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A note on stochastic survival probabilities and their calibration

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Abstract

In this note we use doubly stochastic processes (or Cox processes) in order to model the evolution of the stochastic force of mortality of an individual aged x. These processes have been widely used in the credit risk literature in modelling the default arrival, and in this context have proved to be quite flexible and useful. We investigate the applicability of these processes in describing the individual's mortality, and provide a calibration to the Italian case. Results from the calibration are twofold. Firstly, the stochastic intensities seem to better capture the development of medicine and long term care which is under our daily observation. Secondly, when pricing insurance products such as life annuities, we observe a remarkable premium increase, although the expected residual lifetime is essentially unchanged.

JEL classification: G22, J11.

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1 Introduction

The issue of mortality risk – and, in particular, of longevity risk – has been largely addressed in recent years when dealing with the pricing of insurance products. It is well known from the basics of actuarial science that the price of any insurance product on the duration of life depends on two main basis: demographical and financial assumptions. Traditionally, actuaries have been treating both the demographic and the financial assumptions in a deterministic way, by considering available mortality tables for describing the future evolution of mortality of the individual over time and by setting the so-called "best estimate" of the rate of interest for discounting cash flows over time. However, in recent years solvency problems have emerged in many insurance companies: one of the most remarkable examples has been, for instance, the case of the oldest British insurance company, Equitable, which had to be "closed to new business". This has made evident how demographical and financial risks have been underestimated through the adoption of the deterministic approach. In particular, the dramatic drop in interest rates experienced in the late 1990's, together with a remarkable improvement in mortality rates, led to insufficient mathematical reserves for backing guaranteed annuity options (GAO); the latter are insurance policies that give the right of buying in the future an annuity at guaranteed conditions regarding mortality and financial assumptions. Many articles have appeared recently, that show how differently GAO's should have been priced when allowing for stochastic interest rates (see, for example, Ballotta and Haberman (2003), Biffis and Millossovich (2004)).

A different and more appropriate approach to the pricing of insurance policies includes the adoption of stochastic models to describe the uncertainty linked to mortality and financial factors. In this note, we focus on mortality risk and on modelling the survival function of the individual, leaving to future research the modelling of both mortality and financial risks with a stochastic approach (as in Biffis (2004)).

2 Modelling mortality risk

In the last decades significant improvements in the duration of life have been experienced in most developed countries. Two indicators are typically used to describe the mortality of an individual: the survival function and the "death curve".

The survival function, denoted with $S(t)$, is defined as follows:

$$S(t) = P(T_0 > t) = 1 - P(T_0 \leq t) = 1 - F_{T_0}(t)$$

where T_0 is the random variable that describes the duration of life of a new-born individual, and F_{T_0} is its distribution function. The survival function indicates the probability that a new-born individual will survive at least t years. Via the survival function, one can easily derive the distribution function of the duration of life of an individual aged x, given that he/she is alive at that age (see, for instance, Bowers, Gerber, Hickman, Jones and Nesbitt (1986), Gerber (1997)).

The death curve, ${}_{x/1}q_0$, is defined as follows:

$${}_{x/1}q_0 = \frac{S(x) - S(x+1)}{S(0)}$$

and indicates the probability for a new-born individual of dying in year of age $[x, x+1]$.

An easy way of capturing the mortality trend observed in the past decades consists in looking at the graphs of the survival function and the death curves of a population in different years (for an accurate report about mortality trends, see Pitacco (2004a)). One can notice that the shape of the survival function becomes more and more "rectangular" (and this phenomenon is called "rectangularization") and the mode of the death curve moves towards right (and this phenomenon is called "expansion"). The rectangularization shows that the volatility of the duration of life around the mode of death decreases, leading to lower dispersion of ages of death around the most likely age of death. The expansion shows that the age when death is most likely to occur increases as time passes (due to improvements in economic and social conditions, medicine progresses etc.).

It is clear that continuous improvements in the mortality rates have to be allowed for when pricing insurance products that heavily depend on the duration of life at old ages (like annuities), since strong and unexpected reductions in mortality rates can lead to mispricing of these products and can affect the solvency of the insurance company.

