2 Martingales – theory

2.1 Definitions

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space, let $(E, \mathcal{E})$ be a measurable space and let I be a countable subset of $\mathbb{R}$. A *process in* E is a family $X = (X_t)_{t\in I}$ of random variables in E. A *filtration* $(\mathcal{F}_t)_{t\in I}$ is an increasing family of sub-σ-algebras of $\mathcal{F}$: thus, $\mathcal{F}_s \subseteq \mathcal{F}_t$ whenever $s \le t$. We set $\mathcal{F}_{-\infty} \stackrel{\text{def}}{=} \cap_{t\in I}\mathcal{F}_t$ and $\mathcal{F}_\infty = \sigma(\mathcal{F}_t : t \in I)$. Every process has a *natural filtration* $(\mathcal{F}_t^X)_{t\in I}$, given by

$$\mathcal{F}_t^X \stackrel{\text{def}}{=} \sigma(X_s : s \le t).$$

We will always assume some filtration $(\mathcal{F}_t)_{t\in I}$ to be given. The σ-algebra $\mathcal{F}_t$ is interpreted as modelling the state of our knowledge at time t. In particular, $\mathcal{F}_t^X$ contains all the events which depend (measurably) only on X_s, $s \le t$, that is, everything we know about the process X by time t. We say that X is *adapted* (*to* $(\mathcal{F}_t)_{t\in I}$) if X_t is $\mathcal{F}_t$-measurable for all t. Of course every process is adapted to its natural filtration. We say that X is *integrable* if X_t is integrable for all t.

Unless otherwise indicated, it is to be understood from now on that $E = \mathbb{R}$.

Definition 2.1.1 *A martingale X is an adapted integrable process such that, for all s, $t \in I$ with $s \le t$,*

$$\mathsf{E}\,(X_t \mid \mathcal{F}_s) = X_s \qquad a.s..$$

On replacing the equality in this condition by $\le$ or $\ge$, we get the notions of *supermartingale* and *submartingale*, respectively. Note that every process which is a martingale with respect to the given filtration is also a martingale with respect to its natural filtration.

2.2 Optional stopping

We say that a random variable $T : \Omega \to I \cup \infty$ is a *stopping time* if $\{T \le t\} \in \mathcal{F}_t$ for all t. For a stopping time T, we set

$$\mathcal{F}_T \stackrel{\text{def}}{=} \{A \in \mathcal{F} : A \cap \{T \le t\} \in \mathcal{F}_t \quad \text{for all } t\ \}.$$

It is easy to check that, if $T \equiv t$, then T is a stopping time and $\mathcal{F}_T = \mathcal{F}_t$. Given a process X, we set

$$X_T(\omega) \stackrel{\text{def}}{=} X_{T(\omega)}(\omega) \qquad \text{whenever } T(\omega) < \infty.$$

we also define the *stopped process* X^T by $X_t^T \stackrel{\text{def}}{=} X_{T\wedge t}$.

We assume in the following two results that $I = \{0, 1, 2, \ldots\}$. In this context, we will write n, m or k for elements of I, rather than t or s.

Proposition 2.2.2 *Let S and T be stopping times and let $X = \left(X_n\right)_{n\geq 0}$ be an adapted process. Then*

(a) $S \wedge T$ *is a stopping time;*

(b) if $S \leq T$, *then* $\mathcal{F}_S \subseteq \mathcal{F}_T$;

(c) $X_T \mathbb{1}_{\{T<\infty\}}$ *is an* $\mathcal{F}_T$*-measurable random variable;*

(d) X^T *is adapted;*

(e) if X *is integrable, then* X^T *is integrable.*

Theorem 2.2.3 (Optional stopping theorem) *Let $X = \left(X_n\right)_{n\geq 0}$ be an adapted integrable process. Then the following are equivalent:*

(a) X *is supermartingale;*

(b) for all bounded stopping times T *and all stopping times* S,

$$\mathsf{E}\left(X_T \mid \mathcal{F}_S\right) \leq X_{S\wedge T} \quad \text{a.s.};$$

(c) for all stopping times T, X^T *is a supermartingale;*

(d) for all bounded stopping times S *and* T *with* $S \leq T$,

$$\mathsf{E}\big(X_S\big) \geq \mathsf{E}\big(X_T\big).$$

Proof. For $S \geq 0$ and $T \leq n$, we have

$$\begin{aligned} X_T &= X_{S\wedge T} + \sum_{S\leq k<T} \big(X_{k+1} - X_k\big) \\ &= X_{S\wedge T} + \sum_{k=0}^{n} \big(X_{k+1} - X_k\big)\mathbb{1}_{\{S\leq k<T\}}. \end{aligned} \tag{2.1}$$

