Best estimate calculations of savings contracts by closed formulas

Application to the ORSA

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ABSTRACT

In this paper we present an analytical approximation of the best estimate of a savings contract. This approximation aims to provide a framework for robust and justifiable calculation of the Own Risk Solvency Assessment (ORSA) avoiding the complexity of direct approaches. A numerical application is proposed.

1. INTRODUCTION

One major difficulty in the implementation of Solvency 2 in life insurance is the calculation of the value of best estimate liabilities (fair value) for participating contracts. The complex interaction between the yield of the portfolio of assets $r_A(t)$, the increase rate of savings $r_s(t)$ and the surrender rate $\mu(t)$ leads to complex models for the mathematical reserve $PM(t)$ and the fair valuation of the contract (see PLANCHET and LEROY [2011] for an overview and PLANCHET and *al*. [2011] for a more detailed presentation).

Practitioners are turning to *ad hoc* approaches by projecting the flow of benefits of the contract with Markov models, and obtained numerical results heavily rely on simulations. Though this helps to describe the flow dynamics accurately, cumbersome calculations make these models difficult to use, calibrate and maintain (*cf*. BAUER and *al*. [2010]).

Since the seminal paper of BRIYS and DE VARENNE [2004], many models in the academic literature offer explicit evaluations of best estimates of savings contracts. However, these evaluations are done at the cost of rather restrictive assumptions about the design of the contracts; those assumptions prevent the use of these models for practical assessments. Indeed, the literature only considers contracts with terminal bonus, which is not realistic in the French insurance market (see for instance BACINELLO [2003], HAINAUT [2009], BALLOTTA [2004] and the references therein).

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Moreover, most work on this subject is limited to a one-year projection of the contract. Nevertheless the Own Risk Solvency Assessment (ORSA) implementation requires that the insurer projects its balance sheet over several years. AASE and PERSSON [1996] propose an analytical approach with a projection of the contract over several years. This is to the best of our knowledge the only work that does so, but for a simple contract.

Markov style models mentioned above are poorly suited to do these projections, because of computational complexity and the lack of robustness (which is mainly due to over parameterization). Thus, our goal in this paper is to build a model that is able to take into account complex contracts for computing projected best estimate valuations and which is well-suited to the ORSA framework.

Roughly speaking, the main idea is to use a classical asset-liability model to project the statutory balance sheet and compute the mathematical reserve. Then we develop a simple framework to compute a coefficient (with a closed formula) which when applied to the mathematical reserve gives the associated fair value of the contract. Indeed, in general we observe that the best estimate is near the mathematical reserve (between 95% and 105% of it on most cases considered). Thus, we seek a coefficient to be applied to the mathematical reserve that accounts for the time value of options.

For the Solvency Capital Requirement (SCR) calculation and projection, we adapt here the model described in GUIBERT and *al*. [2012] to life insurance. The framework is built by directly specifying the dynamics of the increase rate of the contract. In our model the best estimate of the contract becomes computable and its application to the ORSA framework allows us to get an explicit expression of the SCR which is easily computable using basic simulation techniques.

2. THE BASIC FRAMEWORK

Consider a savings contract with a surrender value for a policyholder that evolves according to (we denote by $t=0$ the calculation date)

$$VR(t)=VR(0)\times\mathbf{exp}\left(\int_0^t r_s(u)\,du\right)$$

with $r_s(t)$ the instantaneous accumulation rate (including any guaranteed rate). Based on this equation, and the surrender rate at time *t* (conditional to the knowledge of the financial state variables), $\mu(t)$, the value of the mathematical reserve at time *t* is $PM(t)=VR(t)\times S(t)$ with $S(t)=\mathbf{exp}\left(-\int_0^t \mu(u)\,du\right)$.

Under the French GAAP, the surrender value $VR(0)$ equals the mathematical reserve $PM(0)$ in the statutory balance sheet at time zero. In particular, the surrender rate μ is not a compulsory assumption imposed by the regulator (the calculation of the statutory reserve is retrospective in French GAAP) but a projection assumption which will be used to project the mathematical reserve to the future.

The best estimate is required to determine the future stream of benefits (see CEIOPS [2010]). For the current contract, the payment of the mathematical reserve in case of an early withdrawal (surrender or death) and the term T of the contract, assumed to be fixed (non-random), both determine these benefits. The flow of benefit considered here is simply expressed as a function of τ, the release date (random) of the contract (which is the surrender or death time), that is

$$\Lambda = VR(\tau \wedge T) \times \delta(\tau \wedge T),$$

with $\delta(t) = \mathbf{exp}\left(-\int_0^t r(u)\,du\right)$ the discount factor, for a given short rate process r. The random variable τ is supposed to be a stopping time for the natural filtration associated with the process $(VR(t), t \geq 0)$.

a) SOURCES OF RANDOMNESS IN THE ACCUMULATION RATE

The main idea of this paper is to assume that the accumulation rate $r_s(t)$ is affected by two kinds of randomness:

- a hedgeable term that is linked to the market prices of the assets;
- corrections to this return by piloting the accounting result. Even if the management actions are deterministic, we can consider that there is a source of randomness (non hedgeable) associated with the moment the unrealized profit and loss is booked. Indeed, the book yield of a transfer of assets depends on the market price of the assets but also its costs. This second source of randomness must be integrated into the model.

