8TH INTERNATIONAL CONGRESS ON INSURANCE: MATHEMATICS & ECONOMICS

ACTUARIAL THEORY

FOR DEPENDENT RISKS *

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( ★ title of a book co-authored with J. Dhaene, M. Goovaerts, R. Kaas and D. Vyncke)
Risk measures
Stochastic orders
Dependence structures
Credibility models
Bonus-malus scales
Stochastic extrema
Reinsurance pricing
Risk measures

- A risk measure is a functional $\rho$ mapping a risk $X$ to a non-negative real number $\rho[X]$, possibly infinite.
- The meaning of $\rho[X]$ is as follows: $\rho[X]$ represents the minimum extra cash which has to be added to $X$ to make it “acceptable”.
- A large value of $\rho[X]$ indicates that $X$ is “dangerous”.
- Risk measures have been extensively studied in the actuarial literature since 1970, in the guise of premium principles; see e.g. Goovaerts et al. (1984).
A risk measure satisfying

**Translativity:** $\rho[X + c] = \rho[X] + c$ whatever the risk $X$ and the constant $c$;

**Subadditivity:** $\rho[X + Y] \leq \rho[X] + \rho[Y]$ whatever the risks $X$ and $Y$;

**Homogeneity:** $\rho[cX] = c\rho[X]$ whatever the risk $X$ and the positive constant $c$;

**Monotonicity:** $\Pr[X \leq Y] = 1 \Rightarrow \rho[X] \leq \rho[Y]$ whatever the risks $X$ and $Y$;

is said to be coherent in the sense of Artzner et al. (1999).
Comonotonic additivity

- \((X, Y)\) is comonotonic \(\Leftrightarrow \exists Z\) and \(\uparrow\) functions \(t_1\) and \(t_2\) such that

\[
(X, Y) = d \left( t_1(Z), t_2(Z) \right);
\]

see Dhaene et al. (2002a,b) for theory and applications in insurance and finance.

- The risk measure \(\rho\) is comonotonic additive if \(\rho[X + Y] = \rho[X] + \rho[Y]\) whatever the comonotonic risks \(X\) and \(Y\).

- There is no diversification effect for comonotonic risks when the risk measure is comonotonic additive.
Denneberg representation theorem

- Let $\mathcal{B}$ be the set of bounded risks.
- If $\rho : \mathcal{B} \rightarrow \mathbb{R}^+$ is comonotonic additive, monotone and satisfies $\rho[1] = 1$ then there exists a non-decreasing distortion function $g$ satisfying $g(0) = 0$ and $g(1) = 1$, such that

\[
\rho[X] \equiv \rho_g[X] = \int_0^{+\infty} g\left( \Pr[X > t] \right) dt.
\]

- $\rho_g$ is known as a Wang risk measure.
- Moreover,

$\rho_g$ subadditive $\iff g$ concave.
Value-at-Risk (VaR)

- Given a risk $X$ and a probability level $p \in (0, 1)$, the corresponding VaR, denoted as $\text{VaR}[X; p]$, is defined as

$$\text{VaR}[X; p] = F_X^{-1}(p).$$

- Note that any Wang risk measure can be represented as a mixture of VaR’s:

$$\rho_g[X] = \int_0^1 \text{VaR}[X; 1 - p] \, dg(p).$$

- VaR is associated with the distortion function $g(x) = \mathbb{I}[x > 1 - p]$; it is not coherent (it fails to be subadditive).
Tail-VaR

- Given a risk $X$ and a probability level $p$,

$$\text{TVaR}[X; p] = \frac{1}{1-p} \int_p^1 \text{VaR}[X; \xi] \, d\xi, \quad p \in (0, 1).$$

- TVaR is associated with the distortion function $g(x) = \min \left( \frac{x}{1-p}, 1 \right)$; it is coherent.

