Changes of Numeraire for Pricing Futures, Forwards, and Options

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A change of numeraire argument is used to derive a general option parity, or equivalence, result relating American call and put prices, and to obtain new expressions for futures and forward prices. The general parity result unifies and extends a number of existing results. The new futures and forward pricing formulas are often simpler to compute in multifactor models than existing alternatives. We also extend previous work by deriving a general formula relating exchange options to ordinary call options. A number of applications to diffusion models, including stochastic volatility, stochastic interest rate, and stochastic dividend rate models, and jump-diffusion models are examined.

A self-financing portfolio is called a numeraire if security prices, measured in units of this portfolio, admit an equivalent martingale measure. The most commonly used numeraire is the reinvested short-rate process; the corresponding equivalent martingale measure is the risk-neutral measure. Geman, El Karoui, and Rochet (1995) show that other numeraires can simplify many asset pricing problems. In this article, we build on their results and, using the reinvested asset price as the numeraire, unify and extend the literature on option parity, or equivalence, results relating American call and put prices for asset and futures options. The same numeraire change is used to obtain new pricing formulas for futures and forwards that are often simpler to compute in multifactor models. Finally, we use a numeraire change to simplify exchange option pricing, extending a similar result in Geman, El Karoui, and Rochet to dividend-paying assets.

The change of numeraire method is most intuitive in the context of foreign currency derivative securities. As discussed by Grabbe (1983), an American call option to buy 1 DM, with dollar price process S, for K dollars is equivalent to an American put option to sell K dollars, with DM price process K/S, for a strike price of 1 DM. The dollar price of the call must therefore equal the product of the current exchange rate, S_0, and the DM price of the put. The call price is computed using the dollar value of a U.S.

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money market account as the numeraire, while the put price is computed using the dollar value of a German money market account as the numeraire. Corresponding to the change of numeraire is a change in probability measure, from the risk-neutral measure for dollar-denominated assets to the risk-neutral measure for DM-denominated assets.

As suggested in Grabbe (1983), and developed in later articles, an analogous relation applies to any asset option. A call option to buy one unit of an asset, with dollar price process S, for K dollars is the same as a put option to sell K dollars, worth K/S units of asset, for one unit of asset. Multiplying the asset denominated put price by the current asset price converts the price into dollars.

The same numeraire change can be used to obtain the interest parity theorem which expresses the time zero dollar forward price, $G_0(T)$, for time T delivery of one DM as the spot currency rate times the ratio of two discount bond prices:

$$G_0(T) = S_0\tilde{B}_0(T)/B_0(T),$$

where $B_0(T)$ is the time zero dollar price of a discount bond paying \$1 at T, and $\tilde{B}_0(T)$ is the time zero DM price of a discount bond paying 1 DM at T. This result can be extended to forward contracts on any asset.

A key issue examined in this article is the change of measure that corresponds to a change of numeraire. Under the risk-neutral measure, the drift rate of the returns of the asset price S is the short rate minus the dividend rate. In Section 1 we show that the drift rate of the returns of S^{-1} (the price of dollars in units of asset) under the new measure is the dividend rate minus the short rate. The reversal of the roles of the short rate and dividend rate is intuitive because under the new numeraire the asset is riskless while dollars are risky. Example 1 shows that the change of measure can result in more subtle modifications and can change both the intensity and distribution of jumps in jump-diffusion models. In Section 2 we show that the measure change also alters the drift terms of nonprice state variables, such as in stochastic volatility and stochastic interest rate models.

Example 1. *Assume that the short rate and dividend rate are both zero, and the asset price follows a Poisson jump process with intensity λ under the risk-neutral probability measure, Q. At jump time τ_i, $i = 1, 2, \ldots,$ the stock price ratio has the Bernoulli distribution*

$$S(\tau_i) = \begin{cases} uS(\tau_i-), & \text{with } Q\text{-probability } p \\ dS(\tau_i-), & \text{with } Q\text{-probability } 1-p, \end{cases}$$

and between jumps,

$$dS_t/S_t = (1-\mu)\lambda\, dt, \quad \tau_i < t < \tau_{i+1}, \quad i = 0, 1, \ldots,$$

where $\mu \equiv pu + (1-p)d$ is the expected price ratio at jumps, and $\tau_0 \equiv 0$.

A change of numeraire to the underlying asset price is associated with the new measure $\tilde{Q}$, where $d\tilde{Q}/dQ = S_T/S_0$. At jumps, the value of a dollar measured in units of the asset satisfies[1]

$$S(\tau_i)^{-1} = \begin{cases} u^{-1}S(\tau_i-)^{-1}, & \text{with } \tilde{Q}\text{-probability } pu\mu^{-1} \\ d^{-1}S(\tau_i-)^{-1}, & \text{with } \tilde{Q}\text{-probability } (1-p)d\mu^{-1}, \end{cases}$$

and between jumps

$$dS(t)^{-1}/S(t)^{-1} = (\mu - 1)\lambda\, dt, \quad \tau_i < t < \tau_{i+1}, \quad i = 0, 1, \ldots.$$

The intensity of the jump process under $\tilde{Q}$ is $\mu\lambda$, which can be obtained using the martingale property of S under Q:

$$\tilde{Q}(\tau_1 > t) = E_Q(1_{\{\tau_1 > t\}} S_T/S_0) = E_Q(1_{\{\tau_1 > t\}} S_t/S_0) = e^{(1-\mu)\lambda t} Q(\tau_1 > t).$$

The distributions of the returns of S under Q and S^{-1} under $\tilde{Q}$ are identical only in the special case when $u = d^{-1}$ and $\mu = 1$.

We show that subject to some common technical restrictions (Assumptions 1 and 2 below), any American call price formula is the same, after a change of numeraire, to an American put price formula. This result is useful for obtaining prices, derivatives of prices with respect to model parameters, and early exercise boundaries for put option formulas from the properties of the corresponding call option formula. Previous articles derive the correct put-call equivalence formulas only for some special cases. The geometric Brownian motion case (see Example 2 below) is derived in McDonald and Schroder (1990), Bjerksund and Stensland (1993), and, for futures options, in Byun and Kim (1996).[2] Chesney and Gibson (1993) use a change of numeraire to obtain a closed-form European formula for stock-index options when the short rate is stochastic from Jamshidian and Fein's (1990) closed-form European formula for options on assets with a stochastic payout rate. However, the change of measure is incorrect, in part because it neglects to make the appropriate modification to the drift term of the state variable.

[1] The jump probabilities under $\tilde{Q}$ can be verified using the general results in the appendix, or from

$$\tilde{Q}(\{S(\tau_1) = uS(\tau_1-)\} \cap \{\tau_1 \le t\}) = puE_Q(1_{\{\tau_1 \le t\}} S(\tau_1-)/S(0)),$$

for any $t \le T$, and $S(\tau_1)/S(0) = ue^{(1-\mu)\lambda\tau_1}$ on $\{S(\tau_1) = uS(\tau_1-)\}$.

[2] Bjerksund and Stensland (1993) apply a result in Olsen and Stensland (1991) which demonstrates that the current asset price can be factored out in certain control theory problems where the future reward is multiplicative in the price of an asset. Their result could be used to derive the parity result in a diffusion setting when the return volatility is any function of the price, subject to the price process being strictly positive (such as the CEV model below). The Olsen and Stensland (1991) results can be generalized by allowing the payoff in Proposition 1 below to depend on a vector of controls. See also Kholodnyi and Price (1998), who derive equivalence results for geometric Brownian motion and the binomial model. They use no-arbitrage arguments to derive general equivalence results in a setting where each option price is a deterministic function of the current underlying asset price (for example, Markovian S and deterministic r and δ). In the foreign currency context, the equivalence results are in terms of the generators of the domestic and foreign evolution (or present value) operators.

