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Abstract

In this paper, we compare two one-factor short rate models: the Hull White model and the Black-Karasinski model. Despite their inherent shortcomings the short rate models are being used quite extensively by the practitioners for risk-management purposes. The research, as part of students' projects in collaboration with the Asset-Liability-Management (ALM) Group of ABN AMRO Bank, provides detail procedures on the implementation, and assesses model performance from an ALM perspective. In particular, we compare the two models for pricing and hedging Bermudan swaptions because of its resemblance to prepayment option in typical mortgage loans. To our knowledge, the implementation of the two short rate models (and the Black-Karasinski model in particular) is not well documented. We implemented the two models using interest rate derivatives on Euro and US rates over the period February 2005 to September 2007 and with the Hull-White trinomial tree. Our results show that in terms of the in-sample pricing tests, the one-factor Hull-White model outperforms the Black-Karasinski model. The estimated parameters of Hull-White model are also more stable than those of the Black-Karasinski model. On the other hand, the tests for the hedging performance show that the Black-Karasinski model is more effective in hedging the interest rate risk of the at-the-money 10x1 co-terminal Bermudan swaption.

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Short Rate Models: Hull-White or Black-Karasinski?

Implementation Note and Model Comparison for ALM

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June 14, 2008

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Short Rate Models: Hull-White or Black-Karasinski?

Implementation Note and Model Comparison

1 Introduction

In this paper, we study the choice of short rate models for asset-liability management in a global bank. In particular, we compare the performance of two one-factor short rate models, viz. Hull-White and Black-Karasinski, for hedging a 10x1 Bermudan swaption on an annual basis over a one and a half year period. The 10x1 Bermudan swaption is chosen because it resembles a loan portfolio with early redemption feature, an important product for most banks. Unlike the short-term pricing problem, the one-factor model is often preferred for the longer term ALM purpose because of its simplicity. For long horizon hedging, the multi-factor model could produce more noise as it requires more parameters input. Pricing performance measures a model's capability of capturing the current term structure and market prices of interest rate sensitive instruments. Pricing performance can always be improved, in an almost sure sense, by adding more explanatory variable and complexities to the dynamics. However, pricing performance alone cannot reflect the model's ability in capturing the true term structure dynamics. To assess the appropriateness of model dynamics, one has to study model forecasting and hedging performance.

The last few decades have seen the development of a great variety of interest rate models for estimating prices and risk sensitivities of interest rate derivatives. These models can be broadly divided into short rate, forward rate and market models. The class of short-rate models, among others, includes Vasicek (1977), Hull and White (1990), and Black and Karasinski (1991). The Ho-Lee model is an early example of forward-rate modelling. A generalized framework for arbitrage free forward-rate modelling originates from the work of Heath, Jarrow and Morton (HJM, 1992). Market models are a class of models within the HJM framework that model the evolution of rates that are directly observable in the market. Example of such models are the Libor Market Model by Brace, Gatarek, and Musiela (1997) and the Swap Market Model by Jamshidian (1997). All these models have their own strengths and weaknesses. Short-rate models are tractable, easy to understand and implement but do not provide complete freedom in choosing the volatility structure. The HJM framework is popular due to its flexibility in terms of the number of factors that can be used and it permits different volatility structures for different maturity forward rates. Despite these attractions the key problem associated with the HJM is that instantaneous forward rates are not directly observable in the market and hence models under this framework are difficult to calibrate. The market models overcome these limitations but are complex and compositionally expensive when compared with e.g. the short rate models. The question is whether one should use the Gaussian HW model or the lognormal BK model. In this paper, we look at the difference between the Hull-White and the Black-Karasinski short rate models for longer term asset liability management in two interest rate regimes (i.e. Euro and USD). The choice of HW and BK is simple; at the time of writing, they are the most important and popular short rate models used by the industry.

2 Previous Studies

Significant effort has been placed on developing pricing models for interest rate claims. However, the empirical evaluation of these models, especially in the swaption market, has lagged behind the theoretical advances made in this area. Much of the literature on multi-factor term structure models has focused on explaining bond yield and swap rates, as outlined in Dai and Singleton (2003). Therefore, despite the importance of caps and swaptions, there is still wide divergence of opinion on how to best value these claims. It is widely believed that since the term structure of interest rates is driven by multiple factors, interest rate claims should be valued using multi-factor models.

Amin and Morton (1994) implement and test six HJM interest rate models using an implied volatility technique with Eurodollar futures and Eurodollar futures options data. They find that two-parameter models tend to fit prices better, but their parameter estimates are less stable and they earn less from the perceived mispricings. Although the one-parameter models fit slightly less well, their implied parameter values are more stable over time and they are able to earn significantly larger and more consistent abnormal profits from the mispricings they detect. Using caplet data, where maturities ranged from 1 year to 10 years, Ritchken and Chuang (1999) show that generalized Vasicek model captures the hump in the volatility of forward rates, leads to significant improvements on pricing. The interest rate claims are priced in the Heath-Jarrow-Morton paradigm and the structure of volatilities is captured without using time varying parameters. As a result, the volatility structure is stationary. Gupta and Subrahmanyam (2005) examine many one- and two-factor models (HW, HJM and LMM) for pricing and hedging interest rate caps and floors. Unlike Amin and Morton, they conclude that a one-factor lognormal forward rate model outperforms other competing one-factor models in pricing accuracy, with two factor models improving pricing performance only marginally. However, for hedging, they find a significant advantage in moving from one to two-factor models. The caplets (four maturities: 2-, 3-, 4-, and 5-year) in their paper are hedged using Eurodollar futures contracts up to a maturity of 10 years, in increments of 3 months. The hedging rebalancing interval is 5-day and 20-day. Buhler, Uhrig, Walter and Weber (1999) test different one- and two-factor models (four forward rate models and three spot rate models) in the German fixed-income warrants market. They found that the one-factor forward rate model with linear proportional volatility and the two spot-rate models with two factors outperform the other models. Unlike Gupta and Subrahmanyam (2005), they find no advantage in moving beyond a one-factor model. Using 3 years of interest rate caps price data accross strikes, Jarrow, Li, and Zhao (2007) show that even a three-factor model with stochastic volatility and jumps cannot completely capture the smile/skew patterns observed in this market.

In contrast to the cap/floor market, few empirical studies have been conducted on swaptions. Longstaff, Santa-Clara, Schwartz (LSS, 2001a) use a string model to test the relative valuation of caps and swaptions using at-the-money cap and swaptions data, and find support for using model with at least four factors for pricing swaptions. They use 34 swaptions data where the final maturity dates of the underlying swap are less than or equal to 10 years and the interest rate cap data consists of weekly midmarket implied volatility for 2-, 3-, 4-, 5-, 7- and 10 years. Their criterion for evaluating models is based on the sum of squared percentage pricing errors. In other words, their criterion is based on pricing accuracy, not on hedging precision. Peterson, Stapleton and Subrahmanyam (2003) develop an extension of the lognormal model of Black and Karasinski (1991) to multiple factors and provide evidence that the addition of a third factor is helpful in pricing swaptions.

Not all studies, however, indicate that multiple factors are necessary for improving pricing performance for swaptions. For example, Driessen, Klaassen, and Melenberg (2003) (hereafter DKM) investigate the performance of several Gaussian models, where volatility structures are deterministic functions of their maturities. They show that the out-of-sample pricing performance of swaption pricing models does not necessarily improve as the number of factors increases. Indeed, one of their one-factor models prices swaptions no worse than their multi-factor models and to the same degree of accuracy as LSS's multi-factor model. Jagannathan, Kaplin and Sun (2003) investigate the pricing of swaptions using multifactor Cox, Ingersoll and Ross models. Their preliminary conclusions suggest that increasing the number of factors does not necessarily improve pricing performance. Indeed, adding factors makes the pricing of short term contracts worse. Fan, Gupta and Ritchken (2007) compare the pricing performance of several single and multi-factor models with different volatility structures and identify those models that eliminate most of the pricing biases in the swaption market. In this regard, their paper is closely related to DKM and LSS. They find that for pricing swaptions, the benefit of increasing the number of factors beyond one is minor. Their results also show that incorporating level dependence in the volatility structure is extremely important for away-from-the-money caps, and that proportional dependent structures are better than both square root and level independent structures. For at-the-money swaptions, the level dependence issue is minor.

Very few studies have compared the abilities of different models for hedging swaptions. LSS briefly consider hedging, in the context of their four-factor model, relative to the Black model. The most related studies on swaptions hedging are by DKM (2003) and Fan, Gupta and Ritchken (2003). DKM uses the caps with maturities ranging from 1 to 10 years. For the swaptions, the option maturities range from one month to five years while the swaps maturities range from 5 to 10 years. The hedging interval is two weeks, i.e., the value of the hedge portfolio is calculated two weeks after the hedge portfolio is constructed. DKM use the HJM model to demonstrate that if the number of hedge instruments (zero coupon bonds with different maturities) is equal to the number of factors, multi-factor models outperform one-factor models in hedging caps and swaptions. However, they claim that by using a large set of hedge instruments, their one-factor models perform as well as multi-factor models. This last finding is the opposite of what Gupta and Subrahmanyam (2005) find in the cap market. In the context of Unspanned Stochastic Volatility (USV), Fan, Gupta, and Ritchken (2003) show that even swaption straddles can be well hedged using Libor bonds alone if at least a three factor model is used. HJM is used as the basic model in the hedging test. The volatilities of U.S. dollar swaptions with expiration dates of six months and one, two, three, four, and five years, with underlying swap maturities of two, three, four, and five years. Different Libor discount bonds are used as the (factor) hedging instruments. The hedge position is maintained unchanged for one week, and the hedged and unhedged residuals are obtained and stored. They then repeat this analysis for holding periods of two, three, and four weeks. Fan, Gupta and Ritchken (2007) find that multifactor models are essential for reducing the risk in hedged positions. They also demonstrate that allowing additional hedging instruments in a one and two factor model does not improve the results. Their main conclusion is that while accurate swaption prices can be obtained from a one-factor model, one- and even two-factor models cannot hedge swaptions well, and the benefits of multifactor models are significant. The data for the study consists of USD swaption and cap prices. The swaptions data set comprises volatilities of swaptions of maturities 6 months, 1-, 2-, 3-, 4-, and 5-years, with the underlying swap maturities of 1-, 2-, 3-, 4, and 5-years each. The cap prices are for a ten-month period (March 1 - December 31, 1998), across four different strikes (6.5%, 7%, 7.5%, and 8%) and four maturities (2-, 3-, 4-, and 5-year).

Some research has also been done regarding the importance of factors for pricing Bermudan swaptions. Longstaff, Santa-Clara and Schwartz (2001b) show that exercise strategies based on one-factor models understate the true option value for Bermudans. They contend that the current market practice of using one-factor models leads to suboptimal exercise policies and a significant loss of value for the holders of these contracts. However, Andersen and Andreasen (2001) conclude that the standard market practice of recalibrating one-factor models does not necessarily understate the price of Bermudan swaptions. The authors do not investigate any hedging issues however. Pietersz and Pelsser (2005) compare single factor Markov-function and multifactor market models for hedging Bermudan swaptions. They find that on most trade days the Bermudan swaption prices estimated from these two models are similar and co-move together. Their results also show that delta and delta-vega hedging performances of both models are comparable. The delta hedge in their study is set up in terms of discount bonds, one discount bond for each tenor associated with the deal. In the case of 10Y Bermudan with annual coupon, there are 11 such discount bonds. The hedging is carried out daily. For the swaptions, the option maturities range from one month to five years while the swaps maturities range from 1 to 30 years.

3 The short rate models

Short rate model specifies the behaviour of short-term interest rate, r. Short rate model, as it evloved in the literature, can be classified into equilibrium and no-arbitrage models. Equilibrium models are also referred to as “endogenous term structure models” because the term structure of interest rates is an output of, rather than an input to these models. If we have the initial zero-coupon bond curve from the market, the parameters of the equilibrium models are chosen such that the models produce a zero-coupon bond curve as close as possible to the one observed in the market. Vasicek (1977) is the earliest and most famous general equilibrium short rate model. Since the equilibrium models cannot reproduce exactly the initial yield curve, most traders have very little confidence in using these models to price complex interest rate derivatives. Hence, no-arbitrage models designed to exactly match the current term structure of interest rates are more popular. It is not possible to arbitrage using simple interest rate instruments in this type of no-arbitrage models. Two of the most important no-arbitrage short-rate models are the Hull-White model (1990) and the Black-Karasinski (1991) model.

3.1 The Vasicek model

In the Vasicek (1977) model, the risk neutral process for r is

$$dr_t = k[\theta - r_t]dt + \sigma dz,$$

where r_0 (at $t = 0$), k, θ and σ are positive constants. This model is an Ornstein-Uhlenbeck process, where the distribution of the short rate is Gaussian with mean and variance,

$$E(r_t) = r_s e^{-k(t-s)} + \theta\left(1 - e^{-k(t-s)}\right), \quad (1)$$

$$Var(r_t) = \frac{\sigma^2}{2k}\left[1 - e^{-2k(t-s)}\right] \qquad \text{for } s \leq t. \quad (2)$$

From equation (1), we can see that as $t \to \infty$, $E(r_t) \to \theta$. Thus short rate r is mean reverting and θ can be regarded as the long-term average rate. This is the first interest rate model that incorporates mean reversion. With the normal distribution assumption, short rates can be negative with positive probability, which is a major drawback of the Vasicek model. However, the analytic tractability resulting from the Gaussian distribution is the biggest attraction of this model.

3.2 The Hull-White Model

Hull and White (1990) propose an extension of the Vasicek model so that it can be consistent with both the current term structure of spot interest rates and the current term structure of interest-rate volatilities. According to the Hull-White model, also referred to as the extended-Vasicek model, the instantaneous short-rate process evolves under the risk-neutral measure as follows:

$$dr_t = [\theta_t - a_t r_t]dt + \sigma_t dz, \tag{3}$$

where θ, a and σ are deterministic functions of time. The function θ_t is chosen so that the model fits the initial term structure of interest rates. The other two time-varying parameters, a_t and σ_t, enable the model to be fitted to the market derivative prices.[1] Hull and White (1994) note, however, that while a_t and σ_t allow the model to be fitted to the volatility structure at time zero, the resulting volatility term structure could be non-stationary in the sense that the future volatility structure implied by the model can be quite different from the volatility structure today. On the contrary, when these two parameters are kept constant, the volatility structure stays stationary but model's consistency with market prices of e.g. caps or swaptions can suffer considerably. Thus there is a trade-off between tighter fit and model stationarity. We will return to this issue again in Section 8,Page 25.

