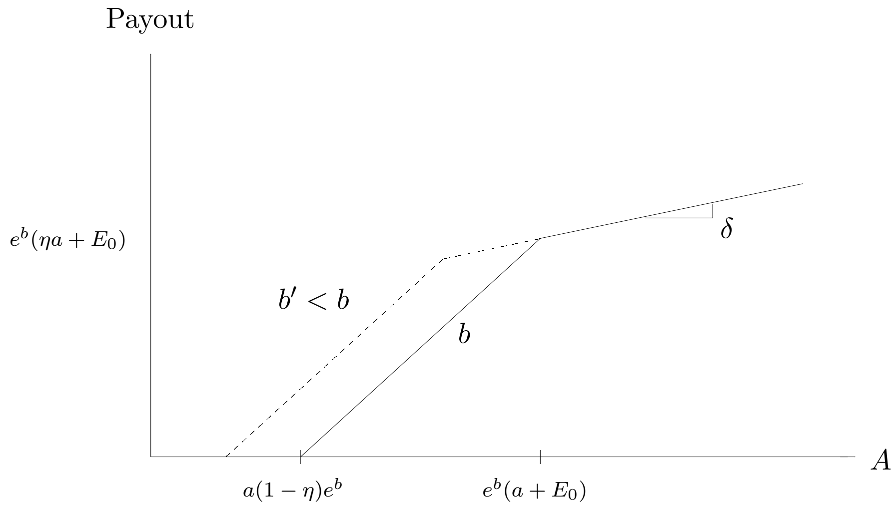

Figure 6.3: Payout to the equity holders in the cooperative case for two different levels of policy rates, $b < b'$.

problem (6.3.1) with $a^1 = 1$ and initial equity equal to $E_0^1 + E_0^2$. In order to compare to the case with competition, we divide the resulting value of equity with two.

6.4 The cooperative solution

In the case without competition one would expect that it would always be optimal to offer the lowest possible policy rate, that is, offer a policy rate equal to the minimum rate of return guarantee. In figure 6.3 the payout to the equity holders is shown for two different levels of policy rates and a given premium percentage, $\eta > 0$. From the way the payout is constructed only the linear part between the two exercise points (i.e. $a(1-\eta)e^b$ and $e^b(a+E_0)$) differs for different levels of policy rates. Hence, other things being equal, it is optimal to set the policy rate equal to the minimum rate of return guarantee, r_g. We therefore have the following lemma:

Lemma 6.4.1. *In the cooperative case the optimal policy rate is equal to the minimum rate of return guarantee for an arbitrary investment strategy, i.e.* $b^* = r_g$ *for any* $\pi \in [0, 1]$.

Let A denote the date one asset value of either of the companies when they cooperate. From the dynamics of the risky asset, we have that the asset value at date one is log-normally distributed. The mean and the variance of the asset value, A, for a given

$\pi \in [0,1]$, are equal to

$$E^Q[A|\pi] = \pi F_0 e^r + (1-\pi)F_0 e^r + a(1-\eta)e^{r_g} \tag{6.4.1}$$
$$= F_0 e^r + a(1-\eta)e^{r_g} \tag{6.4.2}$$
$$= (a+E_0)e^r$$
$$\equiv \bar{A} \tag{6.4.3}$$
$$\mathrm{Var}(A|\pi) = E^Q[A^2|\pi] - (E^Q[A|\pi])^2 = (\pi F_0 e^r)^2(e^{\sigma^2}-1). \tag{6.4.4}$$

The mean is independent of the investment strategy, π, whereas the variance increases with π. Moreover, the mean is equal to the date one asset value for $\pi = 0$. Let $A(0)$ denote the date one asset value with $\pi = 0$, then $\bar{A} = A(0)$.

For any given investment strategy, $\pi \in [0,1]$, the worst outcome of the position in the risky asset is an $\omega \in \Omega$ for which the realization of $\pi F_0 S(1)$ is zero. This worst case yields a lower boundary on the date one asset value. Denote this lower boundary by $\underline{A}(\pi)$ for a given investment strategy, $\pi \in [0,1]$. The lower boundary is equal to the date one value of the position in the risk free asset for the given π, thus

$$\underline{A}(\pi) = (1-\pi)F_0 e^r + a(1-\eta)e^{r_g}$$
$$= (1-\pi)(E_0 + a - a(1-\eta)e^{r_g - r})e^r + a(1-\eta)e^{r_g}$$
$$= (1-\pi)(a+E_0)e^r + \pi a(1-\eta)e^{rg}. \tag{6.4.5}$$

Observe that $\underline{A}(\cdot)$ is monotonically decreasing in π with $\underline{A}(1) = a(1-\eta)e^{r_g}$, and $\underline{A}(0) = (a+E_0)e^r$. Moreover, we have that $\underline{A}(0) = A(0)$.

We have the following proposition:

Proposition 6.4.2. *In a perfect market, i.e. with $\mu = r$, a solution to the cooperative case is given by,*

$$(b^*, \pi^*) = (b, 0), \quad \textit{where} \quad b \in [r_g, r].$$

Proof: The proof of proposition 6.4.2 consists of two parts. First, we show that it is optimal to invest everything in the risk free asset given the optimal policy rate from lemma 6.4.1, i.e. $\pi = 0$ with $b^* = r_g$. Second, we show that with $\pi = 0$, the company is indifferent between policy rates in $[r_g, r]$.

Given that $b^* = r_g$, the lowest possible date one asset value is equal to the first exercise value, i.e. $\underline{A}(1) = a(1-\eta)e^{r_g}$. Therefore the payout function is concave in asset value on the support of the asset value. Furthermore, the mean of the asset value is equal to the asset value with $\pi = 0$, i.e. $\bar{A} = A(0)$. This, combined with an application of *Jensen's* inequality, gives us that $\pi^* = 0$. The details are given in section B of the appendix.

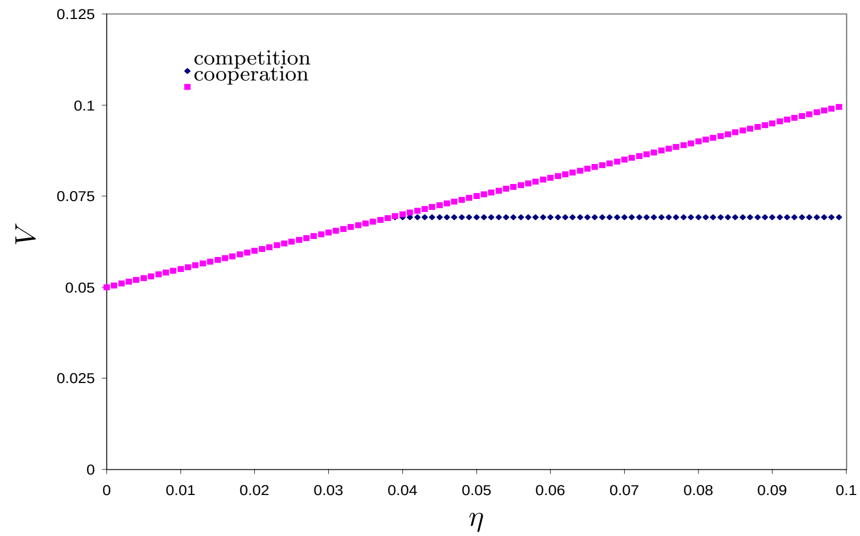

Figure 6.4: Value of equity at date zero for the competitive and the cooperative case as a function of the premium percentage.

Given that the investment strategy is to place everything in the risk free asset, the outcome for the asset value and hence the payout is known. More specifically, the date one asset value is given by $A(0) = e^r(a + E_0)$. Consider again figure 6.3. Let b be equal to the risk free rate of return, r, and b' be equal to r_g. The curve for an arbitrary policy rate in $[r_g, r[$ and the curve for $b = r$ coincide for asset values equal to and above $A(0)$. The payout to the equity holders is equal to the payout attained at $A(0)$ and is therefore the same for any choice of policy rate in $[r_g, r]$. q.e.d.

For a base case set of parameters, $\mu = r = 0.05, \sigma = 0.20, r_g = 0.025, \kappa = 50$, and $E_0^1 = E_0^2 = 0.05.$[78], we have shown optimal policy rate(s)[79] as a function of η in figure 6.5 and the date zero value of equity for either company in figure 6.4. The date zero value of equity is linearly increasing in η. This is a direct result of the fact that the customers are forced into the contracts no matter what the premium is. We have included the results for the case with competition in order to save space. The results for the competitive case will be discussed in the next section.

[78]As our base case we have chosen an initial equity, E_0, that corresponds to an equity position of approximately 10 percent in (symmetric) equilibrium.

[79]All the values in the interval $[r_g, r]$ are equilibrium solutions according to proposition 6.4.2. Therefore the whole interval is marked in figure 6.5.

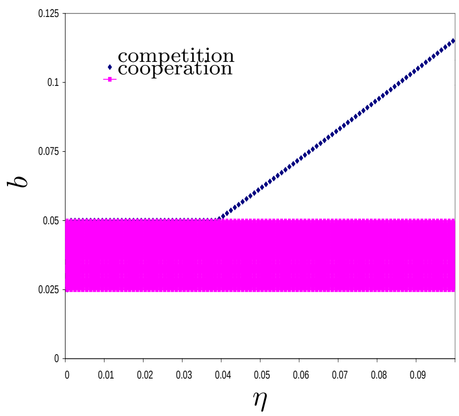

Figure 6.5: Optimal policy rates for the competitive and the cooperative case for different ηs.

