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Valuation of the Minimum Guaranteed Return Embedded in Life Insurance Products

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Valuation of the Minimum Guaranteed Return
Embedded in Life Insurance Products [1]

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Abstract : Consider a traditional life insurance contract paid with a single premium. In addition to mortality factors, the relationship between the fixed amount of benefit and the single premium depends on the interest rate (calculation rate). The calculation rate can be interpreted as the average rate the insurance company must earn on its investment in the insurance period to fulfill its future obligations.

In many countries the traditional life insurance products include a fixed percentage guarantee on each year's return. This annual guarantee comes in addition to the fixed benefit. Furthermore, no extra premium is charged for this guarantee.

In this article we present a model for the valuation of life insurance contracts including a guaranteed minimum return. The model is based on the notion of no arbitrage opportunities from the theories of financial economics.

Numerical examples indicate that these guarantees may have substantial market values.

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1. Introduction

The topic of this paper is the valuation of a periodical minimum return embedded in life insurance contracts. These guarantees are typically included in traditional life insurance products, such as endowment insurances, and are not necessarily connected to more modem products like variable life insurance etc. By issuing traditional life contracts with fixed benefits the insurer guarantees an average rate of return of the contracts. In several countries, e.g., USA and Norway, the insurance companies guarantee the insured an annual minimum return on his policy (in Norway this rate is 3%), in addition to the fixed benefit. Historically, this guarantee was included as a part of the contract at a time when the observed interest rate was high. No extra premium is charged for the guarantee.

In this paper we develop a model for pricing such guarantees. First, a pricing principle for insurance contracts with random interest rates is required. The typical approach in the actuarial literature is to model the interest rate by a stochastic process and price life insurance contracts according to the traditional principle of equivalence, an approach widely accepted under deterministic interest rates. Our approach is different. In a companion paper, Persson (1996), a model for pricing life insurance contracts under stochastic interest rates based on economic theory is developed. The main result of this model is that single premiums of life insurance contracts still may be calculated as expected present values, but in the presence of stochastic interest rates a risk adjusted probability measure must be applied, instead of the original probability measure. This result follows from the theory of arbitrage pricing from financial economics. Compared to the classical principle of equivalence our approach includes a model of the financial market and restricts the insurance companies investment possibilities to the securities traded in this market. For simplicity, the security-market in this model consists only of bonds.

By a participating policy we refer to a contract where the insured is entitled to a share of the surplus if the realized interest rate during the insurance period is above the assumed interest rate. This property is included in many real-life life insurance contracts. Instead of the absolute amount of benefit it is natural to focus on the rate of return of the policy,i.e., we study the amount of insurance available for one currency unit. In addition, the insured must pay a loading for the participating option. The market price of this loading is determined for a certain specification of a participating policy.

By partitioning the insurance period the periodical minimum guarantee described above may be considered as sequence of participating policies. The market based loading for a policy including a minimum guarantee is then determined.

The paper is organized as follows: Section 2 describes the economic model. In section 3 market based loadings for participating policies and participating policies with minimum guarantees are determined. In section 4 these results are compared. Section 5 contains some concluding remarks.

2. Economic Model

A unit discount bond is a financial asset that entitles its owner to one currency unit at maturity without any intermediate coupon payments and without any default risk. We denote by $B_t(s)$ the market price at time t for a bond maturing at a fixed date $s \geq t$, in particular $B_s(s) = 1$.

A finite time horizon T, which later is interpreted as the insurance period, is imposed. There exists a continuum of such bonds maturing at all times $s \in [0,T]$. These bonds are traded in a frictionless market (no transaction costs or taxes and no restrictions on short-sale) with continuous trading opportunities.

We assume the short interest rate, which is the only state variable, is given by the following Ornstein-Uhlenbeck process

$$dr_t = q(m - r_t)dt + v d\hat{W}_t,$$

where m,q,v are positive constants interpretable as the long-range mean to which r_t tends to revert, the speed of adjustment and the volatility factor, respectively. In addition, the initial value r_0 is given and $\hat{W}_t$ is a standard Wiener process defined on a given probability space.

