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Abstract

Variable Annuities represent a major part of the present annuity market. Two main types of Variable Annuities guarantees can be distinguished: the guarantees in case of death (GMDB) and the guarantees in case of survival (GMAB, GMIB, GMWB and GLWB). In the French market the guarantee in case of death is known by the name of "garantie plancher" and is well developed. On the contrary, the guaranties in case of survival are less developed. In particular, the withdrawal guarantees (GMWB and GMLB) have just appeared in France.

The present report presents different analytical approaches to value and hedge the Variable Annuities' guarantees. A keen attention will be given to the GMWB guarantee. We will introduce a new valuation approach for GMWB inspired on previous approaches. We will profit from this approach to study the capital requirement related to this product. The report is composed of three parts:

- First, each GMxB guarantee will be described with some of its variants. The market of Variable Annuities will be introduced throughout some figures, followed by the analysis of this products market.
- Later, we will consider the GMDB, GMAB and GMIB guarantees. We will present some basic results for each of these guarantees, illustrated by numerical examples. We will not pursue to expose these guaranties in a profound manner.
- At last we will consider the case of no surrender, of optimal partial surrender and of optimal total surrender of GMWBs. We will also dedicate a Chapter to give some ideas on the valuation of the GLWBs.

In the total surrender GMWB Chapter we will present a new approach to the GMWB guarantees which is very adequate to this product. This approach will allow us to study the withdrawal behavior of the policyholders. It should be noted that GMWB products are particularly sensible to policyholder behavior. Once our main GMWB framework is presented we will profit from it to study the impact of interest rate, equity, mortality and longevity risk. In order to do so we will follow a QIS4 standard methodology approach. The results will provide us with a keen insight of this guarantee.

Keywords Variable Annuities, GMWB, Optimal Surrender.

Résumé

Aujourd'hui, les contrats "Variable Annuities" représentent une part très importante du marché des rentes. Il s'agit de produits en unités de compte combinés avec des garanties optionnelles. On distingue les garanties en cas de décès (GMDB) et les garanties en cas de vie (GMAB, GMIB, GMWB et GLWB). La garantie GMDB, connue également sous l'appellation "garantie plancher", est bien développée sur le marché Français. En revanche, les garanties en cas de vie le sont moins. En particulier, les garanties de retrait (GMWB et GLWB) sont apparues récemment en France.

Dans ce mémoire, les approches analytiques de valorisation et de couverture des "Variables Annuities" seront présentées. GMWB fera l'objet d'une attention spéciale. On développera alors une stratégie de valorisation de cette garantie en s'inspirant des méthodes existantes. Grâce à cette stratégie, on étudiera le besoin en capital lié à ce produit. Le mémoire se décomposera en trois parties:

- Tout d'abord, chaque garantie GMxB sera décrite ainsi que quelques unes de leurs variantes. On exposera certains montants de vente et d'encours mais également des critiques sur ce marché.
- Ensuite, les modèles des garanties GMDB, GMAB et GMIB seront traités. Quelques résultats de base pour chacune de ces garanties seront démontrés et on les illustrera par des exemples numériques. Toutefois, on n'exposera pas de manière exhaustive ces garanties.
- Enfin, on considérera, dans une dernière partie, les effets de l'absence de rachat, le cas du rachat partiel optimal et du rachat total optimal en GMWBs. On dédiera un chapitre à la valorisation de GLWBs.

Dans le chapitre sur le rachat total optimal des garanties GMWB, on adoptera une nouvelle stratégie de valorisation en adéquation avec le produit. On étudiera le comportement de l'assuré puisque la garantie GMWB y est sensible. Grâce à cette stratégie de valorisation, on analysera l'impact des changements de taux d'intérêt, du cours des actions, de la mortalité et de la longévité. On suivra alors la méthodologie standard du QIS4 nous offrant ainsi une meilleure connaissance de la garantie.

Mots clés Variable Annuities, Unités de Compte, GMWB, Rachat optimal.

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Part I

Introduction

Chapter 1

General Introduction

Variable Annuities were in blossom in the world market, until the beginning of the present financial crisis. Representing $71,5\%$ of the US annuity's 2007 annuity sales, variable annuities have become of great interest to the rest of the world. However, the present financial crisis has reduced the policyholders' interest in investing in a volatile market, as stocks become less profitable policyholders restrain themselves from investing in risky assets. In fact the total Variable Annuity sales for the fourth quarter 2008 reduced in 30.3% in comparison to the sales for the same period 2007 [1]. The crisis has also impacted the insurers that sold these products. Variable Annuities portfolios lost their value due to the crisis but the insurers that had well hedged their portfolios did not suffer considerable losses [2]. This highlights the importance of a product well designed and hedged. However, the interest in variable annuities is still quite vivid in France; most insurers are waiting for the crisis to be over to introduce new annuity products that include these interesting features.

1.1 Why Variable Annuities?

Pension systems around the world are nowadays facing two great challenges. First, the mortality rate has reduced and continues to reduce day by day, which clearly means that a greater monetary provision should be set in order to provide a reasonable pension for life. In fact, even though people are living longer, health is non-granted; many will be confronted with high medical and long-term care costs. Second, the baby-boom, post-war generation, is arriving to the retirement age. This implies a greater rate between the retired population and the total population which might imply that pension systems based in distribution will become short of capital.

Baby-boomers and later generations are faced with the challenge of ensuring an eco-

[1] NAVA Reports Fourth Quarter 2008, Variable Annuity Industry Data

[2] Milliman, Press Release December the 1^{st} 2008.

nomically stable retirement. Personal savings should be well allocated and to do so a wide range of options can be found in the market. These options vary from a very conservative traditional annuity which guarantees a low rate and where no risk is taken, to a risky mutual fund that might be very profitable but can also leave the costumer short in savings. A reasonable mid-point in this spectrum of products are the Variable Annuities. These annuities provide the costumer the opportunity to gain in good market conditions with the guarantee that if the market crashes the costumer savings will not be diminished. A product with such attractive features sells well but is naturally complex. This complexity makes it difficult to value and to manage.

1.2 The present report

The present report introduces the reader to the Variable Annuities' realm; a particular attention will be given to the GMWB guarantee, which is up today the most promising annuity feature. Our approach will be to explore the closed-formulas related to this subject. Insurers' portfolios are in many cases quite complex. To model this complexity it normally best adjusted to use a Monte Carlo approach. From this point of view to work on closed-formula seems only as an academic game but at least the following four reasons should be considered by the practitioner to consider the closed-formula approach

1. In most of the Variable Annuities guarantees the closed-formula can be incorporated inside of a Monte Carlo general portfolio simulation.

2. When an optimization is implicit in the procedure, this optimization can become extremely difficult or impossible to implement in Monte Carlo. Such is the case of the optimal surrender behavior of the policyholder. Closed-formulas can be made in order to deal with this particular issue.

3. Closed-formula models can help to improve the product design. Even though policyholder behavior is not always optimal the insurer should not be too much exposed to a change in the policyholder behavior. Closed-formulas permit to explore different product designs so that the insurer can protect himself from policyholder behavior.

