APPENDIX TO THE ARTICLE

USE OF CIR-TYPE INTEREST RATE MODELS TO ASSESS THE ECONOMIC VALUE OF PARTICIPATING SAVINGS CONTRACTS?

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This document is attached to the paper Armel and Planchet [2020]: "Use of CIR-type interest rate models to assess the economic value of participating savings contracts?". It introduces:

- Some generalities on affine term-structure interest rate models family and their extension by deterministic functions to take into account the initial yield curve;
- The dynamics and analytical properties of (1) one-factor basic and extended CIR models (CIR++ model) and (2) two-factor basic and extended CIR models (CIR2++ model);
- The definition and properties of non-central chi-square distributions.

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Content

1 Affine term-structure interest rate models

The success of models such as Vasicek [1977] and Cox, Ingersoll and Ross [1985] is mainly due to their ability to analytically evaluate bonds and bond options.

The dynamic of Vasicek model ($dr\ (t)\ =\ k\ [\theta - r(t)]dt + \sigma dW(t)$) is interesting from an analytical point of view. The equation is linear and can be solved explicitly. The distribution of the short rate is Gaussian and the prices of bonds and some options can be expressed in analytical form.

In addition, the general equilibrium approach proposed by Cox, Ingersoll and Ross [1985] introduces a "square root" term into the diffusion coefficient of instantaneous short rate dynamic proposed by Vasicek [1977].

The resulting model has been a reference for many years because of its ease of analysis and the fact that, unlike Vasicek [1977] model, the instantaneous short rate is always positive. The dynamic of the model under the risk-neutral measure is written as:

$$dr(t) = k\big(\theta - r(t)\big)dt + \sigma\sqrt{r(t)}\,dW(t)$$

where $r(0) = r_0$ and r_0, k, θ, σ are positive constants.

In order for the instantaneous short rate to remain strictly positive, the parameters of the model must meet the Feller condition:

$$2k\theta > \sigma^2$$

While interesting from an analytical perspective, the initial term structure of the interest rates produced by these models does not necessarily correspond to that observed in the market, regardless of the choice of parameters.

In order for these models to reproduce the term structure of interest rates, the financial literature offers at least two possibilities:

- Make the parameters time-dependent (Hull & White extension, see section 2.1.8.1);
- Introduce additively a deterministic function (see section 1.2).

Note also that the Vasicek [1977] and CIR models are models with an affine term structure. In order to ease the reading of Armel and Planchet [2020], we present in the following some generalities on the family of affine term-structure interest rate models and their extension by deterministic functions in order to take into account the initial yield curve.

We have relied mainly on Brigo and Mercurio [2007] for the writing of this section.

1.1 Affine models: definition and generalities

The continuous compounded spot interest rate evaluated at the date t for maturity T denoted $R(t,T)$ is the constant rate at which an investment of $P(t,T)$ monetary units at the date t accumulates continuously to reach one unit of currency at the date T. If $P(t,T)$ denotes the price of a zero-coupon bond valued at the date t maturing on the date T then: $P(t,T) = e^{-R\,(t,T)(T-t)}$.

Affine term-structure interest rate models are models where the continuous compound spot interest rate evaluated at the date t for maturity T is an affine function of the instantaneous short spot rate, denoted (t) :

$$R(t,T) = \alpha(t,T) + \beta(t,T)r(t)$$

where α and β are deterministic functions.

This condition is always satisfied when the price of the zero-coupon bond valued at the date t maturing on the date T is written:

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}$$

where A and B are deterministic functions.

We can wright in fact:

- $\alpha(t,T) = -ln\big(A(t,T)\big)/(T-t)$;
- $\beta(t,T) = B(t,T)/(T - t)$.

Suppose that the instantaneous short rate follows the following dynamic:

$$dr(t) = b(t,r(t))dt + \sigma(t,r(t))dW(t)$$

The model characterized by this dynamic is affine when the deterministic functions b and σ^2 are affine.

If the coefficients b and σ^2 are of the form:

$$\begin{cases} b(t,x) = \lambda(t) \times x + \eta(t) \\ \sigma^2(t,x) = \gamma(t) \times x + \delta(t) \end{cases}$$

where λ, η, $\gamma$and δ are appropriate deterministic functions, then the model has an affine term structure.

The functions A and B (respectively α and β) can be obtained from the coefficients λ, η, γ and δ by solving the following differential equations:

$$\frac{\partial}{\partial t}B(t,T) + \lambda(t)B(t,T) - \frac{1}{2}\gamma(t)B(t,T)^2 + 1 = 0 \; and \; B(T,T) = 0$$

$$\frac{\partial}{\partial t}\big[\ln\big(A(t,T)\big)\big] - \eta(t)B(t,T) + \frac{1}{2}\delta(t)B(t,T)^2 = 0 \; and \; A(T,T) = 1$$

In the particular case of the CIR model the above equations admit a solution and it is sufficient to take:

$$\begin{cases} \lambda(t) = -k \\ \eta\ (t) = k\theta \\ \gamma(t) = \sigma^2 \\ \delta\ (t) = 0 \end{cases}$$

Therefore, the affinity in the coefficients implies the affinity in the term structure. The opposite is also true in the case where the functions b and σ^2 are time-homogeneous: $b(t,x) = b(x)$ and $\sigma(t,x) = \sigma(x)$.

Indeed, it is possible to prove that if a model has an affine term-structure and time-homogeneous coefficients ($b(t,x) = b(x)$ and $\sigma(t,x) = \sigma(x)$)then these coefficients are necessarily affine as a function of x :

$$\begin{cases} b\ (x) = \lambda x + \eta \\ \sigma^2(x) = \gamma x + \delta \end{cases}$$

for appropriate constants λ, η, γ and δ.

1.2 Extension of affine term-structure interest rate models by deterministic functions

This section presents a method for extending instantaneous short-term rate models with an affine term structure in order to replicate the observed yield curve while preserving the analytical characteristics of the reference model.

1.2.1 Notations and assumptions

Let x^α be a stochastic process whose coefficients are time-homogeneous and whose dynamic under a given measure Q^x is written:

$$dx_t^\alpha = \mu(x_t^\alpha;\alpha)dt + \sigma\ (x_t^\alpha;\ \alpha)dW_t^x$$

where W^x is a standard Brownian motion, x_0^α is a given real number, $\alpha = \{\alpha_1, \dots, \alpha_n\} \in IR^n, n \geq 1$ is a vector of parameters, and μ and σ are appropriate real functions.

We assume that the process x^α describes the evolution of the instantaneous spot interest rate under the measure Q^x. Let F_t^x be the sigma-algebra generated by $\{x_i^\alpha\}_{i\leq t}$.