Example 5 below shows the correct measure change in that model. Carr and Chesney (1996) derive a formula relating call and put prices in a one-factor model in which the volatility of the underlying price obeys a symmetry condition (see Example 3 below). Bates (1991) derives equivalence formulas for American put and call options on futures for some special cases to test classes of option pricing models. Example 8 builds on this idea and derives general conditions under which the equivalence formula takes a particularly simple form: switching the roles of the current futures price and the strike price in the American call option formula gives the price of an otherwise identical American put option.

Section 1 presents the numeraire change method and the general results using the reinvested asset price as the numeraire. Section 2 presents examples of these results. The Appendix derives the numeraire change for a general jump-diffusion model that includes all the Section 2 examples as special cases.

1. The Reinvested Asset Price as the Numeraire

We present the general change of numeraire argument before giving the main results. Fix a finite time horizon $[0, T]$.[3] Let Y denote some reinvested asset price process. That is, Y_t is the time t balance of an investment strategy of buying an asset and reinvesting all dividends into new shares. Let R represent the reinvested short rate with unit initial investment: $R_t = \exp(\int_0^t r_s ds)$, where r is the short rate process. If π is the state price density process, then πY and πR are P-martingales. It follows that Y/R is a Q-martingale, where $dQ/dP \equiv \pi_T R_T$. That is, when measured in units of the numeraire R, Y is a martingale with respect to the risk-neutral probability measure Q. Geman, El Karoui, and Rochet (1995) show that we get the same result when we replace R with another self-financing portfolio V with $V_0 = 1$ (and V/R a Q-martingale). Then Y/V is a $\tilde{Q}$-martingale, where $d\tilde{Q}/dP \equiv \pi_T V_T$ (or, equivalently, $d\tilde{Q}/dQ \equiv V_T/R_T$). That is, when measured in units of the numeraire V, Y is a $\tilde{Q}$-martingale. This simple change of numeraire is the basis for all the pricing results below. The results are very general in that we allow for incomplete markets and price and state variable dynamics which are neither continuous nor Markovian. The main assumption is the existence of a risk-neutral measure.

Assumption 1. *There exists a risk-neutral measure, Q, such that every reinvested price process relative to the reinvested short-rate process is a Q-martingale.*

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[3] We assume throughout the existence of a complete probability space $(\Omega, \mathcal{F}, \mathbf{F}, P)$ and a filtration $\mathbf{F} = \{\mathcal{F}_t;\ t \geq 0\}$ which satisfies the usual conditions. See Protter (1992) for the required conditions and for all the results on semimartingale theory needed in this article. All processes are assumed to be adapted.

The self-financing portfolio that serves as the numeraire for our main results is the reinvested asset price process with unit initial balance. Let S be a semimartingale representing the price process of an asset with a proportional dividend payout rate δ.[4] We assume throughout that S is strictly positive.[5] The value of the numeraire portfolio at any time t is $S_t \exp(\int_0^t \delta_s ds)/S_0$. The probability measure $\tilde{Q}$ that corresponds to the new numeraire is

$$\tilde{Q}(A) = E_Q(1_{\{A\}} Z_T), \quad \forall A \in \mathcal{F}. \tag{1}$$

where Z is defined as the ratio of the new and old numeraires:

$$Z_t \equiv e^{\int_0^t (\delta_s - r_s) ds} S_t/S_0, \quad t \in [0, T]. \tag{2}$$

In other words, the Radon-Nikodym derivative is $d\tilde{Q}/dQ = Z_T$.

All the results of this section hold when S is replaced by a futures price process F (with delivery date $D \geq T$) if we set δ equal to r. To justify this, we construct a numeraire portfolio with value $F_t \exp(\int_0^t r_s ds)/F_0$ at any time t by maintaining a long position of $\exp(\int_0^t r_s ds)/F_0$ futures contracts at t and adding or subtracting mark-to-market gains and losses from a money market account, whose time-zero balance is set to \$1 [this strategy is described in Duffie (1992: chap. 7)]. Alternatively, we can use the fact that F is a Q-martingale and directly define Z_t to be F_t/F_0.

The following proposition provides a general pricing formula under a change of numeraire to the reinvested asset price. The constant K will serve as the strike price in the option pricing applications below. The process $\tilde{S}$ represents the price of KS_0 dollars measured in units of the asset S.

Proposition 1. *Define $\tilde{S}_t = KS_0/S_t$ and $\tilde{Q}$ by (1). Then the time-zero price of an asset with the $\mathcal{F}_\tau$-measurable payoff P_τ at the stopping time $\tau \in [0, T]$ is*

$$E_Q\left(e^{-\int_0^\tau r_s ds} P_\tau\right) = E_{\tilde{Q}}\left(e^{-\int_0^\tau \delta_s ds} P_\tau \tilde{S}_\tau/K\right).$$

Furthermore

$$\begin{aligned} dS_t &= S_{t-}(r_t - \delta_t)dt + dM_t \\ d\tilde{S}_t &= \tilde{S}_{t-}(\delta_t - r_t)dt + d\tilde{M}_t, \quad \tilde{S}_0 = K, \end{aligned}$$

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[4] It is easy to extend the results to discrete dividends. In addition to the proportional dividend rate δ, suppose the asset pays discrete cash dividends C_i at stopping times T_i, $i = 1, 2, \ldots$. All the results are then generalized by adding the term $\sum_{T_i \in [0,t]} \log[1 + C_i/S(T_i)]$ to $\int_0^t \delta_s ds$ throughout. The exponential of this additional term represents the additional shares of asset accumulated by reinvesting the discrete dividends into new shares purchased at the ex-dividend price S.

[5] In many applications the results extend to the case where S has an absorbing boundary at zero. In a diffusion model, for example, construct a modified stock price process whose diffusion term is killed the first time the price hits a small positive constant. The dominated convergence theorem can be used to evaluate the limit of the expectation (in Corollary 1, for example) as this constant goes to zero.

where M and $\tilde{M}$ and local martingales under Q and $\tilde{Q}$, respectively.[6] *The quadratic variations of M and $\tilde{M}$ satisfy $(dM_t)^2/S_t^2 = (d\tilde{M}_t)^2/\tilde{S}_t^2$ between jumps.*[7]

Proof. Assumption 1 implies that the price is given by the first expectation. Applying the numeraire change,

$$\begin{aligned} E_Q\left(e^{-\int_0^\tau r_s ds} P_\tau\right) &= E_Q\left(Z_\tau e^{-\int_0^\tau \delta_s ds} P_\tau \tilde{S}_\tau / K\right) \\ &= E_{\tilde{Q}}\left(e^{-\int_0^\tau \delta_s ds} P_\tau \tilde{S}_\tau / K\right), \end{aligned}$$

where the last equality is obtained using iterated expectations and the martingale property of Z. The equation for the returns of S follows because the ratio of the reinvested price process to the reinvested short rate process is a Q-martingale. The equation for the returns of $\tilde{S}$ follows because the ratio of the short rate price process to the reinvested price process is a $\tilde{Q}$-martingale. The equality, between jumps, of the instantaneous volatilities of returns follows from *Itô*'s lemma and from the Girsanov–Meyer theorem (a generalization of Girsanov's theorem to a non-Brownian setting), which implies that $M_t - \tilde{M}_t$ is absolutely continuous in t between jumps. ■

The proposition shows that the instantaneous return variances of S and $\tilde{S}$ are identical between jumps. Example 1 illustrates that at jumps, the squared returns will generally be different.

