7 Poisson random measures

7.1 Construction and basic properties

For $\lambda \in (0, \infty)$ we say that a random variable $X$ in $\mathbb{Z}^+$ is Poisson of parameter $\lambda$ and write $X \sim \text{Poi}(\lambda)$ if

$$P(X = n) = e^{-\lambda} \frac{\lambda^n}{n!}.$$  

We also write $X \sim \text{Poi}(0)$ to mean $X \equiv 0$ and write $X \sim \text{Poi}(\infty)$ to mean $X \equiv \infty$.

**Proposition 7.1.1 (Addition property)** Let $N_k, k \in \mathbb{N}$, be independent random variables, with $N_k \sim \text{Poi}(\lambda_k)$ for all $k$. Then

$$\sum_k N_k \sim \text{Poi}\left(\sum_k \lambda_k\right).$$

**Proposition 7.1.2 (Splitting property)** Let $N, Y_n, n \in \mathbb{N}$, be independent random variables, with $N \sim \text{Poi}(\lambda), \lambda < \infty$ and $P(Y_n = j) = p_j$ for all $j = 1, \ldots, k$ and all $n$. Set

$$N_j = \sum_{n=1}^N 1_{\{Y_n = j\}}.$$  

Then $N_1, \ldots, N_k$ are independent random variables with $N_j \sim \text{Poi}(\lambda p_j)$ for all $j$.

Let $(E, \mathcal{E}, \mu)$ be a $\sigma$-finite measure space. A *Poisson random measure with intensity $\mu$* is a map

$$M : \Omega \times \mathcal{E} \rightarrow \mathbb{Z}^+$$

satisfying, for all sequences $(A_k : k \in \mathbb{N})$ of disjoint sets in $\mathcal{E}$,

(i) $M(\bigcup_k A_k) = \sum_k M(A_k)$,

(ii) $M(A_k), k \in \mathbb{N}$, are independent random variables,

(iii) $M(A_k) \sim \text{Poi}(\mu(A_k))$ for all $k$.

Denote by $E^*$ the set of integer-valued measures on $\mathcal{E}$ and define

$$X : E^* \times \mathcal{E} \rightarrow \mathbb{Z}^+, \quad X_A : E^* \rightarrow \mathbb{Z}^+, \quad A \in \mathcal{E}$$

by

$$X(m, A) = X_A(m) = m(A).$$

Set $E^* = \sigma(X_A : A \in \mathcal{E})$.

**Theorem 7.1.3** There exists a unique probability measure $\mu^*$ on $(E^*, \mathcal{E}^*)$ such that $X$ is a Poisson random measure with intensity $\mu$.  


Proof. (Uniqueness.) For disjoint sets $A_1, \ldots, A_k \in \mathcal{E}$ and $n_1, \ldots, n_k \in \mathbb{Z}_+$, set

$$A^* = \left\{ m \in \mathcal{E}^* : m(A_1) = n_1, \ldots, m(A_k) = n_k \right\}.$$  

Then, for any measure $\mu^*$ making $X$ a Poisson random measure with intensity $\mu$,

$$\mu^*(A^*) = \prod_{j=1}^k e^{-\mu(A_j)} \mu(A_j)^{n_j}/n_j!.$$  

Since the set of such sets $A^*$ is a $\pi$-system generating $\mathcal{E}^*$, this implies that $\mu^*$ is uniquely determined on $\mathcal{E}^*$.

(Existence.) Consider first the case where $\lambda = \mu(E) < \infty$. There exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which are defined independent random variables $N$ and $Y_n$, $n \in \mathbb{N}$, with $N \sim \text{Poi}(\lambda)$ and $Y_n \sim \mu/\lambda$ for all $n$. Set

$$M(A) \equiv \sum_{n=1}^N \mathbb{I}(Y_n \in A), \quad A \in \mathcal{E}. \quad (7.1)$$  

It is easy to check, by the Poisson splitting property, that $M$ is a Poisson random measure with intensity $\mu$.