Hull and White (1994) introduce a constant parameter version of the model in (3) as follows:

$$dr = [\theta_t - a\ r]dt + \sigma dz, \tag{4}$$

where a and σ are positive constants and, as before, the function θ_t is chosen so that the model fits the initial term structure of interest rates. The parameter θ_t here can be analytically computed as

$$\begin{aligned} \theta_t &= f(0,t) + a\ f(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at}), \qquad (5) \\ P(0,t) &= e^{-f(0,t)}, \end{aligned}$$

[1] The initial volatility of all rates depends on $\sigma(0)$ and $a(t)$. The volatility of short rate at future times is determined by $\sigma(t)$ (Hull and White 1996, p.9).

where $P(0,t)$ is the zero coupon bond price for maturity t.

The mean and variance of $r(t)$ are given by

$$\begin{aligned} E\left(r_t\right) &= r_s e^{-a(t-s)} + \alpha_t - \alpha_s e^{-a(t-s)}, \\ Var\left(r_t\right) &= \frac{\sigma^2}{2a}\left[1 - e^{-2a(t-s)}\right], \end{aligned}$$

for $s \leq t$, and

$$\alpha_t = f(0,t) + \frac{\sigma^2}{2a^2}(1 - e^{-at})^2.$$

The Hull-White (1994) model has a great deal of analytic tractability. Under this model, the price at time t of a discount bond maturing at time T is given by[2]

$$P(t,T) = A(t,T)e^{-B(t,T)r_t},$$

where

$$\begin{aligned} B(t,T) &= \frac{1}{a}\left[1 - e^{-a(T-t)}\right], \\ A(t,T) &= \frac{P(0,T)}{P(0,t)} \exp\left\{B(t,T)\ F(0,t) - \frac{\sigma^2}{4a}\left(1 - e^{-2at}\right)\ B(t,T)^2\right\}. \end{aligned}$$

The price at time t of a European option on a pure discount bond is given by

$$ZBO = z\left\{P(t,s)\ N(zh) - X\ P(t,T)\ N\left[z(h - \sigma_P)\right]\right\}, \tag{6}$$

where $s > T$ is the maturity date of the bond, $T > t$ is the maturity date of the option, X is the strike price, with $z = 1$ for call and $z = -1$ for put, and

$$\begin{aligned} h &= \frac{1}{\sigma_P} \ln \frac{P(t,s)}{P(t,T)\ X} + \frac{\sigma_P}{2}, \\ \sigma_P^2 &= \frac{\sigma^2}{2k}(1 - e^{-2a(T-t)})\ B(T,S)^2. \end{aligned}$$

Equation (6) can also be used to price caplets and floorlets since they can be viewed as option on zero coupon bonds (see Appendix I). In HW model the distribution of short rate is Gaussian. Gaussian distribution leads to a theoretical possibility of short rate going below zero. Like the Vasicek model the possibility of a negative interest rate is a major drawback of this model.

[2]For the corresponding formulas, when the two parameters are time-dependent see Hull-White (1990, p. 577-579).

3.3 The Black-Karasinski Model:

A model that addresses the negative interest rate issue of the Hull-White model is the Black and Karasinski (1991) model. In this model, the risk neutral process for logarithm of the instantaneous spot rate, $\ln r_t$ is

$$d \ln r_t = [\theta_t - a_t \ \ln r_t]\, dt + \sigma_t dz, \tag{7}$$

where r_0 (at $t = 0$) is a positive constant, θ_t, a_t and σ_t are deterministic functions of time. Equation (7) shows that the instantaneous short rate evolves as the exponential of an Ornstein-Uhlenbeck process with time-dependent coefficients. The function θ_t is chosen so that the model fits the initial term structure of interest rates. Functions a_t and σ_t are chosen so that the model can be fitted to some market volatility curves. Based on the arguments given in Section 3.2, we can also have the constant parameter version of (7) by setting $a_t = a$ and $\sigma_t = \sigma$, which leads to

$$d \ln r_t = [\theta_t - a \ \ln r_t] dt + \sigma dz.$$

As before, the coefficient a measures the speed at which $\ln r_t$ tends towards its long-term value, θ_t, and σ is the volatility of the instantaneous spot rate. Unlike the Hull-White model, the Black and Karasinski model does not yield analytical formulas for discount bonds and interest rate options. Therefore, under this model, pricing has to be performed through numerical procedure.

4 Research Design and Data

Let t denote a particular month in the period from February 2005 to September 2007. The procedure for calibrating, hedging and unwinding a 10×1 Bermudan swaption are as follows:

(i) At month t, the short rate model is calibrated to 10 ATM co-terminal European swaptions underlying the 10×1 ATM Bermudan swaption by minimising the root mean square of the pricing errors.

(ii) The calibrated model from (i) is then used to price the 10×1 ATM Bermudan swaption at t, and to calculate the hedge ratios using, as hedge instruments, 1-year, 5-year and 11-year swaps, all with zero initial swap value at time t.

(iii) A delta hedged portfolio is formed by minimising the amount of delta mismatch.

(iv) At time $t + 12$ (i.e. one year later), the short rate model is calibrated to 9 co-terminal ATM European swaptions $(9 \times 1, 8 \times 2, ...)$ underlying the Bermudan swaption from (ii) which is now 9×1.

(v) The calibrated model from (iv) is used to price the 9×1 Bermudan swaption, and the time $t + 12$ yield curve is used to price the three swaps in (ii), which are now 0 year, 4 years and 10 years to maturity.

(vi) The profit and loss is calculated for the delta hedged portfolio formed at t and unwound at $t + 12$.

(vii) Steps (i) to (vi) are repeated every month for $t+1$, $t+2, \cdots$ till $T - 12$ where T is the last month of the sample period.

To perform the valuation and hedging analyses described above, the following data sets are collected from Datastream:

(a) Monthly prices (quoted in Black implied volatility) of ATM European swaptions in Euro and USD. Two sets of implied vol were collected: from February 2005 to September 2006, prices of co-terminal ATM European swaptions underlying the 10×1 Bermudan swaption, and from Fenruary 2006 to September 2007, prices of co-terminal ATM European swaptions underlying the 9×1 Bermudan swaption. These prices are quoted in Black implied volatility. The implied vol matrix downloaded has a number of missing entries especially in the earlier part of the sample period. The missing entries were filled in using log-linear interpolation following Brigo and Morini (2005, p 9, 24 and 25).

(b) Annual yields, $R_{0,t}$, for maturities up to 11 years are downloaded from Datastream.[3] All other yields needed for producing the trinomial tree are calculated using linear interpolation. These annual yields are converted to continuously compounded yields, $r_{0,t} = \ln(1 + R_{0,t})$.

(c) Monthly data of the annual yield curve for the period January 1999 to July 2007, i.e. total 103 observations are downloaded for Euro and USD. This data was transformed into discrete forward rates (as in LMM) for use in the principal component analysis.

(iv) Monthly data of 1-month yield for the period January 2000 to July 2007 was downloaded for estimating the "mean-reversion" parameter. In the implementation, a time step (Δt) of 0.1 year is used for constructing the trinomial tress, which means that the rates on nodes of the tree are continuously compounded Δt-period rates. Here we have used the one-month yield as a proxy for the first 0.1-year short rate.

5 Implementation of the one-factor short-rate models

Hull and White (1994) outline a trinomial tree building procedure for implementing the one-factor constant parameter (a and σ) model in equation (4). This procedure can be adapted conveniently to implement the Ho-Lee model and the Black-Karasinski model (with "a" and "σ" constant). Later, Hull and White (2001) extended the model to incorporate time varying parameters. If the time step used for constructing tree is Δt, then the rates on nodes of the tree are continuously compounded Δt-period rates. Thus the assumption is that the Δt-period rate on any node of the tree, follows the same stochastic short rate process in (4).

[3] The time step (Δt) for the trinomial tress in the C++ program is 0.1 year. The C++ program linearly interpolates all the required yields.

5.1 Building trinomial tree for the one-factor Generalised Hull-White model

In the generalised Hull-White model, the risk neutral process for some function of the short-rate r, $f(r)$, is

$$df(r) = [\theta_t - a_t \; f(r)]dt + \sigma_t dz.$$

When $f(r) = r$, the model becomes the Hull-White and when $f(r) = \ln r$ the model becomes the Black-Karasinski model. Next, define a new deterministic function g and a new variable x,

$$\begin{aligned} dg &= [\theta_t - a_t \; g_t]dt, \\ x(r,t) &= f(r) - g_t. \end{aligned}$$

The process followed by this new variable x is

$$\begin{aligned} dx(r,t) &= df(r) - dg_t \\ &= [\theta_t - a_t \; f(r)]dt + \sigma_t dz - (\theta_t - a_t \; g_t)\, dt \\ &= -a_t \; xdt + \sigma_t dz. \end{aligned}$$

The initial value of g is chosen so that the initial value of x is 0. Since, the process followed by x is mean reverting to 0 and the initial value of x is 0. Therefore, the unconditional expected value of x at all future times is 0.

The construction of the Hull-White tree involves two stages: the first stage builds a tree for x (as opposed to $f(r)$), while the second stage shifts the nodes of x tree by the value of g at each point in time to produce the tree for $f(r)$. Building a tree for x requires the selection of (i) the spacing of the nodes in the time dimension, (ii) the spacing of the nodes in the interest-rate dimension, and (iii) the branching process for $x(r,t)$ through the grid of nodes.

In selecting the time spacing, it is critical to ensure that the tree has placed nodes at all cash flow payment dates. Extra nodes can be added later to increase its precision. Then at time step t_i, nodes are placed at $\pm\Delta x_{i,} \pm 2\Delta x_i, ..., \pm m_i \Delta x_i$, where $\pm m_i$ are the indices of the highest and lowest nodes at each time step with

$$\Delta x_i = \sigma(t_{i-1})\sqrt{3(t_i - t_{i-1})}. \tag{28}$$

This value of Δx_i ensures that the spacing of the nodes at time t_i is wide enough to represent the volatility of x at that time.

Suppose that we are at node $j\Delta x_i$ at time step i and the three successor nodes at time step $i+1$ are $(k+1)\Delta x_{i+1}$, $k\Delta x_{i+1}$, and $(k-1)\Delta x_{i+1}$. with probability of branching p_u, p_m, and p_d respectively. From the diffusion process for x[4]

$$E(dx) = M = -x\ a_{t_i}(t_{i+1} - t_i) = -j\Delta x_i\ a_{t_i}(t_{i+1} - t_i)$$

and

$$E(dx^2) = V + M^2 = \sigma^2_{t_i}(t_{i+1} - t_i) + M^2.$$

Matching the mean and variance gives

$$\begin{aligned} j\Delta x_i + M &= k\Delta x_{i+1} + (p_u - p_d)\Delta x_{i+1}, \\ V + (j\Delta x_i + M)^2 &= k^2 \Delta x^2_{i+1} + 2k(p_u - p_d)\Delta x^2_{i+1} + (p_u + p_d)\Delta x^2_{i+1}. \end{aligned} \tag{29}$$

Solving this gives

$$\begin{aligned} p_u &= \frac{V}{2\Delta x^2_{i+1}} + \frac{\alpha^2 + \alpha}{2}, \\ p_d &= \frac{V}{2\Delta x^2_{i+1}} + \frac{\alpha^2 - \alpha}{2}, \\ p_m &= 1 - \frac{V}{\Delta x^2_{i+1}} - \alpha^2, \end{aligned} \tag{8}$$

where

$$\alpha = \frac{j\Delta x_i + M - k\Delta x_{i+1}}{\Delta x_{i+1}}.$$

Hull and White (2001, p7) show that the following choice of k ensures that all the probabilities stay positive.

$$k = round\left(\frac{j\Delta x_i + M}{\Delta x_{i+1}}\right) \tag{9}$$

[4] When a and σ are constant more accurate values are given by equation (18) and (19).

where $round(\omega)$ denotes ω round to the nearest integer. Using the procedures described above, we determine the branches and branching probabilities for all nodes. The procedure also determines the indices of the highest and lowest nodes at each time step. The indices of the highest/lowest nodes at step $i+1$ i.e. $\pm m_{i+1}$, depends on the branching from the highest/lowest nodes at step i, i.e. on $\pm m_i$. At step 0 there is only one node $m_0 = 0$. Using equations (8) and (9) we determine the branching process for the root node and thereby $\pm m_1$, i.e. the index of highest and lowest nodes at step 1. From this, iteratively, we can determine the indices of the highest/lowest nodes for all the subsequent steps of the tree.

5.1.1 Adjusting the tree

Once the tree for the simplified process is constructed, the next stage is to addg_t back to all the nodes in the x-tree to retrieve the final interest rate tree.[5] From equation (25), g_t is a function of θ_t, and according to model specification the function θ_t is chosen so that the model fits the given term structure of interest rates. Therefore all the nodes of the x-tree are adjusted so that prices of discount bonds of all maturities should be consistent with the initial term structure observed in the market. We illustrate below how g_t is computed for each time step using forward induction technique. To facilitate computations of g_t, Hull and White introduces a new variable $Q(i,j|h,k)$ known as the Arrow-Debreu (AD) price. $Q(i,j|h,k)$ is defined as the value at node (h,k) of a security that pays \$1 at node (i,j) and zero otherwise. To simplify the notations $Q(i,j|0,0)$, referred to as the root AD price for node (i,j) i.e. the value at node (0,0) of a security that pays \$1 at node (i,j),

[5] From equation (26), $f(r) = x(r,t) + g_t$.

is denoted by $Q_{i,j}$. Thus

$$\begin{aligned} Q_{0,0} &= 1, \\ Q(i,j|i-1,k) &= p(i,j|i-1,k)\exp[-r_{i-1,k}(t_i - t_{i-1})] \\ Q_{ij} &= \sum_k Q(i,j|i-1,k)Q_{i-1,k} \\ &= \sum_k p(i,j|i-1,k)\exp[-r_{i-1}(t_i - t_{i-1})]Q_{i-1,k}, \end{aligned} \tag{32}$$

where the summation is over all nodes at time step t_{i-1}. Here, $p(i,j|h,k)$ is the probability of branching from node (h,k) to node (i,j), and $r_{i,j}$ is the interest rate at node (i,j). Thus

$$r_{i,j} = f^{-1}(x_{ij} + g_{t_i})$$

where x_{ij} is the value of x at node (i,j). This show that $Q_{i,j}$ can be calculated once the root AD prices for all the nodes placed at time t_{i-1} have been calculated.