The price at time t, denoted $P^x(t,T)$, of a zero-coupon bond maturing at time T is:

$$P^x(t,T)\ =\ E^{Q^x}\left(exp\left[-\int_t^T x_s^\alpha ds\right]|F_t^x\right)$$

We also assume that there is an analytical form, a real function denoted Π_x, defined on an appropriate subset of IR^{n+3} such as $P^x(t,T) = \Pi^x(t,T,x_t^\alpha;\alpha)$.

The Vasicek [1977] and Cox-Ingersoll-Ross [1985] models are examples of interest rate models for which such a function exists.

Let r_t be the instantaneous short interest rate under the risk-neutral measure Q defined by:

$$r_t = x_t + \varphi(t;\alpha), t \geq 0$$

where x is a stochastic process that has, under Q, the same dynamic as x^α under Q^x and φ is a deterministic function, dependent on the parameter vector (α, x_0) that is integrable over closed intervals.

The process r depends on the parameters $\alpha_1, \dots, \alpha_n, x_0$ and the function φ can be chosen to reproduce the term-structure of interest rates.

Note F_t the sigma-algebra generated by $\{x_i^\alpha\}_{i\leq t}$.

If φ is differentiable, the instantaneous short rate stochastic differential equation is written:

$$dr_t = \left[\frac{d\varphi(t;\alpha)}{dt} + \mu(r_t - \varphi(t;\alpha);\alpha)\right]dt + \sigma(r_t - \varphi(t;\alpha);\alpha)dW_t$$

As discussed in section 1.1, for time-homogeneous coefficients, an affine term structure of short-term interest rates is equivalent to an affine structure of the drift and the squared diffusion coefficients. It follows that if the reference model has an affine term-structure, so does the extended model. We can then anticipate that extended Vasicek model (equivalent to the Hull and White model) and extended CIR (CIR ++) model are affine models.

1.2.2 Reproduction of the initial yield curve

By replacing x_t by $r_t - \varphi(t;\alpha)$ we can prove that the price at time t of a zero-coupon bond with a maturity of T is written:

$$P(t,T) = exp\left(-\int_t^T \varphi(s;\alpha)ds\right)\Pi^x(t,T,r_t - \varphi(t;\alpha);\alpha)$$

Let $f^x(0,t;\alpha)$ be the instantaneous forward rate at time 0 for a maturity t associated with the bond price denoted $P^x(0,t)$ then:

$$f^x(0,t;\alpha) = -\frac{\partial \ln\left(P^x(0,t)\right)}{\partial t} = -\frac{\partial \ln\left(\Pi^x(0,t,x_0;\ \alpha)\right)}{\partial t}$$

Let $f^M(0,t)$ be the instantaneous forward rate of the market observed at time 0 for maturity t:

$$f^M(0,t) = -\frac{\partial \ln\left(P^M(0,t)\right)}{\partial t}$$

then the model reproduces the observed interest rates term structure if and only if:

$$\varphi\ (t;\ \alpha) = \varphi^*(t;\ \alpha) = f^M(0,t) - f^x(0,t;\alpha)$$

which means:

$$exp\left(-\int_t^T \varphi(s;\alpha)ds\right) = \Phi^*(t,T,x_0;\ \alpha) = \frac{\mathrm{P^M(0,T)}}{\Pi^x(0,\mathrm{T},\mathrm{x}_0;\ \alpha)}.\frac{\Pi^x(0,\mathrm{t},\mathrm{x}_0;\alpha)}{\mathrm{P^M(0,t)}}$$

The price of a zero-coupon bond at time t is given by:

$$P(t,T) = \Pi(t,T,r_t;\alpha)$$

where $\Pi(t,T,r_t;\alpha) = \Phi^*(t,T,x_0;\alpha)\Pi^x(t,T,r_t - \varphi^*(t;\ \alpha);\alpha)$.

1.2.3 Explicit formulas for valuing European options

The extension presented in section 1.2.1 is even more interesting when the reference model proposes analytical formulas for valuing European options on zero-coupon bonds. The generalized model can preserve the possibility to price options by closed formulas using analytical correction factors that are functions of φ.

The price at time t of a European call option with maturity T and a strike K on a zero-coupon bond with a maturity τ is:

$$V^x(t,T,\tau,K) = E^x\left\{exp\left[-\int_t^T x_s^\alpha\, ds\right](P^x(T,\tau) - K)^+ \middle| F_t^x\right\}$$

Suppose there is an analytical form, an explicit real function denoted Ψ^x, defined on an appropriate subset of IR^{n+5} such that

$$V^x(t,T,\tau,K) = \Psi^x(t,T,\tau,K,x_t^\alpha;\alpha)$$

The Vasicek [1977] and Cox-Ingersoll-Ross [1985] models are examples of short interest rate models for which such a function exists.

Under the framework described in section 1.2.1 the price at time t of a European call option with a maturity date T and a strike K on a zero-coupon bond with a maturity τ is:

$$\begin{aligned} ZBC(t,T,\tau,K) & \\ &= \exp\left(-\int_t^\tau \varphi\,(s;\ \alpha)ds\right) \\ &\cdot \Psi^x\left(t,T,\tau,Kexp\left[\int_T^\tau \varphi(s;\ \alpha)ds\right],r_t-\varphi(t;\alpha);\ \alpha\right) \end{aligned}$$

The price of a European put option can be obtained via the call-put parity.

Caps and floors can also be priced analytically.

Moreover, if the Jamshidian [1989] decomposition for evaluating swaptions can be applied to the reference model, the same decomposition is also applicable to the extended model. Swaptions can therefore be evaluated by analytical formulas.

2 CIR models: definition, properties and extensions

This section presents the dynamics and analytical properties of:

- The reference one-factor CIR model and the extended one factor CIR++ model;
- The reference two-factor CIR model and the extended two-factor CIR2++ model.

We have mainly relied on Cox, Ingersoll and Ross [1985] and Brigo and Mercurio [2007] for the writing of this section.

2.1 One factor CIR reference model

2.1.1 Model dynamic

The CIR model generalizes the Vasicek [1977] model and introduces a square-root term of the instantaneous short rate into the dynamics allowing the model to produce positive interest rates.

The differential equation of the model under the risk-neutral measure Q is:

$$dr(t) = k\big(\theta - r(t)\big)dt + \sigma\sqrt{r(t)}\, dW(t)$$

with $r(0) = r_0$ and k, θ and σ are positive constants.

