8 Lévy processes

8.1 Definitions and examples

A *Lévy process* is a cadlag process starting from 0 with stationary independent increments. A *Lévy system* is a triple (a, b, K), where $a = \sigma^2 \in [0, \infty)$ is the *diffusivity*, $b \in \mathbb{R}$ is the *drift* and K, the *Lévy measure*, is a Borel measure on $\mathbb{R}$ with $K(\{0\}) = 0$ and

$$\int_{\mathbb{R}} \left(1 \wedge |y|^2\right) K(dy) < \infty.$$

Let B be a Brownian motion and let M be a Poisson random measure with intensity μ on $(0, \infty) \times \mathbb{R}$, where $\mu(dt, dy) = dtK(dy)$, as in the preceding section. Set

$$X_t \overset{\mathsf{def}}{=} \sigma B_t + bt + \int_{(0,t]\times\{|y|\le 1\}} y\, \widetilde{M}(ds, dy) + \int_{(0,t]\times\{|y|>1\}} y\, M(ds, dy).$$

Then $\left(X_t\right)_{t\ge 0}$ is a Lévy process and, for all $t \ge 0$,

$$\mathsf{E}\left(e^{iuX_t}\right) = e^{t\psi(u)}$$

where

$$\psi(u) = ibu - \frac{1}{2}au^2 + \int_{\mathbb{R}} \left(e^{iuy} - 1 - iuy\mathbb{1}_{\{|y|\le 1\}}\right) K(dy).$$

Thus, to every Lévy system there corresponds a Lévy process. Moreover, given $\left(X_t\right)_{t\ge 0}$, we can recover M by

$$M\big((0, t] \times A\big) = \#\big\{s \le t : X_s - X_{s-} \in A\big\}$$

and so we can also recover b and σB. Hence the law of the Lévy process $\left(X_t\right)_{t\ge 0}$ determines the Lévy system (a, b, K).

8.2 Lévy-Khinchin theorem

Theorem 8.2.1 (Lévy-Khinchin theorem) *Let X be a Lévy process. Then there exists a unique Lévy system (a, b, K) such that, for all $t \ge 0$,*

$$\mathsf{E}\left(e^{iuX_t}\right) = e^{t\psi(u)} \tag{8.1}$$

where

$$\psi(u) = ibu - \frac{1}{2}au^2 + \int_{\mathbb{R}} \left(e^{iuy} - 1 - iuy\mathbb{1}_{\{|y|\le 1\}}\right) K(dy). \tag{8.2}$$

Recall that a probability measure μ in $\mathbb{R}^n$ is called *infinitely divisible* if, for each k, there is a probability measure μ_k in $\mathbb{R}^n$ s.t. if X_1, X_2, …, X_k are i.i.d. $\sim \mu_k$ and $X \sim \mu$, then

$$X_1 + X_2 + \cdots + X_k \sim X.$$

If X is a Lévy process, then the law of X_1 is infinitely divisible. By the Lévy-Khinchin theorem, any infinitely divisible law is the law of X_1 for some Lévy process X.

Proof. First we shall show that there is a continuous function $\psi : \mathbb{R} \to \mathbb{C}$ with $\psi(0) = 0$ such that (8.1) holds for all $u \in \mathbb{R}$ and $n \in \mathbb{N}$. Let ν_n denote the law, and ϕ_n the characteristic function, of $X_{1/n}$. Note that ϕ_n is continuous and $\phi_n(0) = 1$. Let I_n denote the largest open interval containing 0 where $|\phi_n| > 0$. There is a unique continuous function $\psi_n : I_n \to \mathbb{C}$ such that $\psi_n(0) = 1$ and

$$\phi_n(u) = e^{\psi_n(u)/n}, \qquad u \in I_n.$$

Since X is a Lévy process, we have $(\phi_n)^n = \phi_1$, so we must have $I_n = I_1$ and $\psi_n = \psi_1$ for all n. Write $I = I_1$ and $\psi = \psi_1$. Then $\phi_n \to 1$ on I as $n \to \infty$ and $\phi_n = 0$ on ∂I for all n. By the argument used in Theorem 5.3.7, $\big(\nu_n : n \in \mathbb{N}\big)$ is then tight, so for some subsequence $\phi_{n_k} \to \phi$ on $\mathbb{R}$, for some characteristic function ϕ. This forces $\partial I = \varnothing$, so $I = \mathbb{R}$.