If P_{i+1} is the price at node (0,0) of a discount bond that pays \$1 at every node at time step i+1 and V_{ij} is the value of this bond at node (i,j), then

$$\begin{aligned} V_{ij} &= \exp[-r_{ij}(t_{i+1} - t_i)] = \exp[-f^{-1}(x_{ij} + g(t_i))(t_{i+1} - t_i)], \text{ and} \\ P_{i+1} &= \sum_j Q_{ij}V_{ij} = \sum_j Q_{ij}\exp\left[-f^{-1}(x_{ij} + g_i)(t_{i+1} - t_i)\right], \end{aligned} \tag{33}$$

where the summation is over all nodes at time step t_{i-1}. The value of g_{t_i} is chosen so that the value of discount bond computed using equation (33), i.e. the model price matches the price of the discount bond computed from the current term structure, i.e.

$$\begin{aligned} P^M(0, t_{i+1}) &= P_{i+1} \\ P^M(0, t_{i+1}) &= \sum_j Q_{ij}\exp[-f^{-1}(x_{ij} + g_i)(t_{i+1} - t_i)], \end{aligned} \tag{34}$$

where $P^M(0, t_{i+1})$ is the market observed price at time 0 for the discount bond of maturity t_{i+1} that can be calculated as

$$P^M(0, t_{i+1}) = e^{-R(0,t_{i+1})t_{i+1}},$$

where $R(t,T)$ is market observed continuously compounded yield at time t on a zero coupon bond (face value 1) that matures at T. In case of Hull-White model

$$f^{-1}(x_{ij} + g_i) = (x_{ij} + g_i).$$

The solution to equation (34) is

$$g_{t_i} = \frac{\ln \sum_j Q_{i,j} e^{-x_{ij}} - \ln P^M(0, t_{i+1})}{(t_{i+1} - t_i)}. \tag{35}$$

For Black-Karasinski model

$$f^{-1}(x_{ij} + g_i) = \exp(x_{ij} + g_i),$$

and thus equation (34) takes the form

$$P^M_{t_{i+1}} = \sum_j Q_{ij}[-\exp(x_{ij} + g_{t_i})(t_{i+1} - t_i)]. \tag{36}$$

This equation can be solved using the Newton-Raphson procedure. Thus the steps in stage two can be summarised as:

(i) Knowing the fact that $Q_{00} = 1$, we apply equations (35) and (36), depending on the model to be implemented, and compute g_0 and therefore r_{00}.

(ii) This allows us to compute Q_{1j}s for every node j at step 1 using equation (33).

Once Q_{1j}s are calculated by solving equations (35) and (36), we can compute g_1, r_{1j}s and so on. This completes the construction of the interest rate tree that has been fitted exactly to the current term structure. In this procedure the functional form of the model, $f(r)$, comes into play only when we fit the tree to the given term structure. Before that step the tree building process is completely generic.

5.2 Pricing interest rate derivatives using trees

Trees are particularly useful for pricing early-exercise products such as the Bermudan-style swaption. Once the tree is constructed we know the value of the payoff at each of the final node, and we can move backward in time, thus updating the value of continuation through discounting. At each node of the tree we compare the "backwardly-cumulated" value of continuation with the payoff evaluated at that node ("immediate-exercise value"), thus deciding whether exercise is to be considered or not at that point. Once this exercise decision has been taken, the backward induction restarts and we continue to propagate backwards. Upon reaching the initial node of the tree (at time 0) we have the approximated price of our early-exerciseproduct.

6 Interest Rate Derivatives

In this Section we discuss the payoffs and important formulae for the European and Bermudan swaptions. A European payer swaption is an option giving the right (and no obligation) to enter a payer interest rate swap [IRS] at a given future time, the swaption maturity. The underlying IRS length ($T_\beta - T_\alpha$ in our case) is called the tenor of the swaption. The payoff of this swaption at time T_α i.e. at time when the option matures is,

$$Swaption_{T\alpha} = A\left(\sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i)\delta(f_{T_\alpha,T_{i-1},T_i} - k)\right)^{+} \tag{40}$$

where A is the notional principal, $(x)^+ \equiv \max(x, 0)$ is the positive-part operator and $f_{t,T,S}$ is the discretely-compounded (Libor) forward rate prevailing at time t for the expiry T and maturity S, and k is the fixed rate for the underlying swap.

According to market convention, the prices of swaptions are quoted in volatility form and Black formula is used to convert swaptions' implied volatilities to its prices. The Black option pricing formula for this payer swaption at time zero is

$$PS_0^{Black} = Az\left[S_{\alpha,\beta}(0)N(zd_1)s - kN(zd_2)\right]\sum_{i=\alpha+1}^{\beta}\delta P(0, T_i) \tag{41}$$

where

$$
\begin{aligned}
d_1 &= \frac{\ln\left(\dfrac{S_{\alpha,\beta}(0)}{k} + \dfrac{\sigma^2_{\alpha,\beta}T_\alpha}{2}\right)}{\sigma_{\alpha,\beta}\sqrt{T_\alpha}} \\
d_2 &= d_1 - \sigma_{\alpha,\beta}\sqrt{T_\alpha} \quad \text{and } z = 1
\end{aligned}
$$

here $\sigma_{\alpha,\beta}$ is the volatility parameter quoted in the market and $S_{\alpha,\beta}(0)$ is the forward start swap rate at time 0. By substituting $z = -1$, we get Black formula for equivalent receiver swaption. A swaption, either payer or receiver is said to be at-the-money (ATM) if and only if

$$
k = k_{ATM} := S_{\alpha,\beta}(0) = \frac{P(0,T_\alpha) - P(0,T_\beta)}{\sum\limits_{i=\alpha+1}^{\beta} \delta P(0,T_i)} \tag{42}
$$

Substituting k from equation (42) in equation (41), we get following expression for the payoff of the ATM payer swaption at time T_α,

$$
Swaption_{T\alpha} = A\left(\sum_{i=\alpha+1}^{\beta} P(T_\alpha,T_i)\delta f_{T_\alpha,T_{i-1},T_i} - \sum_{i=\alpha+1}^{\beta} P(T_\alpha,T_i)\delta k\right)^+ \tag{43}
$$

Putting in the value of f_{T_α,T_{i-1},T_i} from footnote 8, equation (43) becomes

$$
\begin{aligned}
Swaption_{T\alpha} &= A\left(\sum_{i=\alpha+1}^{\beta} P(T_\alpha,T_i)\delta\left\{\frac{1}{\delta}\left(\frac{P(T_\alpha,T_{i-1})}{P(T_\alpha,T_i)} - 1\right)\right\} - \delta k \sum_{i=\alpha+1}^{\beta} P(T_\alpha,T_i)\right)^+ \\
&= A\left(\sum_{i=\alpha+1}^{\beta} \{P(T_\alpha,T_{i-1}) - P(T_\alpha,T_i)\} - \delta k \sum_{i=\alpha+1}^{\beta} P(T_\alpha,T_i)\right)^+ . \qquad (44)
\end{aligned}
$$

Now

$$
\begin{aligned}
& \sum_{i=\alpha+1}^{\beta} \{P(T_\alpha,T_{i-1}) - P(T_\alpha,T)\} \\
=\;& \{P(T_\alpha,T_\alpha) - P(T_\alpha,T_{\alpha+1})\} + \{P(T_\alpha,T_{\alpha+1}) - P(T_\alpha,T_{\alpha+2})\} \\
& +\ldots..\{P(T_\alpha,T_{\beta-1}) - P(T_\alpha,T_\beta)\} \\
=\;& \{P(T_\alpha,T_\alpha) - P(T_\alpha,T_\beta)\} \\
=\;& \{1 - P(T_\alpha,T_\beta)\}
\end{aligned}
$$

Putting the value of

$$\sum_{i=\alpha+1}^{\beta} \{P(T_\alpha, T_{i-1}) - P(T_\alpha, T)\}$$

as calculated above, in equation (44) we get

$$Swaption_{T\alpha} = A\left(1 - \left\{P(T_\alpha, T_\beta) - \delta k \sum_{i=\alpha+1}^{\beta} P(T_\alpha, T_i)\right\}\right)^{+} \tag{45}$$

This is identical to the payoff of a put option on a coupon bond with coupon rate k payable at times T_i i.e. the reset dates of the IRS underlying the swaption, with strike 1. The coupon face value is \$1. An equivalent relation can be derived for receiver swaption using call option on a coupon bond. Therefore, we can conclude that the task of pricing a swaption is identical to that of pricing a suitable option on a coupon bond.

7 Calibration

The "mean reversion rate" parameter for the two models has been extracted from historical interest rate data. Practitioners and econometricians often use historical data for inferring the "rate of mean reversion" (Bertrand Candelon and Luis A. Gil-Alana (2006)). In this study we calibrate the two models with time-dependent short- rate volatility, $\sigma(t)$, and constant rate of mean reversion, a.

7.0.1 Parameterisation of $\sigma(t)$

There can be different ways to parameterise the time-dependent parameter: it can be piecewise linear, piecewise constant or some other parametric functional form can be chosen. In this study, the volatility parameter has been parameterised as follows: The last payoff for the all the instruments that need to be priced in this study would be at 11 years point of their life. We explicitly decided these three points, because values of $\sigma(t)$, $t = 0, 3, 11$ on these three points can be interpreted as instantaneous, short term and long term volatility. These three volatility parameters are estimated through the calibration process. The volatilities for the time periods in between these points are linearly interpolated.

7.0.2 Choosing calibration instruments

A common financial practice is to calibrate the interest rate model using the instruments that are as similar as possible to the instrument being valued and hedged, rather than attempting to fit the models to all available market data. In this study the problem at hand is to price and hedge 10×1 Bermudan. For this 10×1 Bermudan swaption the most relevant calibrating instruments are the 1×10, 2×9, $3 \times 8, \cdots, 10 \times 1$ co-terminal European swaptions. (A $n \times m$ swaption is an n-year European option to enter into a swap lasting for m years after option maturity.) The intuition behind this strategy is that the model when used with the parameters that minimise the pricing error of these individual instruments would price any related instrument correctly. Therefore these 10 European swaptions are used for calibrating the two models for pricing 10×1 Bermudan swaption.[6] When the models are used for pricing 9×1 Bermudan swaption we use nine European swaptions; 1×9, $2 \times 8, \cdots, 9 \times 1$ for calibrating the models. (σ_0, σ_3 and σ_{11} for the 10x1 Bermudan swaption and σ_0, σ_2 and σ_{10} for the 9x1 Bermudan swaption)

7.0.3 Goodness-of-fit measure for the calibration

The models are calibrated by minimising the sum of squared percentage pricing errors between the model and the market prices of the co-terminal European swaptions, i.e. the goodness-of-fit measure is

$$\min \sum_{i=1}^{n} \left(\frac{P_{i,n,\text{model}}}{P_{i,n,\text{market}}} - 1 \right)^2$$

where $P_{i,n,\text{market}}$ is the market price and $P_{i,n,\text{model}}$ is the model generated price of the $i \times (n-i)$ European swaptions, with $n = 11$ when models are calibrated to price 10×1 year Bermudan swaption and $n = 10$ when models are calibrated to

[6] Pietersz and Pelsser (2005) followed the same approach.

price 9×1 year Bermudan swaption. Instead of minimising the sum of squared percentage price errors alternatively we could have minimised the sum of squared errors in prices. However, such a minimisation strategy would place more weight on the expensive instruments. Minimisation of squared percentage pricing error is typically used as a goodness-of-fit measure for similar calibrations in literature and by practitioners.

7.0.4 NAG routine used for calibration

We need to use some optimization technique to solve the minimization problem mentioned in the last section. Various off-the shelf implementations are available for the commonly uses optimization algorithms. We have used an optimisation routine provided by the Numerical Algorithm Group (NAG) C library.

The NAG routine (e04unc) solves the non-linear least-squares problems using the sequential quadratic programming (SQP) method. The problem is assumed to be stated in the following form:

$$\min_{x \in R^n} F(x) = \frac{1}{2} \sum_{i=1}^{n} \{y_i - f_i(x)\}^2$$

where $F(x)$ (the objective function) is a nonlinear function which can be represented as the sum of squares of m sub-functions $(y_1 - f_1(x))$, $(y_2 - f_2(x))$, $\cdots$, $(y_n - f_n(x))$. The ys are constant. The user supplies an initial estimate of the solution, together with functions that define $f(x) = (f_1(x), f_2(x), \cdots, f_n(x))$ and as many first partial derivatives as possible; unspecified derivatives are approximated by finite differences.

In order to use this routine, for our calibration purpose we did following:

(i) Set 's to 1.

(ii) n = 10/9 , depending on the number of calibration instruments

(iii) $f_i(x) = \frac{P_{i,n,\text{model}}}{P_{i,n,\text{market}}}$

(iv) Set initial estimates for all the three parameters as 0.01 i.e. 1%.

(v) Set a lower bound of 0.0001 and upper bound of 1 for the three parameters to be estimated.

Because of the complexity of our objective function partial derivatives are not specified and therefore the routine itself approximates partial derivatives by finite differences.

7.1 Estimating mean reversion parameter (a)

As stated in the start of this section, the mean-reversion parameter has been estimated from historical data of interest rates. To measure the presence of mean reversion in interest rates we need large data set. This study spans over a period of one year, and within this one-year period models are calibrated every month. Hence it does not make sense to estimate this parameter for any economy on a monthly basis. Therefore, using historical data, once we estimate the value of this parameter and then use this value in all the tests.

For the rate of mean reversion parameter 'a' first order autocorrelation of the 1 month interest rate series has been used. Here we present the basic idea behind the estimation procedure used. Under the HW model, the continuous time representation of the short rate process is

$$dr_t = [\theta_t - a\ r_t]dt + \sigma_t dz,$$

The discrete-time version of this process would be

$$\begin{aligned} r_{t+1} - r_t &= [\theta_t - ar_t] + \varepsilon_{t+1}, \\ r_{t+1} &= \theta_t + (1-a)r_t + \varepsilon_{t+1} \end{aligned} \tag{46}$$

where ε_{t+1} is a drawing from a normal distribution. Equation (46) represents an AR(1) process.