The instantaneous short rate remains strictly positive if the parameters of the model meet the Feller condition:

$$2k\theta > \sigma^2$$

2.1.2 Solution of the differential equation

Let p_Y denote the probability density function of the random variable Y, then the density of $r(t)$ conditionally to $r(s)$ is written:

$$p_{r(t)|r(s)}(x) = c_{t-s} \times p_{\chi^2(v,\lambda_{t,s})}(c_{t-s}x) = p_{\chi^2(v,\lambda_{t,s})/c_{t-s}}(x)$$

where:

- $c_{t-s} = \frac{4k}{\sigma^2\left(1-exp\left(-k(t-s)\right)\right)}$;
- $v = 4k\theta/\sigma^2$;
- $\lambda_{t,s} = c_{t-s} r_s exp\big(-k(t-s)\big)$.

The probability density of a non-central chi-square distribution with v degrees of freedom and non-centrality parameter λ is:

$$p_{\chi^2(v,\lambda)}(z) = \sum_{i=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^i}{i!} \mathrm{p}_{\Gamma(i + v/2, 1/2)}(z)$$

where

$$p_{\Gamma\left(i+\frac{v}{2},\frac{1}{2}\right)}(z) = \frac{\left(\frac{1}{2}\right)^{i+\frac{v}{2}}}{\Gamma\left(i+\frac{v}{2}\right)} \times z^{i-1+\frac{v}{2}} \times e^{-\frac{z}{2}} = p_{\chi^2(v+2i)}(z)$$

The function $p_{\chi^2(v+2i)}(z)$ is the probability density of a central chi-square distribution with $v + 2i$ degrees of freedom.

The mean and the variance of $r(t)$ conditionally to F_s are given by:

$$E\{r(t)|F_s\} = r(s)e^{-k(t-s)} + \theta\left(1 - e^{-k(t-s)}\right)$$

$$Var\{r(t)|F_s\} = \frac{r(s)\sigma^2}{k}\left(e^{-k(t-s)} - e^{-2k(t-s)}\right) + \theta\frac{\sigma^2}{2k}\left(1 - e^{-k(t-s)}\right)^2$$

2.1.3 Conditional density of the instantaneous short rate and the dynamic of the compound forward rate

Let Q^T be the T-forward[3] measure and let W^T be the variable defined by: $dW^T(t) = dW(t) + \sigma B(t,T)\sqrt{r(t)}dt$. W^T is a standard Brownian motion under Q^T.

It can be shown that under $Q^T$the distribution of the short rate $r(t)$ conditionally to the rate $r(s)$, $s \leq t \leq T$ is given by:

$$p^T_{\left(r(t)\middle|r(s)\right)}(x) = q(t,s)p_{\chi^2(v,\delta(t,s))}(q(t,s)x)$$

where:

- $q(t,s) = 2[\rho(t-s) + \psi + B(t,T)]$;
- $\delta(t,s) = \frac{4\rho(t-s)^2 r(s) e^{h(t-s)}}{q(t,s)}$.

Let $F(t;T,S)$ be the simply compounded forward rate, observed at the time t whose term is T and the maturity is S, defined by:

$$F\ (t;T,S) = \frac{1}{\gamma(T,S)}\left(\frac{P(t,T)}{P(t,S)} - 1\right)$$

where $\gamma(T,S)$ is the fraction of a year between T and S.

Under the forward measure Q^S the forward rate is written:

$$dF(t;T,S) = \sigma \times \left(F\ (t;\ T,S) + \frac{1}{\gamma(T,S)}\right) \times \sqrt{\left(B(t,S) - B\ (t,T)\right) ln\left[\frac{(\gamma\ (T,S)F\ (t;\ T,S) +\ 1)A(t,S)}{A(t,T)}\right]}\ dW^S(t)$$

Note that this differential equation is quite different from the log-normal dynamic of the forward rate in the LMM model, where typically $dF(t;T,S) = \sigma(t)F(t;T,S)dW^S(t)$ for a deterministic function σ.

2.1.4 Price of a zero-coupon bond

The price at time t of a zero-coupon bond with a maturity T is:

$$P(t,T) = A(t,T)e^{-B\ (t,T)r\ (t)}$$

[3]Q^T is the probability measure defined by the Radon-Nikodym derivative: $\frac{dQ^T}{dQ} = \frac{exp\left(-\int_0^T r(u)du\right)}{P(0,T)}$.

where

$$A\,(t,T) = \left[\frac{2h\,exp\left\{\frac{(k\;+\;h)(T\;-\;t)}{2}\right\}}{2h + (k + h)(exp\{(T - t)h\} - 1)}\right]^{\frac{2k\theta}{\sigma^2}}$$

$$B\;(\;t,T)\;\; = \frac{2(exp\,\{(T\;-\;t)h\} -\;1)}{2h\;+\;(k\;+\;h)(exp\,\{(T\;-\;t)h\} -\;1)}$$

$$h = \sqrt{k^2 + 2\sigma^2}$$

By using Itô's formula one can write:

$$dP(t,T) = r(t)P(t,T)dt - B(t,T)P(t,T)\sigma\sqrt{r(t)}dW(t)$$

By reversing the price formula, $P(t,T)$, thus deducting r from P, we can write:

$$d\ln\big(P(t,T)\big) = \left(\frac{1}{B(t,T)} - \frac{1}{2}\sigma^2 B(t,T)\right)\left[\ln\big(A(t,T)\big) - \ln\big(P(t,T)\big)\right]dt \\ - \sigma\sqrt{B(t,T)[ln\big(A(t,T)\big) - \ln\big(P(t,T)\big)]}\;dW(t)$$

We note that the volatility relative to the zero-coupon bond price is not a deterministic function, but depends on the current price level.

2.1.5 Price of a European zero-coupon bond option

Let $r(t)$ denote the instantaneous short rate at time t. The price at time t of a European call option, with maturity $T >\; t$ and strike X, issued on a zero-coupon bond with a maturity $S > T$, is (cf. Cox, Ingersoll and Ross [1985] and Brigo and Mercurio [2007])[4]:

$$ZBC(t,T,S,X) \\ = P(t,S)F_{\chi^2}\left(2\bar{r}\,[\rho + \psi + B\,(T,S)]; \frac{4k\theta}{\sigma^2}, \frac{2\rho^2 r(t)\exp\{h\,(T\;-\;t)\}}{\rho + \psi + B\,(T,S)}\right) \\ -\; XP(t,T)F_{\chi^2}\left(2\bar{r}\,[\rho + \psi]; \frac{4k\theta}{\sigma^2}, \frac{2\rho^2 r(t)\exp\{h\,(T\;-\;t)\}}{\rho\;+\psi}\right)$$

where

- $\rho \;=\; \rho\,(T - t) = \frac{2h}{\sigma^2(exp[h(T-t)]-1)}$;
- $\psi \;= \frac{k\,+\,h}{\sigma^2}$;
- $\bar{r} \;=\; \bar{r}\,(S - T) = \frac{ln\left(\frac{A(T,S)}{X}\right)}{B(T,S)}$.