The first application of Proposition 1 relates call prices to put prices under a change of numeraire.

Corollary 1. *Define $\tilde{S}_t = KS_0/S_t$ and $\tilde{Q}$ by Equation (1). Then the value of a call option on S is the same, after a change of numeraire, as the value of a put option on $\tilde{S}$:*

$$E_Q\left(e^{-\int_0^\tau r_s ds} \max[S_\tau - K,\ 0]\right) = E_{\tilde{Q}}\left(e^{-\int_0^\tau \delta_s ds} \max[S_0 - \tilde{S}_\tau, 0]\right),$$

for any stopping time $\tau \leq T$.

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[6] A sufficient condition for M and $\tilde{M}$ to be martingales under Q and $\tilde{Q}$, respectively, is that r and δ are bounded processes.

[7] The quadratic variation of any semimartingale Y is denoted by $[Y, Y]$, and can be decomposed into its continuous and jump components:

$$[Y, Y]_t = [Y, Y]_t^c + \sum_{0 \leq s \leq t} (\Delta Y_s)^2,$$

where $[Y, Y]^c = [Y^c, Y^c]$ and Y^c is the path-by-path continuous part of Y. For continuous Y (or for t between jumps), it is common to write $(dY_t)^2$ instead of $d[Y, Y]_t$. The quadratic variation is invariant to changes in measure. See Protter (1992: chap. II) for the formal definitions.

The left-hand side represents the value of a European call option expiring at τ with strike price K and underlying price process S. The right-hand side represents the value of a European put option, also expiring at τ, but with a strike price S_0 and underlying price process $\tilde{S}$. The roles of the short rate and asset payout rate are reversed in the call and put price expressions. Corollary 1 also holds for American options under Assumption 2 below.

Corollary 2. *Define $\tilde{S}_t = KS_0/S_t$ and $\tilde{Q}$ by Equation (1). Then the value of an asset-or-nothing binary option on S is the same, after a change of numeraire, as the value of a cash-or-nothing binary option on $\tilde{S}$:*

$$E_Q\left(e^{-\int_0^\tau r_s ds} S_\tau 1_{\{S_\tau \geq K\}}\right) = S_0 E_{\tilde{Q}}\left(e^{-\int_0^\tau \delta_s ds} 1_{\{S_0 \geq \tilde{S}_\tau\}}\right),$$

for any stopping time $\tau \leq T$.

Another interpretation is obtained if $\tau \equiv \min[T, \inf\{t\colon\ S_t \geq K\}]$ and S has continuous sample paths. Then the left-hand side is the value of a barrier, or first-touch digital, option paying K dollars when the asset price S rises to K; and the right-hand side is the value of a barrier option paying S_0 when $\tilde{S}$ falls to S_0 (the events $\{S_\tau \geq K\}$ and $\{S_0 \geq \tilde{S}_\tau\}$ are identical).[8]

When $\delta \equiv 0$, Corollary 2 can be obtained from Theorem 2 in Geman, El Karoui, and Rochet (1995). The result is derived independently under the assumptions of geometric Brownian motion and constant r and δ by Carr (1993), Dufresne, Keirstead, and Ross (1997), and Ingersoll (1997).

When r and δ are deterministic (extentions to the stochastic case are straightforward), Corollary 2 shows that any European option price can be derived from the probabilities $Q(S_T \geq K)$ and $\tilde{Q}(S_0 \geq \tilde{S}_T)$ [see also Theorem 2 in Geman, El Karoui, and Rochet (1995)].

The next corollary presents a new futures price expression. Let $F_0(T)$ denote the time-zero futures price for delivery of asset S at time T. With continuous marking to market, the futures price equals the risk-neutral expectation of the spot price at delivery [see Duffie (1992: chap. 7)]:

$$F_0(T) = E_Q(S_T). \tag{3}$$

When the interest rate and the payout rate are deterministic, the futures price is simply $F_0(T) = S_0 \exp[\int_0^T (r_s - \delta_s) ds]$. When either the interest rate or the payout rate is stochastic, however, a change of measure to $\tilde{Q}$ gives an expression that is often easier to compute and also more clearly emphasizes the role of the cost of carry in futures pricing.

[8] Reiner and Rubinstein (1991) price a variety of binary and one-sided barrier options assuming the asset price follows geometric Brownian motion.

Corollary 3. *The futures price is the product of the spot price and the expectation, under $\tilde{Q}$, of the exponential of the cost of carry:*

$$F_0(T) = S_0 E_{\tilde{Q}} \left(e^{\int_0^T (r_s - \delta_s) ds} \right).$$

In the general diffusion model in the appendix, for example, the $\tilde{Q}$-expectation on the right-hand side doesn't depend on the stock price process if the instantaneous covariance between asset returns and changes in the state variable is not a function of the asset price. Example 5 shows that the computation of the futures price using Corollary 3 is particularly simple with a constant volatility stock return process and an Ornstein–Uhlenbeck state variable driving either r or δ.

The next corollary presents a new forward price expression. Let $B_t(T)$ denote the time t dollar price of a discount bond paying \$1 at T:

$$B_t(T) = E_Q \left(e^{-\int_t^T r_s ds} \;\middle|\; \mathcal{F}_t \right), \quad t \leq T. \tag{4a}$$

Let $\tilde{B}_t(T)$ denote the time t price, measured in units of asset, of a "discount bond" paying one unit of the asset at T:

$$\tilde{B}_t(T) = S_t^{-1} E_Q \left(e^{-\int_t^T r_s ds} S_T \;\middle|\; \mathcal{F}_t \right) = E_{\tilde{Q}} \left(e^{-\int_t^T \delta_s ds} \;\middle|\; \mathcal{F}_t \right), \quad t \leq T. \tag{4b}$$

Letting $G_0(T)$ denote the time-zero forward price for delivery of asset S at time T, Duffie (1992: chap. 7) shows that

$$G_0(T) = E_Q \left(e^{-\int_0^T r_s ds} S_T \right) \Big/ B_0(T). \tag{5}$$

Corollary 4 follows from Equations (4b) and (5).

Corollary 4. *The forward price is given by the product of the spot price and the ratio of asset and dollar denominated discount bond prices:*

$$G_0(T) = S_0 \tilde{B}_0(T) / B_0(T),$$

where $\tilde{B}_0(T)$ and $B_0(T)$ are defined by Equation (4).

The main advantage of Corollary 4 is in a model where both the short rate and payout rate are stochastic. If the short rate is deterministic, then forward and futures prices are equal and Corollary 3 can be used. If the payout rate is deterministic, then Corollary 4 holds trivially. When we set $\delta \equiv r$ and reinterpret S as a futures price with delivery date T (which implies that the forward on the futures contract is equivalent to a forward on the asset underlying the futures contract), then Corollary 4 provides a simple expression for the ratio of forward and futures prices on the same underlying asset.

The final application of Proposition 1 is to the valuation of exchange options. Let S^a and S^b denote two asset prices and δ^a and δ^b denote their corresponding payout rates. Then

$$dS_t^i = S_{t-}^i(r_t - \delta_t^i)dt + dM_t^i, \quad i \in \{a, b\},$$

where M^i is a Q-local martingale. Corollary 5 expresses the value of an exchange option as an ordinary call option by changing the numeraire to the reinvested price of asset a.