No arbitrage opportunities in the bond market implies the existence of a function $\lambda(r_t,t)$, interpretable as the market price of interest rate risk, see Vasicek (1977). We assume that the market price of interest rate risk equals the constant λ.

As shown in Persson (1996), λ may be used to construct a probability measure Q such that the interest rate process under Q equals

[1] $$dr_t = q(d - r_t)dt + v dW_t,$$

where

$$d = m - \frac{v\lambda}{q}$$

and W_t is a Brownian motion under **Q**. This process is still an Ornstein-Uhlenbeck process, the parameter representing the long-range mean m is now replaced by the parameter d.

The solution to the above stochastic differential equation is

$$r_t = d + (r_0 - d)e^{-qt} + \int_0^t v e^{-q(t-s)} dW_s.$$

Here r_t is normally distributed under Q with mean $d + (r_0 - d)e^{-qt}$ and variance $\frac{v^2}{2q}[1 - e^{-2qt}]$.

For future use we define the quantity

$$R_t = \int_0^t r_s ds,$$

thus, $-R_t$ represents the discount function for the period [0,t]. We solve for R_t from the above equation for r_t and obtain

[2] $$R_t = dt + \frac{1}{q}(r_0 - d)(1 - e^{-qt}) + \int_0^t \frac{v}{q}(1 - e^{-q(t-s)})dW_s.$$

Observe that also R_t is normally distributed under Q with mean

[3] $$\Lambda_t = dt + \frac{1}{q}(r_0 - d)(1 - e^{-qt})$$

and variance

$$\Gamma_t = \frac{v^2}{2q^3}(2qt - 3 + 4e^{-qt} - e^{-2qt}).$$

Let $t_0 = 0 < t_1 < ... < t_n = T$ be a partition of the time horizon. For fixed $t \in (0,T]$ we define $n_t = \min(i : t_i \geq t)$. Then redefine $t_{n_t} = t$. Let

[4] $$\delta_t^i = \int_{t_{i-1}}^{t_i} [d + (r_0 - d)e^{-qs}]ds + \int_{t_{i-1}}^{t_i} \frac{v}{q}(1 - e^{-q(t-s)})dW_s, \ i = 1, .., n_t.$$

Then $R_t = \sum_{i=1}^{n_t} \delta_t^i$. It follows that δ_t^i is normally distributed with mean

[5] $$\Lambda_i = d(t_i - t_{i-1}) + (r_0 - d)\frac{1}{q}(e^{-qt_{i-1}} - e^{-qt_i}),\ i = 1, \ldots, n_t,$$

and variance

$$\Gamma_i(t) = \frac{v^2}{2q^3}[2q(t_i - t_{i-1}) - 4e^{-qt}(e^{qt_i} - e^{qt_{i-1}}) + e^{-2qt}(e^{2qt_i} - e^{2qt_{i-1}})],\ i = 1, \ldots, n_t,$$

which for $i = n_t$ can be simplified to

$$\Gamma_{n_t}(t) = \frac{v^2}{2q^3}[2q(t - t_{n_t - 1}) - 3 + 4e^{-q(t - t_{n_t - 1})} - e^{-2q(t - t_{n_t - 1})}]$$

Observe that $Cov(\delta_t^i, \delta_t^j)$ for $i \neq j$ under Q is zero.

Let the random variable T_x represent the remaining lifetime of the insured (customarily assumed to depend only on x, the insured's age at the point of issue). We assume that the probability density of T_x exists and denote it by f_x. Denote the survival probability $P(T_x > t)$ by ${}_tp_x$. We assume that T_x is independent of $\{W_t,\ t \in [0,T]\}$. That is, mortality is independent of the financial market.

We limit the analysis to single premiums. In principle it is straightforward to generalize the analysis to periodical premiums which is most common in practice.