This results in the following model:

$$r_s(t) = r(t) + \omega(t),$$

with the short rate $r(t)$ the hedgeable part of risk and $\omega(t)$ the non-hedgeable one. In the next section we derive an explicit expression for the best estimate of this contract.

b) GENERAL EXPRESSION FOR THE BEST ESTIMATE

One notes that

$$VR(t) \times \delta(t) = VR(0) \times \mathbf{exp}\left(\int_0^t \left(r_s(u) - r(u)\right) du\right) = VR(0) \times \mathbf{exp}\left(\int_0^t \omega(u)\,du\right).$$

By definition, the best estimate at time $t = 0$ of the contract is calculated by $BEL(0,T) = E^{P^{nh} \otimes Q^h}(\Lambda)$ with the historical probability P^{nh} modeling the non-hedgeable risks and Q^h a risk-neutral probability modeling hedgeable risk. By $P^{nh} \otimes Q^h$ we assume that the non-hedgeable and hedgeable risks can be decomposed independently, and for hedgeable risks we apply the pricing measure Q^h, see also Section 6.2 in WÜTHRICH and MERZ [2013].

Because of the decomposition $r_s(t) = r(t) + \omega(t)$ we assume that we can split the probability P^{nh} (which represents the risk associated with ω) independently into two components, $P^{nh} = P^i \otimes P^\omega$. In this decomposition, P^i is associated with usual insurance risks, mostly mutualizable ones (mortality, structural surrender, etc.) and P^ω stands for the risks associated with ω.

We assume that usual insurance risks (P^i) and other risks ($P^\omega \otimes Q^h$) are independent. We denote $BEL^F(0,T) = E^{P^i}(\Lambda | F)$. Conditioning on F means conditioning on financial risk (that is the risks that affect r_s, hedgeable or non-hedgeable ones) and therefore we set

$$\begin{aligned} BEL^F(0,T) &= E^{P^i}\left(VR(\tau \wedge T) \times \delta(\tau \wedge T) | F\right) \\ &= \int_0^T VR(t) \times \delta(t) \times S(t) \times \mu(t)\,dt + S(T) \times VR(T) \times \delta(T) \end{aligned}$$

Note that $PM(t) = E^{P^i}\left(VR(t) \times 1_{\{\tau > t\}} | F\right)$. We initially assume that $S(t)$ (and also $\mu(t)$) does not depend on the financial environment and is deterministic. This assumption will be relaxed in Section 3 where the issue of inclusion of financial surrender will be discussed. The best estimate of the contract is

$$BEL(0,T) = E^{P^\omega \otimes Q^h}\left(BEL^F(0,T)\right).$$

This involves knowing how to calculate (the last identity defines $\theta(.)$)

$$E^{P^\omega \otimes Q^h}\left(VR(t) \times \delta(t)\right) = VR(0) \times E^{P^\omega}\left(\mathbf{exp}\left(\int_0^t \omega(u)\,du\right)\right) = VR(0) \times \theta(t).$$

Different approaches are possible to model the spread between the yield of the contract and the risk-free rate $\omega(t) = r_s(t) - r(t)$ so that the coefficient $\theta(t)$ becomes computable. Therefore, having an explicit expression for $\theta(t)$, leads to a simple expression for $E^{P^\omega}\left(VR(t) \times \delta(t)\right) = VR(0) \times \theta(t)$ and the best estimate equals

$$BEL(0,T) = VR(0) \times \left(\int_0^T S(t)\mu(t)\theta(t)\,dt + S(T)\theta(T)\right).$$

To avoid the potentially tedious calculation of the integral above, in practice we use the discretized[2] version, where the usual notation l_x is used for $S(x)$ in a discrete time setting, that is, we approximate

$$BEL(0,T) \approx VR(0) \times \left(\sum_{t=1}^{T} \frac{l_{t-1}}{l_0} \times q_{t-1} \times \theta(t) + \frac{l_T}{l_0} \times \theta(T) \right).$$

where q_t denotes the exit probability in the time interval $[t,t+1[$.

So calculating the best estimate of the contract results in knowing how to calculate the coefficients $\theta(t)$. This will be discussed in the next item.

c) Determination of explicit formulas

The process $\omega(t)$ is modeled directly by assuming that the dynamics under the probability P^{ω} of this process is an Ornstein-Uhlenbeck process, that is,

$$d\omega(t) = k \times (\omega_{\infty} - \omega(t))dt + \sigma_{\omega} dB(t),$$

where $k > 0$, $\omega_{\infty} \in R$, $\sigma_{\omega} > 0$ and $(B(t), t \geq 0)$ is a Wiener process under P^{ω}.

The market often retains a target rate of revalorization close to the risk-free rate (TME, 10-year OAT, etc.); this fact motivates our choice. Moreover, the revalorization rate by the contract is determined by the return on assets (its expectation equals to the risk-free rate under a risk-neutral probability) and also smoothing mechanisms induced by accounting principles.

In this model, we know that the variable $z(t) = \int_0^t \omega(u)du$ is Gaussian with expectation and variance as follows (*cf*. Planchet and *al*. [2011], p.434, or Brigo and Mercurio [2006])

$$m(t) = \omega_{\infty} \times t + (\omega_0 - \omega_{\infty}) \frac{(1 - e^{-kt})}{k}, \ v(t) = \frac{\sigma_{\omega}^2}{2k^3}(1 - e^{-kt})^2 + \frac{\sigma_{\omega}^2}{k^2}\left(t - \frac{1 - e^{-kt}}{k}\right)$$

We denote $\varphi(x) = \frac{1 - e^{-x}}{x}$, so that

$$m(t) = (\omega_{\infty} + (\omega_0 - \omega_{\infty}) \times \varphi(kt)) \times t, v(t) = \frac{\sigma_{\omega}^2 t}{2k} \times \left(t\varphi(kt)^2 + \frac{2}{k}(1 - \varphi(kt)) \right).$$