[4] $F_{\chi^2}(.\,; u, v)$ is the cumulative distribution function of a non-centeral chi-square distribution with u degrees of freedom and a non-centrality parameter of v.

The put option price is obtained by the *put-call* parity and is denoted ZBP:

$$ZBP(t,T,\tau,K) = ZBC(t,T,\tau,K) - P(t,\tau) + KP(t,T)$$

2.1.6 Prices of caps and floors

The price, at time t, of a capelet with an expiry date denoted T, a date of payment denoted $T+\tau$, a strike denoted X and a notional amount denoted N is written:

$$Cpl(t,T,T+\tau,N,X) = N(1+X\tau) \times ZBP\left(t,T,T+\tau,\frac{1}{1+X\tau}\right)$$

Note $\zeta = \{t_0,t_1,\dots,t_n\}$ the set of all payments maturities of *caps* or *floors* plus the initialization date t_0. Let τ_i be the difference between t_{i-1} and t_i.

The price at time $t < t_0$ of a cap with a strike denoted X, a nominal denoted N and defined on the set $\zeta = \{t_0,t_1,\dots,t_n\}$ is given by:

$$Cap(t,\zeta,N,X) = N\sum_{i=1}^{n}(1 + X\tau_i) \times ZBP\left(t,t_{i-1},t_i,\frac{1}{1+X\tau_i}\right)$$

The price of the *floor* is given by:

$$Flr(t,\zeta,N,X) = N\sum_{i=1}^{n}(1 + X\tau_i) \times ZBC\left(t,t_{i-1},t_i,\frac{1}{1+X\tau_i}\right)$$

2.1.7 *Swaption* prices

The analytical form of the price of a European *swaption* evaluated using the CIR model can be explicitly formulated using the Jamshidian [1989] decomposition (see Brigo and Mercurio [2007]).

Let's consider a payer *swaption* with a strike X, a maturity T and a nominal N. It gives its holder the right to contract at time $t_0 = T$ an interest rate *swap* with payment dates $\zeta = \{t_1,\dots,t_n\}$, $t_1 > T$ where he pays a fixed rate X and receives the variable rate.

Let τ_i be the fraction of a year from t_{i-1} to t_i, $i = 1,\dots,n$ and let $c_i = X\tau_i$ for $i = 1,\dots,n-1$ and $c_n = 1 + X\tau_n$.

Let r^* be the *spot* rate at time T for which $\sum_{i=1}^{n} c_i\bar{A}(T,t_i) \times e^{-B(T,t_i)r^*} = 1$ and let $X_i = \bar{A}(T,t_i) \times e^{-B(T,t_i)r^*}$.

The price of the payer swaption at time $t < T$ is then given by:

$$PS(t,T,\zeta,N,X) = N\sum_{i=1}^{n} c_i \times ZBP(t,T,t_i,X_i)$$

Symmetrically, the price of the receiver swaption is:

$$RS(t,T,\zeta,N,X) = N\sum_{i=1}^{n} c_i \times ZBC(t,T,t_i,X_i)$$

2.1.8 What is the extension of the CIR model?

The reference CIR model cannot reproduce the term structure of interest rates observed in the market. The financial literature suggests at least two methods of extending this model in order to reproduce the initial market yield curve:

- Make all model parameters time-dependent (Hull & White type extension) ;
- Introduce additively a deterministic function.

Other extensions, which we will not detail here, are proposed by the literature. We can cite, for example, the one presented in Shiu and Yao [1999] who propose closed formulas to value zero coupon bonds assuming that the instantaneous interest rate is described by the following equations:

$$dr(t) = \varphi(t)dt + k[\theta(t) - r(t)]dt + \sigma\sqrt{r(t)}\, dW(t)$$

$$d\theta(t) = \beta(r(t) - \theta(t))dt$$

The deterministic function $\varphi(t)$ allows to replicate the initial yield curve.

2.1.8.1 Extension of the CIR model by Hull & White

In addition to the extension of Vasicek [1977] model, Hull & White [1990] have proposed an extension of Cox, Ingersoll and Ross's [1985] model which is based on the same principle: making the coefficients time-dependent.

The dynamic of the short rate are then given by :

$$dr(t) = [\vartheta(t) - a(t)r(t)]dt + \sigma(t)\sqrt{r(t)}dW(t)$$

where a, ϑ and σ are deterministic functions.

However, the analytical characteristics of such an extension are limited.

Indeed, it can be shown that, for $t < T$, the price of a zero-coupon bond can be written as

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}$$

The function B is a solution of a Riccati equation and A is the solution of a linear differential equation subject to certain conditions.

The same analytical limits are observed for the simplified dynamic where the volatility parameter is constant:

$$dr(t) = [\vartheta(t) - a \times r(t)]dt + \sigma\sqrt{r(t)}dW(t)$$

where a and σ are positive constants and only the function ϑ is assumed to be time-dependent in order to reproduce the term structure of interest rates.

To our knowledge, no general analytical expression of $\vartheta(t)$ has been proposed in the financial literature. When we assume that the report $\vartheta(t)/\sigma^2(t)$ is equal to a positive constant δ above $\frac{1}{2}$ to make the origin inaccessible, the CIR model extended by Hull & White [1990], has more extensive analytical features. These analytical properties are not developed further in this paper, the interested reader can refer to Brigo and Mercurio [2007].

The following section presents an extension of the CIR model that is more interesting from an analytical point of view. It allows in particular to reproduce the observed yield curve and to take into account negative rates.

2.1.8.2 Extension by a deterministic function: CIR++ model

The application of the developments presented in section 1 allows to extend the reference CIR model to the model called CIR++. The instantaneous short rate process r is therefore the sum of a deterministic function and a reference CIR process.

The following section presents the dynamic of the CIR++ model as well as the analytical formulas for valuing zero-coupon bonds, caps, floors and swaptions.

2.2 One factor extended CIR model

2.2.1 Extension of the CIR reference model by a deterministic function: CIR++ model

The application of the developments presented in section 1 allows the CIR model to be extended to the CIR++ model. The instantaneous short rate process r is therefore the sum of a deterministic function φ and a reference CIR process x, whose parameter vector is denoted $\alpha = (k, \theta, \sigma)$, defined as follows:

$$dx(t) = k\big(\theta - x(t)\big)dt + \sigma\sqrt{x(t)}dW(t)\ ;\ x(0) = x_0$$

and we have:

$$r\,(t) = x(t) + \varphi(t)$$

where x_0, k, θ and σ are positive constants such as $2k\theta > \sigma^2$, ensuring that the origin is inaccessible for the variable x, so that this process remains positive.