Suppose that X is a supermartingale and that S and T are stopping times, with $T \leq n$. If $A \in \mathcal{F}_S$, then $A \cap \{S \leq k\}$, $\{T > k\} \in \mathcal{F}_k$, so

$$\mathsf{E}\Big(\big(X_{k+1} - X_k\big)\mathbb{1}_{\{S\leq k<T\}}\mathbb{1}_A\Big) \leq 0.$$

Hence, on multiplying (2.1) by $\mathbb{1}_A$ and taking expectations, we obtain

$$\mathsf{E}\big(X_T \mathbb{1}_A\big) \leq \mathsf{E}\big(X_{S\wedge T}\mathbb{1}_A\big).$$

We have shown that (a) $\Longrightarrow$ (b).

It is obvious that (b) $\Longrightarrow$ (c), (b) $\Longrightarrow$ (d), and (c) $\Longrightarrow$ (a).

Let $m \leq n$ and $A \in \mathcal{F}_m$. Set $T = m\mathbb{1}_A + n\mathbb{1}_{A^c}$, then T is a stopping time and $T \leq n$. We note that

$$\mathsf{E}\big(X_n \mathbb{1}_A\big) - \mathsf{E}\big(X_m \mathbb{1}_A\big) = \mathsf{E}(X_n) - \mathsf{E}(X_T).$$

It follows that (d) $\Longrightarrow$ (a). □

2.3 Doob's inequalities

Let X be a process and let a, $b \in \mathbb{R}$ with $a < b$. For $J \subseteq I$, set

$$U\big([a,b],J\big) \overset{\mathsf{def}}{=} \sup\big\{n : X_{s_1} < a, X_{t_1} > b, \ldots, X_{s_n} < a, X_{t_n} > b \quad \text{for some } s_1 < t_1 < \cdots < s_n < t_n \text{ in } J\big\}.$$

Then $U[a,b] \equiv U([a,b],I)$ is the number of *upcrossings* of $[a,b]$ by X.

Theorem 2.3.4 (Doob's upcrossing inequality) *Let X be a supermartingale. Then*

$$(b-a)\mathsf{E}\big(U[a,b]\big) \le \sup_{t\in I} \mathsf{E}\big((X_t - a)^-\big).$$

Proof. Since $U\big([a,b],I\big) = \lim_{J\uparrow I, J \text{ finite}} U\big([a,b],J\big)$, it suffices, by monotone convergence, to consider the case where I is finite. Let us assume that $I = \{0,1,\ldots,n\}$.

Write $U = U[a,b]$ and note that $U \le n$. Set $T_0 = 0$ and define inductively for $k \ge 0$ (with the usual convention $\inf \varnothing = \infty$):

$$S_{k+1} \overset{\mathsf{def}}{=} \inf\{m \ge T_k : X_m < a\}, \qquad T_{k+1} \overset{\mathsf{def}}{=} \inf\{m \ge S_{k+1} : X_m > b\}.$$

Then $U = \max\{k : T_k < \infty\}$. For $k \le U$, set $G_k \overset{\mathsf{def}}{=} X_{T_k} - X_{S_k}$ and note that $G_k \ge b - a$. Observe that $T_U \le n$ and $T_{U+1} = \infty$. Set

$$R = \begin{cases} X_n - X_{S_{U+1}} & \text{if } S_{U+1} < \infty, \\ 0 & \text{if } S_{U+1} = \infty \end{cases}$$

and note that $R \ge -(X_n - a)^-$.