More generally, if $(E, \mathcal{E}, \mu)$ is $\sigma$-finite, then there exist disjoint sets $E_k \in \mathcal{E}$, $k \in \mathbb{N}$, such that $\cup_k E_k = E$ and $\mu(E_k) < \infty$ for all $k$. We can construct, on some probability space, independent Poisson random measures $M_k$, $k \in \mathbb{N}$, with $M_k$ having intensity $\mu|_{E_k}$. Set

$$M(A) \equiv \sum_{k \in \mathbb{N}} M_k(A \cap E_k), \quad A \in \mathcal{E}.$$  

It is easy to check, by the Poisson addition property, that $M$ is a Poisson random measure with intensity $\mu$. The law $\mu^*$ on $\mathcal{E}^*$ is then a measure with the required properties. $\square$

### 7.2 Integrals with respect to a Poisson random measure

**Theorem 7.2.4** Let $M$ be a Poisson random measure on $E$ with intensity $\mu$ and let $g$ be a measurable function on $E$. If $\mu(E)$ is finite or $g$ is integrable, then

$$X = \int_E g(y) \, M(dy)$$  

is a well-defined random variable with

$$\mathbb{E}(e^{iuX}) = \exp\left\{ \int_E (e^{iug(y)} - 1) \, \mu(dy) \right\}.$$  

Moreover, if $g$ is integrable, then so is $X$ and

$$\mathbb{E}(X) = \int_E g(y) \, \mu(dy), \quad \text{Var}(X) = \int_E g(y)^2 \, \mu(dy).$$
Proof. Assume for now that \( \lambda = \mu(E) < \infty \). Then \( M(E) \) is finite a.s. so \( X \) is well defined. If \( g = 1_A \) for some \( A \in \mathcal{E} \), then \( X = M(A) \), so \( X \) is a random variable. This extends by linearity and by taking limits to all measurable functions \( g \).

Since the value of \( E(e^{iuX}) \) depends only on the law \( \mu^* \) of \( M \) on \( E^* \), we can assume that \( M \) is given as in (7.1). Then

\[
E(e^{iuX} | N = n) = E(e^{iu(g(Y_1))})^n = \left( \int_E e^{iu(g(y))} \frac{\mu(dy)}{\lambda} \right)^n
\]

so

\[
E(e^{iuX}) = \sum_{n=0}^{\infty} E(e^{iuX} | N = n) P(N = n)
\]

\[
= \sum_{n=0}^{\infty} \left( \int_E e^{iu(g(y))} \frac{\mu(dy)}{\lambda} \right)^n e^{-\lambda} \lambda^n / n! = \exp \left\{ \int_E (e^{iu(g(y))} - 1) \mu(dy) \right\}.
\]

If \( g \) is integrable, then formulae for \( E(X) \) and \( \text{Var}(X) \) may be obtained by a similar argument.

It remains to deal with the case where \( g \) is integrable and \( \mu(E) = \infty \). Assume for now that \( g \geq 0 \), then \( X \) is obviously well defined. We can find \( 0 \leq g_n \uparrow g \) with \( \mu(|g_n| > 0) < \infty \) for all \( n \). The conclusions of the theorem are then valid for the corresponding integrals \( X_n \). Note that \( X_n \uparrow X \) and \( E(X_n) \leq \mu(g) < \infty \) for all \( n \).

It follows that \( X \) is a random variable and, by dominated convergence, \( X_n \to X \) in \( L^1(P) \). Further, using the estimate \(|e^{iuX} - 1| \leq |ux|\), we can obtain the desired formulae for \( X \) by passing to the limit. Finally, for a general integrable function \( g \), we have

\[
E \int_E |g(y)| M(dy) = \int_E |g(y)| \mu(dy)
\]

so \( X \) is well defined. Also \( X = X_+ - X_- \), where

\[
X_{\pm} = \int_{\{g > 0\} \times \{\pm g > 0\}} g(y) M(dy)
\]

and \( X_+ \) and \( X_- \) are independent. Hence the formulae for \( X \) follow from those for \( X_{\pm} \). \( \square \)

We now fix a \( \sigma \)-finite measure space \((E, \mathcal{E}, K)\) and denote by \( \mu \) the product measure on \((0, \infty) \times E\) determined by

\[
\mu\left((0, t] \times A\right) = tK(A), \quad t \geq 0, \quad A \in \mathcal{E}.
\]

Let \( M \) be a Poisson random measure with intensity \( \mu \) and set \( \widetilde{M} = M - \mu \). Then \( \widetilde{M} \) is a compensated Poisson measure with intensity \( \mu \).

