ASSESSING THE ECONOMIC VALUE OF LIFE INSURANCE CONTRACTS WITH STOCHASTIC DEFLATORS[1]

Version 2.1 of 17/10/2021

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This article proposes an approach for building an Economic Scenario Generator (ESG) under historical probability, allowing the simulation of interest rates and prices of risky investments, adapted to the process of valuing liabilities of savings contracts with profit-sharing clauses and consistent with the Solvency 2 and IFRS 17 frameworks. It therefore includes the construction of the discounting factor (deflator) used to calculate prices.

It proposes methods for calibrating models and risk premiums based on closed formulas and presents simulation approaches with exact discretization adapted to long-term simulation needs and less computation time consuming.

The article also proposes a study of the sensitivities of the value of a with profit savings liability to the calibrations of the economic scenario generator under historical probability.

Finally, it shows that moving from a "risk neutral" calculation to a deflator approach requires only relatively marginal work to adapt existing models.

[1] A technical appendix is attached to this paper.

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1 Introduction

With a profit savings contracts offer a capitalisation of the investment with a guaranteed rate, increased by a bonus for the policyholder's participation in the financial results of the general fund managed by the insurer. These contracts offer the possibility of the savings redemptions, which can be exercised at any time.

The options included in savings contracts with a profit-sharing clause can be summarised in three categories:

- Financial options: the insurer commits to a minimum return on savings by guaranteeing a minimum revaluation rate or a guaranteed bonus.
- Behavioural options: the insurer offers redemption options, arbitrage options between the guaranteed fund and non-guaranteed assets, free or scheduled payments, loyalty bonuses, etc. The activation of these options is at the discretion of the policyholder.
- Biometric options: are options that depend on mortality (or longevity) risk, such as the insurer's proposal for deferred annuities.

The policyholder therefore has three financial options (see Brys and de Varenne [1994]):

- The technical rate option or guaranteed profit-sharing rate option, similar to a European vanilla option.
- The redemption option, similar to an American put option.
- The forward rate option on free or scheduled payments, like a swaption.

Under the Solvency 2 standard, the "economic" valuation of liabilities corresponds to the best-estimate (discounted future cash flows) and is, if necessary, supplemented by a risk margin to compensate for the immobilisation of the Solvency Capital Requirement for non-hedgeable risks (resulting in a cost of capital for non-financial risks or imperfect hedge for financial risks)[4]. Although the two standards, Solvency 2 and IFRS 17, present significant divergences, the conclusions presented in this paper can naturally be generalised to IFRS 17.

Best-estimate valuation models' implementation for contracts with a profit-sharing clause has led to a form of market consensus articulating a generator of risk-neutral economic scenarios with a cash-flow projection model.

Cash flows must take into account, in particular, two stochastic processes depending on the economy and the behaviour of agents (reaction functions - cf. sections 7 and 8 of the appendix):

- Insurer reaction: the revaluation rate, which varies according to the state of the market and the insurer's investment policy. It is also the consequence of profit optimization under economic constraints and the policyholder behaviour.

[4] Issues related to risk margin calculations will not be addressed in this paper. We will therefore not distinguish between the terms "best-estimate" and "liability value" in the following.

- Policyholder reaction: the dynamic lapse rate reflects a financial arbitrage behaviour aimed at maximizing profitability. It can be negative, that is, policyholders buy back less than "usual". It can be negative when the moneyness of the insurance contract is favourable to them. However, it can also be positive when more profitable risk-contingent investments, compared to insurance contracts, are available.

Armel and Planchet [2020b] show that the use of a risk-neutral probability measure to value savings liabilities with a profit-sharing clause is questionable. The deflator approach, which consists in using an economic scenario generator (ESG) under historical probability, seems relevant in this context. It allows for a better rationalisation of economic valuations (notably the behaviour of agents) and eliminates direct interactions between the construction of cash-flows and the calculation of prices.

The shortcoming of a deflator approach lies in the complexity of its operational implementation. A first step has been made with the approach proposed by Cheng and Planchet [2019]. However, this approach is not appropriate to the Solvency 2 framework.

The objective of this paper is to propose an economic scenario generator under historical probability, allowing to simulate interest rates and prices of risky investments (in equities and real estate)[5], adapted to with profit liabilities valuation process.

To do this, we propose in particular:

- To assume that the interest rate model follows a CIR++ process. This model takes into account negative rates and makes it possible to reproduce the market risk-free yield curve, and in particular that proposed by the EIOPA;
- A calibrating method of the economic scenario generator under historical probability adapted to the Solvency 2 framework.
- A simulation method with exact discretization allowing an optimization of the operational implementation and convergence errors.

We therefore propose an operational model adapted to the Solvency 2 standard, based on a CIR++ interest rate model, which makes the deflator approach applicable in practice.

2 Deflator approach with a CIR++ interest rate model

In this section, we present our approach to constructing an economic scenario generator under historical probability to value savings contracts with a profit-sharing clause. After a brief reminder of the technical framework, we present in the following:

- The interest rate model and zero-coupon bond price.
- The deflator and the likelihood process.

[5] An economic scenario generator for valuing bonds, equities, real estate investments and monetary securities covers 98% of the assets of insurance companies in France in 2016 and allows to simulate risk-free interest rates (see FFA [2017]).

- The price of the risky asset allowing to take into account investments such as equities or real estate.
- Methods for calibrating and simulating models.

2.1 Technical framework [6]

Let $\{r(t)\}_{0\leq t}$ be the instantaneous risk-free short rate and let $\{S(t)\}_{0\leq t}$ be the price of the risky asset (corresponding here to the prices of investments in equities and real estate).

Under historical probability P, assume that the differential equations of $\{r(t)\}_{0\leq t}$and $\{S(t)\}_{0\leq t}$ are written as follows:

$$dr(t) = \alpha\big(t, r(t)\big)dt + \beta\big(t, r(t)\big)dW_{rate}^{P}(t) \quad (1)$$

$$dS(t) = \mu_{risk}(t).S(t).dt + \sigma_{risk}(t).S(t).dW_{risk}^{P}(t) \quad (2)$$

where $W_{rate}^{P}(t)$ and $W_{risk}^{P}(t)$ are two standard Brownian motions whose correlation coefficient is denoted $\rho = Corr\left(dW_{rate}^{P}(t); dW_{risk}^{P}(t)\right)$.

We can wright: $dW_{risk}^{P}(t) = \rho.dW_{rate}^{P}(t) + \sqrt{1-\rho^2}dW_{risk}^{P}(s)^{\perp}$ where $Corr(dW_{rate}^{P}(t); dW_{risk}^{P}(t)^{\perp}) = 0$.