The actuarial literature about modelling and forecasting mortality rates is vaste and has a long history (for a detailed survey of the most significant models proposed in the literature, see Pitacco (2004b)).
Traditionally, a central role has been played by the "force of mortality", defined as the opposite of the derivative of the logarithm of the survival function:

$$\mu_x = -\frac{d}{dx}\log S(x)$$

The force of mortality is a good tool for approximating the mortality of the individual at age x, since it can be shown that:

$$P(x < T_0 \leq x + \Delta x | T_0 > x) = \mu_x \Delta x + o(\Delta x), \tag{2.1}$$

i.e. the probability of dying in a short period of time after x, between age x and age $x + \Delta x$, can be well approximated by $\mu_x \Delta x$ (when Δx is small). The force of mortality is obviously increasing as x increases, as the probability of imminent death increases when ageing (with some exceptions, like very small values of x – due to the infant mortality – and values around 20-25 – due to the young mortality hump).

When allowing for mortality trends over time, it is evident that the force of mortality has to show a dependence also on calendar year, and not only on age. Thus, the force of mortality can be described by a two variable function $\mu_x(y)$, where y indicates the calendar year. As time y increases and the age x remains fixed, the decreasing mortality rates over time translate into a decreasing function $\mu_x(y)$.

Several contributions have been proposed in the last decade in order to model and forecast the year- and age-dependent mortality, i.e. the "dynamic mortality". One of the seminal works is the Lee-Carter method (Lee and Carter (1992) and Lee (2000)), that models the central death rate (an actuarial indicator, similar to the force of mortality) as a two variables function. Many authors have modified the Lee-Carter method, proposing their adjustments. Among these, are the extensions proposed by Renshaw and Haberman (2003) and Brouhns, Denuit and Vermunt (2002). The latter propose a fairly simple model for the force of mortality:

$$\ln(\mu_x(y)) = \alpha_x + \beta_x k_y$$

where the coefficients α_x, β_x and k_y are to be determined by maximization of the log-likelihood based on the assumption that the number of deaths at age x in year y follows a Poisson distribution.

Another way of dealing with mortality trends, largely adopted by insurance companies, is the use of the so-called "projected mortality tables", that incorporate (forecasts of) survival probabilities at any age for different calendar years.

3 Modelling the random duration of life: the mathematical framework

The theory of stochastic intensities, doubly stochastic processes and affine processes underlying the actuarial application presented here is enormous and covered in many texts about stochastic processes. A detailed and thorough treatment is clearly beyond the scope of this note, and we limit ourselves to present a brief summary of the mathematical tools used, sacrificing scientific rigor and omitting all the proofs. However, we refer the interest reader to Brémaud (1981) and Duffie (2001).

The reason why such a sophisticated mathematical framework has been used in describing the mortality risk is the great analytical tractability of the models presented, once some useful and not too restrictive assumptions are made about the processes used. These mathematical tools have been extensively used in the credit risk literature, when modelling the default time. The pioneering works in this field are Lando (1994) and Lando (1998). The similarity with the remaining duration of life is strong, and, although the factors underlying the death of an individual and the default of a firm are obviously completely different, the mathematical tools used are the same. An extensive application of this mathematical framework to dynamic mortality modelling and to insurance products pricing can be found in Biffis (2004).

3.1 Counting processes

In describing the mathematical tools, we will mainly follow Duffie (2002). We are given a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\{\mathcal{G}_t : t \geq 0\}$ of sub-σ-algebras of $\mathcal{F}$ satisfying the usual conditions.

A process Y is said to be *predictable* if $Y : \Omega \times [0, \infty) \rightarrow \mathbf{R}$ is measurable w.r.t. the $\sigma-$algebra on $\Omega \times [0, \infty)$ generated by the set of all left-continuous adapted processes. The intuition behind a predictable processes is that it is possible to "foretell" the value of the process at time t with all the information available at any time *before* t but not including t. Any left-continuous adapted process is predictable, as well as any continuous adapted process.

A *counting* process (or *point* process) N is defined using a sequence of increasing random variables $\{T_0, T_1, ...\}$, with values in $[0, \infty)$, s.t. $T_0 = 0$ and $T_n < T_{n+1}$ whenever $T_n < \infty$, in the following way:

$$N_t = n \qquad for \qquad t \in [T_n, T_{n+1})$$

and $N_t = \infty$ if $t \geq T_\infty = \lim_{n\to\infty} T_n$. It is easy to see T_n as the time of the n^{th} jump of the process N and N_t as the number of jumps occurred up to time t, including time t (hence the definition "counting" process). The counting process is said to be *nonexplosive* if $T_\infty = \infty$ almost surely.