Corollary 5. *Define $\tilde{S}_t^b = S_t^b S_0^a / S_t^a$ and $d\tilde{Q}/dQ = e^{\int_0^T (\delta_s^a - r_s)ds} S_T^a / S_0^a$. Then the value of an option to receive one unit of asset b in exchange for one unit of asset a is the same, after a change of numeraire, as the value of a call option on $\tilde{S}^b$:*

$$E_Q\left(e^{-\int_0^\tau r_s ds}\max[S_\tau^b - S_\tau^a, 0]\right) = E_{\tilde{Q}}\left(e^{-\int_0^\tau \delta_s^a ds}\max[\tilde{S}_\tau^b - S_0^a, 0]\right),$$

for any stopping time $\tau \le T$. Furthermore

$$d\tilde{S}_t^b = \tilde{S}_{t-}^b(\delta_t^a - \delta_t^b)dt + d\tilde{M}_t^b, \quad \tilde{S}_0^b = S_0^b,$$

where $\tilde{M}^b$ is a local martingale under $\tilde{Q}$.

The right-hand side of the first equation is the value of an ordinary call option with underlying asset process $\tilde{S}^b$, short rate process δ^a, and fixed strike price S_0^a. Corollary 5 extends a similar result in Geman, El Karoui, and Rochet (1995) to dividend-paying assets and American-style exercise (under Assumption 2).

To apply Proposition 1 to American options, we need to assume that the price of an American option is the supremum, over all stopping times τ, of the risk-neutral expected discounted payoff from exercising at τ.

Assumption 2. *Let p be the time zero price of an American option allowing the holder to exercise and receive, at any stopping time $\tau \in [0, T]$, the payoff P_τ, where P is an adapted process. Then*

$$p = \sup_{\tau \in [0,T]} E_Q\left(e^{-\int_0^\tau r_s ds} P_\tau\right) \tag{6}$$

Karatzas (1988) proves Equation (6) in a complete markets diffusion setting for American options on assets. When markets are incomplete, this characterization is problematic [see Duffie (1992: chap. 2)]. Because of possible interaction between the state price density and the choice of exercise policy, the two-step procedure of first determining the risk-neutral measure and then computing Equation (6) may not be valid. Nevertheless, it is common in the literature to ignore this interaction and first assign a market price of risk to the relevant state variables (in effect, determining the risk-neutral measure), then price options as in Equation (6).

2. Examples

The examples in this section are all special cases of the general jump-diffusion model presented in the appendix. Throughout the remainder of the article, I let $W \equiv [W^1, \ldots, W^d]'$ and $\tilde{W} \equiv [\tilde{W}^1, \ldots, \tilde{W}^d]'$ be vectors of d independent standard Brownian motions under the measures Q and $\tilde{Q}$, respectively.

Example 2. Constant elasticity of variance (CEV). *The risk-neutral asset price process is*

$$\frac{dS_t}{S_t} = (r_t - \delta_t)\,dt + \nu S_t^{\xi}\,dW_t^1, \quad \xi \in [-1, 1],$$

where ν and ξ are constants, and r and δ are deterministic.[9] *Geometric Brownian motion corresponds to $\xi = 0$. Closed-form solutions for European call and put options in this model have been derived by Cox (1975) [see also Schroder (1989)]. Under the measure $\tilde{Q}$, $\tilde{S}$ is also a CEV process:*

$$\frac{d\tilde{S}_t}{\tilde{S}_t} = (\delta_t - r_t)\,dt + \tilde{\nu}\tilde{S}_t^{\tilde{\xi}}\,d\tilde{W}_t^1, \quad \tilde{S}_0 = K,$$

with an absorbing boundary at zero (see Footnote 7), where $\tilde{\nu} \equiv \nu(KS_0)^{\xi}$ and $\tilde{\xi} \equiv -\xi$. Using Corollary 1, we obtain the pricing formula for the American put from the formula for the American call by exchanging S_0 and K, exchanging r and δ, and replacing ν with $\tilde{\nu}$ and ξ with $\tilde{\xi}$. For the case of geometric Brownian motion, the equivalence formula is particularly simple because $\nu = \tilde{\nu}$ and $\xi = \tilde{\xi}$.

The next example shows that the Carr and Chesney (1996) put-call symmetry result can be obtained from Corollary 1.

Example 3. Carr and Chesney (1996). *Let the risk-neutral asset price process satisfy*

$$\frac{dS_t}{S_t} = (r_t - \delta_t)\,dt + \sigma(S_t)\,dW_t^1,$$

where r and δ are deterministic and $\sigma(\cdot) \equiv f(|\log(\cdot/\sqrt{yK})|)$ for some bounded function f and fixed $y \in \mathbb{R}_+$. The functional form of σ satisfies

[9] The results in this and all the succeeding examples are unchanged if any of the constant parameters are permitted to be deterministic functions of time.

Carr and Chesney's symmetry condition which ensures that $\sigma(S_t) = \sigma(\hat{S}_t)$, $t \geq 0$, where $\hat{S}_t \equiv yK/S_t$.[10] *The dynamics of $\hat{S}$ under $\tilde{Q}$ are therefore*

$$\frac{d\hat{S}_t}{\hat{S}_t} = (\delta_t - r_t)\,dt + \sigma(\hat{S}_t)\,d\tilde{W}_t^1, \quad \hat{S}_0 = yK/S_0.$$

When S represents a futures price process (and $\delta = r$), the return distributions of S and $\hat{S}$ are identical. Applying Corollary 1 and rearranging, we obtain

$$\frac{E_Q\left(e^{-\int_0^\tau r_s ds}\max[S_\tau - K,\ 0]\right)}{\sqrt{S_0 K}} = \frac{E_{\tilde{Q}}\left(e^{-\int_0^\tau \delta_s ds}\max[y - \hat{S}_\tau, 0]\right)}{\sqrt{\hat{S}_0 y}}.$$

The numerator on the left-hand side is the price of a call option on S with strike price K. The numerator on the right-hand side is the price of a put option on $\hat{S}$ with strike price y, and with the roles of r and δ switched. These call and put options have the same "moneyness" in the sense that $\hat{S}_0/y = K/S_0$. For the case of geometric Brownian motion, where f is a constant function, we let $y \equiv S_0$ to reconcile the result with Example 2.

Example 4. Stochastic volatility model of Heston (1991). *The risk-neutral asset price and volatility processes are*

$$\begin{aligned}\frac{dS_t}{S_t} &= (r_t - \delta_t)\,dt + \sqrt{\nu_t}\,dW_t^1,\\ d\nu_t &= (\mu - \kappa\nu_t)dt + \psi\sqrt{\nu_t}\,(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2),\end{aligned}$$

where r and δ are deterministic, and μ, κ, ψ and $\rho \in [-1, 1]$ are constants. Recall that W^1 and W^2 are independent standard Brownian motions under Q, and therefore $\rho W^1 + \sqrt{1-\rho^2}W^2$ is standard Brownian motion with instantaneous correlation of ρ with W^1.

Under the numeraire change,

$$\begin{aligned}\frac{d\tilde{S}_t}{\tilde{S}_t} &= (\delta_t - r_t)\,dt + \sqrt{\nu_t}\,d\tilde{W}_t^1, \quad \tilde{S}_0 = K,\\ d\nu_t &= (\mu - [\kappa - \rho\psi]\nu_t)dt - \psi\sqrt{\nu_t}\,(\rho d\tilde{W}_t^1 + \sqrt{1-\rho^2}d\tilde{W}_t^2).\end{aligned}$$

The numeraire change results in a modification to the mean reversion parameter of the volatility process, and reverses the sign of the covariance between instantaneous asset returns and volatility changes.

[10] Proposition 1 implies that the time t instantaneous return volatility of $\hat{S}$ is $\sigma(S_t) = \sigma(yK/\hat{S}_t)$. The functional form of σ implies $\sigma(yK/\hat{S}_t) = \sigma(\hat{S}_t)$.