Let $\{\lambda(t)\}_{0\leq t}$ an adapted process such as the likelihood process L defined by: $dL(t) = -L(t).\lambda(t)dW_{rate}^{P}(t)$ is integrable and satisfies Novikov condition (in order to apply the Girsanov theorem).

So, the process L is a P-martingale and we have:

$$L(t) = \exp\left(-\int_0^t \lambda(s)dW_{rate}^{P}(s) - \frac{1}{2}\int_0^t \lambda(s)^2 ds\right) \quad (3)$$

The probability measure Q, whose likelihood process is L, is a martingale measure equivalent to P and the process $\left\{W_{rate}^{Q}(t)\right\}_{0\leq t}$ is a Q-standard Brownian motion where:

$$dW_{rate}^{Q}(s) = dW_{rate}^{P}(s) + \lambda(s)ds$$

In the following, the measure Q is called the risk-neutral probability measure.

The deflator is written (see appendix 1):

$$D(t) = L(t).\exp\left(-\int_0^t r(s)ds\right)$$

The deflator $\{D(t)\}_{0\leq t\leq T}$ stochastic differential equation is written:

[6] Appendix 1 provides a brief reminder of the theoretical framework of probability measure change and construction of deflators.

$$dD(t) = -D(t).r(t)dt - D(t)\lambda(t)dW_{rate}^P(t)$$

In addition, a drift and volatility of the risky asset should be chosen such as:

- The process $\left\{S(t).\exp\left(-\int_0^t r(s)ds\right)\right\}_{0\leq t}$ is a martingale under Q ;
- The process $\{D(t)S(t)\}_{0\leq t}$ is a martingale under P.

As in Cheng and Planchet [2019] we assume that $\mu_{risk}(t) = r(t) + \lambda(t)\sigma_{risk}(t)\rho$. We can prove by Itô's lemma that:

$$D(t)S(t) = D(0)S(0)\exp\left(\int_0^t\left(\lambda(s)\sigma_{risk}(s)\rho - \frac{1}{2}\sigma_{risk}(s)^2 - \frac{1}{2}\lambda(s)^2\right)ds\right).\exp\left(\int_0^t\left(\sigma_{risk}(s)\rho - \lambda(s)\right)dW_{rate}^P(s) + \int_0^t \sigma_{risk}(s)\sqrt{1-\rho^2}dW_{risk}^P(s)^{\perp}\right) \quad (4)$$

If the processes $\{\sigma_{risk}(t)\}_{0\leq t}$ and $\{\lambda(t)\}_{0\leq t}$ are constant, it is then obvious that the process $\{D(t)S(t)\}_{0\leq t}$ is a martingale under P.

If both processes $\{\sigma_{risk}(t)\}_{0\leq t}$ and $\{\lambda(t)\}_{0\leq t}$ are stochastic, a sufficient condition for $\{D(t)S(t)\}_{0\leq t}$ to be a martingale under P is to assume that :

$$\lambda(s)\sigma_{risk}(s)\rho - \frac{1}{2}\sigma_{risk}(s)^2 - \frac{1}{2}\lambda(s)^2 = 0$$

This assumption was retained by Cheng and Planchet [2019].

It follows that:

$$\sigma_{risk}(s) = \lambda(s)\rho \pm \lambda(s)\sqrt{\rho^2 - 1} \quad (5)$$

Since $\rho \leq 1$ this equation admits a real solution only if $|\rho| = 1$. In this case $\sigma_{risk}(s) = \lambda(s)\rho$ and $W_{risk}^P(t) = \rho W_{rate}^P(t)$.

We deduce that $\sigma_{risk}(t).W_{risk}^P(t) = \lambda(t).W_{rate}^P(t)$ and $\mu_{risk}(t) = r(t) + \lambda(t)^2$.

So, we can write:

$$D(t)S(t) = D(0)S(0)$$

The dynamics of risky assets under P is therefore written:

$$dS(t) = (r(t) + \lambda(t)^2).S(t).dt + \lambda(t).S(t).dW_{rate}^P(t) \quad (6)$$

Under Q the dynamic of the risky asset is a Black-Sholes type model:

$$dS(t) = r(t).S(t).dt + \lambda(t).S(t).dW_{rate}^Q(t) \quad (7)$$

The solution to this differential equation is written, by applying Itô's lemma:

$$S_T = S_t \exp\left(\int_t^T r(u)du - \int_t^T \frac{\lambda(u)^2}{2} du + \int_t^T \lambda(u) dW_{rate}^Q(u)\right)$$

So :

$$E^Q\left(S_t . \int_0^t -r(u)du\right) = S_0 E^Q\left(\exp\left(-\int_0^t \frac{\lambda(u)^2}{2} du + \int_0^t \lambda(t) dW_{rate}^Q(u)\right)\right)$$

By construction:

$$E^Q\left(\exp\left(-\int_0^t \frac{\lambda(u)^2}{2} du + \int_0^t \lambda(t) dW_{rate}^Q(u)\right)\right)$$
$$= E^P\left(L(t). \exp\left(\int_0^t \frac{\lambda(u)^2}{2} du + \int_0^t \lambda(t) dW_{rate}^P(u)\right)\right) = 1$$

So, the process $\left\{S(t). \exp\left(-\int_0^t r(s)ds\right)\right\}_{0 \leq t}$ is a Q-martingale.

We have described here the technical framework of a scenario generator under historical probability assuming that instantaneous short interest rates follow the model:

$$dr(t) = \alpha(t, r(t))dt + \beta(t, r(t)) dW_{rate}^P(t)$$

In the following sections we present an application of this technical framework if instantaneous short interest rates follow a CIR++ model.

2.2 Interest rate model and zero-coupon bond price

We follow here Brigo and Mercurio [2006] and Cox, Ingersoll and Ross[1985].

2.2.1 Under the risk neutral probability measure

The CIR++ model describes the dynamic of the instantaneous short interest rate r. It is written as a sum of a deterministic function denoted φ and a CIR process denoted x whose vector of parameters is denoted $\alpha = (k, \theta, \sigma)$ and defined under Q as follows:

$$dx(t) = k(\theta - x(t))dt + \sigma_x \sqrt{x(t)} dW_{rate}^Q(t)$$

where: $x(0) = x_0$ and x_0, k, θ, σ are positive constants and we have:

$$r(t) = x(t) + \varphi(t)$$

For the instantaneous short rate to remain strictly positive, the parameters of the model must meet the following Feller condition:

$$2k\theta > \sigma_x^2$$

The function φ enables to reproduce the market yield curve. Let $f^M(0, t)$ be the market instantaneous forward rate at time 0 for maturity t:

$$f^M(0, t) = -\frac{\partial \ln(P^M(0, t))}{\partial t}$$

The two following equations are necessary and sufficient conditions for CIR++ model to fit the initial term structure:

$$\varphi^{CIR}(t;\alpha) = \mathrm{f}^{\mathrm{M}}(0,\mathrm{t}) - \mathrm{f}^{\mathrm{CIR}}(0,\mathrm{t};\ \alpha) \tag{8}$$

$$exp\left(-\int_t^T \varphi(s)ds\right) = \frac{P^M(0,T)A(0,t)exp\{-B(0,t)x_0\}}{P^M(0,t)A(0,T)exp\{-B(0,T)x_0\}} \tag{9}$$