An autoregressive (AR) process is one, where the current values of a variable depends only upon the values that variable took in previous periods plus an error term. A process y_t is autoregressive of order p if

$$y_t = \phi_0 + \phi_1 y_{t-1} + \phi_2 y_{t-2} + ... + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma^2).$$

An ordinary least square (OLS) estimate of coefficient $(1-a)$ in equation (46) $\hat{\beta}$ would be[7]

$$\begin{aligned} 1 - a &= \hat{\beta} = \frac{\rho\sigma(r_{t+1})\sigma(r_t)}{\sigma^2(r_t)} = \rho \\ a &= 1 - \rho \end{aligned}$$

where ρ is the correlation coefficient between r_{t+1} and r_t and can be easily calculated using Excel. For the BK model we perform this regression using time series of $\ln(r)$, where as before r, is 1M interest rate.

8 Hedging Bermudan Swaption

Changes in the term structure can adversely affect the value of any interest rate based asset or liability. Therefore, protecting fixed income securities from unfavourable term structure movements or hedging is one of the most demanding tasks for any financial institutions and for the ALM group in particular. Inefficient hedging strategies can cost big prices to these institutes. In order to protect a liability from possible future interest rate changes first one need to generate realistic scenarios and then need to know how the impacts of these scenarios can be neutralised. Thus, two important issues to be addressed by any interest rate risk management strategy are: (i) how to perturb the term structure to imitate possible term structure movements (ii) how to immunise the portfolio against these movements.

Next sections explain the methodologies applied in this study for

[7]For the regression $y_i = \alpha + \beta x_i + \varepsilon_i$,, the OLS estimate of β is

$$\hat{\beta}_{xy} = \frac{\rho_{xy}\sigma_x\sigma_y}{\sigma_{xy}^2}$$

(i) estimating the perturbations by which the input term structure had been bumped to simulate the possible future changes in the input term structure

(ii) selecting hedge instruments

(iii) delta-hedging the underlying Bermudan swaption using the selected instruments.

(iv) calculating possible profits and losses (P&L).

In each section we have examples from literature have been referred to justify the choices made.

8.1 Perturbing the term structure

Over the years researchers and practitioners have been using duration analysis for interest rate risk management, i.e. they shift the entire yield curve upward and downward in a parallel manner and then estimate how the value of their portfolio is affected as a result of these parallel perturbations. They then hedge themselves against these risks. Parallel shifts are unambiguously the most important kind of yield curve shift but alone cannot explain completely explain the variations of yield curve observed in market. Three most commonly observed term structure shifts are: *Parallel Shift* where the entire curve goes up or down by same amount; *Tilt*, also known as slope shift, in which short yields fall and long yields rise (or vice versa); *Curvature shift* in which short and long yields rise while mid-range yields fall (or vice versa). These three shifts together can explain almost all the variance present in any term-structure and thus and one should not completely rely on duration and convexity measures for estimating the risk sensitivity of a fixed income security. There are numerous examples in literature to support this argument. Here, we mention a few studies that lead to this conclusion. Litterman and Scheikman (1991) performed principal components analysis (PCA) and found that on average three factors, referred to as level (roughly parallel shift), slope, and curvature, can explain 98.4% of the variation on Treasury bond returns. They suggested that "by considering the effects of each of these three factors on a portfolio, one can achieve a better hedged position than by holding a only a zero-duration portfolio".[8] Knez, Litterman, and Scheikman (1994) investigated the common factors in money markets and again they found that on average three-factors can explain 86% of the total variation in most money market returns whereas on average four factors can explain 90% of this variation. Chen and Fu (2002) performed a PCA on yield curve and found that the first four factors capture over 99.99% of the yield curve variation. They too claimed that hedging against these factors would lead to a more stable portfolio and thus superior hedging performance.

Based on the findings of these studies, in this study we performed PCA on historical data for estimating more realistic term structure shifts. In this study we performed PCA on annual changes of the forward rates and used scores of the first three principal components for estimating the shifts by which we bumped the forward rate curves. There are not many examples of estimating price sensitivities w.r.t. multiple factors (like 3 principal components here) with a one-factor term structure model. Generally risk sensitivities are calculated by perturbing only the model intrinsic factors i.e. for the one-factor model, only one-factor is perturbed and so on.

PCA has been performed on annual changes of forward Libor rates. Annual changes have been used because each hedge is maintained for one year. The reason for using forward rates rather than yield curve for doing PCA is two folds: first forward Libor rates are directly observable in market. Second using forward curves easily we can construct zero-curve and swap curve (needed for estimating interest rate sensitivities of swaps that are used for hedging). Without going into the mathematical details of this procedure, here, we briefly describe how PCA has been done for estimating the term structure shifts/bumps in the study:[9]

[8] Litterman and Scheikman (1991, 54)

[9] For details on PCA refer http://csnet.otago.ac.nz/cosc453/student_tutorials/principal_components.pdf

Monthly observations of 11 forward Libor rates ($f_{0,0,1}, f_{0,1,2}, f_{0,2,3}, ..., f_{0,10,11}$) for the period Jan 1999 to Dec 2006 have been used for PCA. Explicitly these rates have been used because we give annual zero curve of maturity till 11 years as input to our short rat models and we want to estimate bumps for all these maturities.[10]

Using these monthly observations of the 11 forward rates we calculate annual changes for each of the 11 forward rates as follows. Suppose we have monthly observations of the 11 forward rates for a period of n years i.e. in total we have $12 \times n$ monthly observations of forward Libor rates $f_{0,\tau_j,\tau_j+1}(t_i)$, where $i = 1, 2, \cdots, 12 \times n$ and $\tau_j = 0, 1, 2, .., 10$. When the observations are arranged in ascending order (by date) then annual change for the forward Libors can be calculated as

$$cL_{i,j} = f_{0,\tau_j,\tau_j+1}(t_i + 12) - f_{0,\tau_j,\tau_j+1}(t_i)$$

now $i = 1, 2, \ldots, 12 \times (n-1)$. These values would form a 12×(n-1) x 11 matrix i.e. we have 12 less entries. To clear this, say t_i= Jan 1999, and $\tau_j = 0$. Then $f_{0,0,1}(t_i)$is value of $f_{0,0,1}$ on Jan 1999; $f_{0,0,1}(t_i + 12)$is the value of $f_{0,0,1}$ on Jan 2000 and cL is the one-year change in the market observed value of $f_{0,0,1}$.

The matrix $cL_{i,j}$ is given as input to the NAG routine (g03aac) that performs "principal component analysis" on the input data matrix and returns principal component loadings and the principal component scores. The other important statistics of the principal component analysis reported by the routine are : the eigen values associated with each of the principal components included in analysis and the proportion of variation explained by each principal component.

We used the scores of first three factors to calculate the three types of shifts for

[10] As we know that this dissertation is a part of a big project. The models to be implemented are LMM and SMM. These maturity forward rates are also needed by these two models.

the 11 forward Libor rates using following regression:

$$\Delta f_{0,\tau_j,\tau_j+1} = \alpha + \beta_{\tau_j,1}P_1 + \beta_{\tau_j,2}P_2 + \beta_{\tau_j,2}P_3 + \varepsilon$$

Where $\tau_j = 0, 1, 2, .., 10$ and P_k is the vector of scores for the kth factor $k = 1, 2, 3$. Nag routine (g02dac) has been employed to perform this regression. The routine computes parameter estimates, the standard errors of the parameter estimates, the variance–covariance matrix of the parameter estimates and the residual sum of squares.

After performing the regressions specified above, the 11 forward rates are bumped by shocks corresponding to the first three factors as:

$$f^{\pm}_{0,\tau_j,\tau_j+1} = f_{0,\tau_j,\tau_j+1} \pm \beta_{\tau_{j,k}}\Delta P_k$$

where $\tau_j = 0, 1, 2, .., 10$, and $k = 1, 2, 3$.

This gives us 6 new sets of term structure, along with the original input term structure for each of the days on which we will hedge the Bermudan swaption

8.2 Choosing the hedge instruments

Selecting appropriate hedge instruments is a critical part of a successful hedging strategy. In literature there are evidences of two hedging strategies, factor hedging and bucket hedging. For *factor hedging*, in a K-factor model, K different instruments (together with the money market account) are used to hedge any derivative. The choice of hedge instruments is independent of the derivative to be hedged i.e., the same K hedging instruments can be used for hedging any derivative in a K-factor model, and depend only on the number of factors in the model. For *bucket hedging* the choice of hedge instruments depend on the instrument to be hedged and not on the factors in the model. In this hedging strategy number of hedge instruments is equal to the number of total payoffs provided by the instrument. The hedge instruments are chosen so their maturities correspond to different payment or decision dates of the underlying derivative.

On using any other criteria for selecting the hedge instruments, the number of hedge instruments will lie between the numbers of hedge instruments for these two hedging strategies. Before discussing the instruments that are used to delta-hedge the Bermudan swaption in this study, in this section first we briefly discuss a few examples from literature. Driessen, Klaassen and Melenberg (2002) used (delta-) hedging of caps and swaptions as criteria for comparing the hedge performance of HJM class models and Libor market models. They used zero coupon bonds as hedge instruments. For each model, they considered factor and bucket hedging strategies. DKM show that when bucket strategies are used for hedging, the performance of the one-factor models improves significantly Fan, Gupta and Ritchken (2006) also used effectiveness of delta neutral hedges (for swaptions) as criteria for comparing the hedging performance of single factor and multi-factor factor term structure models. They used discount bonds to delta-hedge swaptions. In this study they first applied factor hedging for choosing the hedge instruments and found that in context of hedging performance, multifactor models outperform single factor models. Next in light of the DKM results, they repeated their experiments using additional hedging instruments. They found that for the one-factor and two factor models adding more instruments did not result into better hedge results. Pietersz and Pelsser (2005) compared joint delta-vega hedging performance (for 10x1 Bermudan swaption) of single factor Markov-functional and multi-factor market models. They used the bucket hedging strategy and set up hedge portfolios using 11 discount bonds, one discount bond for each tenor time associated with the deal. They found that joint delta-vega hedging performance of both models is comparable

The general implications of these examples are:

(i) Effectiveness of delta neutral hedges is often used to evaluate the hedging performance of term structure models.

(ii) Using multiple instruments can improve the hedge performance of one factor models. Practitioners also favour this practice

Therefore, in this study, we decide to use two different swaps of maturities 5 and 11years as hedge instruments. Maturity of 11-year swap coincides with the maturity of the co-terminal Bermudan swaption to be hedged and the length of other swap is almost half way the life of this Bermudan swaption. We could have used discount bonds to hedge this Bermudan swaption[11] but use of swap is more in line with the general practitioners practice. Studies suggest that large banks tend to use interest rate swaps more intensively for hedging.

8.3 Constructing delta hedged portfolio

To illustrate the idea behind deltas' calculations we first present a simple example that uses a call option on a stock. The delta (Δ) of an option is defined as the rate of change of the option price with respect to the price of the underlying asset. Mathematically, delta is the partial derivative of the option's value with respect to the price of the underlying stock.

$$\Delta = \frac{\partial C}{\partial S}$$

where C is the price of the call option and S is the stock price. From the Taylor expansion

$$C(S+\varepsilon) = C(s) + \varepsilon C'(S) + \frac{1}{2}\varepsilon^2 C''(S) + ...$$

where $C(S+\varepsilon)$ is the option price when value of the underlier has been changed by ε. When ε is small, the second-order term can be ignored and delta can be calculated as

$$\Delta = C'(S) = \frac{C(S+\varepsilon) - C(S)}{\varepsilon} \tag{46}$$

Delta hedging is the process of keeping the delta of a portfolio equal to or as close as possible to zero. Since delta measures the exposure of a derivative to changes in the value of the underlying, the overall value of a portfolio remains unchanged for small changes in the price of its underlying instrument. A delta hedged portfolio is established by buying or selling an amount of the underlier that corresponds to

[11] Pietersz and Pelsser (2005) used discount bonds to hedge Bermudan swaption

the delta of the portfolio. For example if the call option in this example has a delta of 0.25 then a delta neutral portfolio would consist of: Long 1 call and Short 0.25 shares. In this case the value of portfolio $=1 \times C - 0.25 \times S$.

If the price of stock rises by \$1.0, then according to definition of delta, the price of the call will rise by \$0.25. In this case the value of portfolio is

$$1 \times (C + 0.25) - 0.25 \times (S + 1.0) = C - 0.25 \times S.$$

Thus, the total value of the portfolio remains.

The same idea can be applied for estimating the interest rate risk sensitivity of a fixed income security. If the entire initial term structure is perturbed by same amount say ε (parallel shift), then like in equation (46) the risk sensitivity of a fixed income security w.r.t. this perturbation can be estimated as

$$\frac{V(\varepsilon) - V}{\varepsilon}$$

where V is the value of the derivative calculated using initial term structure and $V(\varepsilon)$ is the value of the derivative after the initial term structure is perturbed by ε. If we first increase the entire initial term structure by ε and then next decrease it by ε, then the risk sensitivity can estimated as

$$\frac{V(\varepsilon^+) - V(\varepsilon^-)}{2\varepsilon} \tag{47}$$

where $V(\varepsilon^+)$ is the value of the derivative calculated after initial term structure has been shifted up by ε and $V(\varepsilon^-)$ is the value of the derivative after the initial term structure has been shifted down by ε. In our case we have bumped the initial forward rate curve by three factors. Also for each factor, we have bumped the forward rate curve both up and down. Using the idea presented in equation (47), we estimate the sensitivity (delta) of the Bermudan swaption and the two swaps w.r.t the three factors as follows

$$\begin{aligned}
\Delta_k^{Berm} &= \frac{\partial BSwn}{\partial P_k} = \frac{BSwn_k^+ - BSwn_k^-}{2\Delta P_k} \\
\Delta_k^{S_1} &= \frac{\partial S_1}{\partial P_k} = \frac{(S_1)_k^+ - (S_1)_k^-}{P_k^+ - P_k^-} = \frac{(S_1)_k^+ - (S_1)_k^-}{2\Delta P_k} \\
\Delta_k^{S_5} &= \frac{\partial S_5}{\partial S_k} = \frac{(S_5)_k^+ - (S_5)_k^-}{P_k^+ - P_k^-} = \frac{(S_5)_k^+ - (S_5)_k^-}{2\Delta P_k} \\
\Delta_k^{S_{11}} &= \frac{\partial S_{11}}{\partial P_k} = \frac{(S_{11})_k^+ - (S_{11})_k^-}{P_k^+ - P_k^-} = \frac{(S_{11})_k^+ - (S_{11})_k^-}{2\Delta P_k}
\end{aligned}$$

for $k = 1, 2, 3$, and $(.)_k^+$ and $(.)_k^-$ respectively are the prices of derivative after the initial forward curve has been bumped up and down by the k^{th} factor. Now if we consider a portfolio, consisting of one 10×1 Bermudan swaption, x_{11} units of 11-year swap and x_5 units of 5-year swap, then total delta mismatch of this portfolio w.r.t. k^{th} factor, ε_k is

$$\Delta\varepsilon_k = \Delta_k^{BSwn} - x_{11}\Delta_k^{S_{11}} - x_5\Delta_k^{S_5} - x_1\Delta_k^{S_1} \tag{10}$$

assuming that x_{11}, x_5 and x_1 need not to be whole numbers. From equation (10), we can see that when we use more than one hedging instrument in a one-factor model, the hedge ratios would not be unique, and some rule must be applied for constructing the hedge portfolio using the chosen hedge instruments. Here, to obtain the hedge ratios, x_{11}, x_5 and x_1, we use the basic idea behind the delta hedging. The three hedge ratios x_{11}, x_5 and x_1 are obtained by minimizing the total delta-mismatch of the portfolio with respect to the first three PCA factors, i.e.

$$\min_{x_{11}, x_5, x_1} \sum_{k=1}^{3} (\Delta\varepsilon_k)^2$$

where $\Delta\varepsilon_k$ is given by equation (10).