The next example includes a one-dimensional stochastic state variable driving either the short rate or the dividend rate.

Example 5. Stochastic dividend and stochastic short rate models. *Let the risk-neutral asset price and one-dimensional state variable satisfy*

$$\frac{dS_t}{S_t} = (r_t - \delta_t)\,dt + \sigma\,dW_t^1,$$

$$dX_t = (\mu - \kappa X_t)dt + \nu\,(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2),$$

where the coefficients σ, μ, κ, ν, and $\rho \in [-1, 1]$ are constants. Under the numeraire change,

$$\frac{d\tilde{S}_t}{\tilde{S}_t} = (\delta_t - r_t)\,dt + \sigma\,d\tilde{W}_t^1, \quad \tilde{S}_0 = K,$$

$$dX_t = (\mu + \rho\sigma\nu - \kappa X_t)dt - \nu\,(\rho d\tilde{W}_t^1 + \sqrt{1-\rho^2}d\tilde{W}_t^2).$$

The state variable still is Ornstein–Uhlenbeck under $\tilde{Q}$, but with a different drift parameter.

(a) Stochastic short rate. Let δ be deterministic and $r_t = f(X_t)$ for some $f\colon \mathbb{R} \to \mathbb{R}$. This model includes Ornstein–Uhlenbeck ($f(x) = x, \forall x \in \mathbb{R}$) and lognormal ($f(x) = e^x$) short rate processes. From Corollary 3, the price of a futures contract on S for delivery at T is

$$F_0(T) = S_0 \exp\left(-\int_0^T \delta_s ds\right) E_{\tilde{Q}}\left[\exp\left(\int_0^T r_s ds\right)\right].$$

This is simpler than evaluating the standard Equation (3):

$$F_0(T) = E_Q(S_T) = S_0 \exp\left(-\int_0^T \delta_s ds\right) \times E_Q\left(\exp\left[\int_0^T r_s ds - \frac{1}{2}\sigma^2 T + \sigma W_T^1\right]\right).$$

Corollary 1 implies that the price of a call option on S is equal, after the numeraire change, to the price of a put option on $\tilde{S}$ with a deterministic short rate and a stochastic dividend rate.

(b) Stochastic dividend rate. Let r be deterministic and $\delta_t = f(X_t)$. We obtain the futures and forward prices from either Corollary 3 or Corollary 4:

$$F_0(T) = G_0(T) = S_0 \exp\left(\int_0^T r_s ds\right) E_{\tilde{Q}}\left[\exp\left(-\int_0^T \delta_s ds\right)\right].$$

Again, this is simpler than evaluating the standard Equation (3):

$$F_0(T) = S_0 \exp\left(\int_0^T r_s ds\right) E_Q\left(\exp\left[-\int_0^T \delta_s ds - \frac{1}{2}\sigma^2 T + \sigma W_T^1\right]\right).$$

Corollary 1 implies that a call option on S is equal, after the numeraire change, to the price of a put option on $\tilde{S}$ with a deterministic dividend rate and a stochastic short rate.

Example 5 shows that option pricing models for stochastic dividend models can be obtained from stochastic interest rate models and vice versa. The European call option formula in Jamshidian and Fein's (1990) Ornstein–Uhlenbeck δ and constant r model can be obtained, via some parameter changes, from the European put option formula in Rabinovitch's (1989) Ornstein–Uhlenbeck r and constant δ model. The example also illustrates how Corollary 3 can simplify futures pricing by reducing a two-factor problem to a one-factor problem when the volatility terms of the asset returns and the state variable do not depend on the asset price.

Example 6. Exchange options. *The risk-neutral price processes of assets a and b satisfy*

$$\begin{aligned}\frac{dS_t^a}{S_t^a} &= (r_t - \delta_t^a)\, dt + \sigma^a\, dW_t^1,\\ \frac{dS_t^b}{S_t^b} &= (r_t - \delta_t^b)\, dt + \sigma^b\, (\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2),\end{aligned}$$

where the volatility coefficients, σ^a and σ^b, and the instantaneous correlation between asset returns, $\rho \in [-1, 1]$, are constants. The short rate, r, and the dividend rates, δ^a and δ^b, are functions of the m-dimensional state variable vector, X, which satisfies

$$dX_t = \mu(X_t)dt + \phi(X_t)dW_t,$$

where μ is $m \times 1$, ϕ is $m \times d$, and again $W \equiv [W^1, \ldots, W^d]'$ is standard Brownian motion under Q.

The price ratio $\tilde{S}_t^b \equiv S_t^b S_0^a / S_t^a$ under $\tilde{Q}$ is the constant volatility process

$$d\tilde{S}_t^b / \tilde{S}_t^b = (\delta_t^a - \delta_t^b)\, dt + (\sigma^a - \rho\sigma^b) d\tilde{W}_t^1 - \sqrt{1-\rho^2}\sigma^b d\tilde{W}_t^2, \quad \tilde{S}_0^b = S_0^b,$$

where again $\tilde{W} \equiv [\tilde{W}^1, \ldots, \tilde{W}^d]'$ is standard Brownian motion under $\tilde{Q}$. The drift of X under $\tilde{Q}$ is modified by adding the product of the volatility of asset a and the first column of the state-variable volatility matrix:

$$dX_t = [\mu(X_t) + \sigma^a \phi(X_t)\mathbf{e}]dt - \phi(X_t)d\tilde{W}_t,$$

where $\mathbf{e} = [1, 0, \ldots, 0]'$. *From Corollary 5, the valuation of an exchange option can be reduced to the computation of an ordinary call option on* $\tilde{S}$ *with short rate* δ^a *and dividend rate* δ^b.

The next example is the jump-diffusion model of Merton (1976).

Example 7. Merton (1976). *As in Example 1, the asset price follows a Poisson jump process with intensity* λ *under the risk-neutral probability measure, Q. At jump time* τ_i , $i = 1, 2, \ldots$, *the stock price ratio is lognormally distributed:*

$$\log[S(\tau_i)/S(\tau_i-)] \sim \Theta(\alpha, \gamma^2),$$

where $\Theta(m, v)$ *denotes a normal distribution with mean m and variance* v. *Between jumps the stock price satisfies*

$$\begin{aligned}\frac{dS_t}{S_{t-}} &= [r_t - \delta_t - \lambda(e^{\alpha+\gamma^2/2} - 1)]\,dt\\ &\quad + \sigma\, dW_t^1, \quad \tau_i < t < \tau_{i+1}, \quad i = 0, 1, \ldots,\end{aligned}$$

where $\tau_0 \equiv 0$.