3.2 Stochastic intensity

The definition of random intensity is not uniform in the literature. We will follow Duffie (2001).

If N is a nonexplosive adapted counting process and λ is a nonnegative predictable process s.t. $\int_0^t \lambda_s ds < \infty$ almost surely, then N is said to admit the intensity λ if the compensator of N admits the representation $\int_0^t \lambda_s ds$, i.e. if $M_t = N_t - \int_0^t \lambda_s ds$ is a local martingale. If the stronger condition $E(\int_0^t \lambda_s ds) < \infty$ is satisfied, $M_t = N_t - \int_0^t \lambda_s ds$ is a martingale[1].

The intensity can be related also to stopping times. A stopping time τ has an intensity λ if τ is the first event time of a nonexplosive counting process. This happens only if τ is *totally inaccessible*(see Meyer (1966)): intuitively, if a latent variable cannot "foretell" its arrival. A stopping time can also not have an intensity (see Duffie (2002) and Duffie and Lando (2001)). The crucial point is the filtration with respect to which the process M_t is a (local) martingale. If the filtration is the standard filtration of a Brownian motion B, then M can be represented as a stochastic integral w.r.t B and cannot jump. In order to let M jump at time τ, the filtration under which the counting process has an intensity cannot be too "rich". Usually, the definition of intensity is considered w.r.t. to a filtration $\{\mathcal{F}_t\}$, satisfying the usual conditions and such that $\{\mathcal{F}_t\}$ is "poorer" than $\{\mathcal{G}_t\}$: $\mathcal{F}_t \subset \mathcal{G}_t$.

In what follows, we will be particularly interested in stopping times which have an intensity, because this will allow us to describe the time of death in an analytically tractable way.

From the definition of intensity, one gets:

$$E(N_{t+\Delta t} - N_t|\mathcal{F}_t) = E\left(\int_t^{\Delta t} \lambda_s ds|\mathcal{F}_t\right)$$

which, after a few passages and under technical conditions, leads to:

$$E(N_{t+\Delta t} - N_t|\mathcal{F}_t) = \lambda_t \Delta t + o(\Delta t) \tag{3.1}$$

Equation 3.1 (see the analogy with equation 2.1) stresses the importance of the process λ in giving information about the average number of jumps of the process under observation in a small period of future time. Observe that conditioning is made on the smallest filtration, therefore on the availability of poorer information. The idea is that the information at time t can give insight about the expected number of jumps in the next future or, in other words, about the likelihood of a jump in the immediate future. It cannot predict the actual occurrence of a jump, that comes as a "sudden surprise".

3.3 Doubly stochastic processes

Suppose N is a nonexplosive counting process with intensity λ and $\{\mathcal{F}_t : t \geq 0\}$ is a filtration satisfying the usual conditions, with $\mathcal{F}_t \subset \mathcal{G}_t$. The process N is said to be *doubly stochastic driven by* $\{\mathcal{F}_t : t \geq 0\}$, if λ is $(\mathcal{F}_t)$-predictable and for all t, s, with $t < s$, conditional on the $\sigma-$algebra $\mathcal{G}_t \vee \mathcal{F}_s$, generated by $\mathcal{G}_t \cup \mathcal{F}_s$, the process $N_s - N_t$ has Poisson distribution with parameter $\int_t^s \lambda_u du$.

[1] The requirement of predictability allows us to consider the intensity as essentially unique. In fact, it can be shown (see Brémaud (1981) and Duffie (2001)) that if λ and $\tilde{\lambda}$ are two intensities for N, then $\int_0^\infty |\lambda_s - \tilde{\lambda}_s|\lambda_s ds = 0 \quad a.s.$, which implies that if λ is strictly positive, we have $\lambda = \tilde{\lambda}$ almost everywhere.

As an example, we observe that any Poisson process is a doubly stochastic process driven by the filtration $\mathcal{F}_t = (\emptyset, \Omega) = \mathcal{F}_0$ for any $t \geq 0$, in that the intensity is deterministic.

A stopping time τ is said to be *doubly stochastic with intensity* λ if the underlying counting process whose first jump time is τ is doubly stochastic with intensity λ.