8.4 Calculating P&L

Once hedge has been established on say day t, the hedge error can be evaluated one year later on day $t+1$ as followsOnce the hedging portfolio has been established on date t, the hedging error can be evaluated one year later at $t+1$ as follows:

$$\begin{aligned} P\&L_{t+1} &= (BSwn_t - x_{11,t}S_{11,t} - x_{5,t}S_{5,t} - x_{1,t}S_{1,t}) \times (1+y_{0,t}) \\ &\quad -(BSwn_{t+1} - x_{11,t}S_{10,t+1} - x_{5,t}S_{4,t+1} - x_{1,t}S_{0,t+1}) \qquad (11) \end{aligned}$$

where $BSwn_t$ is the value of the Bermudan swaption on day t; $x_{\tau,t}$ are units of τ-year swap in the hedge portfolio; $S_{\tau,t}$ is the value of τ-year swap on day t; and $y_{0,t}$ is the current 1-year yield at t.

Since, at the point of initiation, the value of any swap is zero, this means that $S_{11,t} = S_{5,t} = S_{1,t} = 0$. Therefore, equation (11) can be written as

$$P\&L_{t+1} = BSwn_t \times (1+y_{0,t}) - BSwn_{t+1} + x_{11,t}S_{10,t+1} + x_{5,t}S_{4,t+1} + x_{1,t}S_{0,t+1}. \quad (12)$$

To calculate the price of $BSwn_{t+1}$, which is now a 9×1 Bermudan swaption, we recalibrate at $t+1$, the interest rate models and use the new calibrated parameter values to calculate the model price of this possibly away-from-money 9×1 Bermudan swaption $BSwn_{t+1.}$ Strike rate for this 9×1 Bermudan swaption is kept the same as it was for the 10×1 Bermudan swaption on day t, as the objective is to find the current value of that old swaption. We also use the $t+1$ term structure to calculate the values of the three swaps at $t+1$.

where B_t is the value of a 10×1 Bermudan swaption on day t; $x_{k,t}$ are units of k-year swap in the hedge portfolio; $S_{k,t}$ is the value of k-year swap on day t; and $i_{0,t:}$ is the forward Libor $f_{0,0,1}$ on day t.

At the point of initiation the value of any swap is zero. This means $S_{11,t} = 0$ and also $S_{5,t} = 0$. Therefore, equation (49) can be written as

$$P\&L_{t+1} = BSwn_t \times (1+y_{0,t}) - BSwn_{t+1} + x_{11,t}S_{10,t+1} + x_{5,t}S_{4,t+1} + x_{1,t}S_{0,t+1}. \quad (13)$$

One year later, on day $t+1$, when hedge portfolio is unwound the 10×1 Bermudan swaption is a 9×1 Bermudan swaption. On day $t+1$, we recalibrate the model and using new parameters calculate the model price of this away-from-money 9×1 Bermudan swaption $B_{t+1.}$ Strike rate for the 9×1 Bermudan swaption is kept same as it was for the 10×1 Bermudan swaption on day t, as objective is to find the current value of that old swaption. We also calculate the values of the two swaps on day $t+1$ Using the example of 5-year swap we show, how the swaps are re-evaluated one year later on day $t+1$.

At the time of swap initialisation, $i_{0,t}$ is known

$$i_{0,t} = \left\{ \frac{1}{P(0,1)} - 1 \right\}$$

and the swap rate $k_{5,t}$ for a 5-year swap on day t and can be calculated as

$$k_{5,t} = \frac{1 - P(0,5)}{P(0,1) + P(0,2) + ... + P(0,5)}$$

At time $t+1$, this 5-year swap is just a 4-year swap and also floating rate $i_{0,t+1}$ for the next reset is now known. The value of this swap at $t+1$ would be

$$\begin{aligned}
S_{5,t+1} &= (i_{0,t} - k_5) + (i_{0,t+1} - k_5)P(0,1) + (\tilde{\imath}_{1,t+1} - k_5)P(0,2) \\
&\quad +(\tilde{\imath}_{2,t+1} - k_5)P(0,3) + (\tilde{\imath}_{3,t+1} - k_5)P(0,4) \\
&= (i_{0,t} - k_5) + \left[\left\{ \frac{1}{P(0,1)} - 1 \right\} - k_5 \right] P(0,1) \\
&\quad + \left[\left\{ \frac{P(0,1)}{P(0,2)} - 1 \right\} - k_5 \right] P(0,2) + \left[\left\{ \frac{P(0,2)}{P(0,3)} - 1 \right\} - k_5 \right] P(0,3) \\
&\quad + \left[\left\{ \frac{P(0,3)}{P(0,4)} - 1 \right\} - k_5 \right] P(0,4) \\
&= (i_{0,t} - k_5) + [1 - P(0,4)] - k_5 [P(0,1) + P(0,2) + P(0,3) + P(0,4)] \\
&= (i_{0,t} - k_5) + \left[\left\{ \frac{1 - P(0,4)}{P(0,1) + P(0,2) + P(0,3) + P(0,4)} \right\} - k_5 \right] \\
&\quad [P(0,1) + P(0,2) + P(0,3) + P(0,4)] \\
&= (i_{0,t} - k_5) + [k_{4,t+1} - k_5] [P(0,1) + P(0,2) + P(0,3) + P(0,4)]
\end{aligned}$$

where $k_{4,t+1}$ would be the swap rate for any 4-year swap that would be initiated on day $t+1$. When the swap is re-evaluated on day $t+1$ all the discount bonds' prices are computed using the yield curve observed on that day.

9 Results

In this section, we report the results of our study. The notional principal is 1 for all the contracts considered in this study.

9.1 Calibration

In this study we have used the two short rate models with time varying volatility parameter. In Section 6 we have discussed the pros and cons of such a parameterisation in detail. In Table 1 and Figure 2 we present one set of pricing errors for the European swaptions obtained by using two alternative parameterisations.

As expected, we can see that introducing an extra time-vary parameter significantly improves the fit. This, thereby support our parameterisation. In the rest of this section we only report results for the models with time–varying σ parameter. The "mean-reversion" parameter is kept constant for the two models.

Though neither of the two models could be tuned to exactly replicate the market prices of the instruments that have been used for the calibration but the RMSE are quite low for both models. But as compared to the BK model, the HW model has been better fitted to the market observed prices of European swaptions.

9.2 Parameter estimates

9.2.1 Rate of mean-reversion

For both models the rate of mean reversion parameter (a) has been estimated using historical data of 1 M yields for Euro and USD. The estimated values of this parameter are:

	HW	BK
EUR	0.01	0.0087
USD	0.006	0.006

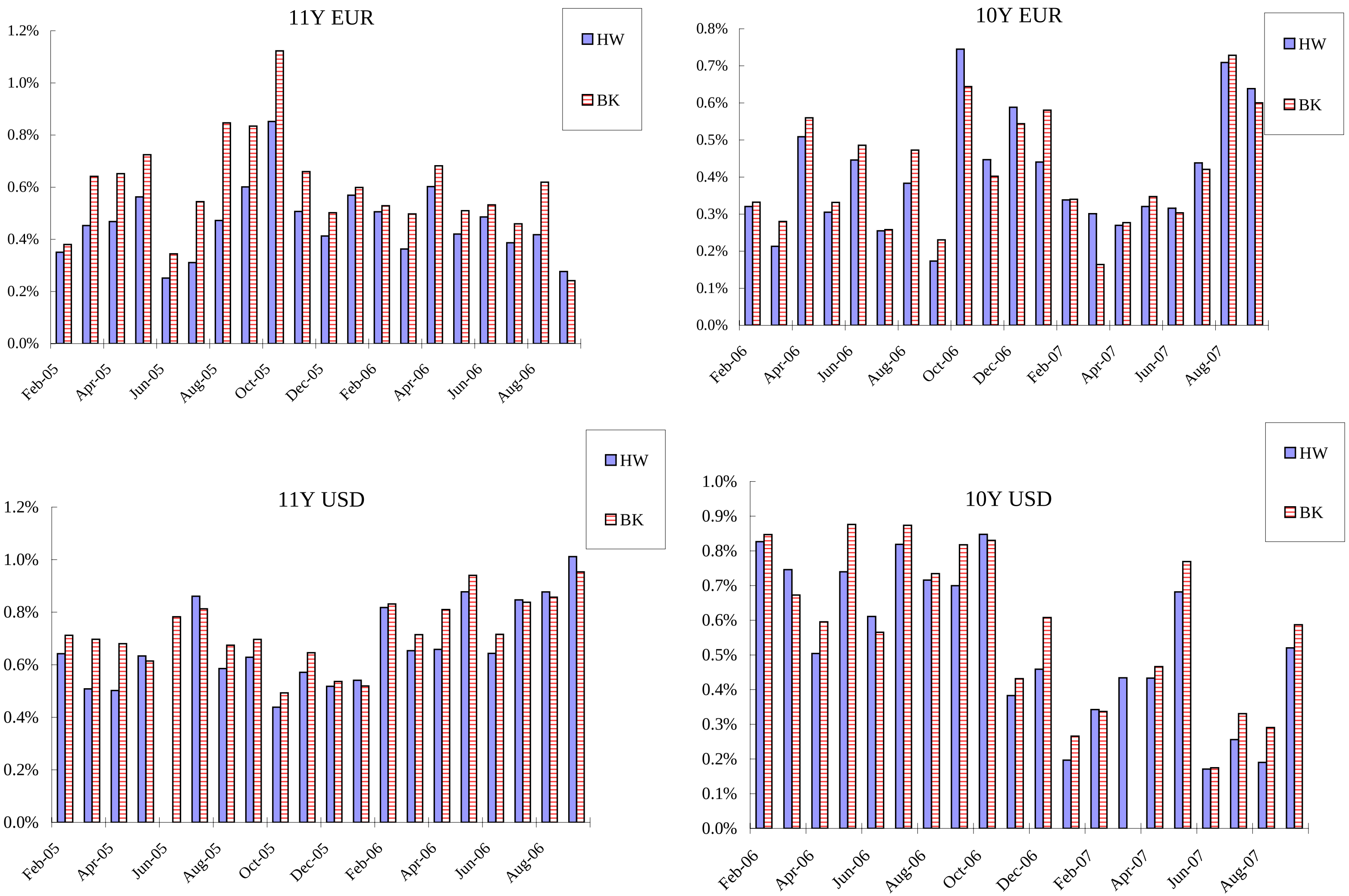
Table 1: Root mean square error to European co-terminal swaptions (11Y and 10Y) in EUR and USD markets from February 2005 to September 2007

	EUR	11Y			USD	11Y
Date	HW	BK		Date	HW	BK
2005-2-28	0.4%	0.4%		2005-2-28	0.6%	0.7%
2005-3-31	0.5%	0.6%		2005-3-31	0.5%	0.7%
2005-4-29	0.5%	0.7%		2005-4-29	0.5%	0.7%
2005-5-31	0.6%	0.7%		2005-5-31	0.6%	0.6%
2005-6-30	0.3%	0.3%		2005-6-30	0.0%	0.8%
2005-7-29	0.3%	0.5%		2005-7-29	0.9%	0.8%
2005-8-31	0.5%	0.8%		2005-8-31	0.6%	0.7%
2005-9-30	0.6%	0.8%		2005-9-30	0.6%	0.7%
2005-10-31	0.9%	1.1%		2005-10-31	0.4%	0.5%
2005-11-30	0.5%	0.7%		2005-11-30	0.6%	0.6%
2005-12-30	0.4%	0.5%		2005-12-30	0.5%	0.5%
2006-1-31	0.6%	0.6%		2006-1-31	0.5%	0.5%
2006-2-28	0.5%	0.5%		2006-2-28	0.8%	0.8%
2006-3-31	0.4%	0.5%		2006-3-31	0.7%	0.7%
2006-4-28	0.6%	0.7%		2006-4-28	0.7%	0.8%
2006-5-31	0.4%	0.5%		2006-5-31	0.9%	0.9%
2006-6-30	0.5%	0.5%		2006-6-30	0.6%	0.7%
2006-7-31	0.4%	0.5%		2006-7-31	0.8%	0.8%
2006-8-31	0.4%	0.6%		2006-8-31	0.9%	0.9%
2006-9-29	0.3%	0.2%		2006-9-29	1.0%	1.0%
Total	9.3%	11.9%		Total	12.8%	14.5%

	EUR	10Y			USD	10Y
Date	HW	BK		Date	HW	BK
2006-2-28	0.3%	0.3%		2006-2-28	0.8%	0.8%
2006-3-31	0.2%	0.3%		2006-3-31	0.7%	0.7%
2006-4-28	0.5%	0.6%		2006-4-28	0.5%	0.6%
2006-5-31	0.3%	0.3%		2006-5-31	0.7%	0.9%
2006-6-30	0.4%	0.5%		2006-6-30	0.6%	0.6%
2006-7-31	0.3%	0.3%		2006-7-31	0.8%	0.9%
2006-8-31	0.4%	0.5%		2006-8-31	0.7%	0.7%
2006-9-29	0.2%	0.2%		2006-9-29	0.7%	0.8%
2006-10-31	0.7%	0.6%		2006-10-31	0.8%	0.8%
2006-11-30	0.4%	0.4%		2006-11-30	0.4%	0.4%
2006-12-29	0.6%	0.5%		2006-12-29	0.5%	0.6%
2007-1-30	0.4%	0.6%		2007-1-30	0.2%	0.3%
2007-2-28	0.3%	0.3%		2007-2-28	0.3%	0.3%
2007-3-31	0.3%	0.2%		2007-3-31	0.4%	0.0%
2007-4-30	0.3%	0.3%		2007-4-30	0.4%	0.5%
2007-5-31	0.3%	0.3%		2007-5-31	0.7%	0.8%
2007-6-30	0.3%	0.3%		2007-6-30	0.2%	0.2%
2007-7-29	0.4%	0.4%		2007-7-29	0.3%	0.3%
2007-8-31	0.7%	0.7%		2007-8-31	0.2%	0.3%
2007-9-28	0.6%	0.6%		2007-9-28	0.5%	0.6%
Total	8.2%	8.3%		Total	10.6%	11.1%

Note: HW stands for Hull-White model, BK stands for Black-Karasinski model. '11Y' denotes calibration results for 11Y co-terminal ATM European swaptions from Feburary 2005 to September 2006; '10Y' denotes calibration results for 10Y co-terminal European swaptions from Feb 2006 to Sep 2007. 'EUR' denotes Euro market and 'USD' denotes US-dollar market.