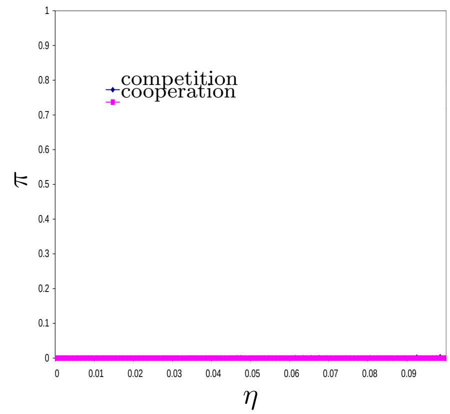

Figure 6.6: Optimal inv. strategies for the competitive and the cooperative case for different ηs.

6.5 The duopoly solution

In the case with competition we cannot arrive at the same kind of straightforward conclusions as we did in the previous section with respect to the optimal choice of policy rate, i.e. figure 6.3. The problem is complicated by the fact that with competition, a company's choice of policy rate depends on the other company's choice of policy rate and both policy rates play a role in determining the number of customers a company receives. We assume that the two competing companies have equal initial equity and are equivalent in all other aspects. In equilibrium they will therefore choose the same policy rate (and investment strategy), and thus each receive half of the customers. We have the following proposition:

Proposition 6.5.1. *In a perfect market,*

(i) *Given the policy rate of company two (one), and a policy rate for company one (two) less than or equal to the risk free rate of return, the optimal investment strategy for company one (two) is to invest everything in the risk free asset, i.e.* $\pi = 0$ *for* $b \leq r$*, for any level of the premium percentage.*

(ii) *In equilibrium, the policy rates of the two companies are equal and larger than or equal to the risk free rate of return, i.e.* $b_1^* = b_2^* = b^*$ *and* $b^* \geq r$*.*

(iii) *Let* $\underline{\eta}(b^1, b^2)$ *be given by*

$$\underline{\eta}(b^1, b^2) = 1 - \frac{a^1 \kappa e^{-\kappa(b^1 - b^2)} e^r}{e^{b^1}(1 + a^1 \kappa e^{-\kappa(b^1 - b^2)})}. \tag{6.5.1}$$

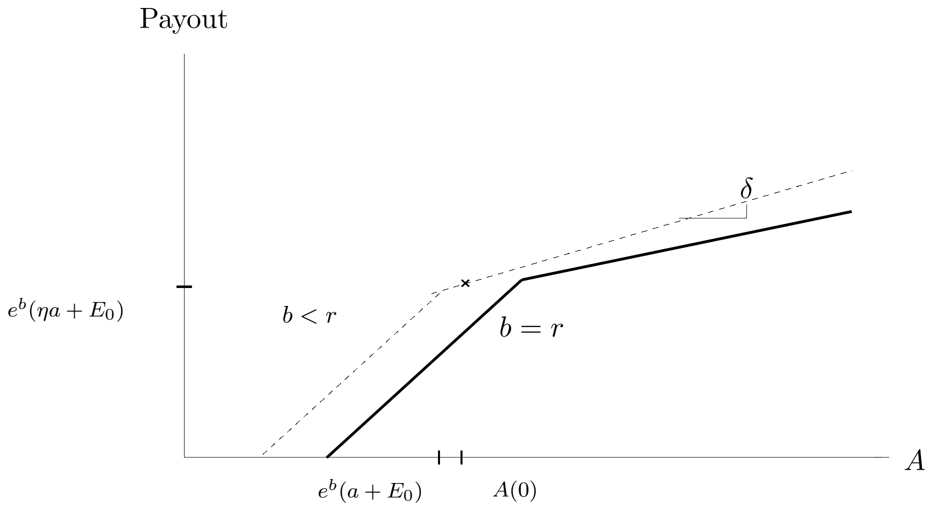

Figure 6.7: Payout curves for equity holders with policy rates below and equal to the risk free rate of return, respectively.

Given that the investment strategy is risk free, i.e. $\pi = 0$, the equilibrium policy rate, b^*, is characterized by

$$b^* > r \quad \textit{if} \quad \eta \geq \underline{\eta}(r, r)$$

and

$$b^* = r \quad \textit{otherwise.}$$

Note that *(i)* holds in and off equilibrium, whereas *(ii)* and *(iii)* are equilibrium results. However, a proposition equivalent to *(iii)* can be shown to hold for the policy rate of a company given the other company's choice of policy rate. The proof of *(iii)* proves the more general result.

Proof:
(i): The proof of *(i)* can be found in section B of the appendix.

(ii): Since the companies are equivalent, a symmetric equilibrium prevails, i.e. $b^1 = b^2 = b^*$ and $\pi^1 = \pi^2 = \pi^*$ in equilibrium. Now assume that $b < r$. From *(i)* it

follows that the investment strategy is the risk free strategy, that is, $\pi = 0$. The date one asset value is therefore known and equal to $A^b(0) = e^r(a(b) + E_0)$. For a policy rate less than the risk free rate of return, the payout to the equity holders is on the part where the extra bonus is shared with the customers. The payout to the equity holders is shown in figure 6.7 (the dashed curve), where it is marked with a cross. Now consider a policy rate equal to the risk free rate of return. For this rate the optimal investment strategy is still the risk free strategy, and the asset value is therefore equal to $A^r(0) = e^r(a(r) + E_0)$. The payout using $b = r$ is also shown in figure 6.7.[80] The payout to the equity holders with $b = r$ is exactly at the upper kink of the payout curve, that is, the highest value without sharing extra bonus with the customers. This payout equals $e^r(\eta a(r) + E_0)$. The payout with $b < r$ is equal to $e^b(a(b)+E_0)+\delta^b(A^b(0)-e^b(a(b)+E_0)) = e^r(\eta a(b)+E_0)$.[81] Since $a(b) < a(r)$ we have that the payout using $b = r$ is higher than the payout arising from the use of a policy rate less than the risk free rate of return. Thus, $b^* < r$ cannot be an equilibrium. This proves *(ii)*.

(iii): Again since $\pi = 0$, the date one asset value is known and equal to $A^b(0) = e^r(a(b) + E_0)$ for a given policy rate b (and given the other company's choice of policy rate). Consider a policy rate above the risk free rate of return, $b > r$. Given such a policy, the date one asset value is always between the two exercise values, that is, on the part of the payout curve, which is linear in asset value with a slope equal to one. We therefore have that payout is equal to the asset value minus the amount promised to the customer, i.e.

$$\text{Payout} = A^b(0) - a(b)(1-\eta)e^b = e^r(a(b) + E_0) - a(b)(1-\eta)e^b \quad \text{for} \quad b > r.$$

We want to find the policy rate which is higher than the risk free rate of return (if any) that yields the highest value of equity. Note, here that since there is no uncertainty ($\pi = 0$) and the risk free rate of return is constant, this is the policy rate that maximizes

[80] Note that contrary to the cooperative case, the number of customers is no longer fixed and this changes the positioning of the two payout curves relative to each other. For a given asset value the $(b < r)$–curve dominates the $(b = r)$–curve. The kink of the $(b = r)$–curve, i.e. the point $(A^r(0), e^r(a(r)+E_0))$ lies below the $(b < r)$–curve. This follows because the slope of the curve connecting the kinks of the two curves, is lower than the slope of the $(b < r)$–curve (after the kink), i.e. δ^b. See section B of the appendix for the calculations.

[81] Recall that $\delta^b = \frac{\eta a(b)+E_0}{a(b)+E_0}$ and $A^b(0) = e^r(a(b) + E_0)$.

the payout. The first order condition for company one is given by

$$
\begin{aligned}
\frac{\partial V^1}{\partial b^1} = \frac{\partial \text{Payout}}{\partial b^1} = e^r \frac{\partial a^1}{\partial b^1} - \frac{\partial a^1}{\partial b^1}(1-\eta)e^{b^1} - a^1(1-\eta)e^{b^1} &= 0 \\
\Leftrightarrow \quad \frac{\partial a^1}{\partial b^1}(e^r - (1-\eta)e^{b^1}) - a^1(1-\eta)e^{b^1} &= 0 \\
\Leftrightarrow \quad a^1\kappa e^{-\kappa(b^1-b^2)}[e^r - (1-\eta)e^{b^1}] - (1-\eta)e^{b^1} &= 0 \\
\Leftrightarrow \quad a^1\kappa e^{-\kappa(b^1-b^2)}e^r - (1 + a^1\kappa e^{-\kappa(b^1-b^2)})(1-\eta)e^{b^1}] &= 0, \qquad (6.5.2)
\end{aligned}
$$

where we have used that $\frac{\partial a^1}{\partial b^1} = (a^1)^2\kappa e^{-\kappa(b^1-b^2)}$. The first order condition for the second company is equivalent to (6.5.2).

If the equilibrium policy rate is to be higher than the risk free rate of return, the first order condition in (6.5.2) must be fulfilled. We can deduct several things from (6.5.2). First of all there is a lower bound on the level of η, for given policy rates, below which the first order condition (6.5.2) can never be satisfied. The premium percentage that solves equation (6.5.2) for given policy rates, $\underline{\eta}(b^1, b^2)$, is given by the expression in (6.5.1). Setting $b^1 = r$ in the equation, we have the value of η below which the optimal policy rate of company one is never greater than the risk free rate of return, and hence according to the arguments above, $b_1^* = r$. That is, $\eta < \underline{\eta}(r, b^2)$ implies that $b_1^* = r$ for a given b^2. Note that the "critical" level of η varies with b^2, so in fact for a given η we have that there is a critical value of b^2 below which $b^1 = r$ is optimal and above which $b^1 > r$ is the best response. For a given premium percentage, η, we have that the equilibrium policy rate(s), b^* is equal to the risk free rate of return if $\eta < \underline{\eta}(r, r)$ and greater than the risk free rate of return otherwise. q.e.d.