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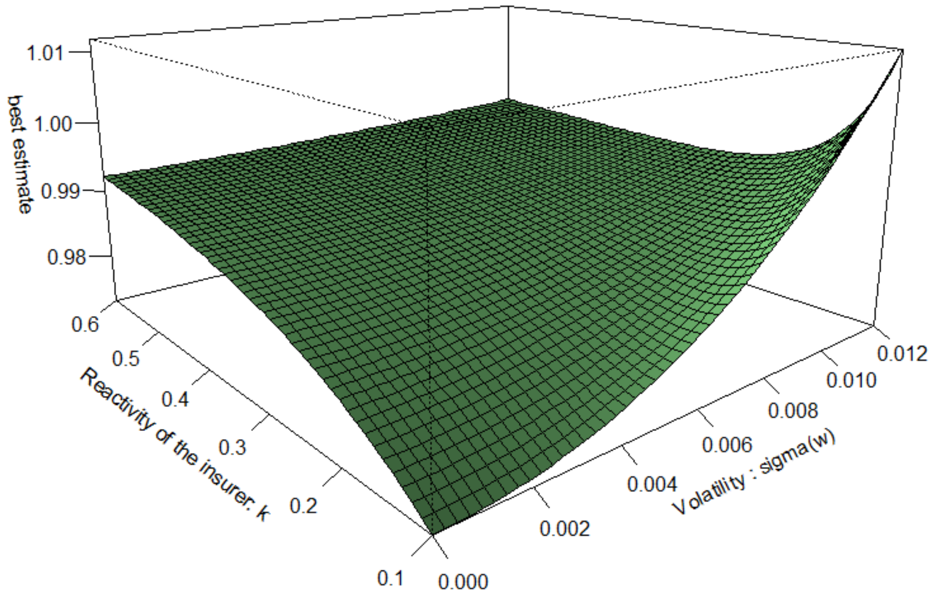
[2] We consider the cash flow paid at the end of period $[t,t+1[$ for the exits before the term.

We thus find $\theta(t) = E^{P^{\varpi}}\left(e^{z_t}\right) = \mathbf{exp}\left(m(t) + \frac{v(t)}{2}\right)$.

Coefficients $(\omega_\infty, k, \sigma_\omega)$ are still to be determined. The parameter k can be seen as the responsiveness of the insurer to adjust its paid rate and σ_ω, the volatility of the spread with the risk-free rate, provides information on the stability of the policy rate paid by the insurer.

The best estimate is an increasing function of σ_ω and decreasing with *k*. We choose $T = 10$ and a constant annual rate of surrender of 4%. Assume that $\omega_0 = -0{,}50\,\%$ and that $\omega_\infty = 0$. Under these conditions we obtain a behavior of the best estimate depending on the volatility of the rate of return of the contract and the reaction speed of the insurer having the following form:

Fig. 1 : <u>Best estimate illustration</u>



We observe that when *k* is large enough (*i.e.* when the responsiveness of the insurer is important), the best estimate is relatively stable in terms of volatility and is slightly less than one. This reflects the fact that the expected earning of the contract is lower than the risk-free rate. In contrast, when the insurer is more constrained (*i.e. k* is small), the impact of volatility is important. The best estimate is, of course, an increasing function of σ_ω.

The calibration of these parameters is not simple and needs further investigation.

3. Modeling of the Surrender

Assume now that the surrender rate is decomposed into the sum of a structural (idiosyncratic) and a cyclical component, $\mu(u) = \mu_i(u) + \mu_c(\omega(u))$. μ is now random.

We recall that the conditional expectation of the sum of the discounted stream of benefits $\Lambda = VR(\tau \wedge T) \times \delta(\tau \wedge T)$ is written as follows

$$BEL^F(0,T) = E^{P^i}(\Lambda | F) = \int_0^T S(t)\mu(t)VR(t)\delta(t)dt + S(T)VR(T)\delta(T).$$

Denoting $S_i(t) = \mathbf{exp}\left(-\int_0^t \mu_i(u)du\right)$ this formula is written (assuming that $VR(0) = 1$ to make the equation more readable)

$$BEL^F(0,T) = \int_0^T S_i(t)\left(\mu_i(t) + \mu_c(\omega(t))\right)\mathbf{exp}\left(\int_0^t \left[\omega(u) - \mu_c(\omega(u))\right]du\right)dt$$
$$+ S_i(T)\mathbf{exp}\left(\int_0^T \left[\omega(u) - \mu_c(\omega(u))\right]du\right).$$

The calculation of $BEL(0,T) = E^{P^\omega \otimes Q^h}\left(BEL^F(0,T)\right)$ not only requires to calculate

$$\theta_1(t) = E^{P^\omega}\left(\mathbf{exp}\left(\int_0^t \left[\omega(u) - \mu_c(\omega(u))\right]du\right)\right),$$

but also

$$\theta_2(t) = E^{P^\omega}\left(\mu_c(\omega(t))\mathbf{exp}\left(\int_0^t \left[\omega(u) - \mu_c(\omega(u))\right]du\right)\right),$$

so that (under the assumption that μ_i is deterministic)

$$BEL(0,T) = VR(0) \times \left(\int_0^T S_i(t)\left(\mu_i(t) \times \theta_1(t) + \theta_2(t)\right)dt + S_i(T) \times \theta_1(T)\right).$$

We use for $\omega(u)$ (*cf.* 2.c)) an Ornstein-Uhlenbeck process and we use the simple linear function for the surrender $\mu_c(\omega(u)) = -\eta \times \omega(u)$ with $\eta > 0$. So we model the effect of declining surrender when the accumulation rate is higher than the risk-free rate and an increasing one otherwise. We choose an unlimited surrender level (upward or downward). This means that we consider an effective risk management system that will reduce the volatility associated with the process $\omega(u)$ when necessary.