It remains to show that ψ can be written in the form (8.2). We note that it suffices to find a similar representation where $1\!\!1_{\{|y|\leq 1\}}$ is replaced by $\chi(y)$ for some continuous function χ with

$$1\!\!1_{\{|y|\leq 1\}} \leq \chi(y) \leq 1\!\!1_{\{|y|\leq 2\}}.$$

We have

$$\int_{\mathbb{R}} (e^{iuy} - 1)\, n\nu_n(dy) = n\big(\phi_n(u) - 1\big) \to \psi(u)$$

as $n \to \infty$, uniformly on compacts in u. Hence

$$\int_{\mathbb{R}} (1 - \cos uy)\, n\nu_n(dy) \to -\Re\psi(u).$$

Now there is a constant $C < \infty$ such that

$$y^2 1\!\!1_{\{|y|\leq 1\}} \leq C(1 - \cos y)$$

$$1\!\!1_{\{|y|\geq \lambda\}} \leq C\lambda \int_0^{1/\lambda} (1 - \cos uy)\, du, \qquad \lambda \in (0, \infty).$$

Set $\eta_n(dy) = n(1 \wedge y^2)\, \nu_n(dy)$. Then, as $n \to \infty$,

$$\begin{aligned} \eta_n(|y| \leq 1) &= \int_{\mathbb{R}} y^2 1\!\!1_{\{|y|\leq 1\}} n\nu_n(dy) \\ &\leq C \int_{\mathbb{R}} (1 - \cos y) n\nu_n(dy) \to -C\Re\psi(1) \end{aligned}$$

and, for all $\lambda \geq 1$,

$$\begin{aligned}\eta_n(|y| \geq \lambda) &= \int_{\mathbb{R}} 1\!\!1_{\{|y|\geq\lambda\}} n\nu_n(dy) \\ &\leq C\lambda \int_0^{1/\lambda} \int_{\mathbb{R}} (1-\cos uy) n\nu_n(dy)du \\ &\to -C\lambda \int_0^{1/\lambda} \Re\psi(u)du.\end{aligned}$$

We note that, since $\psi(0) = 0$, the final limit can be made arbitrarily small by choosing λ sufficiently large. Hence the sequence $\big(\eta_n : n \in \mathbb{N}\big)$ is bounded in total mass and tight. By Prohorov's theorem, there is a subsequence (n_k) and a finite measure η on $\mathbb{R}$ such that $\eta_{n_k}(\theta) \to \eta(\theta)$ for all bounded continuous functions θ on $\mathbb{R}$. Now

$$\begin{aligned}\int_{\mathbb{R}} (e^{iuy}-1)\, n\nu_n(dy) &= \int_{\mathbb{R}} (e^{iuy}-1) \frac{\eta_n(dy)}{1\wedge y^2} \\ &= \int_{\mathbb{R}} \frac{e^{iuy}-1-iuy\chi(y)}{1\wedge y^2}\eta_n(dy) + \int_{\mathbb{R}} \frac{iuy\chi(y)}{1\wedge y^2}\eta_n(dy) \\ &= \int_{\mathbb{R}} \theta(u,y)\eta_n(dy) + iub_n\end{aligned}$$

where

$$\theta(u,y) = \begin{cases} (e^{iuy}-1-iuy\chi(y))/(1\wedge y^2), & \text{if } y \neq 0, \\ -u^2/2, & \text{if } y = 0 \end{cases}$$

and

$$b_n = \int_{\mathbb{R}} \frac{y\chi(y)}{1\wedge y^2}\eta_n(dy).$$

Now, for each u, $\theta(u,\cdot)$ is a bounded function. So, on letting $k \to \infty$,

$$\begin{aligned}\int_{\mathbb{R}} \theta(u,y)\eta_{n_k}(dy) \to \int_{\mathbb{R}} \theta(u,y)\eta(dy) \\ = \int_{\mathbb{R}} (e^{iuy}-1-iuy\chi(y))\, K(dy) - \frac{1}{2}au^2\end{aligned}$$

where

$$K(dy) = (1\wedge y^2)^{-1} 1\!\!1_{\{y\neq 0\}}\eta(dy), \qquad a = \eta(\{0\}).$$

Then b_{n_k} must also converge, say to b, and we obtain the desired formula

$$\psi(u) = ibu - \frac{1}{2}au^2 + \int_{\mathbb{R}} (e^{iuy}-1-iuy\chi(y))\, K(dy).$$

□