The analytical formulas presented in the following sections result directly from the developments presented in section 1.

2.2.2 Price of a zero-coupon bond

By denoting $\varphi(t) = \varphi^{CIR}(t; \alpha)$, we have:

$$\varphi^{CIR}(t; \alpha) = f^M(0, t) - f^{CIR}(0, t;\ \alpha)$$

where

$$f^{CIR}(0, t; \alpha) = \frac{2k\theta(exp\{th\} - 1)}{2h + (k + h)(exp\,\{th\} - 1)} + x_0\,\frac{4h^2 exp\{th\}}{[2h + (k + h)(exp\{th\} - 1)]^2}$$

with $\mathrm{h} = \sqrt{\mathrm{k}^2 + 2\sigma^2}$.

The price at time t of a zero-coupon bond with a maturity T is:

$$P(t, T) = \bar{A}\,(t, T)e^{-B(t,T)r(t)}$$

where

$$\bar{A}(t,T) = \frac{P^M(0,T)A(0,t)exp\{-B(0,t)x_0\}}{P^M(0,t)A(0,T)exp\{-B(0,T)x_0\}} A(t,T)e^{B(t,T)\varphi^{CIR}(t;\alpha)}$$

- $A(t,T)$ and $B(t,T)$ are defined in section 2.1.4.
- $P^M(0,T)$ is the market price of the risk-free zero-coupon bond observed at time 0.

The interest rate at the time t for the maturity T is therefore:

$$R\,(t,T) = \frac{1}{T-t}\left(\ln\left(\frac{P^M(0,t)A(0,T)exp\{-B(0,T)x_0\}}{A(t,T)P^M(0,T)A(0,t)\,exp\{-B(0,t)x_0\}}\right) - B(t,T)\varphi^{CIR}(t;\ \alpha) + B(t,T)r(t)\right)$$

One can notice that the interest rate $R(t,T)$ is an affine function of $r(t)$.

2.2.3 Price of a European zero-coupon bond option

The price at time t of a European call option, expiring on time $T > t$ with a strike K on a zero-coupon bond with a maturity denoted $\tau > T$ is:

$$ZBC(t,T,\tau,K) = \frac{P^M(0,\tau)A(0,t)exp\{-B(0,t)x_0\}}{P^M(0,t)A(0,\tau)exp\{-B(0,\tau)x_0\}} \times \Psi^{CIR}\left(t,T,\tau,K\frac{P^M(0,T)A(0,\tau)exp\{-B(0,\tau)x_0\}}{P^M(0,\tau)A(0,T)exp\{-B(0,T)x_0\}}, r(t)-\varphi^{CIR}(t;\alpha);\alpha\right)$$

where $\Psi^{CIR}(t,T,\tau,X,x;\alpha)$ is the option price evaluated by the CIR model as defined in section 2.1.4.

By simplifying this formula, we can write:

$$ZBC(t,T,\tau,K) = P(t,\tau)F_{\chi^2}\left(2\hat{r}[\rho+\psi+B(T,\tau)];\frac{4k\theta}{\sigma^2},\frac{2\rho^2[r(t)-\varphi^{CIR}(t;\ \alpha)]exp\{h(T-t)\}}{\rho+\psi+B(T,\tau)}\right) - KP(t,T)F_{\chi^2}\left(2\hat{r}[\rho+\psi];\frac{4k\theta}{\sigma^2},\frac{2\rho^2[r(t)-\varphi^{CIR}(t;\ \alpha)]exp\{h(T-t)\}}{\rho+\psi}\right)$$

With

$$\hat{r} = \frac{1}{B(T,\tau)}\left[\ln\left(\frac{A(T,\tau)}{K}\right) - \ln\left(\frac{P^M(0,T)A(0,\tau)exp\{-B(0,\tau)x_0\}}{P^M(0,\tau)A(0,T)exp\{-B(0,T)x_0\}}\right)\right]$$

The put option price is obtained by the put-call parity and is denoted :

$$ZBP(t,T,\tau,K) = ZBC(t,T,\tau,K) - P(t,\tau) + KP(t,T)$$

2.2.4 Prices of caps and floors

Caps and floors can be considered as portfolios of options on zero-coupon bonds. The price on time t of a caplet with an expiry date denoted T, a payment date denoted $T+\tau$, a strike denoted X and a notional amount denoted N is written:

$$Cpl(t,T,T+\tau,N,X) = N(1+X\tau)\times ZBP\left(t,T,T+\tau,\frac{1}{1+X\tau}\right)$$

Let $\zeta = \{t_0, t_1, \dots, t_n\}$ be the set of all payment maturities of caps or floors increased by t_0 corresponding to the initialization time. Let τ_i be the difference between t_{i-1} and t_i.

The price at time $t < t_0$ of a cap with a strike X, a nominal N and defined on the time set $\zeta = \{t_0, t_1, \dots, t_n\}$ is given by:

$$Cap(t,\zeta,N,X) = N\sum_{i=1}^{n}(1 + X\tau_i)\times ZBP\left(t,t_{i-1},t_i,\frac{1}{1+X\tau_i}\right)$$

The price of the floor is given by:

$$Flr(t,\zeta,N,X) = N\sum_{i=1}^{n}(1 + X\tau_i)\times ZBC\left(t,t_{i-1},t_i,\frac{1}{1+X\tau_i}\right)$$

2.2.1 Swaption prices

As with the CIR model, the analytical form of the price of a European swaption valued using the CIR++ model can be explicitly formulated using the Jamshidian decomposition [1989] (see Brigo and Mercurio [2007]).

Let's consider a payer swaption with a strike X, a maturity T and a nominal value N. It gives its holder the right to contract, at time $t_0 = T$, an interest rate swap with payment dates $\zeta = \{t_1, \dots, t_n\}$, $t_1 > T$, where he pays a fixed rate X and receives the variable rate.

We denote by τ_i the fraction of a year from t_{i-1} to t_i, $i = 1, \dots, n$ and let $c_i = X\tau_i$ for $i = 1, \dots, n-1$ and $c_n = 1 + X\tau_n$.

Let r^* be the spot rate at time T for which $\sum_{i=1}^{n} c_i\bar{A}(T,t_i)\times e^{-B(T,t_i)r^*} = 1$ and let $X_i = \bar{A}(T,t_i)\times e^{-B(T,t_i)r^*}$.