In this case, $\theta_1(t) = E^{P^\varpi}\left(\mathbf{exp}\left((1+\eta)\int_0^t \omega(u)\,du\right)\right) = E^{P^\varpi}\left(e^{(1+\eta)Z_t}\right)$ with Z_t a Gaussian random variable $N(m(t), v(t))$. This leads to (the last equation defines m_1 and v_1)

$$\theta_1(t) = \mathbf{exp}\left((1+\eta)\times m(t) + (1+\eta)^2 \times \frac{v(t)}{2}\right) = \mathbf{exp}\left(m_1(t) + \frac{v_1(t)}{2}\right).$$

We still have to compute the term $\theta_2(t) = -\eta \times E^{P^\varpi}\left(\omega(t)\,\mathbf{exp}\left((1+\eta)\int_0^t \omega(u)\,du\right)\right)$. We notice that

$$\begin{aligned}\frac{d\theta_1(t)}{dt} &= E^{P^\varpi}\left(\frac{d}{dt}\mathbf{exp}\left((1+\eta)\int_0^t \omega(u)\,du\right)\right)\\ &= (1+\eta)E^{P^\varpi}\left(\omega(t)\,\mathbf{exp}\left((1+\eta)\int_0^t \omega(u)\,du\right)\right) = -\frac{1+\eta}{\eta}\times\theta_2(t)\end{aligned}$$

and we obtain the expression

$$\begin{aligned}\theta_2(t) &= -\frac{\eta}{1+\eta}\frac{d}{dt}\theta_1(t)\\ &= -\eta\left(\frac{d}{dt}m(t) + \frac{(1+\eta)}{2}\times\frac{d}{dt}v(t)\right)\times\mathbf{exp}\left((1+\eta)\times m(t) + (1+\eta)^2\times\frac{v(t)}{2}\right).\\ &= -\eta\times\theta_1(t)\times\left(m_2(t) + \frac{1}{2}v_2(t)\right),\end{aligned}$$

where m_2 and v_2 are given below.

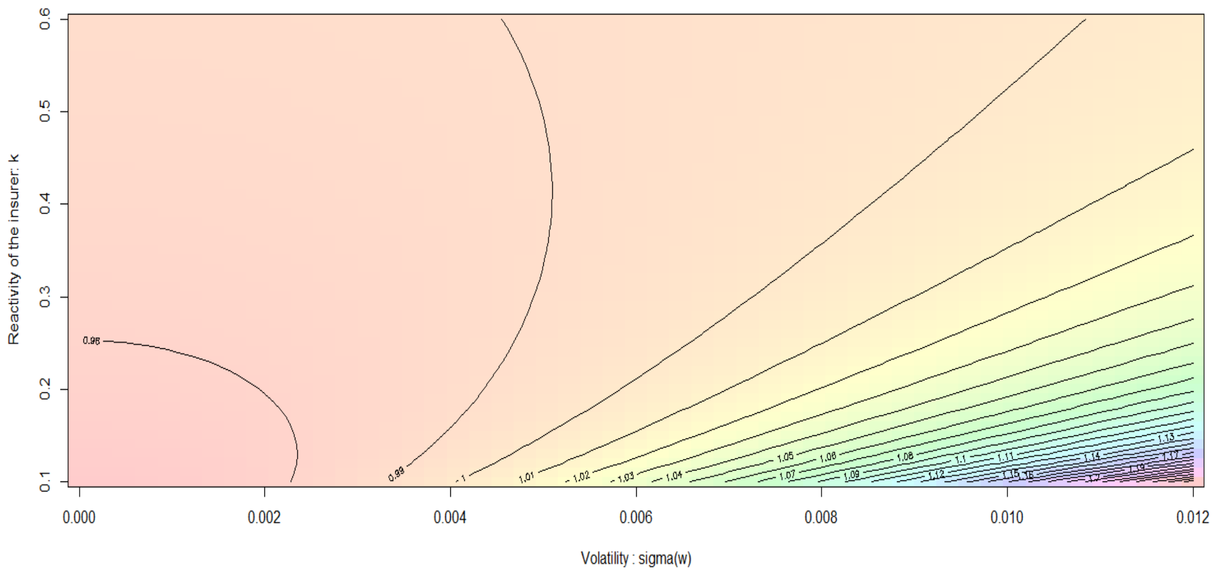
Indeed, denoting $\psi(x) = \frac{d}{dx}\varphi(x) = \frac{xe^{-x} - (1-e^{-x})}{x^2} = \frac{1}{x}\left(e^{-x} - \varphi(x)\right)$ we find the following expressions of m_2 and v_2:

$$\begin{aligned}m_2(t) = \frac{d}{dt}m(t) &= \left(\omega_\infty + (\omega_0 - \omega_\infty)\times\varphi(kt)\right)\\ &\quad + \left((\omega_0 - \omega_\infty)\times k\times\psi(kt)\right)\times t,\end{aligned}$$

$$\begin{aligned}v_2(t) = (1+\eta)\times\frac{d}{dt}v(t) &= (1+\eta)\times\frac{\sigma_\omega^2}{2k}\times\left(t\varphi(kt)^2 + \frac{2}{k}\left(1-\varphi(kt)\right)\right)\\ &\quad + (1+\eta)\times\frac{\sigma_\omega^2 t}{2k}\times\left(\varphi(kt)^2 + \psi(kt)\left(kt\varphi(kt) - 2\right)\right).\end{aligned}$$

With $T = 10$, a constant annual rate of surrender $\mu_i = 4\%$ and $\eta = 2$ we obtain the following values (with *k* varying) for the best estimate:

Fig. 2 : Best estimate illustration – general case



This chart shows that taking the cyclical surrender into account highly reduces the margin of the insurer. We also notice that

- it introduces significant volatility when σ takes values greater than 8%;
- Volatility is neutralized when the insurer is reactive (for *k* greater than 40%).