To summarize what we have found so far: an equilibrium policy rate strictly less than the risk free rate of return is not possible, an equilibrium policy rate equal to the risk free rate of return is always accompanied by a risk free investment strategy, and finally, if the companies can invest only in the risk free asset, the equilibrium policy rate can be greater than or equal to the risk free rate of return depending on the parameter values used. In particular, a policy rate above the risk free rate of return is only possible if the first order condition in (6.5.2) is satisfied.

We would have liked to show that, in general, an equilibrium with a policy rate above the risk free rate of return and an investment strategy allowing for some element of risk, i.e. $\pi > 0$, is not possible. While this has not been possible analytically, all of the numerical results indicate that this is the case. That is, it seems that $\pi = 0$ is the optimal investment strategy for the companies for any choice of policy rates, where we must keep in mind that the policy rates are equal in equilibrium.

Remark 6.5.2. In all of the above it is assumed that $\eta > 0$. If $\eta = 0$, competition has

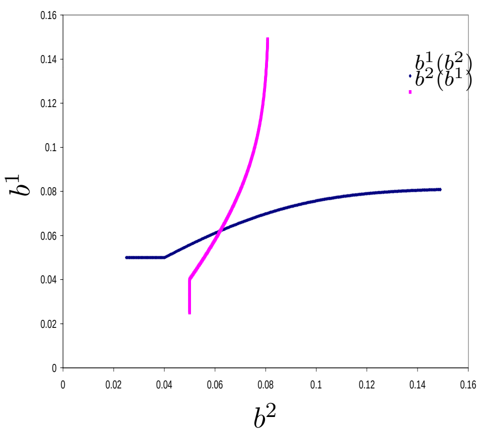

Figure 6.8: Best choices of policy rates given the other company's policy rate given $\eta = 0.05$.

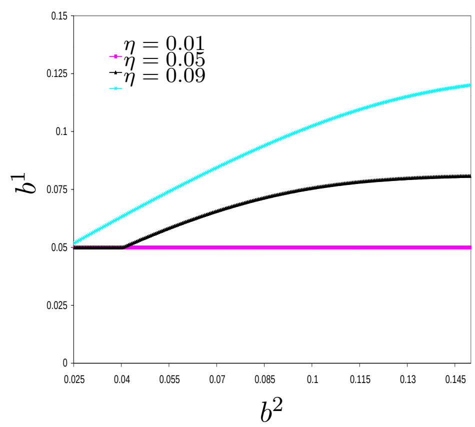

Figure 6.9: Best response curves for a company for different premium percentages.

no effect. The company does not benefit from having more customers since it receives no premium from the customers. Therefore, there is no competition for the customers and hence the solution for the cooperative case is attained. □

6.5.1 Numerical results

We now turn to some of the numerical results. In figure 6.8 we have shown the best choices of policy rates for the two companies given the other company's policy rate and that the investment strategy is chosen optimally. The parameter values are set equal to the values used in the cooperative case, and the premium percentage is set equal to 5 percent, i.e. $\eta = 0.05, r_g = 0.025, r = 0.05, \sigma = 0.20$, and $\kappa = 50$. The companies have an equal amount of initial equity, that is, they are of equal size and hence the two graphs are equivalent. Consider the curve for $b^1(b^2)$. Company one's policy rate is constant for low values of b^2. In particular, the best response to a $b^2 \in [r_g, 0.04021)$ is a policy rate equal to the risk free interest rate, $b^1 = r$. For values of b^2 in $[0.04021, \infty)$, the optimal policy rate of company one is concave in b^2. It can be shown numerically that for the given set of parameter values, b^1 converges to 8.15 percent as b^2 grows very large, i.e. there is an upper boundary for the best response policy rate of company one. The functional form of the best response curves implies that there is at most one equilibrium, i.e. a point where the response functions for the two companies, $b^1(b^2)$ and $b^2(b^1)$, intersect. In figure 6.8 the intersection is at $(b^1, b^2) = (0.0621, 0.0621)$. The corresponding optimal investment strategies are to place everything in the risk free asset for any b^1 and b^2.

The shape of the best response function depends heavily on the percentage premium, η, charged as can be seen in figure 6.9. Here we have depicted the optimal policy rate for company one given the other company's policy rate for the base case, $\eta = 0.05$, and two other choices of η. Recall that the lower boundary, $\underline{\eta}(r, b^2)$, implies that there is a critical level for b^2 below which the best response for company one is the risk free rate of return, given η. As an example consider $\eta = 0.05$. Given the base case parameters, $\eta < \underline{\eta}(r, b^2)$ if and only if $b^2 < 0.04021$, hence, for $b^2 < 0.04021$ the best response is $b^1 = r$ as illustrated in figure 6.9.

In figures 6.5 and 6.6 we have shown the equilibrium policy rates and investment strategies for a duopoly company for different levels of the premium percentage. The equilibrium policy rate(s) is equal to the risk free rate of return for $\eta \in (0, 0.03846)$ and linearly increasing in η hereafter. The value 0.03846 is exactly the critical level of the premium percentage, i.e. $\underline{\eta}(r, r) = 0.03846$, for the base case parameter values. The notion of the critical level of η can be used since the optimal investment strategy is to invest everything in the risk free asset.

Consider the case with $\eta = 0.09$, which is approximately the percentage charged in Denmark on policies offered to individuals. In this case the equilibrium policy rate is 10.5 percent. This is roughly in accordance with the policy rates offered in Denmark for the last couple of years. The optimal investment strategy is to invest everything in the risk free asset for any level of η. This is, on the other hand, not in accordance with empirical evidence. However, we defer the discussion to later.

The critical level of the premium percentage can also be seen in figure 6.4. The value of equity is increasing in the premium percentage until the critical level, $\underline{\eta}(r, r) = 0.03846$, is reached. More specifically, the value of equity with competition is equal to the value of equity without competition until the critical level is reached. Recall, that the optimal strategy without competition is any policy rate in $[r_g, r]$ (they all yield the same level of equity) and $\pi^* = 0$. The optimal solution with competition therefore yields the same level of equity for η below the critical level since here $b^* = r$ and $\pi = 0$. Once the critical level of η is passed, the equity value is constant. The equity holders do not increase the value of their position by increasing the premium percentage above the critical level. This is a major difference from the cooperative case where an increase in the premium percentage is always directly reflected in an increased value of equity. Once the premium percentage has reached the critical level, and competition comes into play, the competition between the two companies competes any additional gains from increasing η away. That is, the gains which the companies would expect to receive from an increase in the premium percentage are exactly matched by the increased costs of a policy rate higher than the risk free rate of return.

Above we have used the results from the case with no risk, i.e. the discussion of a

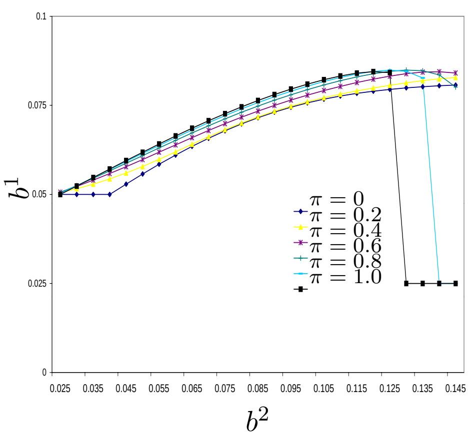

Figure 6.10: Best response as a function of the other company's policy rate for different investment strategies.

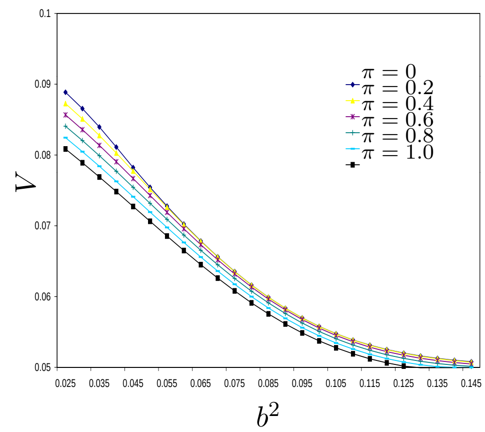

Figure 6.11: Value of equity corresponding to the best responses for the different investment strategies.

critical level for the premium percentage, in the interpretations of the results in general. This is based on our belief that the optimal investment strategy is $\pi = 0$ in general. We cannot prove this analytically, but we have analyzed several different combinations for the parameter values, and every time we ended up with the risk free investment strategy as the optimal one. We are therefore fairly convinced that it holds in general. As an illustration, we have depicted the optimal policy rate as a function of the other company's policy rate for different choices of investment strategies and a premium percentage equal to 5 percent, i.e. $\eta = 0.05$, in figure 6.10.[82] The corresponding value of equity is shown in figure 6.11. Figure 6.11 clearly indicates that $\pi = 0$ is the optimal choice of investment strategy since the value of equity with $\pi = 0$ is constantly above the other curves.

Summary of results:

- Without competition an equilibrium is characterized by $(b^*, \pi^*) = (b, 0)$, where $b \in [r_g, r]$.