The price of the payer swaption at time $t < T$ is then given by:

$$PS(t,T,\zeta,N,X) = N\sum_{i=1}^{n} c_i \times ZBP(t,T,t_i,X_i)$$

Symmetrically, the price of the receiver swaption is:

$$RS(t,T,\zeta,N,X) = N\sum_{i=1}^{n} c_i \times ZBC(t,T,t_i,X_i)$$

2.3 Two-factor extended CIR model

The CIR2++ model is a two-factor short rate model that adds a deterministic function to the sum of two independent CIR processes. This model can be viewed as the natural two-factor extension of the CIR++ model presented in section 2.2.

The CIR2++ model is of the form: $r_t = x_t + y_t + \varphi(t)$ where φ is a deterministic function allowing to reproduce the initial observed yield curve and x and y are two independent CIR processes.

In the following, we first present the reference two-factor CIR model (non-shifted) and then present the CIR2++ model.

2.3.1 The reference two-factor CIR model

2.3.1.1 Model dynamic

The two-factor CIR model defines the instantaneous interest rate as the sum of two independent CIR processes under the risk-neutral measure.

Let x and y be two processes defined by:

$$dx(t) = k_1(\theta_1 - x(t))dt + \sigma_1\sqrt{x(t)}dW_1(t)$$
$$dy(t) = k_2(\theta_2 - y(t))dt + \sigma_2\sqrt{y(t)}dW_2(t)$$

where W_1 and W_2 are independent Brownian motions under the risk neutral measure, and $k_1, \theta_1, \sigma_1, k_2, \theta_2$ and σ_2 are positive constants such as $2k_1\theta_1 > \sigma_1^2$ and $2k_2\theta_2 > \sigma_2^2$.

Positives real numbers $x(0) = x_0$ and $y(0) = y_0$ are respectively the initial values of processes x and y.

The instantaneous short rate is then defined as follows:

$$\xi_t^\alpha = x(t) + y(t)$$

with $\alpha = (\alpha_1, \alpha_2)$, $\alpha_1 = (k_1, \theta_1, \sigma_1)$ and $\alpha_2 = (k_2, \theta_2, \sigma_2)$.

The short rate can therefore be assimilated to a linear sum of two independent, non-central chi-square variables.

2.3.1.2 Price of a zero-coupon bond

Due to the independence of the factors, the price of a zero-coupon bond is directly derived from the analytical pricing formula of the one factor reference CIR model. The price at time t of a zero-coupon bond with a maturity T is:

$$P^\xi(t,T;x(t),y(t),\alpha) = P^1(t,T;x(t),\alpha_1) \times P^1(t,T;y(t),\alpha_2)$$

where P^1 denotes the price of a zero-coupon bond valued by the one-factor CIR model (see section 2.1.4). Recall that if z is a one-factor CIR process with parameters $(k_i, \theta_i, \sigma_i)$, the price of a zero-coupon is given by:

$$P^1(t,T;z(t),k_i,\theta_i,\sigma_i) = A_z(t,T)e^{-B_z(t,T)z(t)}$$

where

- $A_z(t,T) = \left[\frac{2h_i exp\left\{\frac{(k_i+h_i)(T-t)}{2}\right\}}{2h_i+(k_i+h_i)(exp\{(T-t)h_i\}-1)}\right]^{\frac{2k_i\theta_i}{\sigma_i^2}};$
- $B_z(t,T) = \frac{2(exp\{(T-t)h_i\}-1)}{2h_i+(k_i+h_i)(exp\{(T-t)h_i\}-1)};$
- $h = \sqrt{k_i^2 + 2\sigma_i^2}\ ;$
- $z \in \{x,y\}$ and $i = 1$ if $z = x$, $i = 2$ otherwise.

The forward interest rate at time t for maturity T is given by:

$$R^{\xi}(t,T;x(t),y(t),\alpha) = R^1(t,T;x(t),\alpha_1) + R^1(t,T;y(t),\alpha_2)$$

where R^1 denotes the forward interest rate valued by the one-factor CIR model obtained from P^1.

Under the risk neutral measure, the dynamic of bond prices are written:

$$dP^{\xi}(t,T;\alpha) = P^{\xi}(t,T;\alpha)\Big[\xi_t^{\alpha}dt - B(t,T;\alpha_1)\sigma_1\sqrt{x(t)}\,dW_1(t) - B(t,T;\alpha_2)\sigma_2\sqrt{y(t)}dW_2(t)\Big]$$

where deterministic function B is defined as in section 2.1.4.

2.3.1.3 *Price of a European zero-coupon bond option*

The price of a call option evaluated at time t with a maturity $T > t$ and a strike K on a zero-coupon bond with a maturity $S > T$ and a nominal value N, is given by:

$$C^{\xi}(t,T,S,N,K;x(t),y(t),\alpha) = P^{\xi}(t,T;x(t),y(t),\alpha)\int_0^{+\infty}\int_0^{+\infty}[N \times P^1(T,S;x_1,\alpha_1) \times P^1(T,S;x_2,\alpha_2) - K]^+ \times p^T_{x(T)|x(t)}(x_1)p^T_{y(T)|y(t)}(x_2)dx_1dx_2$$

Note the presence in this expression of a double integral on the product of two non-central chi-square densities. The analytical expressions of these conditional densities under the T-forward measure were presented in section2.1.3.

2.3.2 Dynamic of the two-factor extended CIR model

In perfect analogy with the developments presented in section 1.2, used in section 2.2 for the one-factor case, the instantaneous interest rate of the CIR2++ model, under the risk-neutral measure is defined by:

$$r_t = \varphi(t;\alpha) + \xi_t^{\alpha} = \varphi(t;\alpha) + x(t) + y(t)$$

where $x(0) = x_0$, $y(0) = y_0$ and where $\varphi(t;\alpha)$ is a deterministic function depending on the parameter vector $\alpha = (x_0, y_0, k_1, \theta_1, \sigma_1, k_2, \theta_2, \sigma_2)$.

In order to reproduce exactly the yield curve observed in the market, it is sufficient that:

$$\varphi(t;\alpha) = f^M(0,t) - f^1(0,t;x_0,\alpha_1) - f^1(0,t;y_0,\alpha_2)$$

where f^1 is the instantaneous forward *rate* evaluated by the one-factor reference CIR model, as indicated in section 2.2.2, and f^M is the instantaneous forward market interest rate.

In the following, it is useful to define the function:

$$\Phi^{\xi}(u,v;\ \alpha) = exp\left[-\int_u^v \varphi(t;\alpha)ds\right] = \frac{P^M(0,v)P^{\xi}(0,u;\alpha)}{P^M(0,u)P^{\xi}(0,v;\alpha)}$$
$$= \exp\{[R^{\xi}(0,v;\alpha) - R^M(0,v)]v - [R^{\xi}(0,u;\alpha) - R^M(0,u)]u\}$$

which is entirely defined from observed prices ($P^M(0,T)$) and the analytical expression of P^{ξ}.