Figure 1: Root mean square error of prices by date from calibration to 11Y co-terminal European swaptions over the period of February 2005 to September 2006



Note: HW stands for Hull-White model; BK stands for Black-Karasinski model.

9.2.2 Volatility

To compare the stability of the parameters, the results for the HW and BK models in EUR and USD market are displayed in Table 2 and Table 3 respectively. To further facilitate the comparison, summary statistics are reported in Figure 2.

At first glance, we can note that the magnitudes of the volatility parameter for the BK model are a lot higher than for the HW model. This is due to the different functional forms of the two models. In case of the HW model the volatility parameter corresponds to the standard deviation of annual changes in the short-term interest rate, whereas in case of the BK model the volatility parameter is the standard deviation of proportional changes in the rate. As explained by Hull-White (2000, p15), "if interest rates are about 7%, a 1.4% annual standard deviation roughly corresponds to an annual standard deviation of proportional changes of 20%".

For the Hull-While model, except for a few days like date February 28, 2005 and June 2005, the three volatility parameters as seen on each day are very close to each other. Also with time, i.e. on different days the parameters stay relatively constant. In contrast, for the BK model the parameters are not that stable, especially $\sigma(0)$ has been changing by big amounts. For both models, parameters are more stable in year 2006.

The objective of this empirical study is to compare the two models from the ALM perspective. Parameter stability is an important attribute from the ALM's perspective. As far as ALM is concerned, ceteris paribus, a good model should not lead to frequent drastic scenario changes. For our sample period, based on the stability of parameters, we can conclude that the Hull-White model is better that the lognormal BK model.

Table 2: Model calibrated parameter values for EUR market from February 2005 to September 2006

	HW (EUR)					BK (EUR)			
Date	a	σ0	σ3	σ11	Date	a	σ0	σ3	σ11
2005-2-28	0.0100	0.0062	0.0062	0.0072	2005-2-28	0.0087	0.1642	0.1328	0.1521
2005-3-31	0.0100	0.0065	0.0061	0.0067	2005-3-31	0.0087	0.1740	0.1312	0.1405
2005-4-29	0.0100	0.0061	0.0060	0.0064	2005-4-29	0.0087	0.1716	0.1337	0.1351
2005-5-31	0.0100	0.0064	0.0064	0.0059	2005-5-31	0.0087	0.1860	0.1465	0.1229
2005-6-30	0.0100	0.0069	0.0063	0.0064	2005-6-30	0.0087	0.2123	0.1488	0.1412
2005-7-29	0.0100	0.0063	0.0064	0.0066	2005-7-29	0.0087	0.1867	0.1530	0.1401
2005-8-31	0.0100	0.0063	0.0066	0.0065	2005-8-31	0.0087	0.1966	0.1654	0.1406
2005-9-30	0.0100	0.0063	0.0066	0.0064	2005-9-30	0.0087	0.1933	0.1719	0.1348
2005-10-31	0.0100	0.0065	0.0067	0.0071	2005-10-31	0.0087	0.1856	0.1657	0.1562
2005-11-30	0.0100	0.0066	0.0068	0.0065	2005-11-30	0.0087	0.1864	0.1708	0.1397
2005-12-30	0.0100	0.0064	0.0068	0.0064	2005-12-30	0.0087	0.1882	0.1850	0.1425
2006-1-31	0.0100	0.0061	0.0067	0.0058	2006-1-31	0.0087	0.1677	0.1705	0.1257
2006-2-28	0.0100	0.0059	0.0065	0.0059	2006-2-28	0.0087	0.1627	0.1659	0.1310
2006-3-31	0.0100	0.0061	0.0063	0.0060	2006-3-31	0.0087	0.1539	0.1503	0.1300
2006-4-28	0.0100	0.0063	0.0061	0.0065	2006-4-28	0.0087	0.1523	0.1366	0.1354
2006-5-31	0.0100	0.0063	0.0061	0.0062	2006-5-31	0.0087	0.1523	0.1350	0.1213
2006-6-30	0.0100	0.0063	0.0062	0.0062	2006-6-30	0.0087	0.1484	0.1373	0.1231
2006-7-31	0.0100	0.0062	0.0062	0.0060	2006-7-31	0.0087	0.1505	0.1406	0.1205
2006-8-31	0.0100	0.0063	0.0063	0.0060	2006-8-31	0.0087	0.1597	0.1500	0.1266
2006-9-29	0.0100	0.0064	0.0063	0.0060	2006-9-29	0.0087	0.1638	0.1552	0.1283

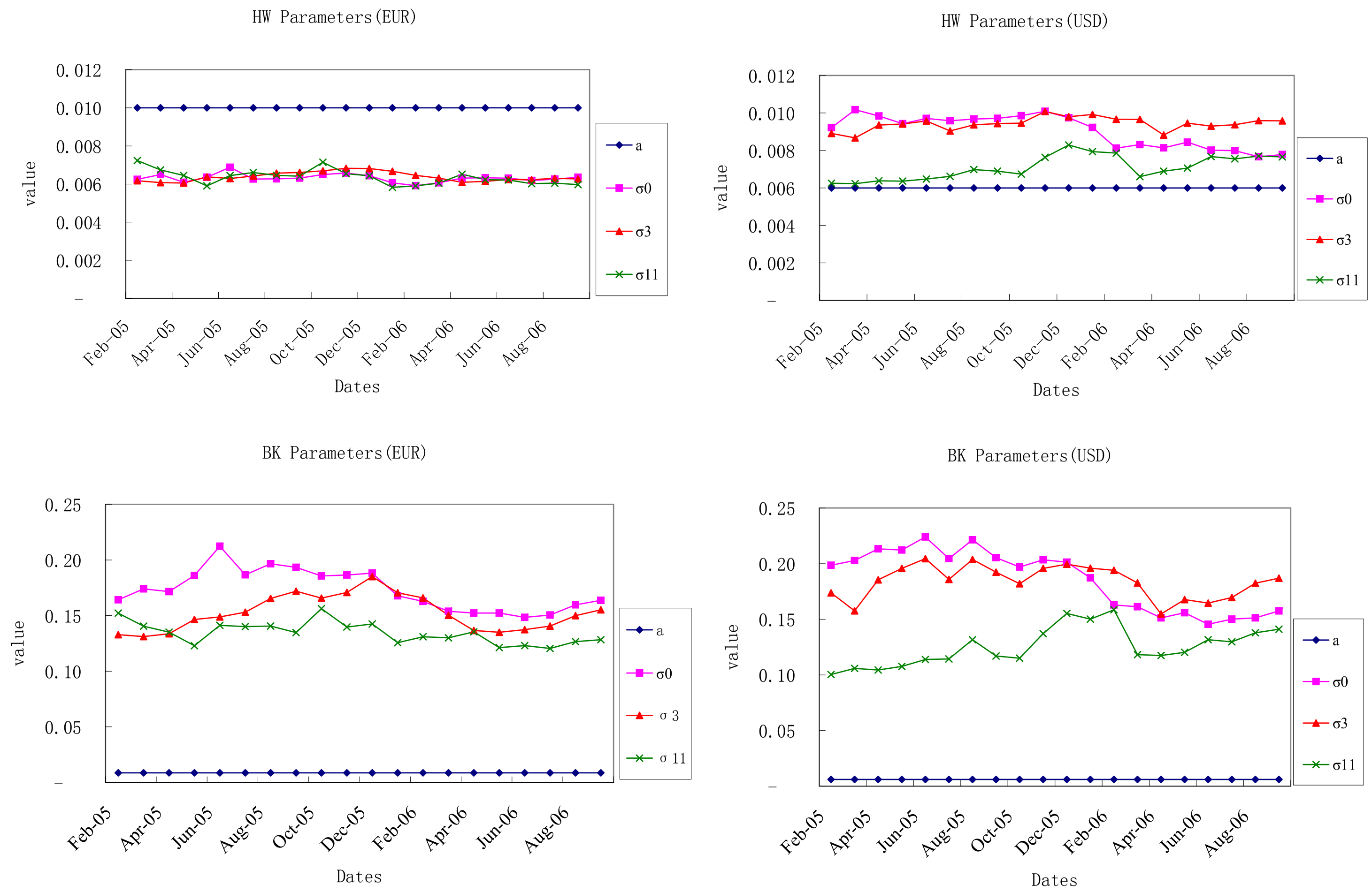
Note: HW stands for Hull-White model, BK stands for Black-Karasinski model. '11Y' denotes calibration results for 11Y co-terminal ATM European swaptions from Feburary 2005 to September 2006; '10Y' denotes calibration results for 10Y co-terminal European swaptions from Feb 2006 to Sep 2007. 'EUR' denotes Euro market and 'USD' denotes US-dollar market.

Table 3: Model calibrated parameter values for USD market from February 2005 to September 2006

	HW (USD)					BK (USD)			
Date	a	σ_0	σ_3	σ_{11}	Date	a	σ_0	σ_3	σ_{11}
2005-2-28	0.0060	0.0092	0.0089	0.0062	2005-2-28	0.0060	0.1985	0.1737	0.1004
2005-3-31	0.0060	0.0102	0.0087	0.0062	2005-3-31	0.0060	0.2028	0.1576	0.1058
2005-4-29	0.0060	0.0098	0.0094	0.0064	2005-4-29	0.0060	0.2133	0.1854	0.1044
2005-5-31	0.0060	0.0094	0.0094	0.0064	2005-5-31	0.0060	0.2123	0.1957	0.1076
2005-6-30	0.0060	0.0097	0.0096	0.0065	2005-6-30	0.0060	0.2239	0.2044	0.1138
2005-7-29	0.0060	0.0096	0.0090	0.0066	2005-7-29	0.0060	0.2045	0.1858	0.1144
2005-8-31	0.0060	0.0097	0.0094	0.0070	2005-8-31	0.0060	0.2214	0.2037	0.1316
2005-9-30	0.0060	0.0097	0.0094	0.0069	2005-9-30	0.0060	0.2053	0.1923	0.1170
2005-10-31	0.0060	0.0099	0.0095	0.0067	2005-10-31	0.0060	0.1969	0.1819	0.1150
2005-11-30	0.0060	0.0101	0.0101	0.0076	2005-11-30	0.0060	0.2034	0.1958	0.1371
2005-12-30	0.0060	0.0097	0.0098	0.0083	2005-12-30	0.0060	0.2013	0.1995	0.1552
2006-1-31	0.0060	0.0092	0.0099	0.0079	2006-1-31	0.0060	0.1872	0.1959	0.1502
2006-2-28	0.0060	0.0081	0.0097	0.0079	2006-2-28	0.0060	0.1628	0.1940	0.1586
2006-3-31	0.0060	0.0083	0.0097	0.0066	2006-3-31	0.0060	0.1613	0.1827	0.1182
2006-4-28	0.0060	0.0081	0.0088	0.0069	2006-4-28	0.0060	0.1513	0.1548	0.1175
2006-5-31	0.0060	0.0084	0.0095	0.0071	2006-5-31	0.0060	0.1559	0.1677	0.1202
2006-6-30	0.0060	0.0080	0.0093	0.0077	2006-6-30	0.0060	0.1455	0.1645	0.1316
2006-7-31	0.0060	0.0080	0.0094	0.0076	2006-7-31	0.0060	0.1501	0.1696	0.1297
2006-8-31	0.0060	0.0077	0.0096	0.0077	2006-8-31	0.0060	0.1513	0.1824	0.1378
2006-9-29	0.0060	0.0078	0.0096	0.0077	2006-9-29	0.0060	0.1574	0.1869	0.1411

Note: HW stands for Hull-White modEL; BK stands for Black-Karasinski model. a in HW and BK models is the mean reversion rate, and is assumed to be sconstant through out the sample period. The parameter values are obtained by calibrating 11-Y co-terminal European swaptoins.

Figure 2: Root mean square error of prices by date from calibration to 11Y co-terminal European swaptions over the period of February 2005 to September 2006



Note: HW stands for Hull-White model; BK stands for Black-Karasinski model. a in HW and BK model is the mean reversion rate. We calibrated to co-terminal 11Y European swaptoins from February 2005 to September 2006. All calibration are performed with both EUR and USD market.

Table 4: PCA factor loadings of Libor rates from January 2000 to September 2007.

		EUR			USD	
Rates	PCA1(EUR)	PCA2(EUR)	PCA3(EUR)	PCA1(USD)	PCA2(USD)	PCA3(USD)
1	0.36	-0.63	0.64	0.47	-0.69	0.49
2	0.43	-0.34	-0.29	0.46	-0.22	-0.39
3	0.41	-0.09	-0.40	0.38	0.03	-0.41
4	0.35	0.05	-0.28	0.30	0.13	-0.34
5	0.32	0.13	-0.14	0.28	0.22	-0.09
6	0.28	0.21	-0.02	0.25	0.23	-0.01
7	0.27	0.26	0.12	0.23	0.25	0.15
8	0.20	0.31	0.14	0.20	0.27	0.21
9	0.20	0.29	0.19	0.20	0.27	0.28
10	0.19	0.29	0.29	0.17	0.27	0.33
11	0.18	0.28	0.32	0.17	0.27	0.26

Table 5: Explainary power of the first three principle components.