Then we have

$$\sum_{k=1}^{n} \big(X_{T_k\wedge n} - X_{S_k\wedge n}\big) = \sum_{k=1}^{U} G_k + R \ge (b-a)U - (X_n - a)^-. \tag{2.2}$$

Now X is a supermartingale and $S_k \wedge n$ and $T_k \wedge n$ are bounded stopping times, with $S_k \wedge n \le T_k \wedge n$. Hence, by optional stopping, $\mathsf{E}\big(X_{T_k\wedge n}\big) \le \mathsf{E}\big(X_{S_k\wedge n}\big)$ and the desired inequality results on taking expectations in (2.2). □

For any process X, for $J \subseteq I$, we set

$$X^*(J) \overset{\mathsf{def}}{=} \sup_{t\in J} |X_t|, \qquad X^* \overset{\mathsf{def}}{=} X^*(I).$$

Theorem 2.3.5 (Doob's maximal inequality) *Let X be a martingale or a non-negative submartingale. Then, for all $\lambda \ge 0$,*

$$\lambda \mathsf{P}(X^* \ge \lambda) \le \sup_{t\in I} \mathsf{E}\big(|X_t|\big).$$

Proof. Note that

$$\lambda \mathsf{P}(X^* \geq \lambda) = \lim_{\nu \uparrow \lambda} \nu \mathsf{P}(X^* > \nu) \leq \lim_{\nu \uparrow \lambda} \Big(\lim_{J \uparrow I, J \text{ finite}} \nu \mathsf{P}\big(X^*(J) \geq \nu\big)\Big).$$

It therefore suffices to consider the case where I is finite. Let us assume then that $I = \{0, 1, \ldots, n\}$. If X is a martingale, then $|X|$ is a non-negative submartingale. It therefore suffices to consider the case where X is non-negative.

Set $T = \inf\{m \geq 0 : X_m \geq \lambda\} \wedge n$. Then T is a stopping time and $T \leq n$ so, by optional stopping,

$$\begin{aligned} \mathsf{E}(X_n) \geq \mathsf{E}(X_T) &= \mathsf{E}\big(X_T 1\!\!1_{\{X^* \geq \lambda\}}\big) + \mathsf{E}\big(X_T 1\!\!1_{\{X^* < \lambda\}}\big) \\ &\geq \lambda \mathsf{P}(X^* \geq \lambda) + \mathsf{E}(X_n 1\!\!1_{\{X^* < \lambda\}}). \end{aligned}$$

Hence,

$$\lambda \mathsf{P}(X^* \geq \lambda) \leq \mathsf{E}(X_n 1\!\!1_{\{X^* \geq \lambda\}}) \leq \mathsf{E}(X_n). \tag{2.3}$$

□

Theorem 2.3.6 (Doob's L^p-inequality) *Let X be a martingale or non-negative submartingale. Then, for all $p > 1$ and $q = p/(p-1)$,*

$$\big\|X^*\big\|_p \leq q \sup_{t \in I} \big\|X_t\big\|_p.$$

Proof. Since $X^* = \lim_{J \uparrow I, J \text{ finite}} X^*(J)$, it suffices, by monotone convergence, to consider the case where I is finite. Let us assume that $I = \{0, 1, \ldots, n\}$. If X is a martingale, then $|X|$ is a non-negative submartingale. So it suffices to consider the case where X is non-negative.

Fix $k < \infty$. By Fubini's theorem, eqn. (2.3) and Hölder's inequality,

$$\begin{aligned} \mathsf{E}\big[(X^* \wedge k)^p\big] &= \mathsf{E} \int_0^k p\lambda^{p-1} 1\!\!1_{\{X^* \geq \lambda\}}\, d\lambda = \int_0^k p\lambda^{p-1} \mathsf{P}(X^* \geq \lambda)\, d\lambda \\ &\leq \int_0^k p\lambda^{p-2} \mathsf{E}(X_n 1\!\!1_{\{X^* \geq \lambda\}})\, d\lambda = q\mathsf{E}\big(X_n (X^* \wedge k)^{p-1}\big) \\ &\leq q\|X_n\|_p \cdot \|X^* \wedge k\|_p^{p-1}. \end{aligned}$$

Hence $\|X^* \wedge k\|_p \leq q\|X_n\|_p$ and the result follows by monotone convergence on letting $k \to \infty$. □

2.4 Convergence theorems

Recall that, for $p \geq 1$, a process X is said to be *bounded in L^p* if

$$\sup_{t \in I} \|X_t\|_p < \infty.$$

Also X is *uniformly integrable* if

$$\sup_{t\in I} \mathsf{E}\big(|X_t| \mathbb{1}_{\{|X_t|>k\}}\big) \to 0 \qquad \text{as } k \to \infty.$$

Recall that, if X is bounded in L^p for some $p > 1$, then X is uniformly integrable. Also if X is uniformly integrable then X is bounded in L^1.