Let C be the benefit payable at time T and C_t the benefit payable at time t, $t \in [0,T]$.In this model the single premiums at time zero, π and π', of a pure endowment insurance, payable if the insured survives the insurance period, and the term insurance, payable upon death before the insurance period expires, are given by

[6] $$\pi = {}_Tp_x E^Q[\exp(-R_T C)]$$

and

[7] $$\pi' = \int_0^T \exp(-R_t)C_t f_x(t)dt,$$

respectively.

3. Insurance policies with guaranteed periodical return

Now we consider different life insurance policies. For the moment we ignore mortality factors and assume first that all policies expire at the fixed future date t. The assumptions of risk neutrality with respect to mortality and independence between mortality and financial factors make it straightforward to incorporate mortality factors.

The first life insurance contract we consider is the standard text book life insurance (abstracting from mortality factors), a fixed amount K is payable at time t. From the assumed economic model the single premium of this policy is

$$\pi = E^Q\left[\exp\left(-\int_0^t r_s ds\right)K\right].$$

For ease of comparison with the other contracts we normalize this contract by dividing by π on both sides,

$$1 = E^Q\left[\exp\left(-\int_0^t r_s ds\right)\exp(gt)\right],$$

where $g = \frac{1}{t}\ln\left(\frac{K}{\pi}\right)$. Here $\frac{K}{\pi} = e^{gt}$ may be interpreted as the "amount of insurance" obtainable for 1 currency unit. Furthermore, g can be interpreted as the promised average rate of return to the policyholder and hence also as the average rate the insurer must earn on his investments in the period [0,t] in order not to suffer a loss.

We next consider a participating policy. That is, the insured is entitled to the maximum of a fixed benefit and the insurer's realized return. Whether or not exactly this policy is observed in real life is not so important. Our prime use of it is as a building block when we study the interest guarantees. We employ the "amount of insurance" formulation and describe the amount of insurance as

$$C_t = \exp\left(\int_0^t r_s ds\right) \vee \exp(gt),$$

that is, the maximum of the promised rate of return g and the realized return. Thus, by this contract the insured receives the entire rate of return exceeding the rate g.

We define $\pi_p(t)$ as the market based loading for this participating policy and obtain

$$[8] \qquad 1 + \pi_p(t) = E^Q\left[\exp\left(-\int_0^t r_s ds\right)C_t\right] = E^Q\left[\exp\left(\left[gt - \int_0^t r_s ds\right] \vee 0\right)\right].$$

The last policy reconsider is a participating policy with a periodically guaranteed minimum return. Whereas g in the above contract is interpreted as an average guaranteed return over the term of the contract, the following policy guarantees a rate of return in each sub-period of the insurance period.

Consider a policy with insurance period [0,T]. Let $t_0 = 0 < t_1 < \dots < t_n = T$ be a partition of the insurance period. The insurer guarantees a constant minimum rate of return g_i in sub-period $[t_{i-1}, t_i]$. The amount of insurance payable at time $t \leq T$ is given by

$$C_t = \exp\left(\sum_{i=1}^{n_t}\left[\int_{t_{i-1}}^{t_i} r_s ds\right] \vee g_i(t_i - t_{i-1})\right),$$

where n_t is as defined previously. The market based loading π_g of this contract is

$$1 + \pi_g(t) = E^Q[e^{-\int_0^t r_s ds} C_t] = E^Q\left[\exp\left(\sum_{i=1}^{n_t}\left[g_i(t_i - t_{i-1}) - \int_{t_{i-1}}^{t_i} r_s ds\right] \vee 0\right)\right]$$

We are now ready to present the market based loadings π_p and π_g in the following two lemmas.

Lemma 1

The market based loading $\pi_p(t)$ for a participating policy expiring at time t is

$$\pi_p(t) = e^{gt - \Lambda_t + \frac{1}{2}\Gamma_t}\Phi\left(\frac{gt - \Lambda_t + \Gamma_t}{\sqrt{\Gamma_t}}\right) + \Phi\left(\frac{\Lambda_t - gt}{\sqrt{\Gamma_t}}\right) - 1,$$

where $\Phi()$ represents the standard cumulative normal distribution.