- If $F_X$ is continuous then

$$\text{TVaR}[X; p] = \mathbb{E} \left[ X \bigg| X > \text{VaR}[X; p] \right], \quad p \in (0, 1),$$

and is the “average loss in the worst $1 - p\%$ cases".
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Most classical stochastic orderings are associated with particular risk measures. Given two risks $X$ and $Y$,

\[ X \preceq_{ST} Y \iff \rho_g[X] \leq \rho_g[Y] \quad \forall \uparrow \text{ distortions } g \]
\[ \iff \text{VaR}[X;p] \leq \text{VaR}[Y;p] \text{ for all } 0 \leq p \leq 1. \]

Given two risks $X$ and $Y$,

\[ X \preceq_{ICX} Y \iff \rho_g[X] \leq \rho_g[Y] \quad \forall \uparrow \text{ concave distortions } g \]
\[ \iff \text{TVaR}[X;p] \leq \text{TVaR}[Y;p] \text{ for all } 0 \leq p \leq 1. \]

See e.g. Denuit et al. (2004).
Convex order

- Given two random variables $X$ and $Y$,

$$X \preceq_{CX} Y \Leftrightarrow X \preceq_{ICX} Y \text{ and } \mathbb{E}[X] = \mathbb{E}[Y].$$

- It can be shown that

$$X \preceq_{CX} Y \Rightarrow \text{Var}[X] \leq \text{Var}[Y]$$

so that $\preceq_{CX}$ expresses the intuitive idea of "$X$ being less variable than $Y$".

- Separation Theorem: $X \preceq_{ICX} Y$ iff $\exists Z$ such that

$$X \preceq_{ST} Z \preceq_{CX} Y.$$
Likelihood ratio order

- Given two random variables \(X\) and \(Y\), \(X\) is said to be smaller than \(Y\) in the likelihood ratio order, denoted as \(X \leq_{LR} Y\), when

\[
\Pr[X \in A] \Pr[Y \in B] \geq \Pr[X \in B] \Pr[Y \in A] \quad \text{for all } A \leq B.
\]

- Let \(X\) and \(Y\) be two rv’s. Then, \(X \leq_{LR} Y\) if, and only if,

\[
[X|a \leq X \leq b] \preceq_{ST} [Y|a \leq Y \leq b] \quad \text{for all } a < b \in \mathbb{R}
\]

or

\[
p \mapsto F_Y(\text{VaR}[X; p]) \text{ is convex.}
\]
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• A copula is (the restriction to the unit square $[0, 1]^2$ of) a joint cdf for a bivariate random vector with unit uniform marginals.

• Let us consider $\boldsymbol{X} = (X_1, X_2)$ with marginals $X_1 \sim F_1$ and $X_2 \sim F_2$.

• Then, there exists a copula $C : [0, 1]^2 \rightarrow [0, 1]$ such that

$$F_{\boldsymbol{X}}(x_1, x_2) = C(F_1(x_1), F_2(x_2)), \; \boldsymbol{x} \in \mathbb{R}^2.$$ 

• $C(\cdot, \cdot)$ is called a copula since it “couples” the marginals $F_1(\cdot)$ and $F_2(\cdot)$ to form the bivariate cdf $F_{\boldsymbol{X}}(\cdot, \cdot)$. 

Conditional increasingness

• The random couple $X$ is said to be CI if
  
  $\Pr[X_2 > x_2|X_1 = x_1]$ is non-decreasing in $x_1$
  $\Pr[X_1 > x_1|X_2 = x_2]$ is non-decreasing in $x_2$.

• This is equivalent to
  
  $[X_2|X_1 = x_1] \leq_{ST} [X_2|X_1 = x'_1]$ for any $x_1 \leq x'_1$
  $[X_1|X_2 = x_2] \leq_{ST} [X_1|X_2 = x'_2]$ for any $x_2 \leq x'_2$.

• CI is a property of the copula, that is, if $C$ is a copula for $X$, $X$ CI $\Leftrightarrow C$ CI.
A function \( \phi : \mathbb{R}^2 \to \mathbb{R} \) is said to be supermodular when

\[
\phi(b_1, b_2) - \phi(a_1, b_2) - \phi(b_1, a_2) + \phi(a_1, a_2) \geq 0
\]

for all \( a_1 \leq b_1, a_2 \leq b_2 \).

Such a function assigns more weight to points \((a_1, a_2)\) and \((b_1, b_2)\) expressing positive dependence.

If \( \phi \) is twice differentiable, it is supermodular iff

\[
\frac{\partial^2}{\partial x_1 \partial x_2} \phi \geq 0
\]

(such a function is called regular supermodular).
Total positivity of order 2 (TP$_2$)

- The random couple $X$ is said to be TP$_2$ if its pdf is log-supermodular, that is, if
  \[
  f_X(a_1, a_2)f_X(b_1, b_2) \geq f_X(a_1, b_2)f_X(b_1, a_2)
  \]
  for any $a_1 \leq b_1$ and $a_2 \leq b_2$.