Theorem 2.4.7 (Almost sure martingale convergence theorem) *Let X be a supermartingale which is bounded in L^1. Then $X_t \to X_\infty$ a.s. for some $X_\infty \in L^1(\mathcal{F}_\infty)$.*

Note that for $I \subseteq [0, \infty)$, a non-negative supermartingale is automatically bounded in L^1.

Proof. By Doob's upcrossing inequality, for all $a < b$,

$$\mathsf{E}(U[a, b]) \le (b - a)^{-1} \sup_{t\in I} \mathsf{E}(|X_t| + |a|) < \infty.$$

Consider for $a < b$ the sets

$$\Omega_{a,b} \overset{\mathsf{def}}{=} \big\{\liminf_{t\to\infty} X_t < a < b < \limsup_{t\to\infty} X_t\big\},$$
$$\Omega_0 \overset{\mathsf{def}}{=} \big\{X_t \text{ converges in } [-\infty, \infty] \text{ as } t \to \infty\big\}.$$

Since $U[a, b] = \infty$ on $\Omega_{a,b}$, we must have $\mathsf{P}(\Omega_{a,b}) = 0$; consequently, the equality

$$\Omega_0 \cup \big(\cup_{a,b\in\mathbb{Q},a<b}\Omega_{a,b}\big) = \Omega$$

implies $\mathsf{P}(\Omega_0) = 1$. Define

$$X_\infty = \begin{cases} \lim_{t\to\infty} X_t & \text{on } \Omega_0, \\ 0 & \text{on } \Omega \setminus \Omega_0. \end{cases}$$

Then X_∞ is $\mathcal{F}_\infty$-measurable and, by Fatou's lemma,

$$\mathsf{E}\big(|X_\infty|\big) \le \liminf_{t\to\infty} \mathsf{E}\big(|X_t|\big) < \infty.$$

So $X_\infty \in L^1$ as required. □

Let us denote by $\mathcal{M}^1$ the set of uniformly integrable martingales and, for $p > 1$, by $\mathcal{M}^p$ the set of martingales bounded in L^p.

Theorem 2.4.8 (L^p martingale convergence theorem) *Let $p \in [1, \infty)$.*

(a) Suppose $X \in \mathcal{M}^p$. Then $X_t \to X_\infty$ as $t \to \infty$, a.s. and in L^p, for some $X_\infty \in L^p(\mathcal{F}_\infty)$. Moreover, $X_t = \mathsf{E}\,(X_\infty \mid \mathcal{F}_t)$ a.s. for all t.

(b) Suppose $Y \in L^p(\mathcal{F}_\infty)$ and set $X_t = \mathsf{E}\,(Y \mid \mathcal{F}_t)$. Then $X = \big(X_t\big)_{t\in I} \in \mathcal{M}^p$ and $X_t \to Y$ as $t \to \infty$, a.s. and in L^p.

Thus the map $X \mapsto X_\infty$ is a one-to-one correspondence between $\mathcal{M}^p$ and $L^p(\mathcal{F}_\infty)$.

Proof for $p = 1$. Let X be a uniformly integrable martingale. Then $X_t \to X_\infty$ a.s. by the almost sure martingale convergence theorem. Since X is uniformly integrable, it follows that $X_t \to X_\infty$ in L^1. Next, for $s \geq t$,

$$\|X_t - \mathsf{E}\,(X_\infty \mid \mathcal{F}_t)\,\|_1 = \|\mathsf{E}\,(X_s - X_\infty \mid \mathcal{F}_t)\,\|_1 \leq \|X_s - X_\infty\|_1.$$

Let $s \to \infty$ to deduce $X_t = \mathsf{E}\,(X_\infty \mid \mathcal{F}_t)$ a.s..

Suppose now that $Y \in L^1(\mathcal{F}_\infty)$ and set $X_t \overset{\mathsf{def}}{=} \mathsf{E}\,(Y \mid \mathcal{F}_t)$. Then $X = \big(X_t\big)_{t\in I}$ is a martingale by the tower property and is uniformly integrable by Lemma 1.5.2. Hence X_t converges a.s. and in L^1, with limit X_∞, say. For all t and all $A \in \mathcal{F}_t$ we have

$$\mathsf{E}\big(X_\infty \mathbb{1}_A\big) = \lim_{s\to\infty} \mathsf{E}\big(X_s \mathbb{1}_A\big) = \mathsf{E}(Y\mathbb{1}_A).$$