- With competition among the companies, the equilibrium policy rate is above or equal to the risk free rate of return, and the equilibrium investment strategy is

[82]The "jumps" in the curve for high levels of π occur when the other company's policy rate reaches a certain level. The level is the policy rate level that would cause company one to lose parts of its initial equity if it were to match the other company's offer. The company therefore drops the "pursuit" with respect to the policy rate and instead merely offers the lowest possible policy rate, r_g. The figure also illustrates that there is an upper boundary on the best response policy rate as suggested in the discussion of figure 6.8.

the risk free strategy. In particular,

$$(b^*, \pi^*) = (r, 0) \quad \text{for} \quad 0 < \eta < \underline{\eta}(r,r)$$
$$(b^*, \pi^*) = (b^e, 0), \quad \text{for} \quad \eta \geq \underline{\eta}(r,r),$$

where $b^e > r$ and $\underline{\eta}(\cdot,\cdot)$ are given by (6.5.1).

- The value of equity is linearly increasing in the premium percentage in the cooperative case.

- The value of equity with competition is equal to the value of equity without competition for $\eta < \underline{\eta}(r,r)$, while it is constant[83] for values of η higher than this critical level. Thus, competition drives any expected gains from an increase in premium to zero once the critical level of the premium percentage is crossed.

Hence, we have a possible explanation of the relatively high policy rates we have seen being offered by life insurance companies in recent years. However, the investment strategy found is clearly at odds with what we observe empirically. In section 6.6 we therefore propose a variation of the model that allows us to arrive at results that are more in accordance with empirical facts.

Remark 6.5.3. If the initial levels of capital from the equity holders in the two insurance companies are different, i.e. $E_0^1 \neq E_0^2$, the equilibrium results remain the same. That is, the optimal policy rates and investment strategies for the two companies are the same as above. See tables 6.2 and 6.3 in section D of the appendix. If the wording of *(ii)* in proposition 6.5.1 is changed so that the policy rates of the two companies are not necessarily equal, then the proposition holds for different levels of initial capital as well. None of the arguments in the proofs of *(i)*-*(iii)* of proposition 6.5.1 depend on the level of E_0. Since only the minimum rate of return must be guaranteed (and not the policy rate), the best solution for a company is always to match the other company's policy rate and get a 'reasonable' number of customers, and hence premiums. □

6.6 The model with an imperfect capital market

In order to produce results that are more in line with empirical evidence in respect to the investment strategy, i.e. that at least a part of the free reserves is placed in the risky asset, we assume that the two life insurance companies are able to outperform all other investors (including equity holders) on the market. That is, the companies are assumed to be able to pick a portfolio of the risky assets that yields an expected return higher

[83]The value of equity is equal to the value attained with $\eta = \underline{\eta}(r,r)$ for all $\eta \geq \underline{\eta}(r,r)$.

than the risk free return under the risk neutral probability measure. We assume that the equity holders cannot outperform the market themselves. They are, however, aware that the companies are able to do so. This would be the case if there is asymmetric information in the sense that the companies receive a private signal enabling them to select a better portfolio than other investors in the market and in particular the equity holders. The equity holders are unable to infer the signals from security prices or the composition of the companies' portfolios.

We do not model the asymmetric information and the price dynamics of all the risky assets explicitly. We simply model the portfolio of risky assets that the companies invest in as a single risky asset with an expected return higher than the risk free return under the risk neutral probability measure, i.e. we use the price dynamics in (6.2.1) with a $\mu > r$.[84] A simple example of an economy where the financial market is complete and an equilibrium can exist even though a company has an expected rate of return under the risk neutral probability measure higher than the risk free rate of return is provided in section C of the appendix.

We consider the case where μ is slightly higher than the risk free rate of return. In particular, we set $\mu = 0.06$ while r remains equal to 0.05.

6.6.1 The cooperative solution

From lemma 6.4.1 we have that the optimal policy rate in the cooperative case is to offer the minimum rate of return guarantee, $b^* = r_g$. With respect to the optimal investment strategy we cannot apply the same argument as in the case with a perfect market. Therefore there is only one optimal value of the policy rate and not a whole range as in the perfect market case. The reason why that argument does not hold is that the mean of the asset value is now neither independent of the investment strategy nor equal to the date one asset value with no risk. In fact, the mean is given by

$$\begin{aligned} E^Q[A|\pi] &= \pi F_0 e^{\mu} + (1-\pi) F_0 e^{r} + a(1-\eta)e^{r_g} \\ &= \pi F_0 (e^{\mu} - e^{r}) + F_0 e^{r} + a(1-\eta)e^{r_g} \\ &\equiv \bar{A}(\pi). \end{aligned} \tag{6.6.1}$$

Note that the mean is increasing in π and strictly greater than the mean under a perfect market assumption for positive π (equality for $\pi = 0$).

The variance with $\mu > r$ is given by

$$\mathrm{Var}(A|\pi) = E^Q[A^2|\pi] - (E^Q[A|\pi])^2 = (\pi F_0 e^{\mu})^2 (e^{\sigma^2} - 1) + 2(\pi F_0)^2 (e^{\mu + r} - e^{\mu} + e^{r}). \tag{6.6.2}$$

[84]The assumption that the companies cannot go short in the risk free asset (they must satisfy the guarantee) prevents the companies from having arbitrage opportunities even though $\mu > r$ under Q.

	Value of Equity		
π	$\eta = 0.01$	$\eta = 0.05$	$\eta = 0.09$
0.0	0.055	0.075	0.095
0.1	0.055007	0.075012	0.095019
0.2	0.055014	0.075024	0.095037
0.3	0.055021	0.075036	0.095053
0.4	0.055027	0.075039	0.095021
0.5	0.055028	0.074996	0.094883
0.6	0.055003	0.074875	0.094635
0.7	0.054932	0.074676	0.094299
0.8	0.054812	0.074412	0.093899
0.9	0.054643	0.074095	0.093450
1.0	0.054433	0.073738	0.092966

Table 6.1: Value of equity as a function of the investment strategy given the optimal policy rate, for the cooperative case in an imperfect market. Parameter values as in the base case and $\mu = 0.06$.

The variance is strictly greater than the variance in the perfect market case for any positive investment strategy, π.

In table 6.1 we have shown the value of equity for different levels of the investment strategy in the cooperative case. We have shown the results for three different premium percentages, $\eta = 0.01$, $\eta = 0.05$, and $\eta = 0.09$. We see that the optimal investment strategy may very well be positive. The value of equity is, however, relatively flat for low values of π, given the base case parameters. If we increase the volatility of the risky asset, the curve is still relatively flat for low πs, whereas it decreases faster for high values of π. The optimal policy rates and investment strategies as a function of the premium percentage are shown in figure 6.12 for the cooperative case. In the same figure we have included the equilibrium results for the competitive case. The corresponding values of equity are given in figure 6.14.

We see that the optimal policy rate is equal to the minimum rate of return guarantee in the cooperative case as stated in lemma 6.4.1. The optimal investment strategy is monotonically decreasing in the premium percentage starting from a level of approximately $\pi = 0.5$, i.e. half of the free reserves invested in the risky asset. Hence, the assumption that the company can outperform the market gives the company a reason to increase the riskiness of their asset portfolio. Moreover, it implies that the company is no longer indifferent between a policy rate equal to r_g and policy rates in $(r_g, r]$.

Remark 6.6.1. The values of equity for the imperfect and perfect market differ only marginally in optimum. For example, the percentage difference is at most 0.056% for premium percentages in $[0, 0.1)$. Of course, for a fixed combination of investment

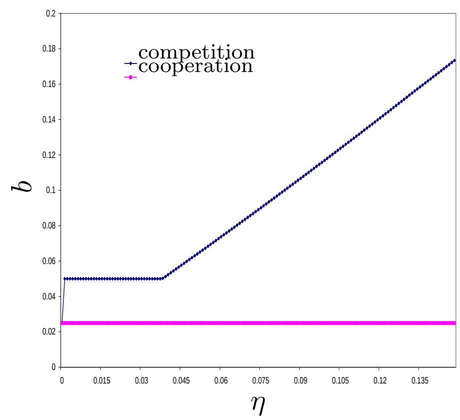

Figure 6.12: Optimal policy rates for the cooperative and the competitive cases in an imperfect market for different levels of η.

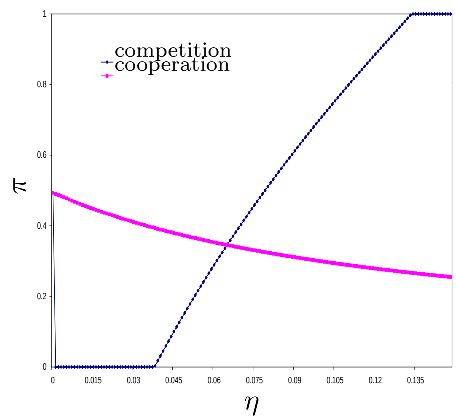

Figure 6.13: Optimal inv. strategies for the cooperative and the competitive cases in an imperfect market for different levels of η.

strategy and policy rate the difference is larger. □

6.6.2 The duopoly solution

The results for the equilibrium policy rate and the corresponding value of equity when there is competition between the two companies are very similar to the perfect market case discussed in in section 6.5.1. The major difference is seen in the investment strategy.

There seems to be a critical level of the premium percentage, η, just as in the case with a perfect market. Note, however, that we cannot think of this critical level as we did in the perfect market case, since we do not have a closed form expression for a critical η when the market is imperfect. The equilibrium policy rate is equal to the risk free rate of return until this certain premium percentage is reached and linearly increasing in η hereafter. The result is shown in figure 6.12. The "critical" level of the premium percentage is marginally lower with the imperfect market assumption. In particular, the "critical" level is at 3.781 percent, as opposed to 3.846 percent in the perfect market case. The level of the equilibrium policy rate is slightly higher under the imperfect market assumption for premium percentages above the critical level of 3.781 percent.