Where:

- $\varphi(t) = \varphi^{CIR}(t;\alpha)$;
- $f^{CIR}(0,t;\alpha) = \frac{2k\theta(exp\{th\}-1)}{2h+(k+h)(exp\,\{th\}-1)} + x_0\ \frac{4h^2exp\{th\}}{[2h+(k+h)(exp\{th\}-1)]^2}$;
- $h\ =\ \sqrt{k^2+2\sigma_x^2}$;
- $P^M(0,T)$ is the market price of the risk-free zero-coupon bond observed at time 0 for maturity T;
- $A(t,T)$ and $B(t,T)$ are defined by:

$$A\,(t,T) = \left[\frac{2hexp\left\{\frac{(k\ +\ h)(T\ -\ t)}{2}\right\}}{2h+(k+h)(exp\,\{(T-t)h\}-1)}\right]^{\frac{2k\theta}{\sigma_x^2}}$$

$$B\ (\ t,T)\ \ = \frac{2(exp\,\{(T\ -\ t)h\}-\ 1)}{2h\ +\ (k\ +\ h)(exp\,\{(T\ -\ t)h\}-\ 1)}$$

- $h = \sqrt{k^2+2\sigma_x^2}$

As stated in Cox, Ingersoll and Ross[1985], the process $x(t)$ conditionally to $x(s)$ follows the probability distribution $\chi^2(v,\lambda_{t,s})/c_{t-s}$:

$$x(t)|x(s) = \chi^2(v,\lambda_{t,s})/c_{t-s}$$

where:

- $\chi^2(v,\lambda_{t,s})$ is a non-central chi-square distribution with v degrees of freedom and whose decentralisation parameter is $\lambda_{t,s}$;
- $c_{t-s} = \frac{4k}{\sigma_x^2\left(1-exp(-k(t-s))\right)}$;
- $v\ =\ 4k\theta/\sigma_x^2$;
- $\lambda_{t,s} = c_{t-s}x_s exp(-k(t-s))$.

Let F_s be the sigma-algebra generated by $\{x_i\}_{i\leq s}$. The mean and variance of $x(t)$ conditionally to F_s are given by:

$$E^Q\{x(t)|F_s\} = x(s)e^{-k(t-s)} + \theta\left(1-e^{-k(t-s)}\right)$$

$$Var\{x(t)|F_s\} = \frac{x(s)\sigma_x^2}{k}\left(e^{-k(t-s)} - e^{-2k(t-s)}\right) + \theta\frac{\sigma_x^2}{2k}\left(1-e^{-k(t-s)}\right)^2$$

The price at time t of a zero-coupon bond with a maturity T can be written in the form (see Brigo and Mercurio [2006]):

$$P(t,T) = \overline{A}'(t,T)e^{-B(t,T)x(t)}$$

where $\overline{A}'(t,T) = \frac{P^M(0,T)A(0,t)exp\{-B(0,t)x_0\}}{P^M(0,t)A(0,T)exp\{-B(0,T)x_0\}}A(t,T)$

The compound interest rate at time t for the maturity T is therefore:

$$R(t,T) = \frac{1}{T-t}\left(\ln\left(\frac{P^M(0,t)A(0,T)exp\{-B(0,T)x_0\}}{A(t,T)P^M(0,T)A(0,t)\ exp\{-B(0,t)x_0\}}\right) + B(t,T)x(t)\right)$$

The price $P(t,T)$ and the rate $R(t,T)$ are functions of the parameters of the one-factor CIR model x and the initial value x_0. We can therefore characterise the dynamics of $R(t,T)$ without having to calculate the function $\varphi(t)$.

Moreover, the simulation of compound interest rates amounts to simulate the process x. This can be simulated from the diffusion of a non-central Chi-square distribution.

2.2.2 Under the historical probability measure

As presented in the previous section, the instantaneous interest rate is assumed to be written in the form $r(t) = x(t) + \varphi(t)$ where φ is a deterministic function and x is a CIR process who's dynamic under Q is as follows:

$$dx(t) = k\big(\theta - x(t)\big)dt + \sigma_x\sqrt{x(t)}dW_{rate}^Q(t)\ ;\ x(0) = x_0$$

and:

$$r(t) = x(t) + \varphi(t)$$

Let us define the risk premium as: $\lambda(t) = \lambda\sqrt{x(t)}/\sigma_x$ where λ is a real number. And let us write: $dW_{rate}^Q(t) = dW_{rate}^P(t) + \lambda(t)dt$.

The choice of the risk premium is such that x is a square root process both under P and Q.

This risk premium depends directly on the CIR process. It therefore depends on the modelled variables and does not follow a process independent of these variables as proposed by Cheng and Planchet [2019].

The form chosen for the risk premium allows to keep the main analytical properties of the model during the probability change. A general framework adapted to affine models and allowing to keep their analytical properties under P and under Q is presented and analysed in Duffee [2002].

The process W_{rate}^P is a Brownian motion under the historical probability measure and the instantaneous short interest rate is written under P as $r(t) = x(t) + \varphi(t)$ with:

$$dx(t) = (k-\lambda)\left(\frac{k\theta}{k-\lambda} - x(t)\right)dt + \sigma_x\sqrt{x(t)}dW_{rate}^P(t)\ ;\ x(0) = x_0$$

The process r therefore follows a CIR++ process under historical probability. It allows to consider negative rates and to reproduce the market yield curve. We can evaluate the prices of zero-coupon bonds by closed formulas and simulate, in exact discretization, the process r by simulating non-central Chi-square distributions.

In addition, the Feller constraint is respected under Q if it is respected under P: $2(k-\lambda)\frac{k\theta}{k-\lambda} = 2k\theta > \sigma_x^2$.

Under the historical probability, the price at time t of a zero-coupon bond with a maturity T is written in the form (see demonstration in appendix 2):

$$P(t,T) = \overline{A}'\,(t,T)e^{-B(t,T)x(t)}$$

The compound interest rate at t for the maturity T is therefore:

$$R\,(t,T) = \frac{1}{T-t}\left(\ln\left(\frac{P^M(0,t)A(0,T)exp\{-B(0,T)x_0\}}{A(t,T)P^M(0,T)A(0,t)\;exp\{-B(0,t)x_0\}}\right) + B(t,T)x(t)\right)$$

The rate $R(t,T)$ is an affine function of $x(t)$ whose coefficients are deterministic and is an affine function of a non-central Chi-square distribution.