- This is equivalent to
  \[
  [X_2|X_1 = x_1] \preceq_{LR} [X_2|X_1 = x'_1] \quad \text{for any } x_1 \leq x'_1
  \]
  \[
  [X_1|X_2 = x_2] \preceq_{LR} [X_1|X_2 = x'_2] \quad \text{for any } x_2 \leq x'_2.
  \]

- $X$ is said to be MTP$_2$ if
  \[
  f_X(x)f_X(y) \leq f_X(x \lor y)f_X(x \land y) \quad \forall \ x \in \mathbb{R}^n.
  \]
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Notation

- Let $N_t$ be the number of claims reported by a given policyholder during period $t$, $t = 1, 2, \ldots, T$.
- Being generated by the same individual, the $N_t$’s may be correlated; this serial correlation justifies a posteriori corrections.
- Let
  \[ N_\bullet = \sum_{t=1}^{T} N_t \]
  be the total number of claims reported during the $T$ observation periods.
Let us denote as $\mathbb{E}[N_t] = \lambda_t$ the expected annual claim number; $\lambda_t$ contains all the information included in the price list about the policyholder in period $t$ (like age, sex, power of the car, and so on).

Let $\Theta$ be a positive random variable with unit mean; it represents the unexplained heterogeneity.

Given $\Theta = \theta$, the random variables $N_t$, $t = 1, 2, \ldots$, are independent and $\sim \text{Poi}(\lambda_t \theta)$, i.e.

$$\Pr[N_t = k | \Theta = \theta] = \exp(-\theta \lambda_t) \frac{(\theta \lambda_t)^k}{k!}, \ k \in \mathbb{N}.$$
Intuitive statements

- In this model, we intuitively feel that the following statements are true:
  S1 $\Theta$ “increases" in the past claims $N$.
  S2 $N_{T+1}$ “increases" in the past claims $N$.
  S3 $N_{T+1}$ and $N$ are “positively dependent".

- The meaning of “increases" in S1 and S2, as well as of “positive dependence" involved in S3 has to be precised.

- These statements are true in the classical Poisson-Gamma model if the increasingness is wrt $\leq_{LR}$ and the positive dependence is $TP_2$. 
The results valid in the Poisson-Gamma model remain true in any Poisson mixture model, that is

\[
\Pr[\Theta \mid N_\bullet = n] \leq_{LR} \Pr[\Theta \mid N_\bullet = n'] \quad \text{for } n \leq n'
\]

\[
\Pr[N_{T+1} \mid N_\bullet = n] \leq_{LR} \Pr[N_{T+1} \mid N_\bullet = n'] \quad \text{for } n \leq n'
\]

but

\[
\mathbb{E}[N_{T+1} \mid N_\bullet = n] = \lambda_{T+1} \psi(n)
\]

where \( \psi \) is increasing but not necessarily linear.

- \((N_{T+1}, N_\bullet)\) as well as each \((N_t, N_s)\) are TP_2. Moreover, \((\Theta, N_1, \ldots, N_T)\) is MTP_2.

- **Shaked & Spizzichino (1998), Purcaru & Denuit (2002a,b, 2003).**
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In practice, bonus-malus scales are enforced in MTPL, and not credibility models.

The model for claim numbers is the same as for credibility theory.

Policyholders are now placed in a scale:

<table>
<thead>
<tr>
<th>Level</th>
<th>Relativities</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s$</td>
<td>$r_s$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$\ell$</td>
<td>$r_\ell$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$0$</td>
<td>$r_0$</td>
</tr>
</tbody>
</table>
Such scales possess a number of levels, $s + 1$ say, numbered from 0 to $s$.

A specified level is assigned to a new driver (often according to the use of the vehicle).

Each claim free year is rewarded by a bonus point (i.e. the driver goes one level down).

Claims are penalized by malus points (i.e. the driver goes up a certain number of levels each time he files a claim).
Bayesian relativities

- Let $L(t)$ be the level occupied by a given policyholder in year $t$; typically,

$$L(t) = \max \left\{ 0, \min\{L(t - 1) - 1 + N_t \times k_{pen}, s\} \right\}.$$

- Let $L(\infty)$ be the level occupied by an “infinitely old” policy (stationary regime).