The mathematical arsenal presented so far is now sufficient to present the first interesting result that will be used in the applications. If τ is a stopping time doubly stochastic with intensity λ, it can be shown, by using the law of iterated expectations, that:

$$P(\tau > s|\mathcal{G}_t) = E\left[e^{-\int_t^s \lambda(u)du}|\mathcal{G}_t\right] \tag{3.2}$$

Readers who are familiar with mathematical finance can easily see in the r.h.s. of equation (3.2) the price at current time t of a unitary default-free zero-coupon bond with maturity at time $s > t$, if the short-term interest rate model is given by the process λ. All the mathematical finance literature about interest rate models can thus be retrieved in this setting.

Another interesting result that can be used relates to the density function of a doubly stochastic stopping time τ. If we let $p(t) = P(\tau > t)$ be the *survival function*, the density function of τ, if it exists, is given by $-p'(t)$. Under certain conditions (see for example Grandell (1976)), that are satisfied in many applications, we have:

$$p'(t) = E\left[-e^{-\int_0^t \lambda(u)du}\lambda(t)\right] \tag{3.3}$$

which can be used in order to find the density function of the stopping time τ.

It is clear how these results can be naturally applied in the actuarial context: if one sees τ as the future lifetime of an individual aged x, T_x, equations 3.2 and 3.3 can be applied to find the survival function and the density function of T_x, so relevant for pricing purposes (see section 2).

3.4 Affine processes

Our next step will be to show how equations like 3.2 and 3.3 can be approached. It turns out that it is convenient to specify the form of the stochastic intensity λ as a function Λ of another process X_t in $\mathbf{R}$, whose dynamics are given by the SDE:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t \tag{3.4}$$

for some Brownian motion B_t in $\mathbf{R}$ and for $\mu(\cdot)$ and $\sigma(\cdot)$ satisfying enough regularity conditions for the equation 3.4 to have a unique strong solution. The survival probability 3.2 becomes:

$$P(\tau > s|\mathcal{G}_t) = E\left[e^{-\int_t^s \Lambda(X(u))du}|X(t)\right] = f(X(t), t) \tag{3.5}$$

Under regularity conditions, equation 3.5 can be tackled with the Feynman-Kac approach, that reduces 3.5 to the solution of the PDE:

$$\mathcal{A}f(x,t) - f_t(x,t) - \Lambda(x)f(x,t) = 0 \tag{3.6}$$

(where $\mathcal{A}$ is the usual infinitesimal generator of X) with the boundary condition $f(x, s) = 1$.

The difficulty inherent in solving the PDE 3.6 can be reduced by choosing appropriately the process X. Here is where affine processes come at help. Interest readers can find a thorough treatment of affine processes in Duffie, Filipovič and Schachermayer (2003). Here we limit ourselves to observe that if X is an affine process, then it is a jump-diffusion process:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t + dJ_t \tag{3.7}$$

where J is a pure jump process and where the drift $\mu(X_t)$, the covariance matrix $\sigma(X_t)\sigma(X_t)'$ and the jump measure associated with J have affine dependence on X_t.

The financial literature on interest rate modelling is full of examples of affine processes: the Ornstein-Uhlenbeck process, used by Vasicek (1977) for modelling interest rates, is affine, as is the Feller process, used by Cox, Ingersoll and Ross (1985).

The convenience of adopting affine processes in modelling the intensity lies in the fact that, under technical conditions (see Duffie et al. (2003)), it yields:

$$E_t\left[e^{\int_t^s -\Lambda(X(u))du + wX(s)}\right] = e^{\alpha(s-t)+\beta(s-t)X(t)} \tag{3.8}$$

where $w \in \mathbf{R}$ and the coefficients $\alpha(\cdot)$ and $\beta(\cdot)$ satisfy generalized Riccati ODEs, that can be solved at least numerically and in some cases analytically. Therefore, the difficult problem of finding the survival function (3.5) can be transformed in a tractable problem, whenever affine processes are employed.

4 The actuarial application

Turning back to our initial problem of modelling adequately the dynamic mortality, we will now use some of the mathematical tools presented in the previous section.

We consider an individual aged x and model his/her random future lifetime T_x as a doubly stochastic stopping time with intensity λ_x driven by the sub-filtration $\{\mathcal{F}_t : t \geq 0\}$, where $\mathcal{F}_t \subset \mathcal{G}_t$[2]. In other words, T_x is the first jump time of a nonexplosive counting process N with intensity λ_x. Intuitively, the counting process N may be seen as a process that jumps whenever the individual dies: for example $N_t = 0$ if $t < T_x$, $N_t = 1$ if $t \geq T_x$.