The symmetric model proposed above simplifies computations and we also find it more realistic. We can change the model by taking into account an asymmetric cyclical component of surrender. It can be done simply by assuming $\mu(u)$ proportional to the negative part of the difference between the accumulation rate and the risk-free rate $\mu_c(u) = -\eta \times [\omega(u)]^-$.

4. Balance Sheet Modeling and SCR Computation

Here we use the formula set out above to calculate the best estimate from the mathematical reserve. For this we use the Markovian character of ω. Then at time t we set $BEL(t,T) = \rho(t,T) \times PM(t)$ with

$$\rho(t,T) = \int_t^T S_{i,t}(u)\left(\mu_i(u) \times \theta_1(\omega(t), u-t) + \theta_2(\omega(t), u-t)\right)du + S_{i,t}(T) \times \theta_1(\omega(t), T-t).$$

We denote $\theta_j(\omega(t), u)$ the coefficient $\theta_j(u)$ computed above with $\omega(t)$ the new initial value for the (Markovian) process $u \to \omega(t+u)$ conditioned on the information at time *t*.

We assume that we have a model that is able to generate paths (under the historical probability) of $r_s(t)$. There is therefore a direct assessment of $BEL(t,T)$ at time *t* using $PM(t)$ and the coefficients θ_i , $i=1,2$. These coefficients depend on the path only through the last value of $\omega(t)$, because of the Markovian assumption. In practice, we will use the approximation

$$\rho(t,T)=\rho(t,T,\omega(t))\approx\sum_{u=t+1}^{T}\frac{l_{u-1}}{l_t}\times\left(q_{u-1}\times\theta_1(\omega(t),u-t)+\theta_2(\omega(t),u-t)\right)$$
$$+\frac{l_T}{l_t}\times\theta_1(\omega(t),T-t).$$

a) DYNAMICS OF THE RISK FACTORS

We use the Vasicek model (*cf*. PLANCHET and *al*. [2011] or BRIGO and MERCURIO [2006]) for the short rate process $r(t)$ and the Black & Scholes model for the insurer's asset value:

$$dr(t)=k_r\times\left(r_\infty-r(t)\right)dt+\sigma_r dB_r(t),$$

$$dr_A(t)=\mu_A dt+\rho\sigma_A dB_r(t)+\sqrt{1-\rho^2}\sigma_A dB_A(t),$$

$$d\omega(t)=k_\omega\times\left(\omega_\infty-\omega(t)\right)dt+\frac{\rho_{s,a}\sigma_\omega}{\sqrt{1-\rho^2}}dB_A(t)+\sqrt{\frac{1-\rho_{s,a}^2-\rho^2}{1-\rho^2}}\sigma_\omega dB_\omega(t),$$

with $\mu_A,k_r,r_\infty,\sigma_\omega,\sigma_A,k_\omega,\omega_\infty,\sigma_\omega>0$ ρ and $\rho_{s,a}\in[-1;1]$. B_A,B_r,B_ω are 3 independent Brownian motions under the historical probability P^h equivalent to the pricing measure Q^h .

In applications, the Ornstein-Uhlenbeck processes are discretized with their exact discretization, following BAUER and *al*. [2010]. With this specification, the correlation between the short rate $r(t)$ and the excess return $\omega(t)=r_s(t)-r(t)$ is zero. This assumption is made for the sake of simplicity and could be relaxed by adding a parameter to the model.

b) PROJECTION MODEL OF THE BALANCE SHEET

Here we describe the dynamics of the asset value $A(t)$ and cash flows of benefits $F(t)$. We assume here that the non-financial risks are perfectly pooled. Initially, we will also assume that no new contracts are signed. The transition from time *t* to $t+1$ is therefore based on the following formula, written under the historical probability,

$$A(t+1)=A(t)\mathbf{exp}\left(\mu_A+\rho\sigma_A\left(B_r(t+1)-B_r(t)\right)+\sqrt{1-\rho^2}\sigma_A\left(B_A(t+1)-B_A(t)\right)\right)-F(t+1),$$

where we denote by $F(t+1)$ the benefit stream in $[t,t+1[$. We assume that the benefits are paid at the end of the year.

Note: in the case of an investment loss we could also use the relationship

$$A(t+1) = A(t)\mathbf{exp}\left(\mu_A + \rho\sigma_A\left(B_r(t+1) - B_r(t)\right) + \sqrt{1-\rho^2}\sigma_A\left(B_A(t+1) - B_A(t)\right)\right)$$
$$\times\left(1 - \frac{F(t+1)}{PM(t)}\right).$$

We will use here the first of these expressions, knowing that the implementation of one or the other is equally complex. Using $\mu(u) = \mu_i(u) + \mu_c(\omega(u))$ for the surrender rate we have

$$F(t+1) = \int_t^{t+1} VR(u)S(u)\mu(u)du = \int_t^{t+1} PM(u)\mu(u)du \approx PM(t)\times q_t .$$

Setting $\mu_c(\omega(u)) = -\eta\omega(u)$ leads to

$$S(t) = \mathbf{exp}\left(-\int_0^t \mu(u)du\right) = S_i(t)\times\mathbf{exp}\left(\eta\int_0^t \omega(u)du\right) \approx S_i(t)\times\mathbf{exp}\left(\eta\sum_{u=0}^{t-1}\omega(u)\right),$$

and $q_t = \dfrac{S(t) - S(t+1)}{S(t)} \approx 1 - \left(1 - q_i(t)\right)\times e^{\eta\times\omega(t)}$, where $q_i(t) = \dfrac{S_i(t) - S_i(t+1)}{S_i(t)}$. We derive from those equations the benefit stream over $[t,t+1[$ as follows

$$F(t+1) \approx PM(t)\times\left(1 - \left(1 - q_i(t)\right)\times e^{\eta\times\omega(t)}\right) .$$