4. Closed-formula models can help to measure the impact on model assumptions. Mortality tables are presented in an annual step. Many times a finer step is needed; the interpolation strategy and the size of the step assumptions can impact the value of the product. Closed-formula models can help to establish the size of this impact. We will explore this particular issue on the GMDB Section.

The remaining of this report will be presented in the following manner. We will begin with the basic definitions; the different GMxB will be exposed with the help of generalized examples. Next the market conditions will be presented; the use of figures will allow us to understand the product's impact in today's market. We will also expose some analysts' positions to try to understand the GMxB phenomenon. Latter we will enter into the main subject of this report's interest: valuation and management of Variable Annuities. The basic valuation mathematical techniques will be presented and illustrated with computational applications; a keen attention will be placed on the GMWB guarantee.

A very brief introduction to stochastic calculus and the presentation of the stochastic control, required to develop the GMWB guarantee, will be placed in the appendix. Proofs to different lemmas and theorems that go beyond the subject of variable annuities will be also placed in the appendix.

In order to approach the valuation and managing of Variable Annuities two perspectives can be followed: a closed-form (analytical) perspective and a Monte Carlo (simulation) perspective. In this report we will work from an analytical point of view. In some cases we will find that the results should be approached with a numerical strategy. For a general simulation approach please consult: Hardy [20], Bauer, Kling and Russ [3] and Sun [43]. A simulation approach for the GLWBs can be found in Holz, Kling and Russ [24]. For a binomial tree approach you can consult Ho, Lee and Choi [22] and Ho and Mudavanhu [23].

1.3 The Surrender Conundrum

Surrender behavior is one of the most complex elements in the actuarial analysis. This is particularly relevant for the Variable Annuity products. In fact the complexity of the surrender behavior relays on its almost psychological character. The goal of a policyholder when she buys a retirement product is to have a secure source of income and if possible she would prefer it to give her a good return. If the market moves and the product becomes not interesting she might surrender her contract, however she is not a trader and will most probably not follow the day-to-day stock market in order to calculate the optimal amount and moment to surrender her policy. In this report we do not suppose that all real life policyholders follow an optimal behavior but a word of caution is given to the insurers about this subject: there is no guarantee that present policyholder behavior will be maintained in the years to come.

Policyholders can and most probably will become more aware of trading opportunities on their insurance products. There exists nowadays consulting firms that advice policyholders when and how to surrender in order to gain from their insurances. Insurers must be aware of this and reduce the risk that they are taking when a product is sold. This risk can be measured as contained inside the range of possibilities between the passive policyholder behavior and an optimal one. As this range is reduced the insurer will know that he is less sensible to policyholder's changes in surrender behavior. It is inside this logic that the insurer should calculate the limits of this range. The insurer should be aware of his exposition to this variable.

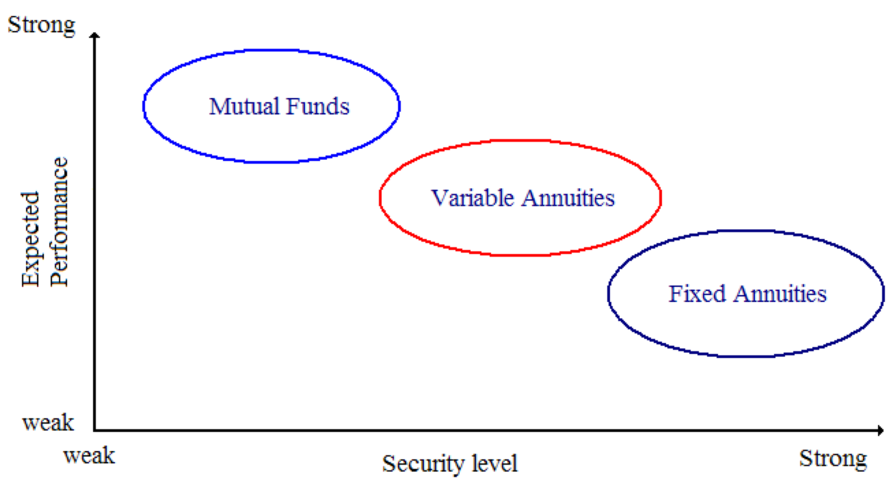
In this report we will present methods to calculate the extreme values of this range. We will study some design elements that allow the insurer to reduce this interval. Simple changes in the product design are enough to reduce this interval. Zhuliang Chen asserts that GMWBs can become the next subprime [12] we consider that this should not be the case, but only if insurers adjust their products to reduce this risk.

Chapter 2

Definition

Variable Annuities are contracts in which in exchange of an initial lump sum the policyholder receives an annuity or a lump sum in a specific date or dates in the future. The value of the future payments depends on the performance of a mutual fund or financial index. This value is protected by a guarantee. Different types of guarantees can be found in the market; in the following paragraphs we will present these guarantees in detail. Variable Annuities are an insurance product and as such usually have tax advantages that are not present in common Mutual Funds.

An important feature of these products is that they represent a mid-point between a strong expected performance without taking great risks. They are a mid-point between mutual funds which have a high expected performance, but with high risk and fixed annuities that have almost non-risk but their expected performance is not as high.



When we speak of Variable Annuities the main interest are the guarantees enclosed to these annuities. The usual American notation to these guarantees is in the form of GMxB: GMDB, GMAB, GMIB, GMWB and GLWB. Where these letters stand for:

- Guaranteed Minimal Death Benefit
- Guaranteed Minimal Accumulation Benefit
- Guaranteed Minimal Income Benefit
- Guaranteed Minimal Withdrawal Benefit
- Guaranteed Lifetime Withdrawal Benefit

Even though the GMxB notation is developed in the United States, similar products under other names can be found in other countries. In France for example the following products can be quoted:

- *Contrat en unité de comptes avec garanties plancher*
- *Contrats avec option de conversion en rentes*
- *"Opération à fenêtre" avec capital garanti au terme*

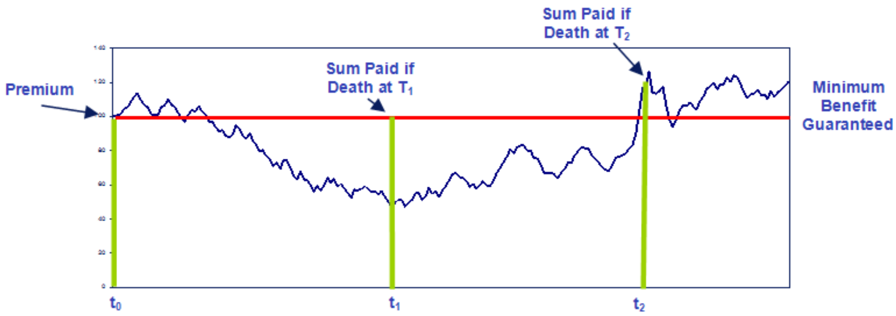
In Canada there are the Segregated Funds which are very similar to the American variable annuities [19] but are focused on a renewal guarantee [40][46]. That is, the policyholder has the right on certain dates to renew her policy keeping her initial conditions. This kind of guarantee is hard to manage and is well developed on the Canadian market.