Now $X_\infty, Y \in L^1(\mathcal{F}_\infty)$ and $\cup_t \mathcal{F}_t$ is a π-system generating $\mathcal{F}_\infty$. Hence, $X_\infty = Y$ a.s.. □

Proof for $p > 1$. Let X be a martingale bounded in L^p for some $p > 1$. Then $X_t \to X_\infty$ a.s. by the almost sure martingale convergence theorem. By Doob's L^p-inequality,

$$\|X^*\|_p \leq q \sup_{t\in I} \|X_t\|_p < \infty.$$

Since $|X_t - X_\infty|^p \leq (2X^*)^p$ for all t, we can use dominated convergence to deduce that $X_t \to X_\infty$ in L^p. It follows that $X_t = \mathsf{E}\,(X_\infty \mid \mathcal{F}_t)$ a.s., as in the case $p = 1$.

Suppose now that $Y \in L^p(\mathcal{F}_\infty)$ and set $X_t = \mathsf{E}\,(Y \mid \mathcal{F}_t)$. Then $X = \big(X_t\big)_{t\in I}$ is a martingale by the tower property and

$$\|X_t\|_p = \|\mathsf{E}\,(X_\infty \mid \mathcal{F}_t)\,\|_p \leq \|Y\|_p$$

for all t, so X is bounded in L^p. Hence X_t converges a.s. and in L^p , with limit X_∞, say, and we can show that $X_\infty = Y$ a.s., as in the case $p = 1$. □

Theorem 2.4.9 (Backward martingale convergence theorem) *Let $p \in [1, \infty)$ and let $Y \in L^p$. Set $X_t = \mathsf{E}\,(Y \mid \mathcal{F}_t)$. Then $X_t \to \mathsf{E}\,(Y \mid \mathcal{F}_{-\infty})$ as $t \to -\infty$, a.s. and in L^p.*

Proof. The argument is a minor modification of that used in Theorems 2.3.4, 2.4.7, and 2.4.8. The process X is automatically uniformly integrable, by Lemma 1.5.2 and is bounded in L^p because $\|X_t\|_p = \|\mathsf{E}\,(Y \mid \mathcal{F}_t)\,\|_p \leq \|Y\|_p$ for all t. We leave the details to the reader. □

In the following result we take $I = \{0, 1, 2, \dots\}$.

Theorem 2.4.10 (Optional stopping theorem – 2) *Let X be a uniformly integrable martingale and let S and T be stopping times. Then*

$$\mathsf{E}\left(X_T \mid \mathcal{F}_S\right) = X_{S\wedge T} \quad \text{a.s..}$$

Proof. We have already proved the result when T is bounded. If T is unbounded, then $T \wedge n$ is a bounded stopping time, so

$$\mathsf{E}\left(X_n^T \mid \mathcal{F}_S\right) = \mathsf{E}\left(X_{T\wedge n} \mid \mathcal{F}_S\right) = X_{S\wedge T\wedge n} = X_{S\wedge n}^T \quad \text{a.s..} \tag{2.4}$$

Now

$$\left\|\mathsf{E}\left(X_n^T \mid \mathcal{F}_S\right) - \mathsf{E}\left(X_T \mid \mathcal{F}_S\right)\right\|_1 \le \left\|X_n^T - X_\infty^T\right\|_1. \tag{2.5}$$

We have $X_n \to X_\infty$ in L^1. So, in the case $T \equiv \infty$, we can pass to the limit in (2.4) to obtain

$$\mathsf{E}\left(X_\infty \mid \mathcal{F}_S\right) = X_S \quad \text{a.s..}$$

Then, returning to (2.5), for general T, we have

$$\left\|X_n^T - X_\infty^T\right\|_1 = \left\|\mathsf{E}\left(X_n - X_\infty \mid \mathcal{F}_T\right)\right\|_1 \le \left\|X_n - X_\infty\right\|_1$$

and the result follows on passing to the limit in (2.4). □

Theorem 2.4.11 (Optional stopping theorem – 3) *Let T be a stopping time with $\mathsf{E}T < \infty$ and let X_n be a supermartingale with uniformly bounded increments, i.e., there exists a finite constant $K > 0$ such that*

$$\left|X_n(\omega) - X_{n-1}(\omega)\right| \le K \qquad \forall (n, \omega).$$

Then X_T is integrable and $\mathsf{E}(X_T) \le \mathsf{E}(X_0)$.