Using the same calculation as in Example 1 (or the general results in the appendix), the intensity under $\tilde{Q}$ *is equal to the product of the intensity under Q and the expected price ratio at jumps:* $\tilde{\lambda} = \lambda \exp(\alpha + \frac{1}{2}\gamma^2)$. *The appendix shows that the distribution functions under Q and* $\tilde{Q}$ *of the stock price ratio, denoted by* $\Psi(\cdot)$ *and* $\tilde{\Psi}(\cdot)$, *respectively, satisfy*

$$\tilde{\Psi}(dy) = \Psi(dy)\exp(-\alpha - \frac{1}{2}\gamma^2)y.$$

From

$$\Psi(dy) = \frac{1}{\sqrt{2\pi}\gamma y}\exp\left[-\frac{1}{2}\left(\frac{\log(y) - \alpha}{\gamma}\right)^2\right]dy,$$

it is straightforward to show that the logarithm of the stock price ratio under $\tilde{Q}$ *is still normally distributed with variance* γ^2, *but with mean* $\alpha + \gamma^2$. *The dynamics of* $\tilde{S}_t \equiv KS_0/S_t$ *are therefore*

$$\log[\tilde{S}(\tau_i)/\tilde{S}(\tau_i-)] \sim \Theta(-\alpha - \gamma^2, \gamma^2), \quad i = 1, 2, \ldots,$$

$$\begin{aligned}\frac{d\tilde{S}_t}{\tilde{S}_{t-}} &= \left[\delta_t - r_t - \tilde{\lambda}(e^{-\alpha-\gamma^2/2} - 1)\right]dt\\ &\quad + \sigma\, d\tilde{W}_t^1, \quad \tau_i < t < \tau_{i+1}, \quad i = 0, 1, \ldots.\end{aligned}$$

When the mean stock return at jumps is zero, that is $\alpha = -\gamma^2/2$, *then* $\lambda = \tilde{\lambda}$ *and the jump returns of S under Q and* $\tilde{S}$ *under* $\tilde{Q}$ *have the same* *distributions. In this special case, Corollary 1 implies that simply switching the roles of the strike price and the current asset price, and switching the roles of the short rate and dividend rate in the American call price formula gives the American put price. Bates (1991) derives this special case for futures options from the partial differential equation for the option price.*

When applied to the case of futures options, several previous examples contain special cases in which the distributions of the returns of F under Q and $\tilde{F}$ under $\tilde{Q}$ are identical.[11] In such cases the equivalence relationship in Corollary 1 takes a particularly simple form: the American put price is obtained from the American call price formula by simply switching the roles of the strike price and current futures price. A change of variables can then be used to relate American call and put prices on the same underlying futures price process. Example 8 shows general conditions under which Corollary 1 can be used to relate call and put prices on the same underlying futures price process, and, using the ideas of Bates (1991), shows how these conditions can be tested. It is easy to show that the same conditions imply that the geometric average of the early exercise boundaries of otherwise identical American calls and puts is equal to the strike price.

Example 8. Empirical implications for futures options. *Let F denote the futures price for delivery at D, where $D \geq T$. The futures price is assumed to follow a jump-diffusion process under Q with intensity $\lambda(X_t)$ and jumps at τ_i, $i = 1, 2, \ldots,$ when the future price ratio has the distribution*

$$Q\left(F(\tau_i)/F(\tau_i-) \leq y\right) = \Psi(y), \quad y \geq 0,$$

and between jumps,

$$\frac{dF_t}{F_{t-}} = \lambda_t(X_t)(1-\mu)\,dt + \sigma(X_t)\,dW_t^1, \quad \tau_i < t < \tau_{i+1}, \quad i = 0, 1, \ldots,$$

where $\tau_0 \equiv 0$ and $\mu \equiv \int_{y \in \mathbb{R}_+} y d\Psi(y)$ is the expected price ratio at jumps. The m-dimensional state variable vector, X, satisfies

$$dX_t = \mu(X_t)dt + \phi(X_t)dW_t,$$

where $W \equiv [W^1, \ldots, W^d]'$ is standard Brownian motion under Q, and the coefficients μ and ϕ have the appropriate dimensions. The short rate may also be a function of X.

We obtain an equivalence formula for calls and puts with the same time zero underlying price by defining $\hat{F}_t = F_0^2/F_t$ $(= \tilde{F}_t F_0/K)$. Using the results in the appendix, $\hat{F}$ is a jump-diffusion process under $\tilde{Q}$ with intensity

[11] See Example 1 when $u = d^{-1}$ and $\mu = 1$; Example 2 when $\xi = 0$; Examples 4 and 5 when $\rho = 0$; and Example 7 when $\alpha = -\sigma^2/2$.

$\tilde{\lambda}_t(\cdot) = \lambda_t(\cdot)\mu$ *and dynamics*

$$\tilde{Q}\left(\hat{F}(\tau_i)/\hat{F}(\tau_i-) \geq y^{-1}\right) = \tilde{\Psi}(y), \quad y > 0, \quad i = 1, 2, \ldots,$$

$$\frac{d\hat{F}_t}{\hat{F}_{t-}} = \tilde{\lambda}_t(X_t)(1-\mu^{-1})\,dt + \sigma(X_t)\,d\tilde{W}_t^1, \quad \tau_i < t < \tau_{i+1}, \quad i = 0, 1, \ldots, \quad \hat{F}_0 = F_0,$$

$$dX_t = [\mu(X_t) + \sigma(X_t)\phi(X_t)\mathbf{e}]dt - \phi(X_t)d\tilde{W}_t,$$

where $\tilde{\Psi}(dy) = \Psi(dy)y\mu^{-1}$*, and, as earlier,* $\mathbf{e} \equiv [1, 0, \ldots, 0]'$ *and* $\tilde{W} \equiv [\tilde{W}^1, \ldots, \tilde{W}^d]'$ *is standard Brownian motion under* $\tilde{Q}$*.*

Sufficient conditions for the distributions of F *under* Q *and* $\hat{F}$ *under* $\tilde{Q}$ *to be identical are (a)* $\phi(\cdot)\mathbf{e} = 0$ *(the instantaneous changes in the state variables and futures price are uncorrelated), and (b)* $\lambda \equiv 0$ *(no jumps) or* $\Psi(y) = \int_{[y^{-1},\infty)} u\Psi(du)$*,* $\forall y > 0$*. Note that the restriction on the distribution function in (b) implies that* $\mu = 1$*. When the futures price ratio at jumps has a discrete distribution, as in Example 1, then the restriction on* Ψ *is equivalent to* $\Delta\Psi(y) = y^{-1}\Delta\Psi(y^{-1})$*,* $\forall y > 0$ *[*$\Delta\Psi(y)$ *and* $\Delta\Psi(y^{-1})$ *are the Q-probabilities of outcomes y and* y^{-1}*, respectively]. When the jump distribution function is differentiable, as in Example 7, then the restriction is equivalent to* $\Psi'(y) = \Psi'(y^{-1})y^{-3}$*,* $\forall y > 0$*.*

Defining $x = K/F_0$*, then Corollary 1 implies the following relationship between calls and puts on futures prices with the same initial value:*

$$E_Q\left(e^{-\int_0^\tau r_s ds}\max[F_\tau - F_0 x,\ 0]\right) = xE_{\tilde{Q}}\left(e^{-\int_0^\tau r_s ds}\max[F_0/x - \hat{F}_\tau, 0]\right), \tag{7}$$

for any $x > 0$*. Under conditions (a) and (b), Equation (7) relates call and put prices on the same underlying futures price process. We can test these conditions by comparing the relative prices of American calls and puts. For example, if both conditions hold, then otherwise identical at-the-money calls and puts should be priced the same.*

Bates (1991) proves Equation (7), using partial differential equation methods, for the cases of geometric Brownian motion, Merton's (1976) jump-diffusion model with zero-mean jump returns ($\alpha = -\gamma^2/2$ in Example 7 above), and for the case of a diffusion stock price process and an uncorrelated one-dimensional state variable representing stochastic volatility.

Appendix: Price and State-Variable Dynamics

This appendix derives price and state-variable processes under a change of numeraire and corresponding change of measure for a general class of diffusion and jump-diffusion processes.