According to (3.2) the survival probability is:

$$S_x(t) = P(T_x > t|\mathcal{G}_0) = E\left[e^{-\int_0^t \lambda_x(u)du}|\mathcal{G}_0\right] \tag{4.1}$$

The similarity with the actuarial survival probability for t years for an individual aged x, ${}_tp_x$, expressed in terms of the force of mortality, is strong (see for instance Gerber (1997)):

$${}_tp_x = e^{-\int_0^t \mu_{x+s}ds}$$

The specification of the intensity process λ_x is now crucial for the solution of equation 4.1.

[2] As above, the uncertainty is described by a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\{\mathcal{G}_t : t \geq 0\}$ of sub-σ-algebras of $\mathcal{F}$ satisfying the usual conditions.

Recent studies on the firm's mortality (as reported in Duffie and Singleton (2003)) indicate the suitability of the following affine processes for modelling the intensity $\lambda_x(t)$:

$$CIR \quad process: \qquad d\lambda_x(t) = k(\gamma - \lambda_x(t))dt + \sigma\sqrt{\lambda_x(t)}dW(t)$$

$$mean\ reverting\ with\ jumps\ (m.r.j.): \qquad d\lambda_x(t) = k(\gamma - \lambda_x(t))dt + dZ(t)$$

where $W(t)$ is a standard Brownian motion and $Z(t)$ is a compound Poisson process with intensity c and jumps exponentially distributed with expected value J (we notice that in both cases, according to the notation introduced before, the choice is $\Lambda(x) = x$: the intensity λ is itself an affine process). The difference between the two models is that the former does not allow for jumps in the mortality rates, while the latter does.

Using the result 3.8 and solving the Riccati ODEs, one gets the survival probabilities in closed form for both specifications of the intensity process (this is a standard result, see Duffie and Singleton (2003)):

$$CIR \qquad {}_tp_x = e^{a(t,k,\gamma,\sigma)+b(t,k,\sigma)\lambda_x(0)} \tag{4.2}$$

$$m.r.j. \qquad {}_tp_x = e^{f(t,k,\gamma,c,J)+g(t,k)\lambda_x(0)} \tag{4.3}$$

where

$$a(t,k,\gamma,\sigma) = \frac{-2k\gamma \ln\left(\frac{1+e^{b_1(k,\sigma)t}}{2}\right)}{b_1(k,\sigma)c_1(k,\sigma)} + k\gamma\frac{t}{c_1(k,\sigma)}$$

$$b(t,k,\sigma) = \frac{1-e^{b_1(k,\sigma)t}}{c_1(k,\sigma)(1+e^{b_1(k,\sigma)t})}$$

$$b_1(k,\sigma) = c_1(k,\sigma) + \frac{\sigma^2}{2c_1(k,\sigma)}$$

$$c_1(k,\sigma) = -\frac{k+\sqrt{k^2+2\sigma^2}}{2}$$

$$f(t,k,J,c,\gamma) = -\gamma(t+g(t,k)) - c\frac{Jt - \ln(1-Jg(t,k))}{J+k}$$

$$g(t,k) = \frac{e^{-kt}-1}{k}$$

It is possible to calibrate the values of the parameters starting from a time series or a cross section of data on the survival probabilities.

4.1 Calibration to the Italian population

As an application, we have calibrated the model to the Italian population. We have considered the survival probability of a male and a female aged $x = 65$ according to the projected mortality table RG48 (which is the most recent generation table available, that considers individuals born in 1948).

The calibration has been done by minimizing the sum of the squared differences between the survival probabilities of the table RG48 and the ones implied by the model. It gives the following values of the parameters:

CIR process		
	Males	Females
k	0.015	0.015
γ	0.518	0.371
σ	0	0

We observe that for both males and females, the calibration gives a value of σ equal to 0, implying a deterministic intensity. With our data, when imposing the constraint $\sigma > 0$, the fit of the model worsens. We therefore accept the solution with $\sigma = 0$ which cancels the diffusive component of the process. The long-run average γ is higher for males than for females, consistently with the higher mortality observed in males. The speed of convergence is the same.