In the British market the Unit-Liked products that had guarantees on the interest rate had a great success during the 1970s and 1980s when the long term interest rates where high but the lack of management techniques combined with a drop of interest rates diminished the product impetus [5].

There is a great range of possible guarantees related to variable annuities. Not only there are variations from country to country, but also from company to company. Even more, simple guarantees can be mixed over in order to produce more complex ones. In order to keep the trace of variable annuities theory we will restrict this report to the classical GMxB in their typical presentation.

2.1 Guaranteed Minimal Death Benefit (GMDB)

GMDB is the most ancient and well known of the variable annuities guarantees. It was introduced to the insurance market in 1980. Known in the French market as the "garantie plancher", GMDB guarantees the maximum between the mutual fund value and a pre-established lump sum in case of death. Consider the following graph:



At an initial time t_0 the policyholder pays a lump sum S_0, 100 Euros in our example. This sum is put into a mutual fund which is random as the financial market. The mutual fund value is represented by the skyline blue line. A minimum benefit is guaranteed, this value normally corresponds to the initial lump sum paid: S_0 and is represented by the horizontal red line. At t_1 the benefit minimum value is greater than the mutual fund value and therefore if the policyholder dies at t_1 then the benefit guaranteed is paid to the beneficiary. At t_2 the mutual fund has achieved a level superior to that of the benefit and in case of death of the policyholder the mutual fund value is paid to the beneficiary. On mathematical terms

$$\text{Sum Paid} = \max(S_T, S_0),$$

where T corresponds to the moment of death, S_0 to the premium (or mutual fund value at moment 0) and S_T the mutual fund value at the moment of death.
The guaranteed as has just been presented is called the **prime return**. There are three other variations to this guarantee:

1. The **Rising floor** (or roll-up): in this case the minimum benefit is capitalized with a given interest rate. That is, it increases in a guaranteed rate determined in the contract. In our example this will correspond to the case in which the horizontal red line is not horizontal anymore and has a slight exponential increase. That is,
$$\text{Sum Paid} = \max(S_T, e^{gT} S_0),$$
where g is the guaranteed interest rate and T the moment of death. Observe that g is a rate stated in the contract and is less than r the risk-free rate.

2. **Look-back** (or rachet[1]): a series of dates are set to redefine the minimum benefit value. Normally this is done during the contracts anniversaries. To redefine the benefit value the maximum value between the actual benefit value and the actual mutual fund value is taken. In our example this will correspond into turning the

[1]The word “rachet” is taken from the French. It corresponds to a mechanism with a wheel with inclined teeth such that once turned, it cannot go back.

horizontal red line into an upward step function that jumps on anniversary dates when the mutual fund is over the benefit value. Therefore

$$\text{Sum Paid} = \max(S_T, S_{t_1}, S_{t_2}, ..., S_{t_n}),$$

where $t_1, .., t_n$ are the dates set to redefine the minimum benefit value prior to T the moment of death.

3. A **rising floor with look-back**. That is

$$\text{Sum Paid} = \max(S_T, e^{g(T-t_1)}S_{t_1}, e^{g(T-t_2)}S_{t_2}, ..., e^{g(T-t_n)}S_{t_n}).$$

The calculation of this product's value and Greeks is not complex and will be presented further on in this report.

2.2 Guaranteed Minimal Accumulation Benefit (GMAB)

The Minimal Accumulation Benefit is the "in case of life" version of the GMDB. That is

$$\text{Sum Paid} = \max(S_T, S_0)$$

but in the accumulation benefit T corresponds to a fixed date in the future, that is, the maturity of the accumulation phase. Observe that the guarantee can be exercised if the policyholder is alive at time T while the guarantee of the GMDB can be exercised at the moment of death. Once the accumulation phase has finished contracts normally become an annuity, this means that the annuity base is guaranteed by this benefit. A roll up version of this guarantee is possible

$$\text{Sum Paid} = \max(S_T, e^{gT}S_0).$$

2.3 Guaranteed Minimal Income Benefit (GMIB)

The GMIB appeared in the market in 1996. In this guarantee the policyholder is given the option to retrieve the Mutual Fund value or to receive an annuity with pre-specified characteristics. This option can be executed on a given day or during a given period. This given day corresponds usually to a contract's anniversary (for example the 10th contact anniversary) or the arrival of the policyholder to a certain age (for example 70 years of age). From this day onwards the policyholder can convert her mutual fund to a fixed annuity that has a minimum guaranteed sum. This product is more complex than the previous two, it has the choice ingredient: the moment of conversion is floored but can be taken any moment later. As well, the interest rate plays a central role in

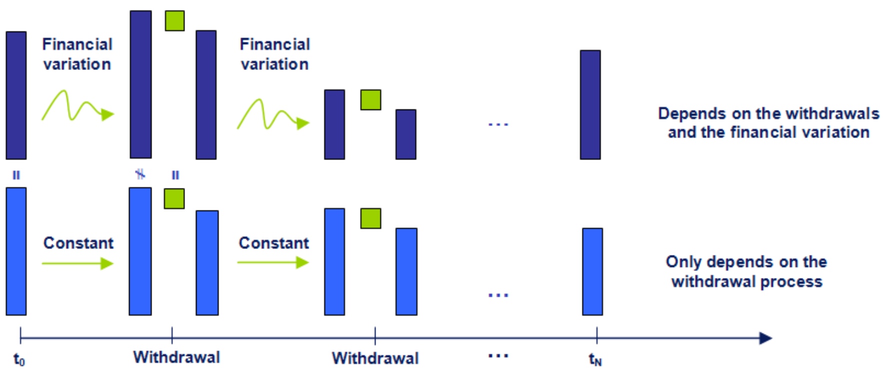
the interest to exercise the option and therefore it is necessary to have an interest rate model.
This guarantee also provides an option of a roll-up guarantee, that is, the minimum sum for the annuity can be increased according to an interest rate.

2.4 Guaranteed Minimal Withdrawal Benefit (GMWB)

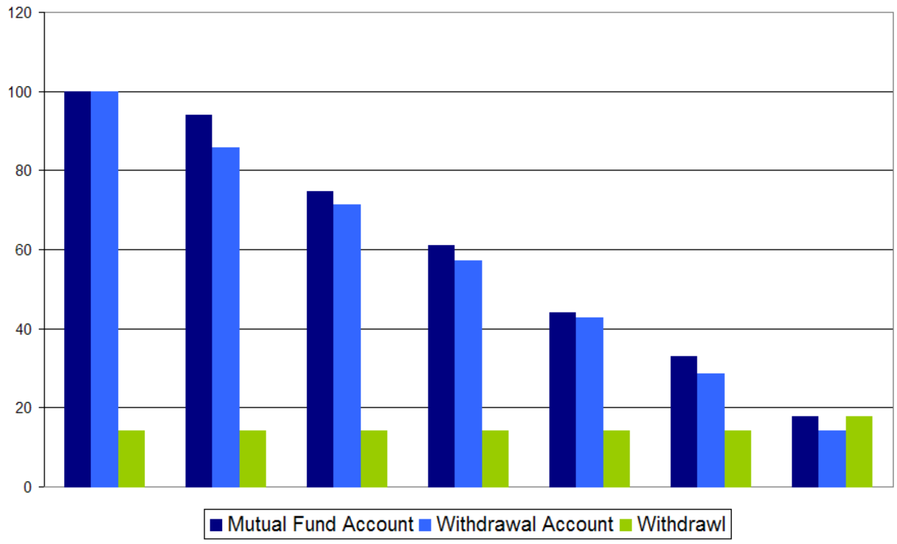
GMWBs and GLWB are the most recent and promising variable annuity guarantees. GMWB where first sold in 2002 and GLWB in 2004. The mechanism is the following, the policyholder pays a lump sum S_0 at t_0. She is entitled to withdraw a fixed amount $\frac{S_0}{N}$ every year for a number N of years. Usually around 15 years. At the end of the contract she will receive the exceeding amount of the mutual fund if there is any.
In order to keep the information of the amount that can still be withdrawn and the amount of mutual fund left after withdrawals, two accounts are defined:

1. Mutual Fund account (W)
2. Withdrawals account (A)

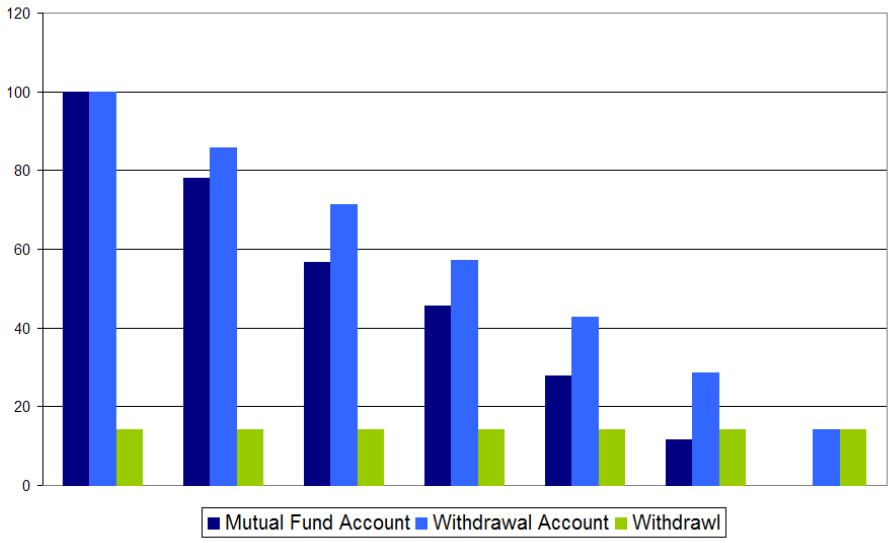
At t_0 both accounts are set to S_0. That is $W_0 = S_0$ and $A_0 = S_0$. Both accounts will decrease the amount withdrawn: $W_t = W_{t-dt} - G_t$ and $A_t = A_{t-dt} - G_t$ where G_t is the amount withdrawn at time t and $t - dt$ represent the instant just before. But while A_t will only vary by the withdrawal process, W_t will also vary with the mutual fund.
For clarity consider the following graph.



If at the end of the contract, after N years, the mutual fund account is superior to the withdrawal account the policyholder can withdraw the mutual fund value.
The GMWB is well illustrated by the following two graphs:



In the first graph the mutual fund performs better than the withdrawal account. At the last year there is more mutual fund than withdrawal account and therefore the last withdrawal corresponds to the value of the mutual fund.



In the second graph the mutual fund has a very poor performance and the mutual fund account diminish before the last year. In this case the guarantee takes place and the policyholder receives what is left in the withdrawal account.
Up to this point this guarantee seems no to be very different from a fixed annuity with a very particular option at the last year, but an important ingredient must be taken into account: the policyholder might follow different withdrawal policies. In fact in a normal GMWB the policyholder is entitled to withdrawal the amount she desires every year up to $\frac{S_0}{N}$ without penalization and up to the maximum between A_t and W_t with

a penalization. This important feature gives the policyholder the ability to perform better than just withdrawing a fixed amount every year. In financial mathematical terms, this feature turns our problem into a stochastic optimization problem. The mathematical instruments to do this kind of optimization and the solutions given to this problem will take an important Part of this report.

In the model considerations of the GMWB two policyholder strategies will be taken into account:

- Static strategy: The policyholder withdrawals $\frac{S_0}{N}$ every year.
- Dynamic strategy: The policyholder withdrawals the exact amount that maximizes the value of her contract.

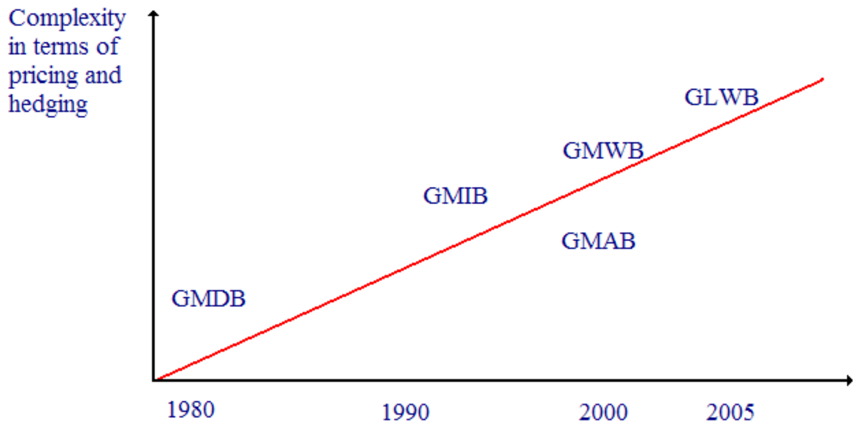
These strategies correspond to the strategies considered in the GMWB literature and represent two extremes in the spectrum of possibilities.
An interest variation to the GMWB is to give the policyholder a 5% bonus for each year no withdrawal is made. Another is to offer the policyholder to step-up her withdrawal amount every certain number of years (3 in Metlife, 5 in Axa) if her mutual fund does well.

2.5 Guaranteed Lifetime Withdrawal Benefit (GLWB)

This guarantee is also kwon as the GMWB-for-life, that is, it consists of a GMWB that do not have a year N when the guarantee finishes. The policyholder will have the right to withdrawal a fixed amount G every year as long as she is alive. In such a guarantee there is no withdrawal account, only a mutual fund that the policyholder can retrieve if she surrenders her contract.