- Denoting as $\Theta$ the unknown (relative) expected claim frequency, Norberg Bayesian relativity attached to level $\ell$ is

$$r_{\ell} = \mathbb{E}[\Theta | L(\infty) = \ell].$$
Dependence in BM scales

- The random vector \((\Theta, L(1), \ldots, L(t))\) is MTP$_2$ for any \(t \geq 1\)

  \[ \Rightarrow (\Theta, L(t)) \text{ and } (\Theta, L(\infty)) \text{ are both TP}_2. \]

- The following stochastic inequalities hold true:

  \[
  \left[ \Theta | L(t) = \ell \right] \preceq_{LR} \left[ \Theta | L(t) = \ell' \right] \text{ for any } \ell \leq \ell', t \geq 1
  \]
  \[
  \left[ \Theta | L(\infty) = \ell \right] \preceq_{LR} \left[ \Theta | L(\infty) = \ell' \right] \text{ for any } \ell \leq \ell'
  \]

  \[ \Rightarrow r_\ell \text{ is increasing with } \ell \]

- Furthermore,

  \[
  \left[ N_{t+1} | L(t) = \ell \right] \preceq_{LR} \left[ N_{t+1} | L(t) = \ell' \right] \text{ for any } \ell \leq \ell'.
  \]

M. Denuit, IME, Rome, June 14-16, 2004 – p. 27/55
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Stochastic bounds

- Often, actuaries act in a conservative way by basing the decision on the worst case compatible with the partial information at their disposal.

- In the univariate case, given the first few moments of the risk $X$, its support, mode, etc., two rv’s $X_-$ and $X_+$ are determined such that

$$X_- \leq X \leq X_+$$

(here $\leq$ can be $\leq_{ST}$, $\leq_{ICX}$ or $\leq_{CX}$ for instance).

- This is closely related to the problem of maximizing/minimizing $\mathbb{E}[\phi(X)]$ for some function $\phi$ when $X$ belongs to a given moment space.
**Example with $\leq_{ICX}$**

\[
\Pr[X_+ \leq x] = \begin{cases} 
0 & \text{if } x < 0, \\
\frac{\sigma^2}{\sigma^2 + \mu^2} & \text{if } 0 \leq x < \frac{\mu^2 + \sigma^2}{2\mu}, \\
\frac{1}{2} + \frac{1}{2} \frac{x-\mu}{\sqrt{(x-\mu)^2 + \sigma^2}} & \text{if } x \geq \frac{\mu^2 + \sigma^2}{2\mu}.
\end{cases}
\]

\[
\Pr[X_- \leq x] = \begin{cases} 
0 & \text{if } x < \mu - \frac{\sigma^2}{b-\mu}, \\
1 - \frac{\mu}{b} & \text{if } \mu - \frac{\sigma^2}{b-\mu} \leq x < \frac{\mu^2 + \sigma^2}{\mu}, \\
1 & \text{if } x \geq \frac{\mu^2 + \sigma^2}{\mu}.
\end{cases}
\]

(see Jansen et al. (1986) and De Vylder & Goovaerts (1982))
In the bivariate case, one could imagine that the marginal distributions are given but the underlying copula is only partially specified (it is PQD, for instance).

Now, two random couples $X_-$ and $X_+$ are determined such that

$$X_- \preceq X \preceq X_+$$

(here $\preceq$ is a suitable bivariate order).

Good candidates for $\preceq$ in the above stochastic inequality are the supermodular order and the directionally convex order.
Supermodular order

- A function \( \phi : \mathbb{R}^2 \to \mathbb{R} \) is said to be supermodular when

\[
\phi(b_1, b_2) - \phi(a_1, b_2) - \phi(b_1, a_2) + \phi(a_1, a_2) \geq 0
\]

for all \( a_1 \leq b_1, a_2 \leq b_2 \).

- Given two random couples \( X = (X_1, X_2) \) and \( Y = (Y_1, Y_2) \), \( X \preceq_{SM} Y \) if \( \mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)] \) for all the (regular) supermodular functions \( \phi \) for which the expectations exist.

- \( \preceq_{SM} \) can only compare random vectors with identical marginals (it is a dependence order).
Extremal elements wrt $\preceq_{SM}$ with given marginals

- Any $X$ satisfies $X^- \preceq_{SM} X \preceq_{SM} X^+$, where $X^-$ (resp. $X^+$) has copula
  \[ C_L(u_1, u_2) = \max\{u_1 + u_2 - 1, 0\} \]
  
  (resp. $C_U(u_1, u_2) = \min\{u_1, u_2\}$)

  and the same marginals as $X$.