2.3 Deflator and likelihood process

The stochastic differential equation of the deflator under the historical probability measure is:

$$\frac{dD(t)}{D(t)} = -r(t)dt - \lambda(t)dW_{rate}^P(t)$$

The stochastic deflator is written (see demonstration in appendix 3):

$$D(T) = D(t)\exp\left(\frac{\lambda k\theta}{\sigma_x^2}(T-t)\right)\exp\left(-\int_t^T \varphi(s)ds\right)\exp\left(-\frac{\lambda}{\sigma_x^2}\big(x(T)-x(t)\big)\right)\exp\left(-\left(1 - \frac{\lambda^2}{2\sigma_x^2} + \frac{\lambda k}{\sigma_x^2}\right)\int_t^T x(s)ds\right)$$

$$D(T) = D(t)\exp\left(\frac{\lambda k\theta}{\sigma_x^2}(T-t)\right)\exp\left(-\int_t^T \varphi(s)ds\right)\exp\left(-\frac{\lambda}{\sigma_x^2}\big(x(T) - x(t)\big)\right)\exp\left(-\left(1 - \frac{\lambda^2}{2\sigma_x^2} + \frac{\lambda k}{\sigma_x^2}\right)\int_t^T x(s)ds\right) \quad (10)$$

The likelihood process of Q is written:

$$L(t) = \frac{D(t)}{D(0)}\exp\left(\int_0^t r(s)ds\right)$$

i.e.

$$L(t) = \exp\left(\frac{\lambda k\theta}{\sigma_x^2} t\right) \exp\left(-\frac{\lambda}{\sigma_x^2}\big(x(t) - x(0)\big)\right) \exp\left(-\left(\frac{\lambda^2}{2\sigma_x^2} + \frac{\lambda(k-\lambda)}{\sigma_x^2}\right) \int_0^t x(s)ds\right) \quad (11)$$

2.4 Risky asset price and risk premium

2.4.1 Risky asset price

As presented in section 2.1, the risky asset process is written under the historical probability measure P:

$$\frac{dS_t}{S_t} = (r(t) + \lambda(t)^2)dt + \lambda(t)dW_{rate}^P(t)$$

The price of the risky asset is written (see appendix 4for a demonstration of this result):

$$S(T) = S(t)\exp\left(-\frac{\lambda}{\sigma_x^2} k\theta(T-t)\right).\exp\left(\int_t^T \varphi(s)ds\right).\exp\left(\frac{\lambda}{\sigma_x^2}\big(x(T) - x(t)\big)\right).\exp\left(\left(1 - \frac{\lambda^2}{2\sigma_x^2} + \frac{\lambda k}{\sigma_x^2}\right)\int_t^T x(s)ds\right) \quad (12)$$

2.4.2 Expected return on risky assets

Note s_t the logarithmic return of the risky asset at time t on a one-year horizon. By definition:

$$s_{t+1} = ln\left(\frac{S(t+1)}{S(t)}\right)$$

The mathematical expectation of the random variable s_{t+1} under the historical probability P is written (see appendix 5 for a demonstration of this result):

$$E^p(s_{t+1}) = R^M(t, t+1) - \left(\ln\left(\frac{A(0,t)}{A(0,t+1)}\right) + x_0\big(B(0,t+1) - B(0,t)\big)\right) + \left(1 + \frac{\lambda^2}{2\sigma_x^2}\right)\left(e^{-(k-\lambda)t}\frac{\big(k\theta - x_0(k-\lambda)\big)}{(k-\lambda)^2}\left(e^{-(k-\lambda)} - 1\right) + \frac{k\theta}{k-\lambda}\right) \quad (13)$$

With:

- $R^M(t, t+1)$ the observed market interest rate between the date t and $t+1$;
- $A(t,T)$ and $B(t,T)$ are deterministic functions defined in section 2.2.1;
- x_0 is the initial value of the process x_t.

The expectation of the excess return of the risky asset over the market risk-free interest rate, denoted e_t, is therefore written:

$$E^p(e_{t+1}) = -\left(\ln\left(\frac{A(0,t)}{A(0,t+1)}\right) + x_0\big(B(0,t+1) - B(0,t)\big)\right) + \left(1 + \frac{\lambda^2}{2\sigma_x^2}\right)\left(e^{-(k-\lambda)t}\frac{\big(k\theta - x_0(k-\lambda)\big)}{(k-\lambda)^2}\big(e^{-(k-\lambda)} - 1\big) + \frac{k\theta}{k-\lambda}\right) \quad (14)$$

This formula allows to link the factor λ of the risk premium to the excess return offered by the risky asset over the risk-free rates.

2.4.3 The expectation of excess return in steady state

In the long term, in steady state ($t >> 0$), the excess return depends only on the risk factor λ and the parameters of the CIR model (k, θ and σ_x) and is written (see appendix 6):

$$E^p(e_\infty) = \frac{k\theta}{\sigma_x^2}(k - h) + \frac{k\theta}{k-\lambda}\left(1 + \frac{\lambda^2}{2\sigma_x^2}\right) \quad (15)$$

This formula explicitly describes the factor λ of the risk premium as a function of the excess return of the risky asset over the risk-free rate in a long-term perspective.

The calibration of the interest rate model allows to define parameters: k, θ and σ_x. The estimation of the expected excess return allows therefore the construction of an estimator of λ.

2.5 Calibration

The calibration of the economic scenario generator under the historical probability requires to define the following parameters:

- The parameters of the interest rate model: k, θ and σ_x;
- The risk premium factor λ.

To do this, we propose the following two-step approach:

- Calibrating parameters k, θ and σ_x using the analytical features offered by the CIR++ model under risk-neutral probability. Caps, floors and swaptions can be valued using closed formulas (see Brigo and Mercurio [2006]). This point is discussed in section 2.5.1.
- Calibrate λ on historical data. This is discussed in section 2.5.2.

2.5.1 Interest rate model calibration

The calibration of a model can be conducted either by using an estimate based on historical data or by an implied evaluation of the parameters based on market prices observed to date.

With the historical approach, the parameters of the model are determined based on a statistical time-series analysis of the relevant market data. With the implied approach, the parameters are evaluated to replicate the observed market prices of the derivatives selected for calibration.

Although the theoretical model assumes that the historical and implied parameters are equal, in practice they are different.

The classic example is the Black-Scholes model implying that the implied volatility of all options on the same underlying must be the same, and equal to the historical volatility of the underlying. However, we can observe smile and skew phenomena and a flattening of the implied volatility as a function of the strike for large maturities. This is discussed for example in Tankov [2015].