The equilibrium investment strategy is to place everything in the risk free asset for premium percentages below the critical level. Once the critical level of η is passed, the investment strategy is increasing in η. This investment strategy is more in line with

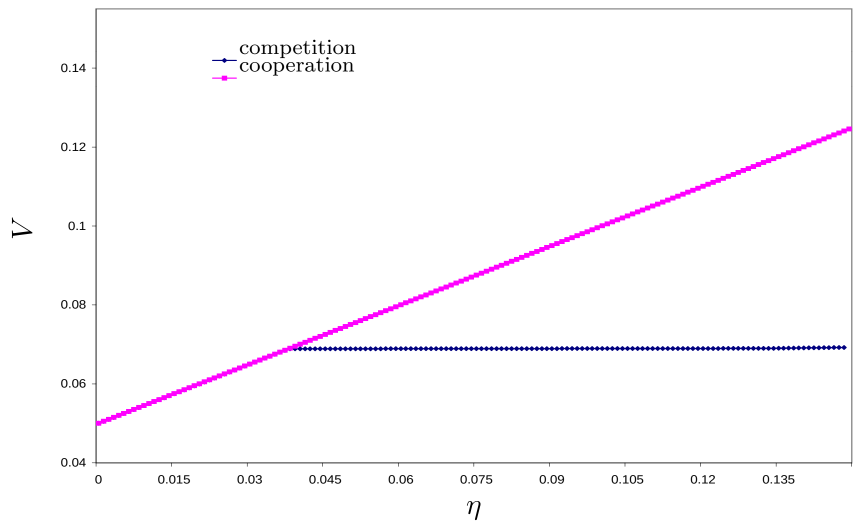

Figure 6.14: Value of equity in the competitive and the cooperative case in an imperfect market for different levels of premium percentage.

what we can expect to see in practice. For instance, given that the premium percentage is equal to 9 percent, the investment strategy is to hold 61.2 percent of the free reserves in the risky asset. This amounts to approximately 12 percent of the initial wealth being held in the risky asset. If we only consider stocks as risky assets and all other assets in a company's investment portfolio, such as bonds and real estate, as risk free assets, the model yields results that are reasonably close to what we observe empirically.

Remark 6.6.2. The investment strategy is of course highly dependent on the companies' abilities to outperform the market, that is, on the level of μ. For instance, with $\eta = 0.09$ and $\mu = 0.07$ the equilibrium investment strategy is $\pi = 79.1$ percent, which amounts to 15.3 percent of the initial wealth being held in the risky asset as opposed to 12 percent when $\eta = 0.09$ and $\mu = 0.06$ (the case mentioned above). The equilibrium policy rate when $\eta = 0.09$ and $\mu = 0.07$ is 10.9 percent and the equity value is 0.06856. The policy rate is higher and the equity value slightly lower than when $\eta = 0.09$ and $\mu = 0.06$, where the policy rate is 9 percent and the equity value is 0.06894. □

Remark 6.6.3. When the initial levels of capital, E_0^1 and E_0^2, in the two companies differ, the equilibrium policy rates of the companies differ only marginally. The effect of different levels of initial capital can been seen in the investment strategies. The company with the lowest level of initial capital will choose a much riskier investment strategy. See tables 6.4 and 6.5 in section D of the appendix. The trade-off between the level of initial capital and the investment strategy follows from their impact on the distribution of the date one asset value, see (6.6.1) and (6.6.2). □

6.7 Concluding remarks

In recent years life insurance companies and pension funds have offered relatively high policy rates to their customers. The companies seem to compete mainly on the announced policy rates. We have provided a simple one-period model that is able to explain these relatively high policy rates. We have shown how competition between companies drive them to offer policy rate well above the risk free rate of return.

Assuming a perfect market, and that the investment strategy is static, the optimal investment strategy for the companies is the risk free strategy. This is not what we observe empirically. In fact, life insurance companies and pension funds are major players on the stock market. We therefore altered the model to allow the companies to be able to outperform the market. More specifically, the drift of the risky asset portfolio that the companies (can) invest in was assumed to be higher than the risk free rate of return under the risk neutral probability measure. Under this assumption, the equilibrium investment strategy is more in line with what is observed in the market. Furthermore, the results for the equilibrium policy rates still holds. In particular, the equilibrium policy rates are only slightly higher than in the perfect market case. Given the payout structure for the insurance contracts considered and that the investment strategy cannot be altered over time, it seems to be the case that the life insurance companies are able to outperform the market. Or at least that the insurance companies and the other investors believe that the insurance companies (as the only ones) can outperform the market.

There are several natural extensions of the model. A multi-period version of the model where the number of customers is based on the level of policy rates and the level of reserves would be interesting to analyze. Especially considering the increased awareness that the level of the reserves in the companies play a significant role in whether the customers actually receive the announced policy rate. Allowing for a dynamic investment strategy would be a desirable extension. Finally, one can always argue that stochastic interest rates would be more appropriate, especially when considering contracts that are offered with a minimum rate of return guarantee.

Appendix

A Calculations

The following section contains the calculations of the value of equity for company k, $k = 1, 2$, in subsection 6.3. Recall, that $X = (1-\pi)F_0 + a(1-\eta)e^{r_g} + a(1-\eta)e^b$. The superscripts indicating which company we are dealing with are left out since the results are the same for both companies.

Let A and B denote the 'exercise' sets of the hypothetical options involved in the payout, i.e. the following sets of ωs

$$A = \Big(0 \leq \pi F_0 e^{\mu - \frac{1}{2}\sigma^2 + \sigma w} + X \leq e^b(\eta a + E_0)\Big) \tag{A.1}$$

$$B = \Big(e^b(\eta a + E_0) < \pi F_0 e^{\mu - \frac{1}{2}\sigma^2 + \sigma w} + X\Big) \tag{A.2}$$

where $w \sim N(0, 1)$.

Define d and l by

$$d = \frac{1}{\sigma}\Big\{\ln\Big(\frac{-X}{\pi F_0}\Big) - (\mu - \frac{1}{2}\sigma^2)\Big\} \tag{A.3}$$

$$l = \frac{1}{\sigma}\Big\{\ln\Big(\frac{e^b(\eta a + E_0) - X}{\pi F_0}\Big) - (\mu - \frac{1}{2}\sigma^2)\Big\}. \tag{A.4}$$

$$\tag{A.5}$$

Then,

$$A = \begin{cases} (d \leq w \leq l) & \text{if } \ X < 0 \\ (w \leq l) & \text{if } \ 0 \leq X < e^b(\eta a + E_0) \\ \emptyset & \text{if } \ X \geq e^b(\eta a + E_0). \end{cases} \tag{A.6}$$

For B we have

$$\begin{aligned} B &= \Big(e^b(\eta a + E_0) \leq (\pi F_0 e^{\mu - \frac{1}{2}\sigma^2 + \sigma w} + X)\Big) \\ &= \Big(\frac{e^b(\eta a + E_0) - X}{\pi F_0} \leq e^{\mu - \frac{1}{2}\sigma^2 + \sigma w}\Big) \\ &= \begin{cases} (l \leq w) & \text{if } \ X < e^b(\eta a + E_0) \\ \mathbf{R} & \text{if } \ X \geq e^b(\eta a + E_0). \end{cases} \end{aligned} \tag{A.7}$$

The value of a company can now be calculated as

$$
\begin{aligned}
V &= E^Q\Big[e^{-r}\min\Big\{\max\Big\{\pi^i F_0^i e^{\mu-\frac{1}{2}\sigma^2+\sigma W}+X,0\Big\}, e^{b^i}(\eta a^i+E_0^i)\Big\} \\
&\quad + \delta\max\Big\{\pi F_0 e^{\mu-\frac{1}{2}\sigma^2+\sigma w}+X-e^b(\eta a+E_0),0\Big\}\Big] \\
&= e^{-r}\Big\{E^Q\Big[(\pi F_0 e^{\mu-\frac{1}{2}\sigma^2+\sigma w}+X)1_A\Big]+E^Q\Big[(e^b(\eta a+E_0)1_B\Big] \\
&\quad + \delta E^Q\Big[(\pi F_0 e^{\mu-\frac{1}{2}\sigma^2+\sigma W}+X-e^b(\eta a+E_0))1_B\Big]\Big\} \\
&= e^{-r}\Big\{E^Q\Big[(\pi F_0 e^{\mu-\frac{1}{2}\sigma^2+\sigma w}+X)1_A\Big]+(1-\delta)e^b(\eta a+E_0)E^Q\Big[1_B\Big] \\
&\quad + \delta E^Q\Big[(\pi F_0 e^{\mu-\frac{1}{2}\sigma^2+\sigma w}+X)1_B\Big]\Big\}
\end{aligned}
$$