- If $X$ is known to be PQD, that is if
  \[ \Pr[X_1 > t_1, X_2 > t_2] \geq \Pr[X_1 > t_1] \Pr[X_2 > t_2] \text{ for all } t_1, t_2, \]

  then $X^-$ can be taken with independent components.
\( \preceq_{ICX} \)-ordering of functions of dependent risks

- For any non-decreasing supermodular function \( \Psi \), Müller (1997) established that \( X^- \preceq_{SM} X \preceq_{SM} X^+ \) implies

\[
\Psi(X_1^-, X_2^-) \preceq_{ICX} \Psi(X_1, X_2) \preceq_{ICX} \Psi(X_1^+, X_2^+).
\]

- True e.g. for

\[
\Psi(x_1, x_2) = \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2,
\]

with \( \alpha_0 \in \mathbb{R}, \alpha_1 > 0, \alpha_2 > 0 \), so that

\[
X_1^- + X_2^- \preceq_{CX} X_1 + X_2 \preceq_{CX} X_1^- + X_2^-.
\]
Directionally convex order

- A function $\phi : \mathbb{R}^2 \to \mathbb{R}$ is directionally convex, if it is supermodular, and in addition convex in each component, when the other component is held fixed.

- $X \preceq_{\text{DIR-CX}} Y$ if $\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)]$ for all the directionally convex functions $\phi$ for which the expectations exist.

- Directional convex order allows to compare random vectors with different marginals (and allows for shift in both the copula and the marginal cdf’s).
A sufficient condition for $\preceq_{\text{DIR-CX}}$

- If $X$ expresses less PQD than $Y$, in the sense that

$$\Pr[X_1 > t_1, X_2 > t_2] - \Pr[X_1 > t_1] \Pr[X_2 > t_2] \leq \Pr[Y_1 > t_1, Y_2 > t_2] - \Pr[Y_1 > t_1] \Pr[Y_2 > t_2]$$

for all $t_1, t_2$, then

$$X_1 \preceq_{\text{CX}} Y_1 \text{ and } X_2 \preceq_{\text{CX}} Y_2 \Rightarrow X \preceq_{\text{DIR-CX}} Y.$$

- See Rüschendorf (2004) for further results in that vein.
Comparing random vectors with a common copula

- Let $X$, $X^-$ and $X^+$ have the same CI copula $C$, and $X_i^- \preceq_{CX} X_i \preceq_{CX} X_i^+$, $i = 1, 2$, MÜLLER & SCARSINI (2001) proved that

$$X^- \preceq_{\text{DIR-CX}} X \preceq_{\text{DIR-CX}} X^+.$$ 

- DENUIT, GENEST & MESFIoui (2004) suggest to proceed in two steps:
  - first, the copula is replaced with a worse/better CI one (in the $\preceq_{\text{SM}}$-sense)
  - second, the marginals are replaced with worse/better ones (in the $\preceq_{\text{CX}}$-sense)
  giving bounds in the $\preceq_{\text{DIR-CX}}$-sense on $X$. 

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Loss-ALAE data set

- Data set provided by Insurance Services Office, Inc.
- ALAE’s: expenses that are specifically attributable to the settlement of individual claims such as lawyers’ fees and claims investigation expenses.
- The data consist of 1500 observed values of the pair (loss, ALAE), as well as a corresponding Policy Limit.
Losses and ALAE’s in reinsurance

- Let us consider a reinsurance treaty on a policy with unlimited liability and insurer’s retention $R$.
- Assuming a prorata sharing of expenses, the reinsurer’s payment for a given realization of $(\text{LOSS}, \text{ALAE})$ is described by

$$g(\text{LOSS}, \text{ALAE}) = \begin{cases} 
0 & \text{if } \text{LOSS} \leq R, \\
\text{LOSS} - R + \frac{\text{LOSS} - R}{\text{LOSS}} \text{ALAE} & \text{if } \text{LOSS} > R.
\end{cases}$$
ISO Loss-ALAE data

- Particularity of the data: some losses were censored because the claim amount cannot exceed the policy limit.
- Specifically,

\[
(T, ALAE_i), \quad i = 1, \ldots, n \quad \text{where} \quad T = \min(loss_i, \ell_i),
\]

\[
\delta_i = \mathbb{I}[T = \ell_i] = \begin{cases} 
1, & \text{if } loss_i > \ell_i \Rightarrow \text{censored claim} \\
0, & \text{if } loss_i \leq \ell_i \Rightarrow \text{uncensored claim}
\end{cases}
\]
### Summary statistics of the Loss-ALAE data