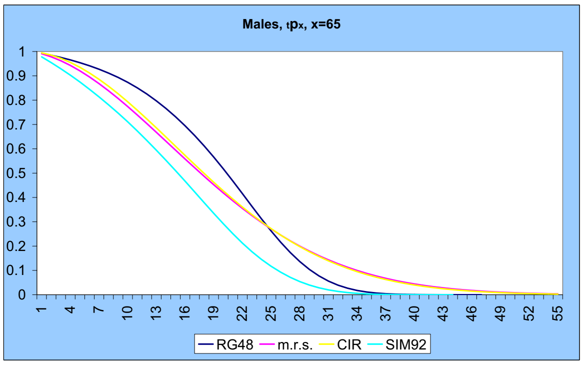
m.r.j. process		
	Males	Females
k	0.0037	0.006
γ	1.002	0.473
c	0.0025	0.0025
J	-0.0025	-0.0025

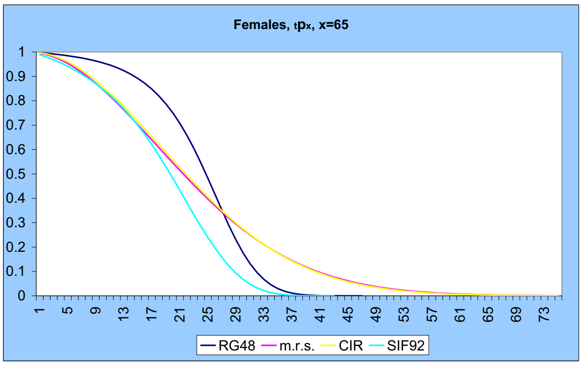
Again, the long-run average is higher for males than for females, while the speed of convergence is higher for females than for males. This latter result is interesting and might imply that further improvements that expected to occur in the future are likely to be more substantial for males than for females. This interpretation can be supported by the evidence that every census of the Italian population produces mortality tables that show improvements for males that are superior to the corresponding improvements for females. This phenomenon is not captured by the CIR model.

The negative value for J is an expected result, for it shows the improvement in mortality rates over time. The same parameter values for the two sexes in the jump part of the process (c and J) can be explained by the fact that jumps correspond to discontinuity points of the intensity process, which may be related to medicine progresses or other general factors, that affect in the same way the global population.

4.2 Survival function, expected residual lifetime and annuity price

Graphs 1 and 2 show the value of the survival probability for males and females ${}_tp_x$ with $x = 65$, respectively. They report the survival probability for the two processes analyzed (CIR and m.r.j.) and the tables RG48 and SIM/F92, to facilitate the comparison.



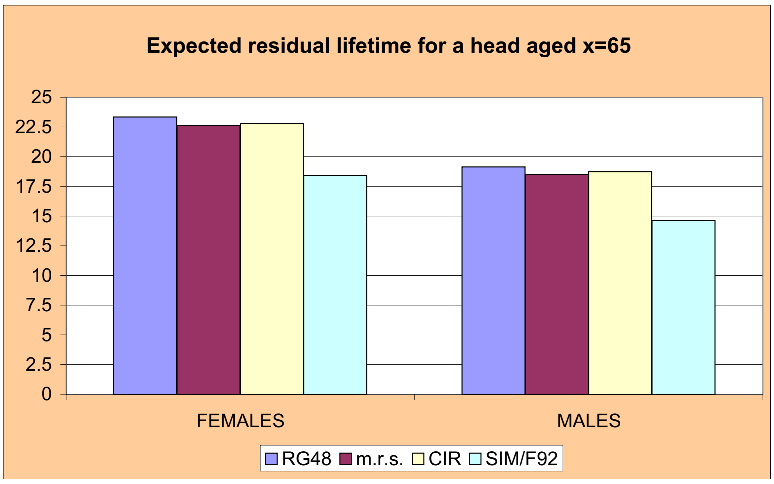


Graph 1 Graph 2

Observing the graphs, one can notice that the survival probability with the CIR and the m.r.j. processes at very old ages is higher than in both the RG48 and SIM/F92: in this sense, the stochastic intensities seem to better capture the development of medicine and long term care which is under our daily observation. With respect to the SIM/F92 we observe improvement at all ages. This does not happen w.r.t. the RG48, which has higher survival probabilities at younger ages, and seems to better describe the rectangularization phenomenon.