For interest rate models, parameters calibrated on historical data depend on the choices of this data (index, size, frequency, etc.). The implied parameters depend on the price of the financial instruments, the strike, the risk-free interest rate and the shift factors used to enable some models to take negative rates into account. Also, the implied volatility surfaces do not necessarily result from a direct price measurement but from a reconstruction by the price provider (e.g. *via a* SABR model for Bloomberg).

In the context of derivatives valuation, it can be observed that academics and practitioners tend to use an implied calibration approach which seems more appropriate to produce prices consistent with market observations. This is discussed in Rebonato [2004].

Furthermore, in the Solvency 2 framework, the economic scenarios used for the best-estimate valuation must be consistent with market prices (Market-Consistent). A mark-to-market valuation consists of valuing quantities of interest with reference to the values of assets and liabilities traded.

Applying a Mark to Market approach to valuate liabilities at fair value implies that the prices of options and guarantees of insurance policies are observable. As this information is not available in an organised and liquid market, the calculation is therefore carried out using a Marked-to-Model framework.

The QIS technical specifications [2010] provide a framework for this valuation by requiring the use of the risk-free yield curve published by EIOPA and by referring to a calibration of the models considering implied volatilities[7].

We therefore propose here a calibration approach (1) implied for the parameters of the interest rate model and (2) historical for the risk premium. In a normative valuation framework, this approach leads to a clear separation between the determination of the cost of options, included in the deflator, and the production of contract cash-flows. It also complies with regulatory requirements.

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[7] See the Q&A of QIS 5, published by EIOPA, question 76 of the document:
https: //eiopa.europa.eu/Publications/QIS/CEIOPS-Q-and-A-document-20101104.pdf.

Brigo and Mercurio [2006] present the analytical properties of the CIR++ model and Armel and Planchet [2020a] propose an approach for calibrating this model on caps and swaptions that is consistent with the Solvency 2 framework.

2.5.2 Risk premium calibration

The approach we propose to estimate the factor λ is based on the average historical excess return generated by the risky asset over the risk-free rate. The idea is to inject into the model of the risky asset an expected long-term excess return equal to the historical long term excess return.

Let $[\![1, d]\!]$ be an interval of historical data of interest of d points and let us denote $\{e_i^M\}_{i\in[\![1,d]\!]}$ the historical annual excess returns of the risky asset.

According to section 2.4.3, the mathematical expectation of long-term excess return is written:

$$E^p(e_\infty) = \frac{k\theta}{\sigma_x^2}(k - h) + \frac{k\theta}{k-\lambda}\left(1 + \frac{\lambda^2}{2\sigma_x^2}\right)$$

We propose to estimate the risk factor λ by solving the equation:

$$\frac{1}{d}\sum_{i=1}^{d} e_i^M \approx E^P(e_\infty) = \frac{k\theta}{\sigma_x^2}(k - h) + \frac{k\theta}{k-\lambda}\left(1 + \frac{\lambda^2}{2\sigma_x^2}\right) \quad (16)$$

This equation is equivalent to a second order polynomial equation and can admit complex solutions. If this is the case, we propose the following real solution:

$$\hat{\lambda} = \underset{\lambda}{\operatorname{argmin}}\left(\left|\frac{k\theta}{\sigma_x^2}(k - h) + \frac{k\theta}{k-\lambda}\left(1 + \frac{\lambda^2}{2\sigma_x^2}\right) - \frac{1}{d}\sum_{i=1}^{d} s_i^M\right|\right)$$

2.6 Simulation under historical probability

The dynamics of the instantaneous short rate follows the CIR++ model as defined in section 2.2.2. The diffusion of $\{x(t)\}_{0\le t}$ (following a non-central Chi-square distributions) allows the simulation of zero-coupon bond prices and compound interest rates. Indeed, recall that:

$$P(t,T) = \exp\big(-(T-t)R(t,T)\big) = \bar{A}'(t,T)e^{-B(t,T)x(t)}$$

In addition, we propose to evaluate the quantity $\int_0^T x(s)ds$ by relying on Riemann integration. We can indeed write for an appropriate integer N:

$$\int_t^T x(s)ds \approx \frac{T-t}{N}\sum_{i=0}^{N-1} x\left(t + \frac{i}{N}(T-t)\right)$$

The simulation of $\{x(t)\}_{0\le t}$ and the evaluation of $\int_t^T x(s)ds$ allow to simulate the price of the risky asset and the deflator whose dynamics are written, as a reminder, as follows:

$$S(T) = S(0)\exp\left(-\frac{\lambda}{\sigma_x^2}k\theta T\right).\exp\left(\int_0^T \varphi(s)ds\right).\exp\left(\frac{\lambda}{\sigma_x^2}(x(T)-x_0)\right).\exp\left(\left(1-\frac{\lambda^2}{2\sigma_x^2}+\frac{\lambda k}{\sigma_x^2}\right)\int_0^T x(s)ds\right)$$

$$D(T) = D(0)\exp\left(\frac{\lambda k\theta}{\sigma_x^2}T\right)\exp\left(-\int_0^T \varphi(s)ds\right)\exp\left(-\frac{\lambda}{\sigma_x^2}(x(T)-x_0)\right)\exp\left(-\left(1-\frac{\lambda^2}{2\sigma_x^2}+\frac{\lambda k}{\sigma_x^2}\right)\int_0^T x(s)ds\right)$$

In this section we have presented an approach for building, calibrating and simulating an economic scenario generator, under historical probability, adapted to the valuation process of savings contracts with a profit-sharing clause and consistent with the Solvency 2 standard.

We present in the following section (section 3) an application of this approach including studies of the sensitivity of best-estimate to model and data choices.

3 Application: model calibration and best-estimate sensitivities

In this section we complete the sensitivity tests carried out in Armel and Planchet ([2018], [2019] and [2020a]) by assessing the impact of the choice of an economic scenario generator under the historical probability measure, whose interest rate model is the CIR++ model, on the best-estimate of savings contracts. Sensitivities to the choice of data and the shift factor of the Black model, used in the calibration process, are also presented.

3.1 Data

We have used the same market-data for model calibration and the same valuation parameters for liabilities presented in our previous work.

For the calibration of the risk premium, we have used historical equity and property investment returns from 2011 to 2018 to take into account an economic situation close to that which exists at the projection and calibration date used here (end 2018). We have thus excluded the 2007 crisis and considered the monetary easing policy following the Greek debt crisis in 2011.

If equities represent 67% of the risky assets and property represents 33%, the average excess return is 2.7%.