2.6 The relation between the GMxBs

The following graph represents the relation between the year the guarantee was released and its level of complexity:



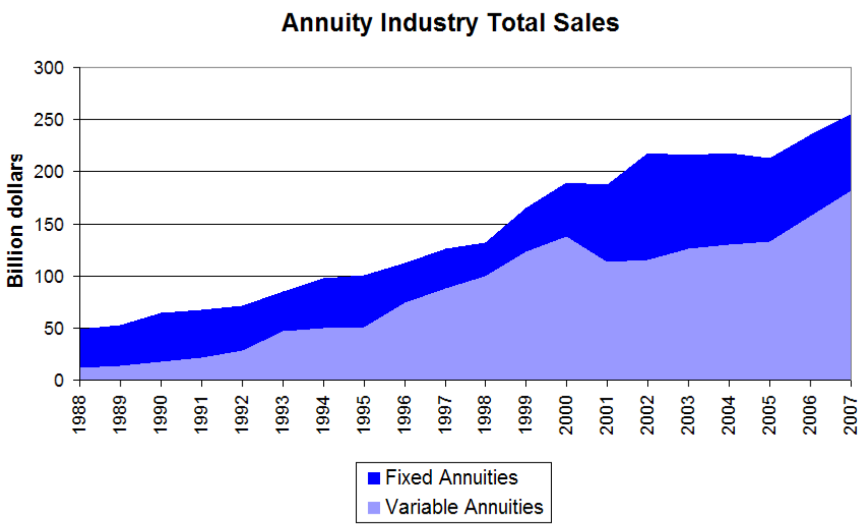
It is clear that with the pass of time, the benefit's complexity has enormously increased.

Chapter 3

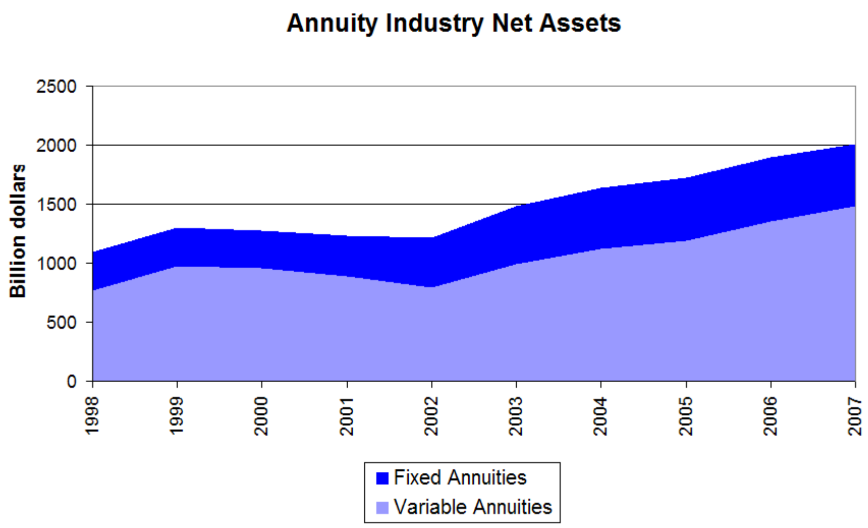
The Variable Annuity Market

3.1 Some figures on Variable Annuities before the present crisis

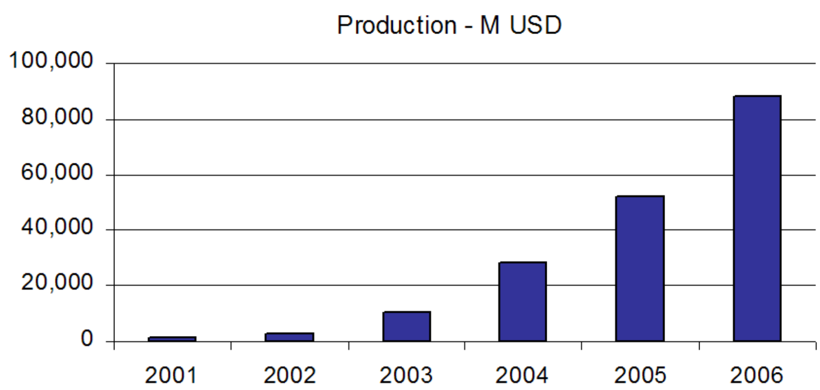
According to the NAVA (formerly the National Association for the Variable Annuities) [36] the total annual sales of Variable Annuities in the United States of America in 2007 was of 182,2 billion dollars a new high record in the history of this product. In 2006 the total annual sales was of 157,3 billion dollars, a record up to that year. In the following graph the amazing growth of Annuities products can be seen since 1988. In 1988 from the total of 48,9 billion dollars sold only 24,1% was due to variable annuities. As the variable annuities became very popular the product arrived to similar levels to that of fixed annuities in 1993 and from then on it has become the major product in the annuity market. In 2007 variable annuities represented 71,5% of the US annuity market.



Even though 2001 and 2002 represent a decline in the product sales, the product has now regain the 2000 levels and is blooming. Similar results can be inferred from the following graph, which represents the annuities net assets.



Once again the product is in record values with 1485,2 billion dollars in assets, which represents 73,9% of the US annuity market. The variable annuities have also had a remarkable growth in the Asian market. Over the 40 life insurance companies established in Japan more than 20 sell Variable Annuities. The guarantee with greatest success is the GMAB which might be explained by the Asiatic low interest rates. Leading Asiatic companies in this product are: Hartford, Sumitomo and ING. The following graph represents the product's growth in the Japanese market:



Presently the different types of variable annuities are entering the European market. In the UK AEGON introduced a GMAB in 2006. In 2007 different Variable Annuities entered the British market with Metlife, Hartford, AIG and Lincoln. In continental

Europe in 2006 Axa introduced a GMIB product in Germany. In 2006 Generali also introduced a Variable Annuity in Switzerland. In 2007 Axa launched a GMWB in Italy, France and Spain.

3.2 Understanding the products market

Though a stochastic simulation research, Milevsky and Panyagometh (2001) compared the performance of Variable Annuities with respect to Mutual Funds. These authors found that the performance depended on the time to maturity of the annuity. According to these authors, although low-cost variable annuities are superior to low-cost mutual funds for investors with a long time horizon, the critical threshold is at least 10 years for typical levels of risk aversion. And that for those more risk averse than average, the break-even horizon is even larger.

According to Brown and Poterba (2004), there are at least three reasons for why individuals demand variable annuities. These reasons are not necessarily exclusive. The first reason is to accumulate wealth at favourable after-tax rates of return. Variable annuities are purchased in Part to avoid the tax burden on investments in traditional taxable accounts. A second reason is that these products provide various forms of insurance. That is, the policyholder has guarantees that protect her from the financial market's risks. When the policyholder buys a Variable Annuity she pays some basic points to insure that the market will not diminish her savings. The third reason is that these products can become fix life annuities that give security for the retirement.

From the total amount of variable annuities sold in the US market in 2005, 82% had GMDB guarantees, 75% had GMWB guarantees, 46% GMIB guarantees and 37% GMAB guarantees. The fact that the sum of percentages is not 100% follows from the fact that guarantees can be combined. In fact from the percentages is clear that guarantees are usually combined.

From the figures just showed it is clear that today GMWB represents by far the most sold living guarantee in the US. According to Ayers and Sholder (2006) "the most commonly-offered (and popular) GLB at this time is the guaranteed minimum withdrawal benefit (GMWB). Based on stated company plans, versions of GMWBs will continue to outpace the other GLB choices, and represent the focus of most product enhancements that are currently under development. (...) Among the 15 executives interviewed, at least 7 were launching, or planning to launch, a GMWB product with "for life" component (...)".

Even though they are very recent, GMWB for life have become very strong in the US market. According to Herschler (2006) three key factors explain this:

1. Simple and flexible vs. complex and rigid: GMWB and GMWB for life are easy to understand and policyholder finds interesting to be able to surrender if she has gain in the mutual fund. Lifetime annuitization is commonly perceived as too inflexible.