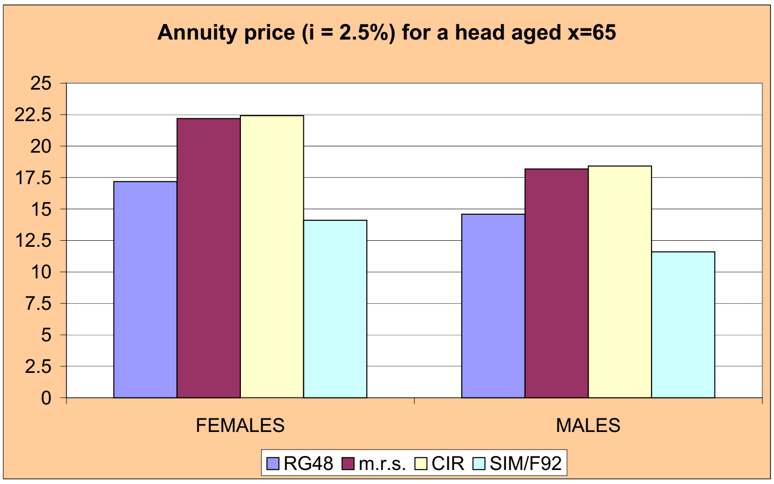
Other two indicators of the mortality trend are the expected residual lifetime and the price of the lifetime annuity. We have computed these indexes for an individual aged 65, adopting the different survival probabilities derived from the calibration process and comparing them with the corresponding values from RG48 and SIM/F92. In the calculation of the price of the annuity we have taken the interest rate equal to $i = 2.5\%$ (the technical rate usually adopted these days by Italian insurance companies).

Graph 3 shows the values of the expected residual lifetime for an individual aged $x = 65$, for both sexes and for the four mortality assumptions.



Graph 3

Graph 4 shows the values of the price of a lifetime annuity for an individual aged $x = 65$, with interest rate $i = 2.5\%$ for both sexes and for the four mortality assumptions.



Graph 4

The expected residual lifetime with the CIR and the m.r.j. processes does not differ too much from the value of the RG48, the mortality table used for the calibration. A remarkable difference is to be noticed with the mortality tables SIM/F92. These results are to be explained with the observations made on the survival probabilities of graphs 1 and 2. Comparing with the RG48, the high probability at very old ages is counterbalanced by low probabilities at younger ages and gives almost the same result in terms of expected duration of life. Comparing with the SIM/F92, the higher probabilities at all ages lead to a higher expected residual lifetime.

However, considering the annuity price, the mortality described with the CIR and the m.r.j. processes leads to significantly higher prices for the lifetime annuity than with both RG48 and SIM/F92. As for females, the price increases by around 30% w.r.t. RG48, by 60% w.r.t. SIF92. As for males, the corresponding figures are 20% and 40%. The higher probability of surviving at very old ages seems to dominate the lower probability of surviving at lower ages. This result underlines the risk of underestimating the mortality (longevity) risk when pricing insurance products on the duration of life based on deterministic mortality tables, such as the RG48 and SIM/F92 ones. Since the final interest of insurance companies is the correct pricing (and the consequent hedging) of annuities and other life insurance products, this is the point on which the usefulness of the Cox approach has to be appreciated.

5 Conclusions

In this note, we have described the evolution of mortality by using doubly stochastic (or Cox) processes. The time of death has been modelled as a doubly stochastic stopping time, or the first jump time of a doubly stochastic counting process. The intensity has been described as an affine process, with two different specifications: the CIR model and a mean reverting process with jumps. For these two models the survival probabilities are known in closed form.

The model has been calibrated to the Italian population, considering an individual aged 65 and using the table RG48 for the calibration. The survival probabilities have been calculated and compared with the corresponding ones taken from the tables RG48 and SIM/F92. The expected residual lifetime has been computed, as well as the price of a lifetime annuity. Results show that, although the expected residual lifetime is essentially unchanged when describing the mortality using Cox processes, the annuity price shows a significant increase. This highlights the importance of mortality risk when pricing insurance policies on the duration of life and the extent to which this risk can be underestimated.

For future research, the robustness of results w.r.t. the interest rate chosen for the annuity price can be interesting. Furthermore, the investigation of other affine processes can be worth, as well as the calibration to different mortality tables. Finally, the inclusion of stochastic interest rates in the model is certainly of the greatest importance.

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