**Proposition 7.2.5** Let \( g \) be an integrable function on \( E \). Set

\[
X_t \overset{\text{def}}{=} \int_{(0, t] \times E} g(y) \widetilde{M}(ds, dy).
\]
Then \((X_t)_{t \geq 0}\) is a cadlag martingale with stationary independent increments. Moreover,

\[
\mathbb{E}(e^{iuX_t}) = \exp \left\{ t \int_E \left( e^{iug(y)} - 1 - iug(y) \right) K(dy) \right\},
\]

\[
\mathbb{E}(X_t^2) = t \int_E g(y)^2 K(dy).
\]

**Theorem 7.2.6** Let \(g \in L^2(K)\) and let \((g_n : n \in \mathbb{N})\) be a sequence of integrable functions such that \(g_n \to g\) in \(L^2(K)\). Set

\[
X^n_t \overset{\text{def}}{=} \int_{(0,t] \times E} g_n(y) \tilde{M}(ds, dy).
\]

Then there exists a cadlag martingale \((X_t)_{t \geq 0}\) such that

\[
\mathbb{E}\left( \sup_{s \leq t} |X^n_s - X_n|^2 \right) \to 0
\]

for all \(t \geq 0\). Moreover, \((X_t)_{t \geq 0}\) has stationary independent increments and

\[
\mathbb{E}(e^{iuX_t}) = \exp \left\{ t \int_E \left( e^{iug(y)} - 1 - iug(y) \right) K(dy) \right\}.
\]

The notation \(\int_{(0,t] \times E} g(y) \tilde{M}(ds, dy)\) is used for \(X_t\) even when \(g\) is not integrable with respect to \(K\). Of course \((X_t)_{t \geq 0}\) does not depend on the choice of approximating sequence \((g_n)\). This is a simple example of a stochastic integral.

**Proof.** Fix \(t > 0\). By Doob’s \(L^2\)-inequality and Proposition 7.2.5,

\[
\mathbb{E}\left( \sup_{s \leq t} |X^n_s - X^m_s|^2 \right) \leq 4\mathbb{E}((X^n_t - X^m_t)^2) = 4t \int_E (g_n - g_m)^2 K(dy) \to 0
\]
as \(n, m \to \infty\). Hence \(X^n_s\) converges in \(L^2\) for all \(s \leq t\). For some subsequence we have

\[
\sup_{s \leq t} |X^n_{s_k} - X^n_{s_j}| \to 0 \quad \text{a.s.}
\]
as \(j, k \to \infty\). The uniform limit of cadlag functions is cadlag, so there is a cadlag process \((X_s)_{s \leq t}\) such that

\[
\sup_{s \leq t} |X^n_{s_k} - X_s| \to 0 \quad \text{a.s.}
\]

Since \(X^n_s\) converges in \(L^2\) for all \(s \leq t\), \((X_s)_{s \leq t}\) is a martingale and so by Doob’s \(L^2\)-inequality

\[
\mathbb{E}\left( \sup_{s \leq t} |X^n_s - X_s|^2 \right) \leq 4\mathbb{E}((X^n_t - X_t)^2) \to 0.
\]
Note that $|e^{iu} - 1 - iug| \leq u^2g^2/2$. Hence, for $s < t$ we have

$$E \left( e^{iut}X_t - X_s \mid \mathcal{F}_s \right) = \lim_n E \left( e^{iunt}X_t^n - X_s^n \mid \mathcal{F}_s \right)$$

$$= \lim_n \exp \left\{ (t - s) \int_E e^{iug(y)} - 1 - iug(y) K(dy) \right\}$$

$$= \exp \left\{ (t - s) \int_E e^{iug(y)} - 1 - iug(y) K(dy) \right\}$$

which shows that $(X_t)_{t \geq 0}$ has stationary independent increments with the claimed characteristic function.