EUR	Percentage variance	Cumulative variance	USD	Percentage vararíance	Cumulative vararíance
PCA1	82.37%	82.37%	PCA1	74.22%	74.22%
PCA2	13.87%	96.24%	PCA2	20.59%	94.81%
PCA3	2.81%	99.05%	PCA3	2.96%	97.77%

9.3 Principal Component Analysis

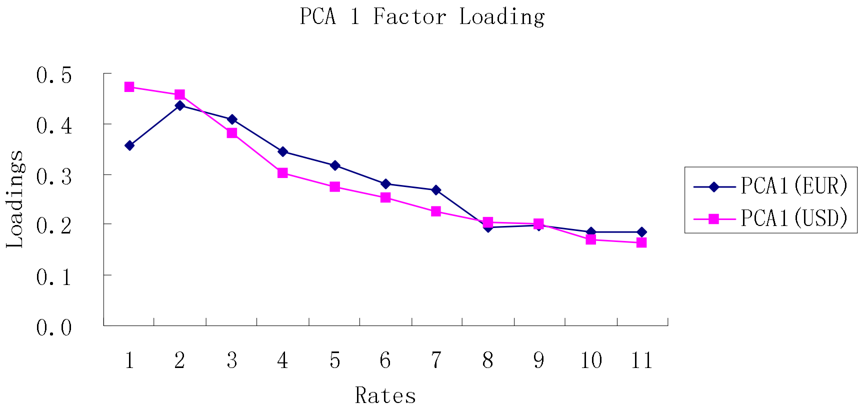
The statistics indicate that the first three factors explain about 99.24% of the forward rates curve changes, and the first five factors explain about 97.77% of the total variance of the forward rates curve.

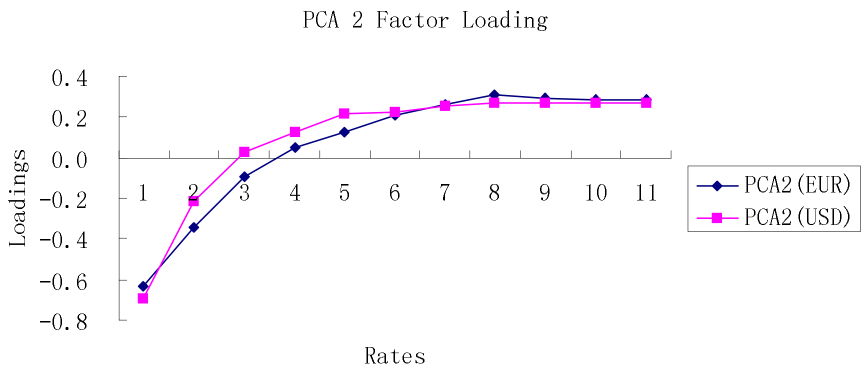
Table 4 and 5 and Figure 3 and 4 display the loadings of the first three factors. We have used absolute of mean of scores for computing the shocks. We could have used mean of scores or standard deviation of scores or some other statistical criteria. Litterman and Scheikman (1991) used one standard deviation to compute the shocks. The values show that for our data set the standard deviations are quite large and can estimate rather unrealistic shocks, thereby we used mean absolute of scores for estimating the forward rate bumps.

9.4 Bermudan Swaption Prices and Hedging Results

In this section, first we report the Bermudan swaption prices computed on the first seven trading days of our sample period. The Bermudan swaption used for the

Figure 3: Factor loadings for the first three principle components of forward rates term structure in the EUR and USD markets





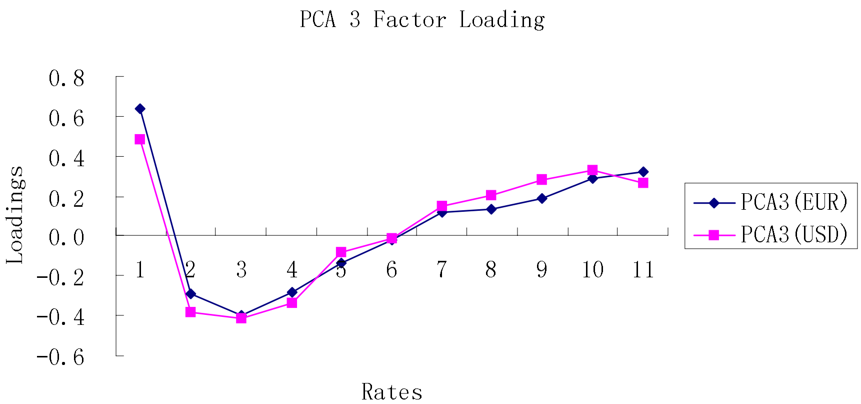
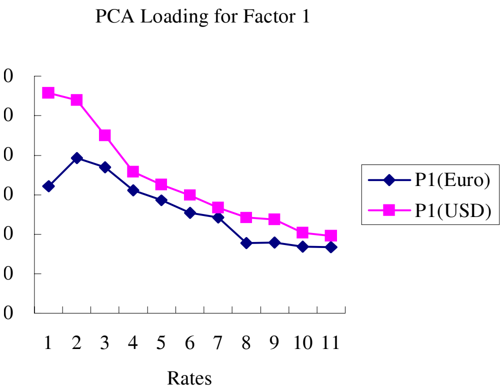
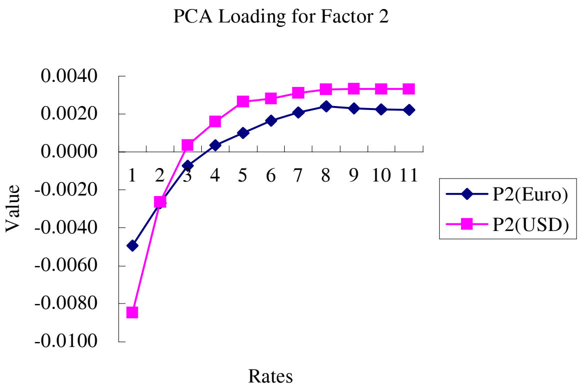
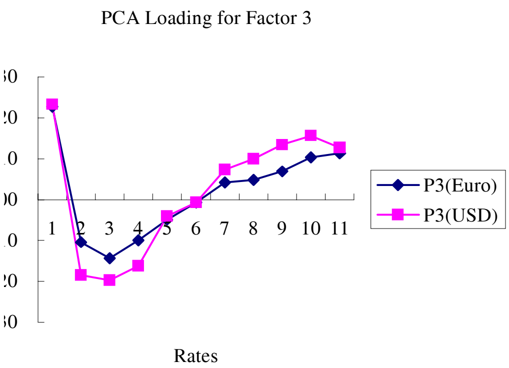


Figure 4: PCA factor loadings (mean of the absolute change) for the first three principle components in EUR and USD markets.







Mean of first Three Principal Components

	P1	P2	P3
EUR	0.0180562	0.00783554	0.00357607
USD	0.0236124	0.0122442	0.00478759

comparison test is a 10×1 payer's Bermudan swaption which is exercisable yearly, i.e. it can be exercised at $\{1yr,\ 2yr,\ \cdots,\ 10yr\}$ points at its life. Notional principal is 1. When started these Bermudan swaptions are at-the-money, so strike equals the "rate on a forward start swap starting in year one and terminating in year eleven" i.e. $S_{1,11}(0)$.

The Bermudan swaption prices are given in Table 6 for Euro market and 7 for USD market. As can be seen, the prices produced from the two models, stay quite close to each other on all trade days. Also in all the observations, the Hull-White model prices are higher than the Black-Karasinski model prices. A price comparison has also been displayed in Figure 5.

As explained in Section 8.1, on all the seven trade days the forward curve is bumped up and down by the shocks corresponding to first three factors. From the bumped forward curves, we calculate bumped discount factors and from there we extract the continuously compounded zero yields. Using these six new sets of zero curves, for each day, we calculate the six bumped Bermudan swaption prices. We also compute the sensitivities of the two swaps w.r.t. the three factors. To do this we re-evaluate 5 year and 11 year swaps on each of these days, corresponding to these six bumps. The idea of delta-hedging the underlying Bermudan swaption implies that we need to construct a swaps' portfolio that can offset the Bermudan swaption value change caused by the perturbations. Using the price sensitivities of Bermudan swaption, and of the two swaps with respect to the three factors, hedge ratios are computed using the procedure explained in Section 8.3.

Each of the seven 10×1 Bermudan swaptions are re-evaluated one year after their trade day. At this time all these Bermudan swaptions are away-from-money and are now 9×1 Bermudan swaptions. In order to compute these 9 x 1 Bermudan swaptions' prices models are re-calibrated using the latest market prices of $\{1 \times 9, 2 \times 8, \cdots, 9 \times 1\}$ ATM European swaptions. These Bermudan swaptions are reported in Table 6, 7 and Figure 5.

A year later, swaps used for the hedging are also re-evaluated. To calculate P&L

Table 6: 11Y and 10Y Bermudan swaption prices in EUR market from February 2005 to September 2007.

DATE(11Y)	HW	BK	AVERAGE
2005-2-28	0.0465	0.0441	0.0453
2005-3-31	0.0467	0.0441	0.0454
2005-4-29	0.0476	0.0447	0.0462
2005-5-31	0.0491	0.0460	0.0475
2005-6-30	0.0508	0.0474	0.0491
2005-7-29	0.0485	0.0455	0.0470
2005-8-31	0.0486	0.0455	0.0471
2005-9-30	0.0464	0.0438	0.0451
2005-10-31	0.0459	0.0437	0.0448
2005-11-30	0.0452	0.0432	0.0442
2005-12-30	0.0429	0.0414	0.0422
2006-1-31	0.0411	0.0398	0.0404
2006-2-28	0.0396	0.0384	0.0390
2006-3-31	0.0386	0.0376	0.0381
2006-4-28	0.0387	0.0377	0.0382
2006-5-31	0.0396	0.0384	0.0390
2006-6-30	0.0386	0.0376	0.0381
2006-7-31	0.0387	0.0376	0.0382
2006-8-31	0.0389	0.0379	0.0384
2006-9-29	0.0381	0.0373	0.0377

DATE(10Y)	**HW**	**BK**	**Strike**
2006-2-28	0.0271	0.0272	0.0402
2006-3-31	0.0379	0.0367	0.0395
2006-4-28	0.0534	0.0510	0.0377
2006-5-31	0.0610	0.0582	0.0363
2006-6-30	0.0744	0.0720	0.0344
2006-7-31	0.0610	0.0584	0.0356
2006-8-31	0.0600	0.0573	0.0339
2006-9-29	0.0545	0.0522	0.0344
2006-10-31	0.0417	0.0404	0.0369
2006-11-30	0.0390	0.0378	0.0371
2006-12-29	0.0563	0.0543	0.0357
2007-1-31	0.0526	0.0508	0.0377
2007-2-28	0.0443	0.0427	0.0378
2007-3-30	0.0379	0.0332	0.0409
2007-4-30	0.0308	0.0301	0.0431
2007-5-31	0.0391	0.0380	0.0433
2007-6-29	0.0476	0.0461	0.0442
2007-7-31	0.0487	0.0471	0.0428
2007-8-31	0.0510	0.0492	0.0408
2007-9-28	0.0578	0.0557	0.0401

Note: HW stands for Hull-White model, BK stands for Black-Karasinski model. 11-y Bermudan swaption are priceed as at-the-money from February 2005 to September 2006; 10 year Bermudan swaptions are priced with the corresponding 11-y ATM strikes one year ago.10-y Bermudan swaption are priced from February 2006 to September 2007.

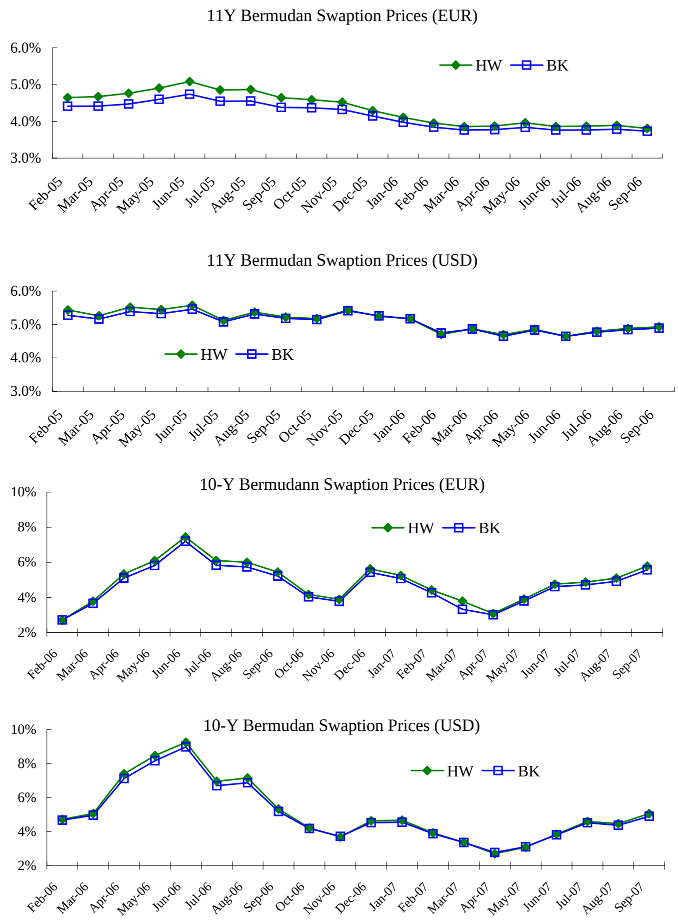
Table 7: 11Y and 10Y Bermudan swaption prices in USD market from February 2005 to September 2007.