For $0 \le X < e^b(\eta a+E_0)$, we have

$$
\begin{aligned}
V &= e^{-r}\Big\{E^Q\Big[(\pi F_0 e^{\mu-\frac{1}{2}\sigma^2+\sigma w}+X)1_{(w\le l)}\Big]+(1-\delta)e^b(\eta a+E_0)E^Q\Big[1_{(w\ge l)}\Big] \\
&\quad + \delta E^Q\Big[(\pi F_0 e^{\mu-\frac{1}{2}\sigma^2+\sigma w}+X)1_{(w\ge l)}\Big]\Big\} \\
&= e^{-r}\Big\{\pi F_0 e^{\mu}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\sigma^2+\sigma w}1_{(w\le l)}\frac{1}{\sqrt{2\pi}}e^{-\frac{w^2}{2}}dw + XN(l)+(1-\delta)e^b(\eta a+E_0)(1-N(l)) \\
&\quad + \delta\Big(\pi F_0 e^{\mu}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\sigma^2+\sigma w}1_{(l\le w)}\frac{1}{\sqrt{2\pi}}e^{-\frac{w^2}{2}}dw + X(1-N(l))\Big)\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= e^{-r}\Big\{\pi F_0 e^{\mu}\int_{-\infty}^{l}\frac{1}{\sqrt{2\pi}}e^{-\frac{(w-\sigma)^2}{2}}dw + XN(l)+(1-\delta)e^b(\eta a+E_0)(1-N(l)) \\
&\quad + \delta\pi F_0 e^{\mu}\int_{l}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{(w-\sigma)^2}{2}}dw + \delta X(1-N(l))\Big\} \\
&= e^{-r}\Big\{\pi F_0 e^{\mu}N(l-\sigma)+XN(l)+(1-\delta)e^b(\eta a+E_0)(1-N(l))) \\
&\quad + \delta\pi F_0 e^{\mu}(1-N(l-\sigma))+\delta X(1-N(l))\Big\} \\
&= e^{-r}\Big\{(1-\delta)(\pi F_0 e^{\mu}N(l-\sigma)+XN(l)+e^b(\eta a+E_0)(1-N(l)))+\delta(\pi F_0 e^{\mu}+X)\Big\} \\
&=: g.
\end{aligned}
$$

For $X < 0$, equivalent calculations yield

$$
\begin{aligned}
V &= e^{-r}\Big\{E^Q\Big[(\pi F_0 e^{\mu-\frac{1}{2}\sigma^2+\sigma w}+X)1_{(d\le w\le l)}\Big]+(1-\delta)e^b(\eta a+E_0)E^Q\Big[1_{(w\ge l)}\Big] \\
&\quad + \delta E^Q\Big[(\pi F_0 e^{\mu-\frac{1}{2}\sigma^2+\sigma w}+X)1_{(w\ge l)}\Big]\Big\} \\
&= e^{-r}\Big\{\pi F_0 e^{\mu}(N(l-\sigma)-N(d-\sigma))+X(N(l)-N(d))+(1-\delta)e^b(\eta a+E_0)(1-N(l)) \\
&\quad + \delta\pi F_0 e^{\mu}(1-N(l-\sigma))+\delta X(1-N(l))\Big\} \\
&=: h.
\end{aligned}
$$

For $X \geq e^b(\eta a + E_0)$,

$$V = e^{-r}\Big\{(1-\delta)e^b(\eta a + E_0) + \delta(\pi F_0 e^{\mu} + X)\Big\} =: f.$$

Hence, the present value of profits to a company is given by

$$V = f1_{(X \geq \exp(b)(\eta a + E_0))} + g1_{(0 \leq X < \exp(b)(\eta a + E_0))} + h1_{(X<0)}.$$

Note that V is continuous. For $X \to e^b(\eta a + E_0)$, we have that $l \to -\infty$ and hence $N(l-\sigma)$ and $N(l)$ both converge to zero, which implies $g \to f$. For $X \to 0$, $d \to -\infty$, which implies that $N(d-\sigma)$ and $N(d)$ converge to zero and thus $h \to g$.

B Proofs

Proof of proposition 6.4.2: Let $H(\cdot)$ denote the equity holders' payout function as a function of asset value A. The function H is concave in A on the support of A.

Let $E^Q[\cdot|\pi]$ denote the conditional expectation under the risk neutral probability measure, Q, given the investment strategy π.[85] For a given arbitrary investment strategy, π, and the optimal choice of policy rate, $b^* = r_g$, the application of *Jensen's* inequality yields that the date zero value of equity, $V(\pi, b = r_g)$, satisfies the following:

$$\begin{aligned} V(\pi, b = r_g) &= e^{-r} E^Q[H(A)] \\ &\leq e^{-r} H(E^Q[A]) \\ &= e^{-r} H(A(0)) && \text{(B.1)} \\ &= V(0, b = r_g). && \text{(B.2)} \end{aligned}$$

(B.1) follows from the fact that the mean is equal to the certain outcome for the asset value when $\pi = 0$, $A(0)$. From (B.2) it is clear that $\pi = 0$ is a solution to the problem of maximizing $V(\pi, b = r_g)$ w.r.t. π.

q.e.d.

Proof of proposition 6.5.1: Recall that for given policy rates the mean of the asset value is equal to the asset value with $\pi = 0$, i.e. $E^Q[A] = A^b(0) = e^r(a(b) + E_0)$, where the superscript b indicates the dependence on the company's choice of policy rate, b. Let the policy rate of company two be given. With a policy rate for company one, $b \leq r$, the point $A^b(0)$ is to the right of the second exercise value, i.e. $A^b(0) \geq e^b(a(b) + E_0)$. Let $H(A)$ denote the equity holders' payout function as a function of asset value A given policy rate b, and let $h(A)$ denote the affine function which is equal to $H(A)$ for A above $e^b(a(b) + E_0)$ and extended along the line with slope δ on the rest of A. See figure 6.15.[86] With this definition of h, $h(A) = H(A)$ for $A \geq e^b(a(b) + E_0)$ and

[85] Recall that the distribution of A under Q depends on π, see (6.4.3) and (6.4.4).

[86] The intercept of the h-curve, say y, is at the origin. This follows since, $y = e^b(\eta a(b) + E_0) - \delta^b\big(e^b(a(b) + E_0)\big) = e^b(\eta a(b) + E_0) - \frac{\eta a(b) + E_0}{a(b) + E_0}\big(e^b(a(b) + E_0)\big) = 0$.

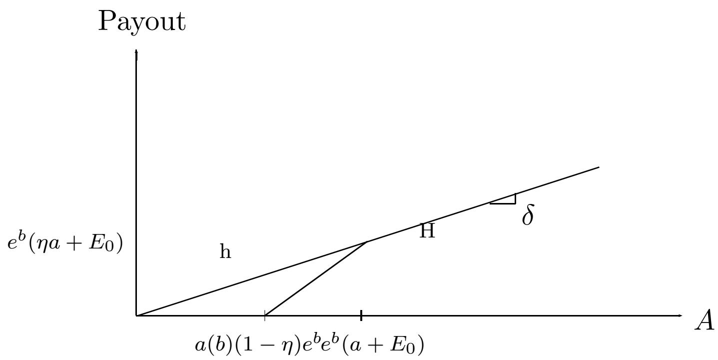

Figure 6.15: Illustration of h and H.

$h(A) > H(A)$ for $A < e^b(a(b) + E_0)$. We have that

$$\begin{aligned} V(\pi, b \leq r) &= e^{-r} E^Q[H(A)] \\ &\leq e^{-r} E^Q[h(A)] \\ &= e^{-r} h(E^Q[A]) \qquad \text{(B.3)} \\ &= e^{-r} h(A^b(0)) \\ &= e^{-r} H(A^b(0)) \qquad \text{(B.4)} \\ &= V(0, b \leq r). \end{aligned}$$

The equality sign in (B.3) follows because h is an affine function in A. In (B.4) we have used that the mean of A is higher than the second exercise value and hence that h and H are equal when evaluated at the mean. Again it is clear that $\pi = 0$ is a solution to the problem of maximizing $V(\pi, b \leq r)$ w.r.t. π given the other company's policy rate.

q.e.d.

Proof of the positioning of the payout curves with competition: First of all we know that the fraction of surplus to the company is smaller the higher the policy rate is (given the other company's policy rate). That is $\delta^b > \delta^{b'}$ for $b < b'$. This follows from the definition of δ ($\delta^b = \frac{\eta a(b)+E_0}{a(b)+E_0}$) since $a(b) < a(b')$ for $b < b'$. Now consider the payout curves arising from policy rates equal to b and $b' > b$, respectively. The kink of the b–curve is at $(e^b(a(b) + E_0), e^b(\eta a(b) + E_0))$ while the kink of the b'–curve is at $(e^{b'}(a(b') + E_0), e^{b'}(\eta a(b') + E_0))$. Denote the slope of the line connecting these two point by α. This slope is equal to $\alpha = \frac{\exp(b')(\eta a(b')+E_0)-\exp(b)(\eta a(b)+E_0)}{\exp(b')(a(b')+E_0)-\exp(b)(a(b)+E_0)}$. It should be clear that the b'–curve is always below the b–curve if $\alpha < \delta^b$ or equivalently if $\alpha - \delta^b < 0$. For $b < b'$ and $\eta \in (0, 1]$, we have

$$\begin{aligned}
\alpha - \delta^b &= \frac{e^{b'}(\eta a(b') + E_0) - e^b(\eta a(b) + E_0)}{e^{b'}(a(b') + E_0) - e^b(a(b) + E_0)} - \frac{\eta a(b) + E_0}{a(b) + E_0} \\
&= \frac{\big(e^{b'}(\eta a(b') + E_0) - e^b(\eta a(b) + E_0)\big)(a(b) + E_0) - (\eta a(b) + E_0)\big(e^{b'}(a(b') + E_0) - e^b(a(b) + E_0)\big)}{\big(e^{b'}(a(b') + E_0) - e^b(a(b) + E_0)\big)(a(b) + E_0)} \\
&= \frac{e^{b'}(\eta a(b') + E_0)(a(b) + E_0) - (\eta a(b) + E_0)e^{b'}(a(b') + E_0)}{\big(e^{b'}(a(b') + E_0) - e^b(a(b) + E_0)\big)(a(b) + E_0)} \\
&= \frac{e^{b'}\big[\eta a(b')E_0 + E_0 a(b) + E_0^2 - (\eta a(b)E_0 + E_0 a(b') + E_0^2)\big]}{\big(e^{b'}(a(b') + E_0) - e^b(a(b) + E_0)\big)(a(b) + E_0)} \\
&= \frac{e^{b'}E_0(1 - \eta)(a(b) - a(b'))}{\big(e^{b'}(a(b') + E_0) - e^b(a(b) + E_0)\big)(a(b) + E_0)} < 0. \qquad \text{(B.5)}
\end{aligned}$$

C Example

In order to support the story of the companies being able to outperform the market in the sense that they can pick a portfolio which yields an expected rate of return higher than the risk free rate of return under the risk neutral probability measure, we provide a simple example.