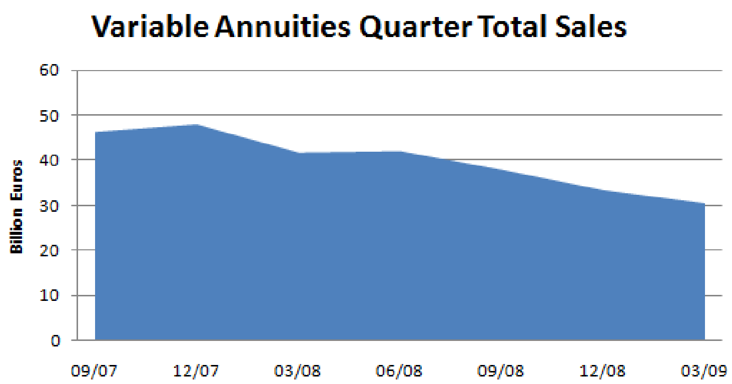
2. Control loss aversion trumps bigger monthly checks: most of policyholders are loss averse, this means that they weight more heavily the loss of value that the gain of value. A guarantee that floors the losses but do not cap the winnings goes well with this trend.

3. Investors' dim view of pooling mortality risk: policyholders don't like the idea that if they die early their savings will be used to pay someone else's annuities as happens with fix annuities for life. In the GMWB for life if the policyholder dies early, the mutual fund account value will be given to her beneficiaries.

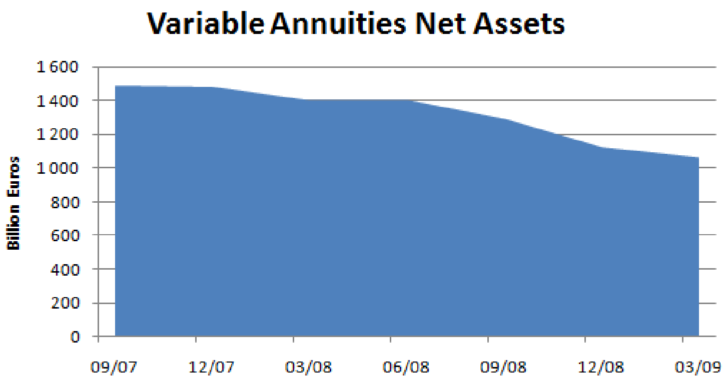
This third point coincides with the 2007 Nava Consumer Survey, which found that fewer than two out of ten old Americans would consider purchasing a product with guaranteed payments for life, which do not continue to be paid after they die because they would like to leave inherence onto their heirs.

3.3 Some figures on Variable Annuities during the present crisis

The present financial crisis has impacted the whole of the financial market, where variable annuities are not an exception. In what follows we will present the actual impact on the US market as reported by the NAVA association. First we will observe the chart of quarter net sales of variable annuities during the last two years. Please notice that the previous sales graph was annually based while the following is quarterly.



Now let's observe the evolution of the underlying assets during the same period.



From these graphs it is quite clear that the Variable Annuity market, as the financial market, has been strongly impacted by the present financial market.

Part II

Valuation and Managing of GMDB, GMAB and GMIB Variable Annuities

Chapter 4

GMDB valuation

The following Section is based on the arguments of Milevsky's article on Titanic Options [32]. We will keep Milevsky's parameters, that is we will work with a flat interest rate curve.

We will consider the usual asset model

$$dS_t = (r_t - \alpha)S_t dt + \sigma(S_t, t)S_t dB_t \qquad S_0 = 1$$

where r_t is the interest process, α is the insurance risk charge, $\sigma(S_t, t)$ the asset's volatility and B_t a Brownian motion. Observe that it consists of a classical geometric Brownian motion model. Define

$$R_t = e^{\int_0^t r_s ds}$$

the value of a monetary unit at time t. In fact R_t could be seen the value of a money market account and R_t^{-1} as the discount factor.
The idea of the model is to find the value of α such that the insurer's engagement is equal to the policyholder's engagement. Denote τ the stopping time that represents either the moment T of death of the policyholder or a minimum $\min(T, K)$ between T and K, where K is a moment when the contract expires. Let F_t denote the stochastic discounted value of fees collected at time t and V_t the value of the GMDB guarantee at time t. Therefore aim of the model is to find α such that

$$E_x[F_\tau] = E_x[V_\tau].$$

4.1 The policyholder's engagement

Lets first consider the policyholder's engagement F_t. By construction, we have

$$dF_t = R_t^{-1}\alpha S_t dt.$$

By chain rule we have that

$$\begin{aligned}
d(R_t^{-1}S_t) &= -r_t R_t^{-1} S_t dt + R_t^{-1} dS_t \\
&= -r_t R_t^{-1} S_t dt + R_t^{-1}((r_t - \alpha) S_t dt + \sigma(S_t, t) S_t dB_t) \\
&= -R_t^{-1} \alpha S_t dt + R_t^{-1} \sigma(S_t, t) S_t dB_t \\
&= -dF_t + R_t^{-1} \sigma(S_t, t) S_t dB_t,
\end{aligned}$$

therefore

$$\begin{aligned}
F_\tau &= \int_0^\tau dF_t dt \\
&= -\int_0^\tau d(R_t^{-1} S_t) + \int_0^\tau R_t^{-1} \sigma(S_t, t) S_t dB_t \\
&= R_0^{-1} S_0 - R_\tau^{-1} S_\tau + \int_0^\tau R_t^{-1} \sigma(S_t, t) S_t dB_t \\
&= 1 - R_\tau^{-1} S_\tau + \int_0^\tau R_t^{-1} \sigma(S_t, t) S_t dB_t.
\end{aligned}$$

Which means that, if r and σ are constant

$$\begin{aligned}
E_x[F_\tau] &= 1 - E_x\left[e^{-r\tau} S_\tau\right] + E_x[\text{Martingale}] \\
&= 1 - E_x\left[e^{-r\tau} S_\tau\right] \\
&= 1 - E_x\left[e^{(-\alpha - \frac{1}{2}\sigma^2)\tau + \sigma B_\tau}\right] \\
&= 1 - E_x\left[E\left[e^{(-\alpha - \frac{1}{2}\sigma^2)t + \sigma B_t} \middle| \tau = t\right]\right] \\
&= 1 - E_x\left[e^{-\alpha\tau}\right].
\end{aligned}$$

Observe that $E_x\left[e^{-\alpha\tau}\right]$ is a Laplace transform of the random variable τ valued at α. We now consider two examples of possible values of $E_x[F_\tau]$ according to the distribution of τ. For the first example we can let $\tau = T$ the moment of death of the policyholder and consider that the mortality distribution is exponential. Therefore

$$E_\lambda[F_T] = 1 - \lambda \int_0^\infty e^{-(\alpha+\lambda)t} dt = \frac{\alpha}{\lambda + \alpha}.$$

A second example is when we consider $\tau = \min(T, K)$ where K is the expiration of the contract. Therefore

$$\begin{aligned}
E_x[F_\tau] &= 1 - E_x[e^{-\alpha \min(T,K)}] \\
&= 1 - E_x[e^{-\alpha K} \mathbf{1}_{K<T}] - E_x[e^{-\alpha T} \mathbf{1}_{T<K}] \\
&= 1 - e^{-\alpha K} P_x[K < T] - \int_0^K e^{-\alpha t} dF_x(t) \\
&= 1 - e^{-\alpha K}(1 - F_x(K)) - \int_0^K e^{-\alpha t} dF_x(t),
\end{aligned}$$

where F_x is the mortality cumulative distribution function. In the exponential distribution case this turns out to be

$$E_\lambda[F_T] = 1 - e^{-(\alpha+\lambda)K} - \lambda \int_0^K e^{-(\alpha+\lambda)t} dt = \frac{\alpha}{\lambda+\alpha} \left(1 - e^{-(\alpha+\lambda)K}\right).$$

4.2 The insurer's engagement

Now let's consider the insurer's engagement V_t. This engagement depends on the guarantee offered.