2.3.3 Valuation of a zero-coupon bond using the two-factor extended CIR model

The two-factor CIR process ξ^{α} allows zero-coupon bond pricing by closed formulas. This analytical property is preserved in the CIR2++ model.

The price at time t of a zero-coupon bond with a maturity T is written as the product of the exponential of the primitive of the shift function φ and the price of a zero-coupon bond valued by the reference non-shifted two-factor CIR model (see Section 2.3.1.2). This price is:

$$P(t,T;x(t),y(t),\alpha) = \Phi^{\xi}(t,T;\alpha) \times P^{\xi}(t,T;x(t),y(t),\alpha)$$

2.3.4 Valuation of caps and floors by the two-factor extended CIR model

The price at time t of a European call option with a maturity denoted $T > t$ and a strike denoted K on a zero-coupon bond with a nominal denoted N and a maturity denoted $S > T$ is:

$$ZBC(t,T,S,N,K;x(t),y(t),\alpha)$$
$$= N \times \Phi^{\xi}(t,S;\alpha) \times C^{\xi}\left(t,T,S,N,\frac{K}{\Phi^{\xi}(T,S;\alpha)};x(t),y(t),\alpha\right)$$

where C^{ξ} is the price function of a call option valued by the two-factor CIR model (see section 2.3.1.3).

The price of a put option is obtained from the put-call parity and is written as follows:

$$ZBP(t,T,S,N,K;x(t),y(t),\alpha)$$
$$= ZBC(t,T,S,N,K;x(t),y(t),a) - N \times P(t,S;x(t),y(t),a)$$
$$+ K \times P(t,T;x(t),y(t),\alpha)$$

As with the reference two-factor CIR model, the valuation of an option on a zero-coupon bond requires the resolution of a double integral.

Caps and floors are written as a series of options on zero-coupon bonds (see for example section 2.1.6). The valuation of these instruments can therefore be achieved using the semi-closed formula presented in section 2.3.1.3 or by other methods such as Monte Carlo simulation.

2.3.5 Valuation of swaptions using the two-factor extended CIR model

Unlike the one-factor CIR model, the Jamshidian [1989] decomposition to value swaptions is not applicable for the case of the two-factor CIR model. Therefore, the price of swaptions is valued by other methods, such as Monte Carlo simulation.

3 Non-central chi-square distributions: definition and properties

The purpose of this section is to:

- Define the family of non-central chi-square distributions and present their characteristics;
- Present a method for simulating non-central chi-square distributions;
- Present some Gaussian approximations to the non-central chi-square distributions.

We have relied on the following three references to write this section: Johnson et al [1970], Devroye [1986] and Patel & Read [1982].

3.1 Definition and properties

A random variable X follows a central chi-square distribution χ^2 with $\mathrm{v} > 0$ degrees of freedom if the probability density of X is given by:

$$p_{\chi^2(v)}(x) = \frac{e^{-\frac{x}{2}}}{2\Gamma\left(\frac{v}{2}\right)}\left(\frac{x}{2}\right)^{\frac{v}{2}-1}; x > 0$$

where Γ is the Gamma function.

When $v = 0$ then $p_{\chi^2(0)}(x) = 0$ and the distribution function $F_{\chi^2(0)}(x) = 1$ for every $x > 0$.

The distribution of a central χ^2 is a special case of Gamma distributions. Indeed, if X follows a Gamma distribution with parameters (a, b) then its probability density is written:

$$p_{a,b}(x) = \frac{x^{a-1}e^{-\frac{x}{b}}}{b^a\Gamma(a)}; x > 0$$

For $a = v/2$ and $b = 2$ we find exactly the probability density of a central χ^2 distribution with v degrees of freedom.

The random variable X follows a non-central χ^2 distribution with $v \geq 0$ degrees of freedom and non-centrality parameter λ if its distribution function is written:

$$F_{\chi^2(v,\lambda)}(x) = \sum_{k=0}^{+\infty} \frac{\exp\left(-\frac{\lambda}{2}\right)}{k!} \left(\frac{\lambda}{2}\right)^k F_{\chi^2(v+2k)}(x)$$

The density is written as follows:

$$P_{\chi^2(v,\lambda)}(x) = \sum_{k=0}^{+\infty} \frac{\exp\left(-\frac{\lambda}{2}\right)}{k!} \left(\frac{\lambda}{2}\right)^k P_{\chi^2(v+2k)}(x)$$

Note that the function $P_{\chi^2(v,\lambda)}$ is written as the sum of density-functions of central χ^2 distributions weighted by Poisson's distribution probabilities.

When v is a positive integer, the cumulative distribution function of a non-central χ^2 distribution with v degrees of freedom and non-centrality parameter λ is naturally written as the cumulative distribution function of the quadratic sum of normal distributions. More precisely, let $X_1, \dots, X_v$ be independent normal distributed random variables with means μ_k, $k = 1, \dots, v$ and unit variances. Then the probability density of the random variable $\sum_{k=1}^{v} X_k^2$ is $p_{\chi^2(v,\lambda)}$ with: $\lambda = \sum_{k=1}^{v} \mu_k^2$.

The distributional properties of a non-central χ^2 distribution may be difficult to obtain because the density is not in a closed form. Another expression of the density $p_{\chi^2(v,\lambda)}$ which is not necessarily simpler, is:

$$p_{\chi^2(v,\lambda)}(x) = \frac{1}{2} \exp\left(-\frac{x+\lambda}{2}\right) \left(\frac{x}{\lambda}\right)^{\frac{v-2}{4}} I_{\frac{v-2}{2}}(\sqrt{\lambda x})$$

The function $I_v(x)$ is the modified Bessel function of the first kind defined by:

$$I_v(x) = \sum_{k=0}^{+\infty} \frac{\left(\frac{x}{2}\right)^{2k+v}}{k!\,\Gamma(v+k+1)}$$

Moreover, if N is a Poisson-distributed random variable with a mean $\lambda/2$ whose cumulative distribution function is defined by:

$$F(k) = \frac{\exp\left(-\frac{\lambda}{2}\right)}{k!} \left(\frac{\lambda}{2}\right)^k ; k = 0, 1, \dots$$

then the random variable following a central χ^2 distribution with $v + 2N$ degrees of freedom follows a non-central χ^2 distribution with v degrees of freedom and a non-centrality parameter of λ.