Let $W \equiv [W^1, \ldots, W^d]'$ be a vector of d independent standard Brownian motions under the risk-neutral measure Q. The asset price, S, and the state variables, $X \equiv [X^1, \ldots, X^m]'$, satisfy

$$\begin{aligned} \frac{dS_t}{S_t} &= (r_t - \delta_t)\,dt + \sigma(S_t, X_t)\,dW_t^1, \\ dX_t &= \mu(X_t)dt + \phi(X_t)dW_t, \end{aligned}$$

where μ is $m \times 1$, and ϕ is $m \times d$. To simplify notation (and without loss of generality), the differential of S is defined as a function of the differential of W^1 only, and thus the volatility process σ is a scalar. The short rate, r, and payout rate, δ, are given by $r_t = \beta(X_t)$ and $\delta_t = \kappa(X_t)$, where β and κ are real-valued functions. It is easy to generalize the model to allow the parameters of X, as well as r and δ, to depend on S also.

The Radon–Nikodym derivative [Equation (2)] has the explicit solution

$$Z_t = \exp\left[-\frac{1}{2}\int_0^t \sigma(S_s, X_s)^2 ds + \int_0^t \sigma(S_s, X_s)dW_s^1\right].$$

Define the d-length column vector $\mathbf{e} = [1, 0, \ldots, 0]'$. By Girsanov's theorem,

$$\tilde{W}_t = \mathbf{e}\int_0^t \sigma(S_s, X_s)ds - W_t$$

is d-dimensional standard Brownian motion under $\tilde{Q}$, where $d\tilde{Q}/dQ = Z_T$. Defining $\tilde{S}_t = KS_0/S_t$ and applying *Itô*'s lemma and Girsanov's theorem, we obtain

$$\begin{aligned} \frac{d\tilde{S}_t}{\tilde{S}_t} &= (\delta_t - r_t)\,dt + \sigma(KS_0/\tilde{S}_t, X_t)\,d\tilde{W}_t^1, \quad \tilde{S}_0 = K, \\ dX_t &= [\mu(X_t) + \phi(X_t)\sigma(KS_0/\tilde{S}_t, X_t)\mathbf{e}]\,dt - \phi(X_t)d\tilde{W}_t. \end{aligned}$$

The modification to the drift, $\phi(X_t)\sigma(KS_0/\tilde{S}_t, X_t)\mathbf{e}$, represents the instantaneous covariance between asset returns and the increments in the vector of state variables.

We now introduce d jump processes, each indexed by i, $i = 1, \ldots d$. Each jump process is characterized by the double sequence $(T_n^i, J_n^i; n = 1, 2, \ldots)$, where T^n represents the time and J_n^i the amount of the nth jump.[12] Let $\mathcal{B}(\mathbb{R})$ denote the Borel σ-algebra of subsets of the real line. For each set $A \in \mathcal{B}(\mathbb{R})$ and $i \in \{1, \ldots d\}$, the counting process $N_t^i(A)$ represents the number of jumps with a magnitude in the set A by time t. The jump processes are assumed to be independent of W and are assumed to satisfy $[N^i((0,\infty)), N^1((0,\infty))] = 0$, a.s., $i \neq 1$; that is, the jumps of the first process do not

[12] See Brémaud (1981), for all the needed results on point processes. This discussion borrows heavily from chapter VIII.

coincide with the jumps of the other processes.[13] The counting measure $p^i(dt \times dy)$ is defined as $p^i((0,t] \times A) = N_t^i(A)$, $A \in \mathcal{B}(\mathbb{R})$, $i = 1, \ldots, d$.

Let $\lambda_t^i(dy)$ denote the intensity kernel of $p^i(dt \times dy)$, $i = 1, \ldots, d$; for each $A \in \mathcal{B}(\mathbb{R})$, $\lambda_t^i(A)$ is an $\mathcal{F}_t$-predictable process. Write the intensity kernel as

$$\lambda_t^i(dy) = \lambda_t^i \Phi_t^i(dy), \quad i = 1, \ldots, d,$$

where $\lambda_t^i \equiv \lambda_t^i(\mathbb{R})$ and $\Phi_t^i(dy) = \lambda_t^i(dy)/\lambda_t^i$ on $\{\lambda_t^i > 0\}$. The process Φ_t^i is a distribution function for each t. Loosely speaking, $\lambda_t^i dt$ can be interpreted as the probability, conditional on $\mathcal{F}_{t-}$, of a jump in the next dt units of time; $\Phi_t^i(A)$ can be interpreted as the probability of a jump with magnitude in the set A conditional on $\mathcal{F}_{t-}$ and given that a jump occurs at t.

Define the compensated point processes $q \equiv [q^1, \ldots, q^d]'$, where

$$q^i(dt \times dy) = p^i(dt \times dy) - \lambda_t^i \Phi_t^i(dy)dt, \quad i = 1, \ldots, d.$$

For any bounded and $\mathcal{F}_t$-predictable process $f(\cdot, y)$, the process M^i defined by

$$M_t^i = \int_0^t \int_{\mathbb{R}} f(s, y) q^i(ds \times dy), \quad t > 0, \quad i = 1, \ldots, d$$

is a martingale [see Brémaud (1981) for less restrictive conditions on f]. At the jump times,

$$M^i(T_n^i) = M^i(T_n^i-) + f^i(T_n^i, J_n^i), \quad n = 1, 2, \ldots,$$

and between jumps,

$$dM_t^i = -\int_{\mathbb{R}} f^i(t, y) \lambda_t^i \Phi_t^i(dy) dt, \quad T_{n-1}^i < t < T_n^i, \quad n = 1, 2, \ldots.$$

The asset price and the state variables satisfy

$$\begin{aligned}
\frac{dS_t}{S_{t-}} &= (r_t - \delta_t)\, dt + \sigma(S_t, X_t)\, dW_t^1 + \int_{\mathbb{R}} g(S_{t-}, X_{t-}, y) q^1(dt \times dy),\\
dX_t &= \mu(X_t)dt + \phi(X_t)dW_t + \int_{\mathbb{R}} G(X_{t-}, y) q(dt \times dy),
\end{aligned}$$

where σ and g are real-valued functions, μ is $m \times 1$, and ϕ and G are each $m \times d$. We allow λ_t^i and Φ_t^i to be functions of S_{t-} and X_{t-}.

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[13] The Brownian motion W introduced above is defined on the probability space $(\Omega^W, \mathcal{F}^W, P^W)$. The jump processes are defined on the probability space $(\Omega^p, \mathcal{F}^p, P^p)$ where the filtration is that generated by the history of the processes:

$$\mathcal{F}_t^p = \sigma\left(N_s^i(A); s \in [0, t], A \in \mathcal{B}(\mathbb{R}), i \in \{1, \ldots d\}\right).$$

On the product space,

$$(\Omega, \mathcal{F}, P) \equiv (\Omega^W \times \Omega^p, \mathcal{F}_t^W \otimes \mathcal{F}_t^p, P^W \otimes P^p),$$

the counting jump processes and W are independent.