4.2.1 The Premium Return

First we will consider the premium return. In such a case the death payment is

$$\max(S_T, S_0) = S_T + \max(S_0 - S_T, 0),$$

which clearly corresponds to the asset value at T plus a put option with strike S_0. In our model we take for simplicity $S_0 = 1$. Therefore the guarantee is worth [39]:

$$\begin{aligned} E_x[V_T] &= E_x[\max(1 - S_T, 0)] = E_x[E[\max(1 - S_t, 0)|T = t]] \\ &= E_x[Put(1, 1, T)] = \int_0^\infty Put(1, 1, t) dF_x(t), \end{aligned}$$

where P corresponds to the Black-Sholes/Merton price of a put option. That is,

$$Put(1, 1, t) = e^{-rt}\mathcal{N}(-d_2\sqrt{t}) - e^{-\alpha t}\mathcal{N}(-d_1\sqrt{t}),$$

where $\mathcal{N}$ is the Gaussian normal cumulative distribution function, and

$$d_1 = \frac{r - \alpha + \frac{1}{2}\sigma^2}{\sigma} \qquad d_2 = d_1 - \sigma.$$

4.2.2 The Rising Floor

Now, in the case of a rising floor GMDB the guarantee is worth

$$\begin{aligned} E_x[V_T] &= E_x[\max(e^{gT} - S_T, 0)] = E_x[E[\max(e^{gT} - S_t, 0)|T = t]] \\ &= E_x[BSM(T, g|\sigma, r, \alpha)] = \int_0^\infty BSM(t, g|\sigma, r, \alpha) f(t) dt, \end{aligned}$$

where BSM corresponds to the strike modified Black-Sholes/Merton price of a put option. That is,

$$BSM(t, g|\sigma, r, \alpha) = e^{(g-r)t}\mathcal{N}(-\xi_2\sqrt{t}) - e^{-\alpha t}\mathcal{N}(-\xi_1\sqrt{t}),$$

where $\mathcal{N}$ is the gaussian normal cumulative distribution function, and,

$$\xi_1 = \frac{r - g - \alpha + \frac{1}{2}\sigma^2}{\sigma} \qquad \xi_2 = \xi_1 - \sigma.$$

4.2.3 The Look Back

Through a similar argument it can be shown that the look-back GMDB has an option value of

$$E_x[V_T] = \int_0^\infty GSG(t|\sigma, r, \alpha) f(t)dt,$$

where GSG is the Goldman, Sosin and Gatto look-back option valuation which is worth

$$GSG(t|\sigma, r, \alpha) = e^{-rt}\mathcal{N}(-\xi_2\sqrt{t}) - e^{-\alpha t}\mathcal{N}(-\xi_1\sqrt{t}) - \eta(e^{-rt}\mathcal{N}(\xi_3\sqrt{t}) - e^{-\alpha t}\mathcal{N}(\xi_1\sqrt{t})),$$

where,

$$\eta = \frac{\sigma^2}{2(r-\alpha)} \qquad \xi_1 = \frac{r - \alpha + \frac{1}{2}\sigma^2}{\sigma} \qquad \xi_2 = \xi_1 - \sigma \qquad \xi_3 = \xi_1 - \frac{2(r-\alpha)}{\sigma}.$$

4.2.4 The Exponential Mortality Example

Explicit closed formulas can be found for the exponential mortality. In order to do so we will calculate the value of

$$\begin{aligned}
\int_0^K e^{-at}\mathcal{N}(-b\sqrt{t})dt &= \int_0^K e^{-at} \int_{-\infty}^{-b\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dxdt \\
&= \int_0^K e^{-at} \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dxdt + \int_0^K e^{-at} \int_{0}^{-b\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dxdt \\
&= \int_0^K e^{-at}\frac{1}{2}dt + \int_0^{-b\sqrt{K}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \int_{(\frac{x}{b})^2}^K e^{-at} dtdx \\
&= \frac{1-e^{-aK}}{2a} + \int_0^{-b\sqrt{K}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \left(\frac{e^{-aK} - e^{-a(\frac{x}{b})^2}}{-a} \right) dx \\
&= \frac{1-e^{-aK}}{2a} - \frac{e^{-aK}}{a} \int_0^{-b\sqrt{K}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx + \frac{1}{a} \int_0^{-b\sqrt{K}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2\left(1+\frac{2a}{b^2}\right)} dx \\
&= \frac{1-e^{-aK}}{2a} - \frac{e^{-aK}}{a} \left(\mathcal{N}\left(-b\sqrt{K}\right) - \frac{1}{2} \right) \\
&\quad + \frac{1}{a\sqrt{1+\frac{2a}{b^2}}} \left(\mathcal{N}\left(-b\sqrt{K}\sqrt{1+\frac{2a}{b^2}}\right) - \frac{1}{2} \right) \\
&= \frac{1}{2a} - \frac{e^{-aK}}{a}\mathcal{N}\left(-b\sqrt{K}\right) + \frac{1}{a\sqrt{1+\frac{2a}{b^2}}} \left(\mathcal{N}\left(-b\sqrt{K}\sqrt{1+\frac{2a}{b^2}}\right) - \frac{1}{2} \right).
\end{aligned}$$

Therefore if $K \to \infty$ and if $a > 0$ and $b > 0$ we have that

$$\int_0^\infty e^{-at}\mathcal{N}(-b\sqrt{t})dt = \frac{1}{2a} - \frac{1}{2a\sqrt{1+\frac{2a}{b^2}}}$$
$$= \frac{1}{2a}\left(1 - \frac{b}{\sqrt{b^2+2a}}\right)$$

and if $K \to \infty$ and if $a > 0$ and $b < 0$ we have that

$$\int_0^\infty e^{-at}\mathcal{N}(-b\sqrt{t})dt = \frac{1}{2a}\left(1 + \frac{b}{\sqrt{b^2+2a}}\right).$$

Let

$$M_K(a,b) = \frac{1}{2a} - \frac{e^{-aK}}{a}\mathcal{N}\left(-b\sqrt{K}\right) + \frac{1}{a\sqrt{1+\frac{2a}{b^2}}}\left(\mathcal{N}\left(-b\sqrt{K}\sqrt{1+\tfrac{2a}{b^2}}\right) - \frac{1}{2}\right).$$

Let's do the calculations for the rising floor formula with exponential mortality using the preceding lemma. For $K < \infty$ we have that

$$\begin{aligned} E_\lambda[V] &= \int_0^K BSM(t,g|\sigma,r,\alpha)\lambda e^{-\lambda t}dt \\ &= \lambda\int_0^K e^{(g-r-\lambda)t}\mathcal{N}(-\xi_2\sqrt{t})dt - \lambda\int_0^K e^{-(\alpha+\lambda)t}\mathcal{N}(-\xi_1\sqrt{t})dt \\ &= \lambda M_K(r+\lambda-g,\xi_2) - \lambda M_K(\alpha+\lambda,\xi_1), \end{aligned}$$

where $\xi_1 = \dfrac{r-g-\alpha+\frac{1}{2}\sigma^2}{\sigma}$ and $\xi_2 = \xi_1 - \sigma$.