Indeed, we can note that:

$$\sum_{k=0}^{+\infty} P(N=k)\, P(\chi^2(v+2N) < x | N=k) = \sum_{k=0}^{+\infty} \frac{\exp\left(-\frac{\lambda}{2}\right)}{k!} \left(\frac{\lambda}{2}\right)^k F_{\chi^2(v+2k)}(x) = F_{\chi^2(v,\lambda)}(x)$$

3.2 Simulation of a non central χ^2 distribution

Let $X_{(v,\lambda)}$ be a non-central χ^2 distributed random variable. Thus the variable $X_{(v,\lambda)}$ can be written as the sum of two independent random variables X_v and X_λ: $X_{(v,\lambda)} = X_v + X_\lambda$ with (Johnson et al [1970]):

- The variable X_v follows a central χ^2 distribution with v degrees of freedom;
- The variable X_λ follows a non-central χ^2 distribution with 0 degrees of freedom and a non-centrality parameter equal to λ. This is the purely eccentric part of the variable $X_{(v,\lambda)}$. The variable X_λ follows therefore a central χ^2 distribution with $2N$ degrees of freedom, where N is a Poisson-distributed random variable with a mean of $\lambda/2$. Its cumulative distribution function is written:

$$F_{\chi^2(0,\lambda)}(x) = \sum_{k=0}^{+\infty} \frac{\exp\left(-\frac{\lambda}{2}\right)}{k!} \left(\frac{\lambda}{2}\right)^k F_{\chi^2(2k)}(x)$$

This decomposition of a non-central χ^2 random variable in two variables, isolating the degree of freedom in a central χ^2 distribution and the non-centrality parameter in a non-central χ^2 distribution with 0 degrees of freedom, allows to simulate the non-central χ^2 distribution using Gamma and Poisson distributed random variables. The Gamma or Poisson random number generators are generally available in classical statistical tools and software.

Indeed:

- The variable X_v follows a Gamma distribution with parameters $(v/2,2)$ and can be generated directly by simulating a Gamma distribution;
- The variable X_λ follows a central χ^2 with $2N$ degrees of freedom, where N is Poisson-distributed with a mean of $\lambda/2$. It can be generated by first drawing a random number K following a Poisson distribution and then drawing a Gamma distributed number with parameters $(K, 2)$.

3.1 Approximating a non-central χ^2 distribution by normal distributions

The distributional properties of a non-central χ^2 distribution may be difficult to obtain because the density is not in a closed form. The approximation of a non-central χ^2 distribution by normal distributed random variables may be of interest. Patel and Read [1982] synthesize a set of approximations which we present in the following.

Let us denote by $y \to F(y\,; v\,, \lambda)$ the cumulative distribution function of a non-central χ^2 distribution with parameters (v, λ). The following list presents some approximating methods of the function F by the cumulative distribution function of a centred reduced normal distribution denoted Φ.

1. Linear approximations: two simple normal approximations, having an error of the order of $O(1/\sqrt{\lambda})$ when $\lambda \to +\infty$:
   a. $F(y\,;v\,,\lambda) \approx \Phi\left(\frac{y-v-\lambda}{\sqrt{2(v+2\,\lambda)}}\right)$;
   b. $F(y\,;v\,,\lambda) \approx \Phi\left(\frac{y-v-\lambda+1}{\sqrt{2(v+2\,\lambda)}}\right)$.
2. Non-linear approximations:
   a. This approximation is more suitable when the non-centrality parameter is small (and therefore when the distribution is more like the distribution of the central χ^2 distribution) and deteriorates as the parameter λ increases:

$$F(y\,;v,\lambda) \approx \Phi(u)$$

   with:

$$u = \frac{\left(\left\{\frac{y}{v+\lambda}\right\}^{\frac{1}{3}} - 1 + \frac{2(v+2\lambda)}{9(v+\lambda)^2}\right)}{\left(\frac{2(v+2\lambda)}{9(v+\lambda)^2}\right)^{\frac{1}{2}}}$$

   b. An approximation whose error is comparable to that of linear approximations:

$$F(y\,;v\,,\lambda) \approx \Phi(u)$$

   with:

$$u = \sqrt{\frac{2y(v+\lambda)}{v+2\lambda}} - \left(\frac{2(v+\lambda)^2}{v+2\lambda} - 1\right)^{\frac{1}{2}}$$

   c. Approximation without constraints on the degree of freedom and the non-centrality parameter. It remains appropriate even when the degree of freedom is low. The error is of the order of $O(1/\lambda^2)$. Although complicated, this approximation is the best of all the approximations listed here. Armel and Planchet [2020] illustrate its quality for a set of parameters. The distribution function is written:

$$\mathrm{F}(\mathrm{y}\,;\mathrm{v}\,,\lambda) \approx \Phi\left(\frac{\left(\left(\frac{\mathrm{y}}{\mathrm{v}+\lambda}\right)^{\mathrm{h}} - \mathrm{a}\right)}{\mathrm{b}}\right)$$

   with:

$$h = 1 - \frac{2(v+\lambda)(v+3\lambda)}{3(v+2\lambda)^2}$$

$$a \; = \; 1 + \frac{h(h-1)(v+2\lambda)}{(v+\lambda)^2} - \frac{h(h-1)(2-h)(1-3h)(v+2\lambda)^2}{2(v+\lambda)^4}$$

$$b \; = \frac{h\sqrt{2(v+2\lambda)}}{v+\lambda}\left(1 - \frac{(1-h)(1-3h)(v+2\lambda)}{2(v+\lambda)^2}\right)$$

d. A similar approximation as 2.c but the error is of the order of $O(1/\lambda)$. Indeed, the distribution function is written:

$$F(y\,;v\,,\lambda) \approx \Phi\left(\frac{\left(\left(\frac{y}{v\;+\;\lambda}\right)^h - a'\right)}{b'}\right)$$

with:

$$h \; = \; 1 \; - \frac{2(v+\lambda)(v+3\lambda)}{3(v+2\lambda)^2}$$

$$a' \; = \; 1 + \frac{h(h-1)(v+2\lambda)}{(v+\lambda)^2}$$

$$b' \; = \frac{h\sqrt{2(v+2\lambda)}}{v+\lambda}$$

3. Quantile approximations: let y_p and z_p be the quantiles of order $1-\mathrm{p}$ such as $F\left(y_p\,,v\,,\lambda\right) = 1-p = \Phi(z_p)$. Using the notations in point 2, we can approximate y_p by z_p as follows:

   a. $y_p \approx (v+\lambda)\left(z_p\sqrt{C} + 1 - C\right)^3$; $C = \frac{2(v+2\lambda)}{9(v+\lambda)^2}$;

   b. $y_p \approx (v+\lambda)\left(a + bz_p\right)^{\frac{1}{h}}$;

   c. $y_p \approx (v+\lambda)\left(a' + b'z_p\right)^{\frac{1}{h}}$.

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