The Radon–Nikodym derivative [Equation (2)] is

$$\begin{aligned} Z_t &= \exp\left[-\frac{1}{2}\int_0^t \sigma_s^2 ds + \int_0^t \sigma_s dW_s^1 - \int_0^t \int_{\mathbb{R}} g^1(S_{s-}, X_{s-}, y)\lambda_s^1 \Phi^1(dy)ds\right] \\ &\quad \times \prod_{n\geq 1}[1 + g(S(T_n^1-), X(T_n^1-), J_n^1)]1_{\{T_n^1 \leq t\}}. \end{aligned}$$

The Girsanov–Meyer theorem implies

$$\tilde{W}_t = \mathbf{e}\int_0^t \sigma(S_s, X_s)ds - W_t$$

is d-dimensional standard Brownian motion under $\tilde{Q}$. The intensity kernel of p^1 under $\tilde{Q}$ is characterized by

$$\tilde{\lambda}_t^1 = \lambda_t^1 \int_{\mathbb{R}} [1 + g(S_{t-}, X_{t-}, y)]\Phi_t^1(dy),$$

and

$$\tilde{\Phi}_t^1(dy) = \Phi_t^1(dy)\frac{1 + g(S_{t-}, X_{t-}, y)}{\int_{\mathbb{R}}[1 + g(S_{t-}, X_{t-}, y)]\Phi_t^1(dy)}.$$

The following simple heuristic derivation can be used to obtain the intensity kernel under $\tilde{Q}$. Suppose there have been exactly $n-1$ jumps in the asset price before time t. Then

$$\begin{aligned} \tilde{\lambda}_t^1(dy)dt &= \tilde{Q}\left(T_n^1 \in [t, t+dt], J_n^1 \in [y, y+dy] \;\middle|\; \mathcal{F}_{t-}\right) \\ &= Z_{t-}^{-1} E_Q\left(Z_{t+dt} 1_{\{T_n^1 \in [t,t+dt],\, J_n^1 \in [y,y+dy]\}} \;\middle|\; \mathcal{F}_{t-}\right), \end{aligned}$$

where the martingale property of Z and iterated expectations have been used to get the second equality. Now substitute $Z_{t+dt} = Z_{t-}[1+g(S_{t-}, X_{t-}, y)]$ on $\left\{T_n^1 \in [t, t+dt]\right\}$ (ignoring smaller-order terms) to get

$$\begin{aligned} \tilde{\lambda}_t^1(dy)dt &= [1 + g(S_{t-}, X_{t-}, y)]Q\left(T_n^1 \in [t, t+dt], J_n^1 \in [y, y+dy] \;\middle|\; \mathcal{F}_{t-}\right) \\ &= [1 + g(S_{t-}, X_{t-}, y)]\lambda_t^1(dy)dt. \end{aligned}$$

The intensity kernels of $(p^2, \ldots, p^d)$ are unaltered by the measure change. The compensated point processes under $\tilde{Q}$ are therefore $\tilde{q} \equiv [\tilde{q}^1, \ldots, \tilde{q}^d]'$, where

$$\tilde{q}^1(dt \times dy) = p^1(dt \times dy) - \tilde{\lambda}_t^1 \tilde{\Phi}_t^1(dy)dt$$

and $\tilde{q}^i = q^i, i = 2, \ldots, d$.

The processes under the numeraire change satisfy

$$\begin{aligned} \frac{d\tilde{S}_t}{\tilde{S}_{t-}} &= (\delta_t - r_t)\, dt + \sigma(KS_0/\tilde{S}_t, X_t)\, d\tilde{W}_t^1 \\ &\quad - \int_{\mathbb{R}} \frac{g(KS_0/\tilde{S}_{t-}, X_{t-}, y)}{1 + g(KS_0/\tilde{S}_{t-}, X_{t-}, y)}\tilde{q}^1(dt \times dy), \end{aligned}$$

$$dX_t = \tilde{\mu}_t(KS_0/\tilde{S}_t, X_t)dt - \phi(X_t)d\tilde{W}_t + \int_{\mathbb{R}} G(X_{t-}, y)\tilde{q}(dt \times dy),$$

where

$$\begin{aligned}\tilde{\mu}_t(KS_0/\tilde{S}_t, X_t) \equiv\ & \mu(X_t) + \phi(X_t)\sigma(KS_0/\tilde{S}_t, X_t)\mathbf{e} \\ & + \int_{\mathbb{R}} G(X_t, y)\mathbf{e}[\tilde{\lambda}_t^1\tilde{\Phi}_t^1(dy) - \lambda_t^1\Phi_t^1(dy)].\end{aligned}$$

References

Bates, D., 1991, “Option Pricing Under Asymmetric Processes, with Applications to Options on Deutschemark Futures,” working paper, University of Pennsylvania.

Bjerksund, P., and G. Stensland, 1993, “American Exchange Options and a Put-Call Transformation: A Note,” *Journal of Business Finance and Accounting*, 20, 761–764.

Brémaud, P., 1981, *Point Processes and Queues*, Springer-Verlag, New York.

Byun, S., and I. Kim, 1996, “Relationships Between American Puts and Calls on Futures Contracts,” working paper, Korea Advanced Institute of Science and Technology.

Carr, P., 1993, “Deriving Derivatives of Derivative Securities,” working paper, Cornell University.

Carr, P., and M. Chesney, 1996, “American Put Call Symmetry,” working paper, Morgan Stanley, Groupe H.E.C.

Chesney, M., and R. Gibson, 1993, “State Space Symmetry and Two Factor Option Pricing Models,” in J. Janssen and C. H. Skiadas (eds.), *Applied Stochastic Models and Data Analysis*, World Scientific Publishing, River Edge, N.J.

Cox, J., 1975, “Notes on Option Pricing I: Constant Elasticity of Variance Diffusions,” working paper, Stanford University.

Duffie, D., 1992, *Dynamic Asset Pricing Theory*, Princeton University Press, Princeton, N.J.

Dufresne, P., W. Keirstead, and M. Ross, 1997, “Martingale Pricing: A Do-It-Yourself Guide to Deriving Black-Scholes,” in *Equity Derivatives: Applications in Risk Management and Investment*, Risk Books, London.

Geman, H., N. El Karoui, and J. Rochet, 1995, “Changes of Numeraire, Changes of Probability Measure and Option Pricing,” *Journal of Applied Probability*, 32, 443–458.

Grabbe, J., 1983, “The Pricing of Call and Put Options on Foreign Exchange,” *Journal of International Money and Finance*, 2, 239–253.

Heston, S., 1991, “A Closed-Form Solution for Options with Stochastic Volatility, with Application to Bond and Currency Options,” working paper, Yale University.

Ingersoll, J., 1997, “Digital Contracts: Simple Tools for Pricing Complex Derivatives,” working paper, Yale University.

Jamshidian, F., and M. Fein, 1990, “Closed Form Solutions for Oil Futures and European Options in the Gibson Schwartz Model: A Note,” working paper, Merrill Lynch Capital Markets.

Karatzas, I., 1988, “On the Pricing of American Options,” *Applied Mathematics and Optimization*, 17, 37–60.

Kholodnyi, V., and J. Price, 1998, *Foreign Exchange Symmetry*, World Scientific Publishing, River Edge, N.J.

McDonald, R., and M. Schroder, 1990, "A Parity Result for American Options," working paper, Northwestern University.

Merton, R., 1976, "Option Pricing when Underlying Stock Returns are Discontinuous," *Journal of Financial Economics*, 3, 125–144.

Olsen, T., and G. Stensland, 1991, "Invariant Controls in Stochastic Allocation Problems," in D. Lund and B. B. Øksendal (eds.), *Stochastic Models and Option Values*, Elsevier Science, New York.

Protter, P., 1992, *Stochastic Integration and Differential Equations*, Springer-Verlag, New York.

Rabinovitch, R., 1989, "Pricing Stock and Bond Options when the Default-Free Rate is Stochastic," *Journal of Financial and Quantitative Analysis*, 24, 447–457.

Reiner, E., and M. Rubinstein, 1991, "Unscrambling the Binary Code," *RISK*, 4, 75–83.

Schroder, M., 1989, "Computing the Constant Elasticity of Variance Option Pricing Formula," *Journal of Finance*, 44, 211–219.

Schroder, M., 1992, "Some Option Pricing Results Obtained Using a Change of Numeraire," Ph.D. summer paper, Northwestern University.