3.2 Model calibration results

3.2.1 Calibration of the interest rate model on the prices of *caps* and *swaptions*

Here we retain the calibration results of the CIR++ model carried out in Armel and Planchet [2020a]. This paper also studied the impact of the initial value of the CIR model on the distributional characteristics of the CIR++ model under Q. Given the negligible impact of the initial value of the CIR model, we focus from section 3.2.2 on studying the impact of the shift factor of the Black model and we set the initial value of the CIR++ models at 1%. For more material on the calibration process and on the analysis of the results, the reader can refer to this article.

Table 1 presents the results of the calibration of the CIR++ model (also noted CIR1F in the following) on ATM cap prices and Table 2 presents the results of the calibration on ATM swaption prices.

The meta-parameter x_0 of the CIR++ model representing the initial value of the CIR process must be set upstream of the calibration process. This parameter has no impact on the reproduction of the initial yield curve but may have an impact on the dynamics of the simulated interest rates. Three levels of the meta-parameter x_0 are assessed: 0.4%, 1% and 2%.

By the notation $CIR1F(i, j)$ we refer to the CIR++ model calibrated on caps or swaptions with the i^{th} shift factor of Black and the j^{th} meta-parameter of the CIR++ model, both belonging to (0.4%; 1%; 2%; 0.4%; 1%; 2%).

Table 1: calibration results of the CIR++ model on Caps.

Parameters	CIR1F(1,1)	CIR1F(1,2)	CIR1F(1,3)	CIR1F(2,1)	CIR1F(2,2)	CIR1F(2,3)	CIR1F(3,1)	CIR1F(3,2)	CIR1F(3,3)
k	1.95%	2.91%	3.89%	2.2%	3.12%	4.09%	2.62%	3.45%	4.37%
θ	99.39%	99.22%	99.16%	98.77%	99.98%	99.24%	99.05%	99.34%	99.24%
σ	2.57%	2.1%	1.81%	3.68%	3.06%	2.68%	5.45%	4.69%	4.14%
Total relative squared error	**1.05%**	**1.19%**	**1.35%**	**1.00%**	**1.15%**	**1.34%**	**0.88%**	**1.03%**	**1.21%**

Table 2: calibration results of the CIR++ model on Swaptions

Parameters	CIR1F(1,1)	CIR1F(1,2)	CIR1F(1,3)	CIR1F(2,1)	CIR1F(2,2)	CIR1F(2,3)	CIR1F(3,1)	CIR1F(3,2)	CIR1F(3,3)
k	2.63%	2.99%	3.45%	3.52%	3.85%	4.31%	4.95%	5.19%	5.5%
θ	99.95%	99.99%	100,00%	99.97%	99.99%	100,00%	99.9%	99.96%	99.98%
σ	5.76%	5.31%	4.87%	7.05%	6.65%	6.2%	9.53%	9.16%	8.7%
Total relative squared error	**2.92%**	**3.19%**	**3.6%**	**3.45%**	**3.73%**	**4.15%**	**4.89%**	**5.16%**	**5.6%**

The total error is calculated as the sum of squared errors (objective function) divided by the sum of squared Black prices.

Calibrated parameters respect Feller's constraint.

3.2.2 Estimation of the risk premium factor

The calibration of CIR++ models to market prices allows to derive the risk premium parameter from the historical excess return as explained in section 2.5.2.

In summary, the calibration of the models studied here is presented in the *Table 3*.

Table 3: CIR++ model calibration results for $x_0 = 1\%$ and for different Black shift factors (0.4%; 1%; 2%)

	Cap			Swaption		
Parameters	CIR1F(1,2)	CIR1F(2,2)	CIR1F(3,2)	CIR1F(4,2)	CIR1F(5,2)	CIR1F(6,2)
k	2.91%	3.12%	3.45%	2.99%	3.85%	5.19%
ϴ	99.22%	99.98%	99.34%	99.99%	99.99%	99.96%
σ	2.10%	3.06%	4.69%	5.31%	6.65%	9.16%
λ	-0.70%	-1.36%	-2.58%	-3.30%	-4.09%	-5.68%

Note that since we have, under the historical probability:

$$\frac{dP(t,T)}{P(t,T)} = \varphi(t)dt + x(t)\big(1 - \lambda B(t,T)\big)dt \; - \; B(t,T)\sigma_x \sqrt{x(t)}dW_{rate}^P(t)$$

The excess yield of the zero-coupon bond is $(-\lambda\, x(t)B(t,T))$. Thus, if λ is negative, the yield under historical probability is greater than the risk-neutral yield which means that the risk premium is positive.

3.2.3 Martingality tests

In this section we present graphical illustrations to validate martingality tests performed with a number of paths of 2000 and a simulation step for Riemann integration presented in section 2.6of 1/500 (N=500).

We have retained 2000 trajectories for the validation of the tests because this is the number of simulations that we use for the evaluation of the best-estimate in section 3.3. In addition, the report published by the Institute of Actuaries [2016] specifies that, commonly, the number of scenarios used by practitioners, to value with profit savings contracts using Monte Carlo methods, is of the order of a thousand.

The objective here is to verify that the deflated prices of zero-coupon bonds and risky assets are martingales. That means that for real numbers t and T with $t \leq T$ and by assuming that $D(0) = 1$:

$$E^P\big(D(t).P(t,T)\big) = P^M(0,T)$$

$$E^P(D(t).S(t)) = S(0)$$

Zero-coupon prices with maturity T published by the EIOPA are noted $P^M(0,T)$.

In the following, we present five martingality tests performed:

- Test 1 consists of choosing $T = t$ and aims to verify that the deflator converges towards observed market prices: $E^P\big(D(t)\big) = P^M(0,t)$;
- Test 2 consists of choosing $T = t + 5$ and aims to verify that the deflated prices of 5-year maturity zero-coupon bonds are martingales: $E^P\big(D(t) * P(t,t+5)\big) = P^M(0,t+5)$;
- Test 3 consists of choosing $T = t + 10$ and aims to verify that the deflated prices of 10-year maturity zero-coupon bonds are martingales: $E^P\big(D(t) * P(t,t+10)\big) = P^M(0,t+10)$;

- Test 4 consists of choosing $T = t + 20$ and aims to verify that the deflated prices of 20-year maturity zero-coupon bonds are martingales: $E^P\big(D(t) * P(t, t+20)\big) = P^M(0, t+20)$.
- Test 5 consists of verifying the martingality of the risky asset: $E^P(D(t).S(t)) = S(0)$

We observe in the following figures that the deflated prices converge well towards the market prices initially observed.