Proof?

From expression *[8] we* may write

$$E^Q[\exp([gt - R_t]\vee 0)] = \int_{-\infty}^{gt} e^{gt - R_t} h_x dx + \int_{gt}^{\infty} e^0 h_x dx,$$

where h_x denotes the probability density function of R_t under Q. Now $e^{-x}h_x = h_y e^{-\Lambda_t + \frac{1}{2}\Gamma_t}$, where $y \sim N(\Lambda_t - \Gamma_t, \Gamma_t)$. Substituting to $z = \dfrac{x - \Lambda_t}{\sqrt{\Gamma_t}}$ in the second integral and $w = \dfrac{x - \Lambda_t + \Gamma_t}{\sqrt{\Gamma_t}}$ in the first integral and using the symmetry property of the normal

Lemma 2

The market based loading for a participating policy expiring with a guaranteed minimum return at time t is

$$\pi_g(t) = \prod_{i=1}^{n_t} \left| e^{g_i(t_i - t_{i-1}) - \Lambda_i + \frac{1}{2}\Gamma_i(t)} \Phi\left(\frac{g_i(t_i - t_{i-1}) - \Lambda_i + \Gamma_i(t)}{\sqrt{\Gamma_i(t)}}\right) + \Phi\left(\frac{\Lambda_i - g_i(t_i - t_{i-1})}{\sqrt{\Gamma_i(t)}}\right)\right| - 1.$$

Proof:

The Gaussian properties of δ_t^i and zero covariance between different δ_t^i's imply independence. Hence

$$\pi_g(t) = E^Q\left[\exp\left(\sum_{i=1}^{n}[g_i(t_i - t_{i-1}) - \delta_t^i]\vee 0\right]\right. = \prod_{i=1}^{n} E^Q[\exp([g_i(t_i - t_{i-1}) - \delta_t^i]\vee 0].$$

The result follows by the same arguments as above on each factor in the last product.

We now incorporate mortality factors and calculate the proper loadings for a pure endowment contract and a term insurance.

Let [0,T] be the insurance period. Let ${}_x\pi_p$ and ${}_x\pi_g$ denote the loadings for the participating and the guaranteed version of the pure endowment policy, respectively. Let π_p^1 and π_g^1 denote the loadings for the participating and the guaranteed version of the pure endowment policy, respectively.

We obtain from expressions [6] and [7], the assumptions of risk neutrality with respect to mortality and independence between mortality and financial factors, and the previous two lemmas, we obtain

[9]

$$_x\pi_p = {}_Tp_x\pi_p(T),$$

$$_x\pi_g = {}_Tp_x\pi_g(T),$$

[10]

$$\pi_p^1 = \int_0^T \pi_p(t)f_x(t)dt$$

and

$$\pi_g^1 = \int_0^T \pi_g(t)f_x(t)dt,$$

for the four different policies, respectively. For the pure endowments contracts the loadings are just the loadings found earlier weighted by the probability for payment. This interpretation carries roughly over to the term insurances as well.

4. Numerical examples and comparisons

In the numerical calculations we use an annual grid. We consider policies with from 1 to 10 years insurance period. The assumed parameters of the interest rate process [1] are given in table 1.

Table 1. Interest Rate Parameters and Market Price of Interest Rate risk.

q	0,1
m	0,06
v	0,05
λ	−0,2
r_0	0,06

.

It then follows that d = 0,16. We have assumed that both types of guarantees, g and g_1 for all i, equal g = ln(1,04). This corresponded to the guarantee previously used in Norway. As for the parameters of the interest rate process and the market price of risk the numbers used are of approximate the same magnitude as examples found in standard finance textbooks, e.g., Hull (1989). This reference can also be consulted for arguments explaining why the market price of interest rate risk is negative.

Using these parameters we obtain the following market based loadings in percent from Lemmas 1 and 2.