DATE(11Y)	HW	BK	AVERAGE
2005-2-28	0.0543	0.0528	0.0535
2005-3-31	0.0526	0.0516	0.0521
2005-4-29	0.0552	0.0539	0.0545
2005-5-31	0.0545	0.0532	0.0539
2005-6-30	0.0557	0.0546	0.0552
2005-7-29	0.0512	0.0508	0.0510
2005-8-31	0.0537	0.0531	0.0534
2005-9-30	0.0522	0.0518	0.0520
2005-10-31	0.0517	0.0515	0.0516
2005-11-30	0.0543	0.0541	0.0542
2005-12-30	0.0525	0.0526	0.0525
2006-1-31	0.0517	0.0517	0.0517
2006-2-28	0.0470	0.0475	0.0472
2006-3-31	0.0487	0.0486	0.0487
2006-4-28	0.0470	0.0465	0.0467
2006-5-31	0.0485	0.0483	0.0484
2006-6-30	0.0465	0.0464	0.0465
2006-7-31	0.0480	0.0477	0.0478
2006-8-31	0.0488	0.0484	0.0486
2006-9-29	0.0493	0.0489	0.0491

DATE(10Y)	HW	BK	Strike
2006-2-28	0.0472	0.0468	0.0496
2006-3-31	0.0507	0.0497	0.0522
2006-4-28	0.0741	0.0713	0.0482
2006-5-31	0.0848	0.0817	0.0464
2006-6-30	0.0926	0.0899	0.0454
2006-7-31	0.0696	0.0670	0.0486
2006-8-31	0.0716	0.0688	0.0456
2006-9-29	0.0534	0.0519	0.0492
2006-10-31	0.0422	0.0419	0.0520
2006-11-30	0.0369	0.0372	0.0517
2006-12-29	0.0464	0.0453	0.0503
2007-1-31	0.0467	0.0455	0.0512
2007-2-28	0.0393	0.0388	0.0514
2007-3-30	0.0337	0.0336	0.0548
2007-4-30	0.0271	0.0277	0.0573
2007-5-31	0.0308	0.0311	0.0577
2007-6-29	0.0385	0.0381	0.0585
2007-7-31	0.0460	0.0453	0.0566
2007-8-31	0.0447	0.0437	0.0538
2007-9-28	0.0506	0.0491	0.0525

Note: HW stands for Hull-White model, BK stands for Black-Karasinski model.11-y Bermudan swaption are priceed as at-the-money from February 2005 to September 2006; 10 year Bermudan swaptions are priced with the corresponding 11-y ATM strikes one year ago. 10-y Bermudan swaption are priced from February 2006 to September 2007.

Figure 5: 11Y and 10Y Bermudan swaption prices in EUR and USD markets from February 2005 to September 2007.



Note: HW stands for Hull-White model, BK stands for Black-Karasinski model.

values we also need to know the one-year market observed interest rates on the days when the delta-hedge portfolios were computed. The numbers in Table 8 are the seven P&L values computed for the two models. A perfectly hedged portfolio should have a zero P&L value. Non-zero values, whether positive or negative indicates hedge error. A descriptive statistics of absolute values of P&Ls are recorded in Figure 6.

Table 8 together with Figure 6 present the hedging results for the two models. Overall, in terms of descriptive statistics Black-Karasinski model has produced better hedge results than the Hull-White model. Descriptive statistics appear more favourable for the BK model because of one large difference in results for the two models observed on Feb 28, 2006. Therefore, based on our results we can say that in terms of hedging BK model performs marginally better than the HW model. Though BK model show better hedging performance, but if we look at the individual results we can see that the P&Ls for both models are really small. Thus our results that show that both models can be adequately used to risk manage this Bermudan swaption. However, while making this claim we cannot ignore the fact that Bermudan swaption prices reported in this study are very small (due to the principal) and thus the P&Ls estimates are relative to these Bermudan swaption prices.

10 Conclusions

The need for the management of banks' interest rate risk stemming from a mismatch between assets and liabilities has driven bank mangers to use new financial tools. In the past few decades, interest rate derivatives market has seen an enormous expansion. Consequently, past few decades have seen the development of a large variety of models and techniques that can be used for estimating prices and risk sensitivities of these interest rate derivatives. When a wide selection of models is available it become important to compare analyse and compare their pros and cons.

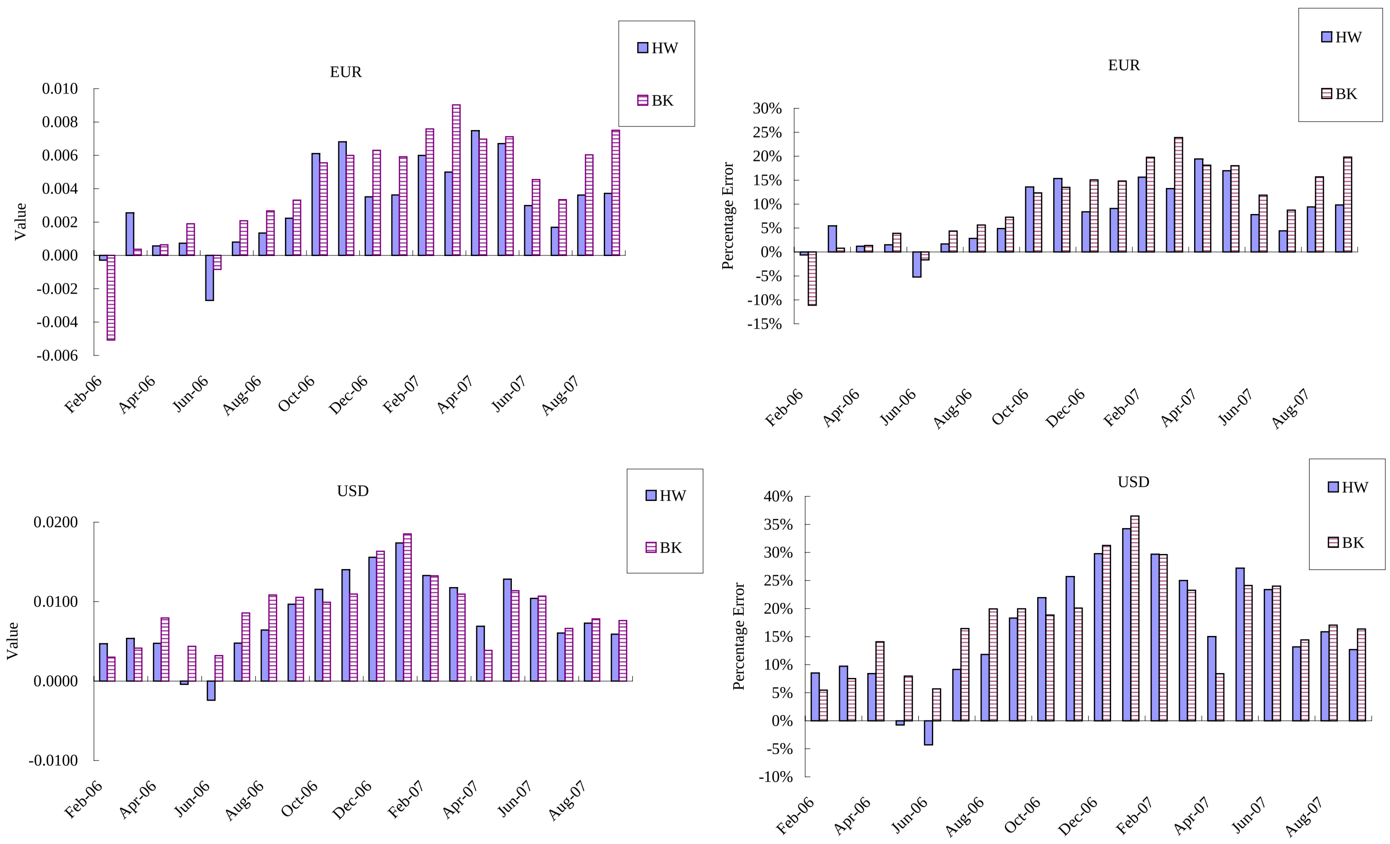
Table 8: Hedging profit and loss in EUR and USD Markets.

EUR	HW	BK
2006-02-28	-0.0003	-0.0051
2006-03-31	0.0026	0.0004
2006-04-28	0.0006	0.0006
2006-05-31	0.0007	0.0019
2006-06-30	-0.0027	-0.0008
2006-07-31	0.0008	0.0021
2006-08-31	0.0013	0.0027
2006-09-29	0.0022	0.0033
2006-10-31	0.0061	0.0056
2006-11-30	0.0068	0.0060
2006-12-29	0.0035	0.0063
2007-01-31	0.0036	0.0059
2007-02-28	0.0060	0.0076
2007-03-30	0.0050	0.0090
2007-04-30	0.0075	0.0070
2007-05-31	0.0067	0.0071
2007-06-29	0.0030	0.0045
2007-07-31	0.0017	0.0033
2007-08-31	0.0036	0.0060
2007-09-28	0.0037	0.0075
RMSS	0.0041	0.0053

USD	HW	BK
2006-02-28	0.0047	0.0030
2006-03-31	0.0054	0.0041
2006-04-28	0.0047	0.0079
2006-05-31	-0.0004	0.0044
2006-06-30	-0.0024	0.0032
2006-07-31	0.0048	0.0086
2006-08-31	0.0064	0.0108
2006-09-29	0.0097	0.0105
2006-10-31	0.0115	0.0099
2006-11-30	0.0140	0.0110
2006-12-29	0.0156	0.0163
2007-01-31	0.0174	0.0185
2007-02-28	0.0133	0.0132
2007-03-30	0.0118	0.0109
2007-04-30	0.0069	0.0039
2007-05-31	0.0128	0.0114
2007-06-29	0.0104	0.0107
2007-07-31	0.0060	0.0066
2007-08-31	0.0073	0.0078
2007-09-28	0.0059	0.0076
RMSS	0.0433	0.0443

Note: HW stands for Hull-White modle, BK stands for Black-Karasinski model. All numbers are in real value. RMSS is the root mean sum of sqaure of all P&L in each period.

Figure 6: Hedging profit and loss in EUR and USD markets



Note: HW stands for Hull-White modle, BK stands for Black-Karasinski model. All the values are in real monetary value and the percentage error is the hedging P&L in percentage of the Bermudan swaption prices.

In this paper, we presented an empirical comparison of two one-factor short rate models. Despite their inherent shortcomings these models attract the attention of practitioners and are being used quite extensively for risk-management purposes The emphasis was on comparing the performance of these two models from ALM perspective. Therefore, we compared one factor Hull-White and Black Karasinski models for pricing and hedging an at-the-money 10x1 co-terminal Bermudan swaption. Because of its long maturity and resemblance to prepayment option this instrument is an appropriate candidate for the test from ALM perspective. The two models are examined based on (i) accuracy of in-sample pricing of European swaptions (ii) stability of parameters and (iii) their ability to hedge the chosen Bermudan swaption. We used Euro data for the period of Feb 2005 to September 2007 to conduct our test. We implemented the two models using Hull-White generalised trinomial tee building procedure and calibrated the two models to a set of "core European swaptions" for the 10×1 Bermudan swaption. For both models we kept the volatility parameter time-dependent and the mean-reversion parameter constant. We delta-hedged this 10×1 co-terminal Bermudan swaption using two different swaps: (i) 5-year swap and (ii) 11 year swap. Overall, in terms of the in-sample price tests, the one-factor Hull-White model outperformed the lognormal Black-Karasinski model. Also the estimated parameters of this model are more stable than of the Black-Karasinski model. On the other hand, the tests for the hedging performance of these two models show that the Black-Karasinski model is more effective in hedging the interest rate risk of the at-the-money 10x1co-terminal Bermudan swaption. Our results also show that pricing errors grew significantly when we kept the volatility parameter constant. Our results clearly show that for our data and our instrument both models, in terms of delta hedging, the performance of both models is quite good.

So what are the implications of these results? The final result has to be based on the joint results. From ALM perspective, hedging is an important criterion for judging the goodness of some model. But in addition to hedging performance, model and parameter stability are important attributes of a model from the ALM's perspective. As far as ALM is concerned, ceteris paribus, a good model should not lead to frequent drastic scenario changes.

Therefore putting the three metrics: pricing performance, stability of parameters and hedging performance together, based on our results we can say that Hull-White model is marginally better than the Black-Karasinski model. Also based on our findings we conclude that these one-factor short rate models can be used to risk manage Bermudan swaption. There are evidences available in literature that supports this conclusion.[12]

Conclusions are based on a very small set of data. We are judging the hedging performance of two models based on mere seven observations. The results are based on one market. As mentioned in the introduction section large global banks have huge interest rate businesses around the world. Therefore, it would be really interesting to see that a model that performs well in one market would be how much stable in completely different economic conditions like Brazilian or Turkish or maybe Australian market. These markets are considered to be good test cases because they are high interest rate environments and people are interested in finding out how different models will react under such environments.

We could not verify the Bermudan swaption prices estimated using the two models, as we did not have their market prices or any other benchmarks. The hedge re-balancing period is too long. Both researchers and practitioners rebalance their hedge portfolios more frequently. The results are based on a very specific instrument. Therefore based on our findings we cannot ensure whether these results would be applicable to some other class of fixed income securities or to some other type of Bermudan swaption (say a 10 year co-terminal Bermudan swaption exercisable monthly) or not. We have tried to immunise a single liability at time. Such a test can be used as an indicative of the relative performance of two models but from ALM perspective the scope of such a test is very limited. A more practical approach

[12] See Pietersz & Pelsser (2005) and Andersen & Andreasen (2001)

would be to hedge a collection of several types of liabilities.

It would be interesting to see the hedge performance of the both two models when both "a" and "σ" are kept constant. It is well proven fact in literature that adding more time-varying parameters improve the pricing performance of one factor models, however there are evidences in literature that making the parameters time-varying leads to larger hedging errors.[13] We used only "core" European swaptions to calibrate the two models. Andersen and Andreasen (2005) show that the set of core swaptions is not sufficient to adequately calibrate a one-factor model. They recommend including caps or swaptions on non-core rates in the calibration set. The results of our principal component analysis reveal that there are multiple factors that driving the evolution of the term structure. Therefore, it is definitely worth testing the hedge performance of multi-factor short rate models.

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Appendix: Caplet as Option on Zero Coupon Bond

Let's assume following caplet with cap rate to be R_K, which will prevail over a period $[(T+\delta)-T]>0$. This caplet leads to a payoff at time $T+\delta$ of

$$A\delta \max(R_{T,\delta}-R_K,0) \tag{14}$$

where $R_{T,\delta}$ is the interest rate for the period between T and $T+\delta$. A is the principal.

The payoff in equation (14) at time $T+\delta$ is equivalent to

$$\frac{A\delta}{1+\delta R_{T,\delta}} \max(R_{T,\delta}-R_K,0) \tag{15}$$

at time T.

Equation (15) can be written as

$$A\times \max\left[1-\frac{(1+\delta R_K)}{1+\delta R_{T,\delta}},0\right] \tag{16}$$

The expression

$$\frac{(1+\delta R_K)}{1+\delta R_{T,\delta}}$$

is the value at time T of a zero-coupon bond that pays off $(1+\delta R_K)$ at time $T+\delta$. The expression in equation (16) is therefore the payoff from a put option with maturity T on a zero-coupon bond with maturity $T+\delta$, when the face value of the bond is $(1+R_K\delta)$ multiplied by the principal. It follows that an interest rate caplet can be regarded as European put options on zero-coupon bond. Likewise an interest rate floorlet can be regarded as European call options on zero-coupon bond.