Figure 1: test 1 - deflator convergence to market prices : $E^P\big(D(t)\big) = P^M(0, t)$

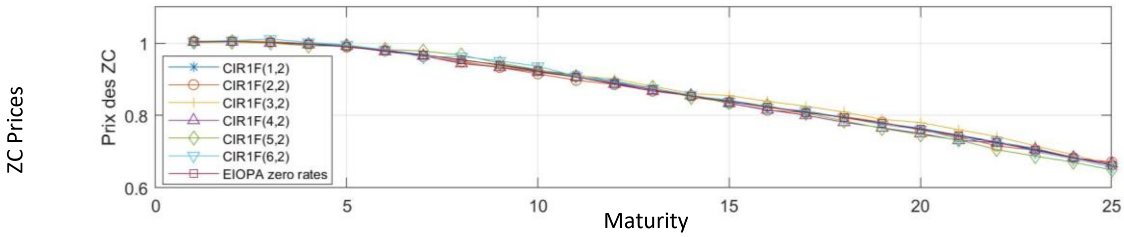


Figure 2: test 2 - deflated 5-year ZC convergence towards market prices: $E^P\big(D(t) * P(t, t+5)\big) = P^M(0, t+5)$

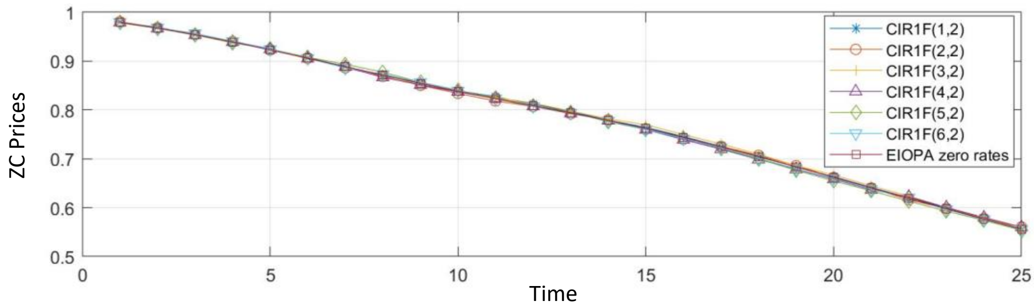


Figure 3: test 3 – deflated 10-year ZC convergence towards market prices: $E^P\big(D(t) * P(t, t+10)\big) = P^M(0, t+10)$

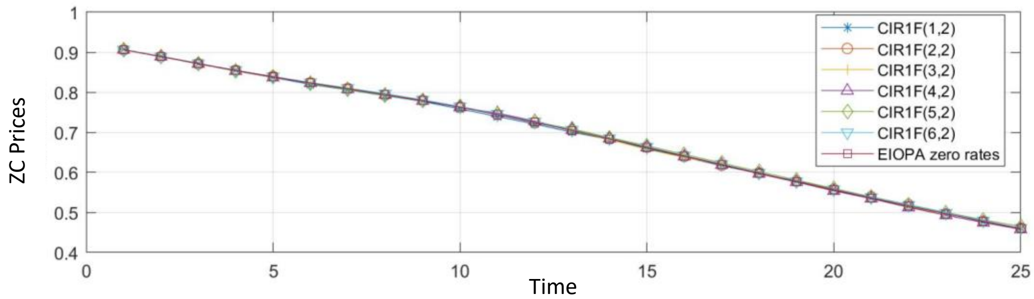


Figure 4: test 4 - deflated 20-year ZC convergence towards observed prices: $E^P\big(D(t) * P(t, t+20)\big) = P^M(0, t+20)$

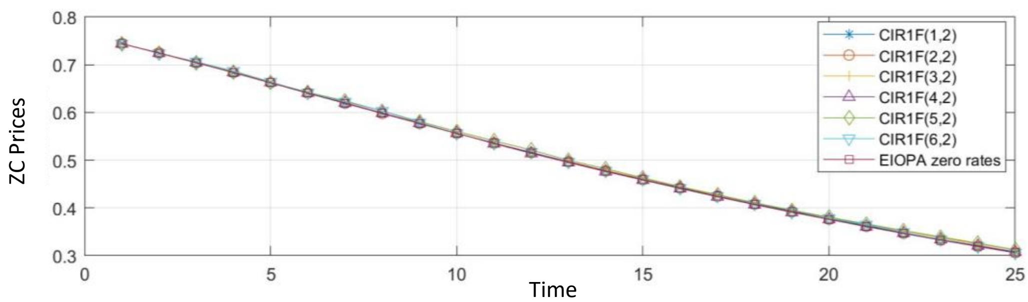
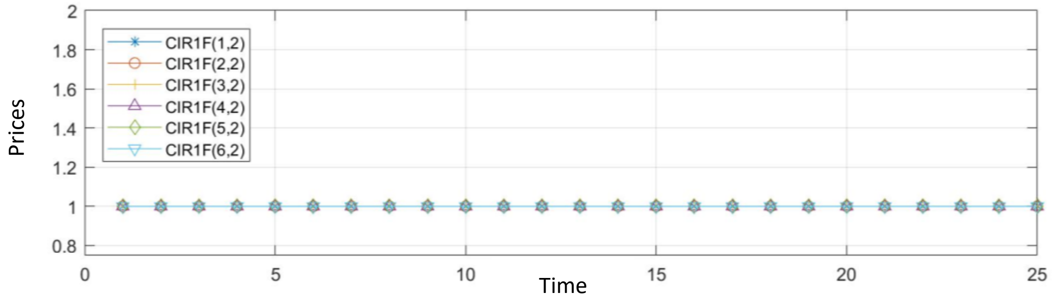


Figure 5: test 5 - risky asset martingality test: $E^P(D(t).S(t)) = S(0)$



3.3 Impact study on the best-estimate

In this section, we assess the impact of the choice of an economic scenario generator under the historical probability measure, whose interest rate model is the CIR++ model, on the best-estimate of French with profit savings contracts (euro-denominated contracts).

We have used the SimBEL R package[8] fed with modified data from an insurer. The market value of the asset is €100 million, the mathematical reserve is €70 million and the projection horizon is 20 years.

We present in Table 4 and Table 5 best-estimates evaluated by an ESG under historical probability and a risk-neutral ESG whose interest rate models are CIR++.

We observe that the differences between best-estimates vary between 4% and 7% and are explained by variations in best-estimates net of expenses. Best-estimates of expenses (discounted future expenses) are indeed stable. The decrease in the best-estimate net of expenses, when a deflator approach is used, is explained by the decrease in the BEG and the FDB[9].

The impact on the value of the best-estimate can appear to be limited. However, this impact is substantial when compared to shareholders' equity. In France, equity capital represents on average 6.1% of with profit savings reserves at the end of 2018 (FFA [2019]).