The approximation used for the asset side is

$$A(t+1) \approx A(t)\times\mathbf{exp}\left(\mu_A + \rho\sigma_A\varepsilon_r(t+1) + \sqrt{1-\rho^2}\sigma_A\varepsilon_A(t+1)\right)$$
$$- PM(t)\times\left(1 - \left(1 - q_i(t)\right)\times e^{\eta\times\omega(t)}\right)$$

where $\varepsilon_r(t+1)$ and $\varepsilon_A(t+1)$ are independent and standard Gaussian.

The mathematical reserve evolves as follows

$$PM(t+1) = PM(t)\times\mathbf{exp}\left(\int_t^{t+1}\left(r(u) + \omega(u) - \mu(u)\right)du\right)$$
$$\approx PM(t)\times\mathbf{exp}\left(r(t) + (1+\eta)\omega(t)\right)\times\left(1 - q_i(t)\right),$$

which allows us to compute $BEL(t+1,T)=\rho(t+1,T)\times PM(t+1)$, which is a random variable, conditionally on the available information at time *t*. We can now compute the net asset value at time *t* +1, $E_{t+1}=A_{t+1}-BEL(t+1,T)$ and we get

$$E_{t+1}=e^{r(t)}\times\begin{pmatrix} A(t)\times e^{\mu_A-r(t)+\rho\sigma_A\varepsilon_r(t+1)+\sqrt{1-\rho^2}\sigma_A\varepsilon_A(t+1)} \\ -PM(t)\times e^{-r(t)}\times\left(1-\left(1-q_i(t)\right)\times e^{\eta\times\omega(t)}\right) \\ -PM(t)\times e^{(1+\eta)\omega(t)}\times\left(1-q_i(t)\right)\times\rho\left(t+1,T,\omega(t+1)\right)\end{pmatrix}.$$

Once this value is calculated, it remains to compute the SCR. It takes no account of the risk margin and it leads to the following equation (*cf*. BAUER and *al*. [2010])

$$SCR_t=E_t-VaR_t\left(E_{t+1}\times e^{-\int_t^{t+1}r(u)du};0.5\%\right)$$

where $VaR_t\left(X_{t+1};0.5\%\right)$ denotes the Value-at-Risk of the variable X_{t+1} conditional on the available information at *t* for the 0.5 % level.

We approximate the above expression by $SCR_t\approx E_t-VaR_t\left(E_{t+1}\times e^{-r(t)};0.5\%\right)$.

Since conditional on information available at time t $\rho\left(t+1,T,\omega(t+1)\right)$ is a sum of log-normal variables, it is not possible to analytically compute the quantiles of this distribution (*cf*. GUIBERT and *al*. [2012] for a discussion of this point).

Note: in the above equation we considered only one generation of contracts. In the more realistic case of aggregation of different generations of contracts, the term $PM(t)\times\left(e^{-r(t)}\times q_t+e^{(1+\eta)\omega(t)}\times\left(1-q_i(t)\right)\times\rho\left(t+1,T,\omega(t+1)\right)\right)$ in the equation defining the SCR is replaced by a term of the form

$$\sum_{j\in I}PM_j(t)\times\left(e^{-r(t)}\times q_t+e^{(1+\eta)\omega(t)}\times\left(1-q_i(t)\right)\times\rho\left(t+1,T_j,\omega(t+1)\right)\right).$$

Conditionally on the risk factors at time *t*, we are able to calculate the empirical distribution of the net value of assets at time *t* +1 and derive an empirical estimator of the SCR and of the coverage rate $\pi_t=\dfrac{A_t}{SCR_t}$. One also notes that if an amount of contributions $C(t)$ is set at time *t*, the assets are increased by $C(t)$ and liabilities grow with the amount of the best estimate. Consideration of future subscriptions is thus not too difficult. This case will not be considered in the sequel, which simplifies the illustration.

c) NUMERICAL APPLICATION IN THE CASE OF A RUN-OFF PORTFOLIO

Numerical applications are built with the R software (R DEVELOPMENT CORE TEAM [2012]). We use the following exact discretization for the stochastic processes:

$$r_{t+1} = r_t \times e^{-k_r} + r_\infty \times \left(1 - e^{-k_r}\right) + \sigma_r \sqrt{\frac{1 - e^{-k_r}}{2k_r}} \times \varepsilon_r ,$$

$$\omega_{t+1} = \omega_t \times e^{-k_\omega} + \omega_\infty \times \left(1 - e^{-k_\omega}\right) + \frac{\rho_{s,a}}{\sqrt{1-\rho^2}} \sigma_\omega \sqrt{\frac{1 - e^{-k_\omega}}{2k_\omega}} \times \varepsilon_A + \sqrt{\frac{1 - \rho_{s,a}^2 - \rho^2}{1-\rho^2}} \sigma_\omega \sqrt{\frac{1 - e^{-k_\omega}}{2k_\omega}} \times \varepsilon_\omega ,$$

where $\varepsilon, \varepsilon_A, \varepsilon_s$ are 3 independent and identically distributed standard Gaussian variables. We do not need to discretize r_A, because the asset value is directly projected with the equations presented in the section above.