Consider a market with two risky assets, indexed by 1 and 2, and one risk free asset. The risk free interest rate is without loss of generality set equal to zero, that is, $r = 0$. The possible outcomes for the price of risky asset 1, S^1, is given in figure 6.16. The market is complete since there are three possible states and there are three assets trading. The risk neutral probabilities of the three states are denoted q_1, q_2, and q_3. The companies are restricted from short sales in the risky assets and they cannot invest

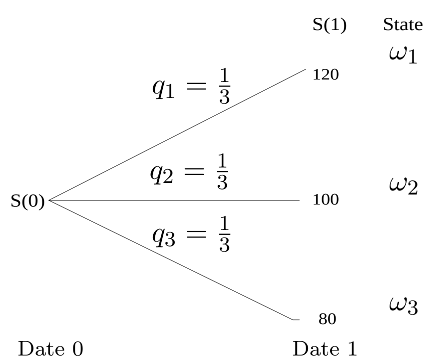

Figure 6.16: Dynamics of the price of asset 1, S^1, under the risk neutral probability measure.

more than a specific amount, namely the free reserves, in risky assets.[87] The market price of the asset at date 0 is

$$E^Q[S^1(1)] = q_1 120 + q_2 100 + q_3 80 = 100,$$

which is equivalent to an expected rate of return equal to the risk free interest rate, $r = 0$. The information structure for the company is given by $\{\{\omega_1, \omega_2\}, \{\omega_3\}\}$. That is, at date 0 the companies receive a signal, y_1 or y_2, which tells them whether $\{\omega_1, \omega_2\}$ or $\{\omega_3\}$ will occur.[88] Only the companies can observe the signals and therefore actually

[87]The life insurance companies must be able to satisfy the guarantee.
[88]That is, $y_1 \mapsto \{\omega_1, \omega_2\}$ and $y_2 \mapsto \{\omega_3\}$.

use the signals in the decision to buy or sell the asset. However, everyone knows the information structure of the company and can perform the calculations that follow below.

The risk neutral probabilities of receiving y_1 and y_2 are $q(y_1) = 2/3$ and $q(y_2) = 1/3$, respectively. Conditional on the signal, the risk neutral probabilities are,

$$q_1|_{y_1} = 1/2, \quad q_2|_{y_1} = 1/2, \quad q_3|_{y_1} = 0 \quad \text{and} \quad q_1|_{y_2} = 0, \quad q_2|_{y_2} = 0, \quad q_3|_{y_2} = 1. \tag{C.1}$$

Given the signal y_1, the value of the asset is $E^Q[S^1(1)|y_1] = q_1|_{y_1}120 + q_2|_{y_1}100 + q_3|_{y_1}0 = 110$. If the companies receive the signal y_2, the value is $E^Q[S^1(1)|y_2] = 0 + 0 + q_3|_{y_2}80 = 80$. Hence, if the companies observe y_1, they will buy the asset and receive an expected return of $\mu^1|y_1 = 10$ percent, whereas if they observe y_2, they will not buy the asset and receive an expected return of $\mu^1|y_2 = 0$ percent. If there are no short sale restrictions, they would of course take a short position in asset 1 if they observe y_2. Since this is not possible, they simply choose not to buy asset 1. Note that infinite arbitrage is also precluded if the companies observe y_1. The companies cannot borrow an infinite amount of money and invest in asset 1 because they must be able to satisfy a guarantee with certainty. To sum up, the companies' expected rate of return on asset 1 given that they receive a signal y_1 or y_2 is

$$\mu^1 = q(y_1)\mu^1|_{y_1} + q(y_2)\mu^1|_{y_2} = \frac{2}{3}0.1 + \frac{1}{3}0 = 0.0667 > r = 0.$$

A similar argument can be made for asset 2. Note that asset 2 could be positively or negatively correlated with asset 1: either way the expected rate of return on asset 2 given the signal will be higher than the risk free rate of return under the risk neutral probabilities. Thus, the company can choose a portfolio of assets that yields a rate of return higher than the risk free rate of return under the risk neutral probabilities.

The above is of course only one example of an information structure that allows an equilibrium to exist in a market as the one described. Another possibility is that the companies receives signals that change their risk neutral probabilities instead of separating the state space.

D Tables

This section contains the equilibrium results with competition for the case where the initial levels of capital in the two companies differ. The results are shown for two levels of the premium percentage, $\eta = 0.05$ and $\eta = 0.09$, for the perfect and the imperfect market cases.

E_0^1	$b^1 = b^2$	$\pi^1 = \pi^2$	V^1	V^2
0.00	0.0621	0.00	0.01923	0.06923
0.01	0.0621	0.00	0.02923	0.06923
0.02	0.0621	0.00	0.03923	0.06923
0.03	0.0621	0.00	0.04923	0.06923
0.04	0.0621	0.00	0.05923	0.06923
0.05	0.0621	0.00	0.06923	0.06923
0.06	0.0621	0.00	0.07923	0.06923
0.07	0.0621	0.00	0.08923	0.06923
0.08	0.0621	0.00	0.09923	0.06923
0.09	0.0621	0.00	0.10923	0.06923
0.10	0.0621	0.00	0.11923	0.06923

Table 6.2: Perfect market case: Equilibrium results for different levels of E_0^1 for $E_0^2 = 0.05$ and $\eta = 0.05$.

E_0^1	$b^1 = b^2$	$\pi^1 = \pi^2$	V^1	V^2
0.00	0.1051	0.00	0.01923	0.06923
0.01	0.1051	0.00	0.02923	0.06923
0.02	0.1051	0.00	0.03923	0.06923
0.03	0.1051	0.00	0.04923	0.06923
0.04	0.1051	0.00	0.05923	0.06923
0.05	0.1051	0.00	0.06923	0.06923
0.06	0.1051	0.00	0.07923	0.06923
0.07	0.1051	0.00	0.08923	0.06923
0.08	0.1051	0.00	0.09923	0.06923
0.09	0.1051	0.00	0.10923	0.06923
0.10	0.1051	0.00	0.11923	0.06923

Table 6.3: Perfect market case: Equilibrium results for different levels of E_0^1 for $E_0^2 = 0.05$ and $\eta = 0.09$.

E_0^1	b^1	b^2	π^1	π^2	V^1	V^2
0.00	0.06295	0.06298	0.3442	0.1655	0.01890	0.06892
0.01	0.06296	0.06298	0.2778	0.1655	0.02890	0.06892
0.02	0.06297	0.06299	0.2349	0.1656	0.03890	0.06891
0.03	0.06298	0.06300	0.2049	0.1656	0.04890	0.06891
0.04	0.06300	0.06300	0.1827	0.1657	0.05890	0.06890
0.05	0.06301	0.06301	0.1658	0.1658	0.06890	0.06890
0.06	0.06302	0.06301	0.1523	0.1658	0.07890	0.06889
0.07	0.06303	0.06302	0.1414	0.1659	0.08890	0.06889
0.08	0.06304	0.06302	0.1324	0.1659	0.09890	0.06888
0.09	0.06305	0.06303	0.1249	0.1660	0.10890	0.06888
0.10	0.06306	0.06303	0.1184	0.1660	0.11890	0.06887

Table 6.4: Imperfect market case ($\mu = 0.06$): Equilibrium results for different levels of E_0^1 for $E_0^2 = 0.05$ and $\eta = 0.05$.

E_0^1	b^1	b^2	π^1	π^2	V^1	V^2
0.00	0.10758	0.10718	1.0000	0.6142	0.01878	0.06857
0.01	0.10680	0.10680	0.8991	0.6122	0.02889	0.06894
0.02	0.10676	0.10678	0.7921	0.6121	0.03891	0.06896
0.03	0.10677	0.10678	0.7180	0.6121	0.04892	0.06895
0.04	0.10678	0.10679	0.6595	0.6122	0.05893	0.06895
0.05	0.10679	0.10679	0.6122	0.6122	0.06894	0.06894
0.06	0.10681	0.10680	0.5731	0.6122	0.07895	0.06893
0.07	0.10682	0.10680	0.5403	0.6122	0.08896	0.06893
0.08	0.10683	0.10681	0.5124	0.6123	0.09897	0.06892
0.09	0.10684	0.10682	0.4884	0.6123	0.10898	0.06892
0.10	0.10685	0.10682	0.4675	0.6123	0.11899	0.06891

Table 6.5: Imperfect market case ($\mu = 0.06$): Equilibrium results for different levels of E_0^1 for $E_0^2 = 0.05$ and $\eta = 0.09$.

Summary

In this dissertation we analyze various types of problems, all of a financial nature, which are present in life and pension insurance. Common for the issues that are addressed is that they are closely connected to the so-called interest rate guarantees. A typical life or pension insurance contract is issued with some sort of guarantee. Whether the guarantee is an annual guarantee or a maturity guarantee is often debated. In the latter case the customer is guaranteed a rate of return on average over the life time of the contract and not each year.