Table 2. Marked Based Loadings for the Base Scenario.

expiration	participating	guaranteed
t	$\pi_p(t)$	$\pi_g(t)$
1	0,27%	0,27%
2	0,95%	1,93%
3	1,83%	5,09%
4	2,81%	9,78%
5	3,86%	16,09%
6	4,94%	24,17%
7	6,04%	34,22%
8	7,16%	46,48%
9	8,27%	61,30%
10	9,39%	79,09%

Casual experiments indicate that these market based loading are increasing in the parameters v, g, and t, and decreasing in q, m, and r_0.

We have also studied 3 other cases. The first case is the situation where the initial interest rate is above the long term average rate, $r_0 > m$. The second case presents the prices one would have obtained by the classical principle of equivalence, i.e. $\lambda = 0$. In the last example we doubled the volatility v to see if the loadings are sensitive to this parameter.

Table 3. Market based loadings for Alternative Scenarios.

expiration	high initial	interest rate	actuarial	method	high	volatility
	$r_0 = 0,12$	$r_0 = 0,12$	$\lambda = 0$	$\lambda = 0$	$v = 0,1$	v - 0,1
t	$\pi_p(t)$	$\pi_g(t)$	$\pi_p(t)$	$\pi_g(t)$	$\pi_p(t)$	$\pi_g(t)$
1	0,00%	0,00%	0,37%	0,37%	1 ,05%	1,05%
2	0,04%	0,43%	1,45%	2,36%	3,15%	5,25%
3	0,15%	1,95%	3,03%	6,15%	5,84%	13,07%
4	0,34%	4,76%	5,04%	11 ,87'%	9,01%	25,19%
5	0,61%	8,97%	7,44%	19,69%	12,68%	42,69%
6	0,94%	14,69%	10,23%	29,88%	16,88%	67,17%
7	1,32%	22,06%	13,42%	42,79%	21,70%	101,03%
8	1,75%	31,25%	17,01%	58,91%	27,25%	14,78%
9	2,20%	42,49%	21,05%	78,85%	33,68%	212,56%
10	2,68%	56,07%	25,56%	103,39%	41,18%	302,91%

Incorporating mortality factors is done by the 1983 Individual annuity mortality table found in Black and Skipper (1987). The insurance period are assumed to be 10 years. The loadings are calculated from our base case presented in Tables 1 and 2. The market based loadings for the pure endowment policies follow from the expressions labeled [9]. The similar loadings for the term insurances are discretized the natural way

$$\pi_j^1 = \sum_{i=1}^{10} \pi_j(i) q_x(i), \, j = p,g,$$

where $q_x(i)$ represents the probability for an x year old insurer to die in the period (i – 1, i].

From this table we calculate the survival probabilities as ${}_{10}p_{30}$ = 99,07% and ${}_{10}p_{50}$ = 94,31%. The loadings for the two pure endowment contracts are given in following table.

Table 4. Market Based Loadings for Pure Endowment Contracts

age	participating	guaranteed
30	9,30%	78,32%
50	8,86%	74,56%

For the term insurances the similar loadings are given in Table 5.

Table 5. Marked Based Loadings for term Insurance Contracts.

age	participating	guaranteed
30	0,97%	1,22%
50	5,98’%	7,54%

If we consider a guaranteed term insurance with benefit \$50 000 and insurance period 10 year, the single premium would have been \$306 and \$2356 for a 30 and a 50 year old male, respectively, given our pricing framework. The two market based loadings for an annual guarantee rate of ln(1,04) would then have been \$18 and \$178, respectively.

5. Concluding Remarks

In this paper we present a model which is a suitable framework for valuation of periodically guaranteed returns. Since we are concerned about rates of return, we adopt an “amount of insurance formulation” instead of studying absolute amounts. Our model, including a financial market where bonds are traded, leads to two formulas for the marked based loadings of participating and guaranteed insurance contracts.

Our numerical examples indicate that such guarantees may have substantial market values.

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