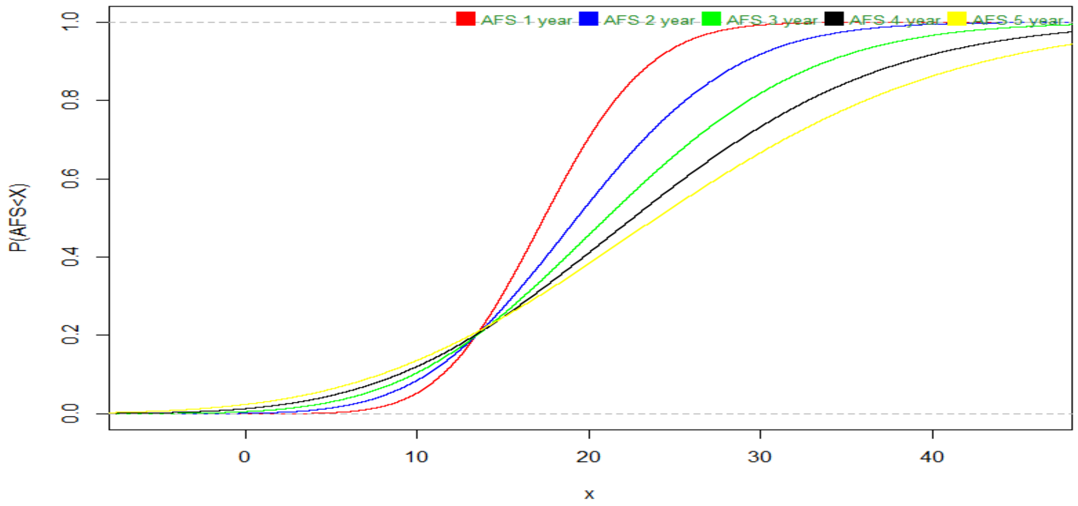
A numerical application is performed by retaining the settings listed in the annex (the parameters used for the asset allocation are equivalent to consider a 12% equity and 88% bonds allocation). On this basis the projection of the following variables is performed over the next 5 years:

- the value of the mathematical reserve;
- the benefit stream;
- the market value of the assets;
- the simulated paths of financials variables (this information is required for the ORSA process).

At last, this allows projecting the evolution over the next 5 years of the Available Financial Surplus[3] (AFS) represented *via* the following graph

[3] AFS is equal ti the net asset value.

Fig. 3 : Evolution of the cdf of the AFS for a run-off portfolio



Based on the distribution of the balance sheet of the company over the next 5 years, we put in place an ORSA process. To do this, we retain an annual 5% quantile (for the j^{th} year we will thus be positioned on the quantile level $1-0{,}95^{j}$). This leads to empirically estimate two quantities:

- The empirical $1-0{,}95^{j}$ quantile of the AFS for every year j of the next 5 years;
- SCR value associated with each quantile.

The estimation is done in three steps:

- Based on the knowledge of the dynamics of the interest variables we simulate 10 000 realizations of the balance sheet over the next five years ;
- based on the knowledge of the distribution of the balance sheet relative to the j^{th} year, we select the trajectory corresponding to the empirical $1-0{,}95^{j}$ quantile;
- conditionally on the information on the selected path, we calculate the empirical quantile at 0.5% of the AFS for the year j +1. This provides the SCR associated with the trajectory withholding for the j^{th} year. We then deduce the quantile coverage ratio of the j^{th} year.

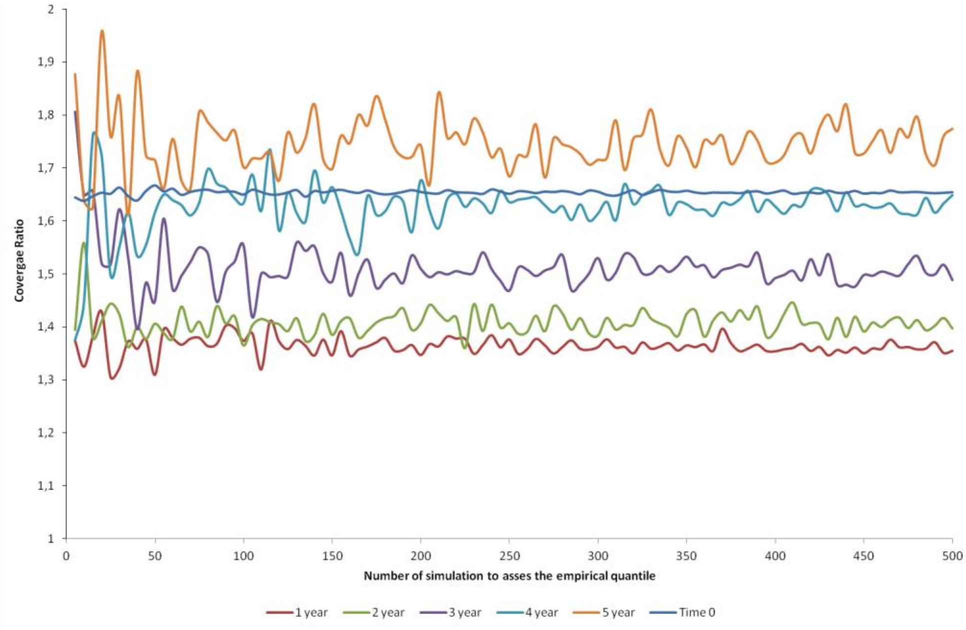
Withholding a quantile based solely on the value of AFS leads to unstable results. The AFS is in fact the imperfect synthesis of the two main variables of interest, the assets and liabilities. Therefore, the three steps above are followed a hundred times, and ultimately we compute the empirical mean of the different simulated quantiles. The following table shows the results we find:

Tab. 1 : Evolution of the coverage ratio (AFS/SCR)

Time 0	1 year	2 years	3 years	4 years	5 years
165 %	137 %	141 %	150 %	161 %	171 %

The $1-0{,}95^j$ quantile being empirical, we test the convergence of the result by gradually increasing the number of simulations. The following graph reflects this convergence (results are obtained on the basis of 500 simulations for the quantile and 10 000 for the SCR).