Besides the guarantee a typical life or pension insurance contract gives the customer the possibility of receiving a return which is higher than what the guarantee prescribes. This extra return is called bonus. The distribution of bonus to the customers must be done according to the so-called contribution principle. This principle states that bonus must be distributed in a way that reflects each party's contribution to the bonus. In real life, however, it is, very difficult to find out exactly what rules for distributing bonus the various life and pension insurance companies apply.

In the dissertation we consider various types of contracts that have a form of guarantee and a specific bonus distribution rule. In particular, we are interested in *fair* contracts. By fair we mean that the terms of a contract (share of bonus, etc.) are initially set such that the present value of the premium payments from the holder of the contract is equal to the present value of the payout to him. This condition is very similar to the so-called principle of equivalence known from actuarial science. The difference is that in this dissertation we operate with present values that incorporate financial risk, that is, we use the risk neutral probability measure.[89]

In most of the dissertation, parts I–IV, it is assumed that the insurance market is characterized by perfect competition and that the fair price therefore equals the market price. Part I contains an introduction to the dissertation. Chapters 2 and 3 in part II provide a brief survey of work which is concerned mainly with the pricing of various life and pension insurance contracts. Chapter 2 deals with so-called equity-linked policies that are equipped with a guarantee as opposed to pure equity-linked or unit-linked contracts where the policy holder carries the financial risk alone. The common factors

[89] Traditionally the principle of equivalence is formulated using expectations under the physical probability measure.

of the models presented in chapter 2 are that the payout to the customer is based on a fixed reference portfolio (for instance, a stock index) and that the customer receives all of the bonus that might be generated by the contract. In chapter 3 models that use a different type of bonus distribution are considered. In the first part contracts with a maturity guarantee and a relatively simple bonus distribution rule are considered. Next, some models that are concerned with contracts with an annual rate of return guarantee and a relatively advanced bonus distribution rule are presented. The types of contracts that chapter 3 deals with are called *participating* policies. A participating policy has a payout, and hence bonus, which depends on the performance of the issuing company's own investment portfolio. The dynamics of this investment portfolio is typically modeled as a fixed portfolio, and the contract therefore resembles the equity-linked contract quite a bit.

In the paper presented as part III the attention is drawn to the issue of modeling the insurance company's investment portfolio. The paper takes portfolio choice into consideration when analyzing fair participation contracts. This has, to the best of the authors' knowledge, not been done previously. The fact that the underlying portfolio can be changed over time affects the fair terms of a contract in a way that cannot be ignored. The paper considers two cases: a situation where the issuing company can default on the guarantee and a case where the guarantee is binding, that is, the company must invest in such a way that the guarantee can always be honored.

With the paper in part IV we return to modeling the underlying portfolio as a fixed reference portfolio. Contracts with annual rate of return guarantees and a fairly complex bonus distribution rule are analyzed. The paper can be divided into two parts: the first part deals with fair pricing of the contracts on an individual level, that is, where each customer has his own bonus account. The second part is concerned with an investigation of a situation where two different types of customers share a bonus account. The customers could for example have different minimum rate of return guarantees. In particular, the focus is on the redistributions of bonus that might occur between the two customers.

While parts I-IV are based on the assumption that the insurance market is perfectly competitive, the paper presented in part V uses a Cournot model of duopoly. In the paper the Cournot model is used in an attempt to explain the relatively high policy rates that have been offered by insurance companies during the last few years. It seems to be the case that the companies compete mainly on policy rates. A situation where two companies, through their choice of policy rates, compete for a group of customers is analyzed. The competition clearly drives the policy rates up—how high depends on how large a premium the companies collect from each customer. In the model the companies can choose a static investment strategy together with the policy rate. They must, however, always be able to satisfy the guarantee which they have given their customers.

Summary in Danish—Resumé

I denne afhandling beskæftiger vi os med forskellige finansielle problemstillinger inden for livs– og pensionsforsikring. Fælles for de problemstillinger, der analyseres, er at de er tæt forbundet med de såkaldte rentegarantier. En typisk livs– eller pensionsforsikringskontrakt er udstedt med en form for rentegaranti. Det diskuteres ofte, om der er tale om en årlig garanteret rente, eller om der er tale om en ydelsesgaranti. For sidstnævnte type gælder det, at kunden er garanteret en bestemt forrentning i gennemsnit over hele kontraktens løbetid og ikke hvert enkelt år.

Ud over garantielementet giver en typisk livs- eller pensionsforsikringskontrakt mulighed for, at kunden (kontraktholderen) modtager et højere afkast end det, som rentegarantien giver anledning til. Dette merafkast kaldes bonus. Tildelingen af bonus til kunderne skal ske efter det såkaldte kontributionsprincip, der siger, at bonus skal tilfalde den enkelte bidragsyder i et omfang, der svarer til hans bidrag til fremkomsten af bonus. I praksis kan det dog være svært at få klarhed over, nøjagtig hvilke regler de enkelte selskaber benytter, når der skal fordeles bonus.

I afhandlingen ser vi på forskellige typer af kontrakter med et garantielement samt en bonusfordelingsregel. Vi er især interesseret i at betragte *fair* kontrakter, hvor der med fair menes, at kontraktbetingelserne (andel af bonus o.lign.) initialt er fastsat således, at nutidsværdien af indbetalingerne fra en kunde er lig med nutidsværdien af udbetalingerne til kunden. Denne betingelse minder meget om det såkaldte ækvivalensprincip kendt fra forsikringslitteraturen. Forskellen ligger i, at der i denne afhandling arbejdes med nutidsværdier, som inddrager finansiel risiko, og der arbejdes således under det risikoneutrale sandsynlighedsmål.[90]

I størstedelen af afhandlingen, del I-IV, arbejdes der ud fra en antagelse om, at forsikringsmarkedet er præget af fuldkommen konkurrence, og at den fair pris således er lig med markedsprisen. Del I indeholder en introduktion til afhandlingen. Kapitel 2 og 3 i del II indeholder en kort fremstilling af, hvad der er lavet inden for hovedsagelig prisfastsættelse af diverse livs- og pensionsforsikringskontrakter. Kapitel 2 fokuserer på såkaldte equity-linked kontrakter, der er udstyret med en garanti i modsætning til rene equity-linked eller unit-linked kontrakter, hvor kunden bærer hele den finansielle

[90] I dets oprindelige formulering er ækvivalensprincippet formuleret ved hjælp af forventninger under det fysiske sandsynlighedsmål.

risiko. Fælles for modellerne, som beskrives i kapitel 2, er, at kontraktens payout er baseret på en fast referenceportefølje fx et indeks, og at kunden får al bonus, der måtte blive genereret i forbindelse med kontrakten. I kapitel 3 behandles modeller med en anden bonusfordeling. Først kigges der på kontrakter med en ydelsesgaranti og en relativt simpel bonusfordelingsregel. Dernæst betragtes kontrakter med en årlig rentegaranti og en relativt kompliceret bonustildelingsregel. Kontrakttyperne i kapitel 3 benævnes *participating*-kontrakter. En sådan kontrakt har et payout—og derved bonus—der er afhængig af, hvordan det udstedende selskabs egen investeringportefølje klarer sig. Typisk modelleres udviklingen i denne investeringsportefølje som værende fast og kontrakten, der analyseres minder derfor umådeligt meget om en equity-linked kontrakt.

Netop modelleringen af et selskabs egen investeringsportefølje tages op i artiklen, der indgår som del III. Her inddrages porteføljevalg i analysen af fair kontrakter af 'participation'-typen. Dette er efter forfatternes opfattelse ikke gjort tidligere. Det, at den underliggende portføjle kan lægges om løbende, har ikke neglicérbare effekter på de fair betingelser for en kontrakt. Der betragtes to tilfælde: Et tilfælde hvor det udstedende selskab ikke nødvendigvis skal opfylde garantien over for kunden—garantien opfattes som et løfte, der kan vise sig ikke at være muligt at indfri, samt et tilfælde hvor det kræves at selskabet investerer på en sådan måde, at garantien altid kan honoreres.

I artiklen i del IV går vi tilbage til at modellere med en fast referenceportefølje. Kontrakter med årlige rentegarantier samt en relativ advanceret bonusfordelingsmekanisme analyseres. Artiklen kan opdeles i to dele: Første del omhandler fair prisfastsættelse af kontrakterne på individuelt niveau, dvs. hvor den enkelte kunde har en individual bonuskonto. Anden del betragter et tilfælde, hvor to forskellige kundetyper (fx. kunder med forskellige rentegarantier) deles om en bonuskonto. Der lægges især vægt på at undersøge den omfordeling af bonus mellem kundetyperne, som en fælles bonuskonto kan give anledning til.

Mens del I–IV er baseret på en antagelse om, at der er fuldkommen konkurrence på livs- og pensionsforsikringsmarkedet, arbejdes der i artiklen i del V med en Cournot duopol model. Ved hjælp af denne model gøres et forsøg på at forklare de relativt høje kontorenter, der er blevet observeret de senere år. Når man betragter de kontorenter som selskaber, typisk i slutningen af et år, har annonceret for det kommende år, ser det ud til, at kontorenten er en betydelig konkurrenceparameter for de forskellige selskaber. Artiklen i del V analyserer en situation, hvor to selskaber konkurrerer om kunder, netop gennem deres valg kontorente. Konkurrencen driver tydeligvis kontorenterne op—størrelsen af kontorenterne er dog meget afhængig af, hvor stor en præmieindtægt der modtages når man får en ny kunde. I modellen har selskaberne mulighed for samtidigt med deres valg af kontorente at vælge en statisk investeringstrategi. Selskaberne skal dog altid være i stand til at opfylde rentegarantien, som de har givet deres kunder.

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