<table>
<thead>
<tr>
<th></th>
<th>Loss (uncensored)</th>
<th>Loss (censored)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Total N</strong></td>
<td>1,500</td>
<td>1,466</td>
</tr>
<tr>
<td><strong>Min</strong></td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td><strong>1st Qu.</strong></td>
<td>4,000</td>
<td>3,750</td>
</tr>
<tr>
<td><strong>Mean</strong></td>
<td>41,208</td>
<td>37,110</td>
</tr>
<tr>
<td><strong>Median</strong></td>
<td>12,000</td>
<td>11,049</td>
</tr>
<tr>
<td><strong>3rd Qu.</strong></td>
<td>35,000</td>
<td>32,000</td>
</tr>
<tr>
<td><strong>Max</strong></td>
<td>2,173,595</td>
<td>2,173,595</td>
</tr>
<tr>
<td><strong>Std Dev.</strong></td>
<td>102,748</td>
<td>92,513</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>ALAE (uncensored)</th>
<th>ALAE (censored)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Total N</strong></td>
<td>1,500</td>
<td>34</td>
</tr>
<tr>
<td><strong>Min</strong></td>
<td>15</td>
<td>5,000</td>
</tr>
<tr>
<td><strong>1st Qu.</strong></td>
<td>2,333</td>
<td>50,000</td>
</tr>
<tr>
<td><strong>Mean</strong></td>
<td>12,588</td>
<td>217,941</td>
</tr>
<tr>
<td><strong>Median</strong></td>
<td>5,471</td>
<td>100,000</td>
</tr>
<tr>
<td><strong>3rd Qu.</strong></td>
<td>12,577</td>
<td>300,000</td>
</tr>
<tr>
<td><strong>Max</strong></td>
<td>501,863</td>
<td>1,000,000</td>
</tr>
<tr>
<td><strong>Std Dev.</strong></td>
<td>28,146</td>
<td>258,205</td>
</tr>
</tbody>
</table>
Scatterplot of the Loss-ALAE data
Testing for PQD

- Empirical investigations carried out
  - by Denuit & Caillet (2004), distance tests
  - by Caillet (2004), Kolmogorov-type tests
  strongly support PQD between Losses and their ALAE’s.

- PQD means that large (resp. small) values of Loss and ALAE tend to occur simultaneously.

- Both methodologies only deal with complete data, and were thus applied to the 1466 uncensored pairs (loss,ALAE).
Archimedean copulas: definition

- Consider a function \( \phi : [0, 1] \rightarrow \mathbb{R}^+ \) satisfying
  \[ \phi(1) = 0, \; \phi^{(1)}(\tau) < 0 \text{ and } \phi^{(2)}(\tau) > 0 \text{ for all } \tau \in (0, 1). \]
- Every such function \( \phi \) generates a copula \( C_\phi \) given by

\[
C_\phi(u_1, u_2) = \begin{cases} 
\phi^{-1}\{\phi(u_1) + \phi(u_2)\} & \text{if } \phi(u_1) + \phi(u_2) \leq \phi(0), \\
0 & \text{otherwise}; 
\end{cases}
\]

the copula \( C_\phi \) is called an archimedean copula.
Nonparametric estimation of $\phi$

- In the literature,
  2. Wang & Wells (2000) for doubly censored data
- The nonparametric estimation of $\phi$ serves as a benchmark for selecting an appropriate parametric archimedean model.
Selection of the parametric generator on the basis of $\lambda = \phi / \phi^{(1)}$
Selection of the parametric generator: QQ-plot of \( K(z) = z - \lambda(z) \)

- \( \hat{\alpha} \) omnibus
- \( \int_0^1 (K\hat{\alpha}(z) - \hat{K}(z))^2 dz \)

<table>
<thead>
<tr>
<th>Generator</th>
<th>Value</th>
<th>Value</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clayton</td>
<td>0.517</td>
<td>0.0001123993</td>
<td>0.00009302016</td>
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<td>Frank</td>
<td>3.077</td>
<td>0.0001477749</td>
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<tr>
<td>Gumbel-Hougaard</td>
<td>1.444</td>
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</tbody>
</table>
To have an idea of the behavior of ALAE for some given Loss level, the next figure displays the graph of \( x_2 \mapsto \Pr[\text{ALAE} \leq x_2 | \text{Loss}] \):
Application to Loss-ALAE

- We also provide the quantile regression curves (i.e. the $q$th quantiles of ALAE for some given Loss level):