Fig. 4 : Convergence of the coverage ratio



5. Conclusion

Having a closed formula to go from the mathematical reserve to the best estimate evaluation of the reserve improves the performance of calculations. Being easily reproducible, it facilitates the process of audit and control.

We propose in this work a model based on the idea that a (French) saving contract is mainly non-hedgeable, because of the accounting rules effect on the revalorization rate of the contract. With this observation, the hedgeable part of the flows is « absorbed » by the discounting process, which leads to very simple calculations.

This approach models the behavior of the insurer with a parameter k - representative of its ability to react to the market - and that of the insured with a parameter η – representing its responsiveness. We can make an implicit computation of these behavioral parameters according to the results given by an evaluation as part of a traditional ALM model initially to calibrate the model.

This approach also provides us with a powerful tool for making projections of SCR along a « critical path ». This is especially interesting when seen as part of an ORSA process, like time dependent stress scenario analysis.

This first analytical framework can then be expanded to capture more complex effects, such as the wealth effect of the insurer through its management of unrealized losses.

6. REFERENCES

AASE K.K., PERSSON S.A. [1996] « Valuation of the Minimum Guaranteed Return Embedded in Life Insurance Products », *Wharton School Working Paper*, 96-20.

BACINELLO A.R. [2003] « Fair Valuation of the surrender option embedded in a guarantee life insurance participating policy », Trieste University Working Paper.

BALLOTTA L. [2004] « Alternative framework for the fair valuation of participating life insurance contracts », Cass Business School, Actuarial Research Paper n°157.

BAUER D., BERGMANN D., REUSS A. [2010] « Solvency II and Nested Simulations – a Least-Squares Monte Carlo Approach », *Proceedings of the 2010 ICA congress*.

BONNIN F., PLANCHET F., JUILLARD M. [2010] « Applications de techniques stochastiques pour l'analyse prospective de l'impact comptable du risque de taux. », *Bulletin Français d'Actuariat*, vol. 11, n°21.

BRIGO D., MERCURIO F. [2006] *Interest Rate Models - Theory and Practice*. 2nd Edition. Springer.

BRIYS E., de VARENNE F. [1994] « Life insurance in a contingent claim framework: pricing and regulatory implications », The Geneva Papers on Risk and Insurance Theory 19, 53-72.

CAJA A., PLANCHET F. [2010] « La mesure du prix de marché du risque : quels outils pour une utilisation dans les modèles en assurance ? », Assurances et gestion des risques, Vol.78 (3/4).

CEIOPS [2010] *Technical Specifications for QIS5*, European Commission.

FILIPOVIC D. [2009] *Term-Structure Models*. Springer.

GERBER H. [1997] *Life Insurance Mathematics*, Third Edition, Springer.

GUIBERT Q., JUILLARD M., PLANCHET F. [2012] « Measuring uncertainty of solvency coverage ratio in ORSA for Non-Life Insurance », *European Actuarial Journal*, To appear.

HAINAUT D. [2009] « Profit sharing: a stochastic control approach », *Bulletin Français d'Actuariat*, vol. 9, n°18.

PLANCHET F., LEROY G. [2011] « Solvabilité 2 : quels standards pour le risque de marché ? », la Tribune de l'Assurance (rubrique « le mot de l'actuaire »), n°156 du 01/03/2011.

PLANCHET F., THEROND P.E., JUILLARD M. [2011] *Modèles financiers en assurance. Analyses de risque dynamiques - seconde édition*, Paris : Economica.

R DEVELOPMENT CORE TEAM [2012] *R: A Language and Environment for Statistical Computing*, R Foundation for Statistical Computing, Vienna, Austria. URL: http://www.R-project.org.

WÜTHRICH M., MERZ M. [2013] *Financial Modeling, Actuarial Valuation and Solvency in Insurance*, Springer.

7. APPENDIX: COMPLETE PARAMETERS SET

Parameters	
Risk free rate	
$r_0 = 0.03$	Initial short rate
$r_1 = 0.05$	Long term risk free rate
$k_r = 0.18$	Speed of reversion of the risk free rate
$\sigma_r = 0.02$	Volatility of the risk free rate
Rate of remuneration for the contract	

$\omega_0 = 0.005$	difference between the yield of the contract and the risk-free rate
$\omega_1 = 0.005$	Long term mean level of the difference between the yield of the contract and the risk-free rate under real word measure
$k = 0.3$	Speed of reactivity of the insurer
$\sigma_\omega = 0.008$	Volatility of the difference between the yield of the contract and the risk-free rate under real word measure
Log return of the asset	
$\mu_a = 0.04$	Mean
$\sigma_a = 0.06$	Volatility
Correlation	
$\rho_{a,s} = 0.95$	Linear correlation between the rate served and the rate of return of the asset
$\rho_{a,r} = 0.25$	Linear correlation between the risk free rate and the rate of return of the asset
Parameters of projection	
$T = 10$	Duration of the contract
$\mu_i = 0.04$	Rate of structural lapse
$\eta = 2$	Slope of cyclical lapse
$N_s = 10 \wedge 4$	Number of simulations
$plan = 5$	Duration of the strategic plan
$q_{ORSA} = 0.95$	Level of the annual Quantile for the ORSA
$N_0 = 500$	Number of simulation allowing to calculate empirically the profits ORSA