Table 4: Estimated Best-Estimate under Risk Neutral Probability - ESG with CIR++ IR Model

GSE - CIR++ under Q (M€)	*Calibration on ATM Caps*			*Calibration on ATM Swaptions*		
Shift of Black model for CIR++ calibration	*0.4%*	*1.0%*	*2.0%*	*0.4%*	*1.0%*	*2.0%*
BE	**91**	**91**	**91**	**91**	**92**	**92**
BE net of fees	*83*	*83*	*83*	*84*	*84*	*85*
BEG net of fees	*72*	*72*	*72*	*73*	*73*	*73*
FDB	*11*	*11*	*11*	*11*	*11*	*12*
Fees	*8*	*8*	*8*	*8*	*8*	*8*

[8] See http://www.ressources-actuarielles.net/simbel

[9] We define BEG and FDB in the following.

Table 5: Estimated Best-Estimate under Historical Probability - ESG with CIR++ IR Model

GSE - CIR++ under P (M€)	*Calibration on ATM Caps*			*Calibration on ATM Swaptions*		
Shift of Black model for CIR++ calibration	*0.4%*	*1.0%*	*2.0%*	*0.4%*	*1.0%*	*2.0%*
BE	***87***	***86***	***85***	***86***	***86***	***86***
BE net of fees	*79*	*78*	*77*	*78*	*78*	*78*
BEG net of fees	*71*	*70*	*69*	*70*	*69*	*70*
FDB	*8*	*8*	*8*	*9*	*8*	*9*
Fees	*8*	*8*	*8*	*8*	*8*	*8*

Recall that liability out-flow at time t can be written as an exit probability (α_t)[10] multiplied by the initial investment (PM_0) plus the cumulative revaluation $(\sum_{i=0}^{t-1} c_{i+1})$[11]:

$$F_t = \alpha_t . PM_0 . exp\left(\sum_{i=0}^{t-1} c_{i+1}\right)$$

This flow can be split into two flows: $F_t = F_t^{gar} + F_t^{discr}$ with:

- F_t^{gar} takes into account the contractually guaranteed minimum revaluation rate. If $(tmg_i)_{i\in[\![1,T]\!]}$ are the guaranteed rates, then:

$$F_t^{gar} = \alpha_t . PM_0 . exp\left(\sum_{i=0}^{t-1} tmg_{i+1}\right)$$

- F_t^{discr} represents the flow of the additional revaluation corresponding to the surplus that the insurer distributes at its discretion as profit sharing:

$$F_t^{discr} = F_t - F_t^{gar} = \alpha_t . PM_0 . \left(exp\left(\sum_{i=0}^{t-1} c_{i+1}\right) - exp\left(\sum_{i=0}^{t-1} tmg_{i+1}\right)\right)$$

The guaranteed best-estimate (BEG) is the expectation of the sum of discounted flows F_t^{gar}. The Future Discretionary Benefits (FDB) is the expectation of the sum of discounted flows F_t^{discr}. The best-estimate is then the sum of the BEG and the FDB.

All other parameters/inputs being equal (mortality rates, structural lapses, expense rates, loading rates, etc.), deflator approach use can have an impact on the exit probability and on the revaluation rate of savings.

Indeed, we can note that the probability of exit α_t is the only stochastic factor dependent on the economy of the guaranteed flow F_t^{gar} because it integrates dynamic lapses. Adopting a deflator approach has an impact on F_t^{gar} only through the factor α_t.

The decrease in BEG evaluated in historical probability is explained by the spread of lapses over the projection horizon following the decrease in dynamic lapses (see appendix 8). When we adopt a deflator approach, we observe that the expected flows F_t^{gar} are lower

[10] $\alpha_{t+1} = \left(\prod_{j=0}^{t-1}(1-q_j)(1-v_j)\right)(q_t + v_t - q_t . v_t)$ where:

- q_t the mortality rate between t and $t+1$ and $q_{-1} = 0$.
- v_t the lapse rate between t and $t+1$ and $v_{-1} = 0$.

[11] See Armel and Planchet [2019].

over most of the projection time horizon and that their duration is longer, as shown in *Table 6*.

Table 6: Flow durations and lapses

Duration of guaranteed cash-flows in years	*Calibration on ATM Caps*			*Calibration on ATM Swaptions*		
Shift of Black model for CIR++ calibration	*0.4%*	*1.0%*	*2.0%*	*0.4%*	*1.0%*	*2.0%*
Under Q	*8.2*	*8.2*	*8.1*	*8.2*	*8.2*	*8.4*
Under P	*9,0*	*9.1*	*9.3*	*9.2*	*9.2*	*9.0*
Gap	*0.8*	*1.0*	*1.1*	*1.0*	*0.9*	*0.5*

We deduce that dynamic lapses are more important when liabilities' valuation is carried out under the risk-neutral probability.

In addition, the analysis of the variation in FDB flows involves studying variations in exit probabilities and revaluation rates, in particular the discretionary part.

If the drop in exit probabilities α_t naturally implies a drop in *FDB* flows, it is difficult to have a more detailed analysis of the movements in discretionary revaluation rates given the large number of parameters involved in their evaluation.

Furthermore, under the historical probability measure, at each projection step, the financial income resulting from the insurer's assets accounting management policy implemented in the model, is different from that recorded under the risk neutral probability. The insurer's buy and sale transactions are in fact different, the resulting asset is different and accounting reserves related to the asset are different.

Also, the decrease in the number of scenarios where policyholders' reaction is more pronounced, under the historical probability, means that the revaluation algorithm (see appendix 7) is less constrained to distribute a surplus of available wealth to reduce dynamic lapses.

Finally, the decrease in lapses allows the insurer to improve its margin. Having fewer lapses, the reserve's loadings base is greater, which results in a transfer of a part of the wealth to its own funds.

4 Conclusion

In this article, we propose an approach of building an economic scenario generator under historical probability, allowing the diffusion of interest rates and prices of risky investments (in equities and real estate). This ESG is adapted to the process of valuing the liabilities of savings contracts with profit-sharing clauses and is consistent with the normative Solvency 2 standards.

This paper proposes methods for calibrating models and risk premiums based on closed formulas and presents simulation approaches with exact discretization that are optimal and adapted to long-term simulation needs.

It assesses also the impact of the choice of an economic scenario generator under the historical probability measure, whose interest rate model is the CIR++ model, on the best-estimate of with profit French savings contracts.

The best-estimate gaps observed between the deflator approach and the risk-neutral assessment are explained by the reaction functions implemented in the model that reflect the actions of the insurer and the policyholder. The justification of the behaviour of these functions is delicate under the risk-neutral probability measure and a deflator approach seems more appropriate.

In a normative valuation framework, the deflator approach presented in this article also leads to a clear separation between the determination of the cost of options, included in the deflator, and the production of contract cash-flows. It also allows to meet regulatory requirements.

Finally, the work presented in this article shows that the deflator approach is operational for insurers and is not limited to an academic style exercise. It also shows that moving from a "risk neutral" calculation to a deflator approach requires only relatively marginal work to adapt existing models.

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