For $K \to \infty$ we have that

$$\begin{aligned} E_\lambda[V] &= \int_0^\infty BSM(t,g|\sigma,r,\alpha)\lambda e^{-\lambda t}dt \\ &= \lambda\int_0^\infty e^{(g-r-\lambda)t}\mathcal{N}(-\xi_2\sqrt{t})dt - \lambda\int_0^\infty e^{-(\alpha+\lambda)t}\mathcal{N}(-\xi_1\sqrt{t})dt \\ &= \frac{\lambda}{2(r-g+\lambda)}\left(1 - \frac{\xi_2}{\sqrt{\xi_2^2+2(r-g+\lambda)}}\right) - \frac{\lambda}{2(\alpha+\lambda)}\left(1 - \frac{\xi_1}{\sqrt{\xi_1^2+2(\alpha+\lambda)}}\right), \end{aligned}$$

where $\xi_1 = \dfrac{r-g-\alpha+\frac{1}{2}\sigma^2}{\sigma}$ and $\xi_2 = \xi_1 - \sigma$.

Observe that for our demonstration to work well we require ξ_1, ξ_2, $r+\lambda-g$ and $\alpha+\lambda$ to be positive, which is often the case since r, λ, g, α and σ are positive and normally

$r > g + \alpha + \frac{1}{2}\sigma^2$.

A very similar demonstration can be done for the look-back case, if $K < \infty$

$$
\begin{aligned}
E_\lambda[V] =& \int_0^K GSG(t|\sigma, r, \alpha)\lambda e^{-\lambda t} dt \\
=& \lambda \int_0^K e^{-(r+\lambda)t}\mathcal{N}(-\xi_2\sqrt{t})dt - \lambda \int_0^K e^{-(\alpha+\lambda)t}\mathcal{N}(-\xi_1\sqrt{t})dt \\
&- \eta\lambda \int_0^K e^{-(r+\lambda)t}\mathcal{N}(\xi_3\sqrt{t})dt + \eta\lambda \int_0^K e^{-(\alpha+\lambda)t}\mathcal{N}(\xi_1\sqrt{t})dt \\
=& \lambda M_K(r+\lambda, \xi_2) - \lambda M_K(\alpha+\lambda, \xi_1) \\
&- \eta\lambda M_K(r+\lambda, -\xi_3) + \eta\lambda M_K(\alpha+\lambda, -\xi_1),
\end{aligned}
$$

and if $K \to \infty$

$$
\begin{aligned}
E_\lambda[V] =& \frac{\lambda}{2(r+\lambda)}\left(1 - \frac{\xi_2}{\sqrt{\xi_2^2 + 2(r+\lambda)}}\right) - \frac{\lambda}{2(\alpha+\lambda)}\left(1 - \frac{\xi_1}{\sqrt{\xi_1^2 + 2(\alpha+\lambda)}}\right) \\
&- \frac{\eta\lambda}{2(r+\lambda)}\left(1 + \frac{\xi_3}{\sqrt{\xi_3^2 + 2(r+\lambda)}}\right) + \frac{\eta\lambda}{2(\alpha+\lambda)}\left(1 + \frac{\xi_1}{\sqrt{\xi_1^2 + 2(\alpha+\lambda)}}\right),
\end{aligned}
$$

where

$$
\eta = \frac{\sigma^2}{2(r-\alpha)} \qquad \xi_1 = \frac{r - \alpha + \frac{1}{2}\sigma^2}{\sigma} \qquad \xi_2 = \xi_1 - \sigma \qquad \xi_3 = \xi_1 - \frac{2(r-\alpha)}{\sigma}.
$$

Observe that we have supposed again that $r > g + \alpha + \frac{1}{2}\sigma^2$.

4.2.5 The Greeks

In mathematical finance the greeks correspond to the infinitesimal sensitivity of a price with respect to one its variables. In mathematical terms these correspond to the first and second derivatives of the price function. Observe that for the GMDB the price function is always an integral of a known put option times the mortality density with respect of the time variable. Explicit formulae are known for the greeks of these put options, for explicit formulas see for example Hull [25]. If the derivate is done with respect to a variable x different to t we have the following

$$
\begin{aligned}
\frac{\partial}{\partial x} GMDB(x,t) &= \frac{\partial}{\partial x}\int_0^\infty Put(x,t)f(t)dt = \int_0^\infty \frac{\partial}{\partial x} Put(x,t)f(t)dt \\
&= \int_0^\infty Greek_x(x,t)f(t)dt.
\end{aligned}
$$

Let's consider a simple example to illustrate the idea, take the Δ (derivate with respect to the underlying asset) of a pure premium GMDB with exponential mortality, in such a case we have

$$\begin{aligned} GMDBDelta &= \int_0^\infty PutDelta(x,t)\lambda e^{-\lambda t} dt = \int_0^\infty (\mathcal{N}(d_1\sqrt{t}) - 1)\lambda e^{-\lambda t} dt \\ &= \int_0^\infty e^{-\lambda t}\mathcal{N}(d_1\sqrt{t})\lambda dt - 1 = \frac{1}{2}\left(1 - \frac{d_1}{\sqrt{d_1^2 + 2\lambda}}\right) - 1 \\ &= -\frac{1}{2}\left(1 + \frac{d_1}{\sqrt{d_1^2 + 2\lambda}}\right), \end{aligned}$$

where $d_1 = \dfrac{r - \alpha + \frac{1}{2}\sigma^2}{\sigma}$.

4.3 Numerical examples

Numerical examples are calculated using a mortality table by generation. Four age where consideres: 30, 40, 50 and 60 years, such that the policiholder would have her age in 2009. We considered a 15 year contract for each age.

In order to implement the GMDB analytical results two approches are possible: 1) Fit the mortality curve with a continous parametrical model, or 2) discretizise the time step and approach mortality from a discrete perspective. The first approach requires that the model fits well the data which is not always the case. In fact a one parameter model such as the exponential model fits poorly the data while two or thee parameter models have a better fit but the time integral

$$E[V] = \int_0^K Put(t)dt$$

has no longer a analitical expression and must be approched numerically [32]. The second approach is straight forward to implement but a time step must be chosen to be small enough not to distort the continous property of mortality. Normally this time step is smaller than a year and mortality tables are yearly then an interpolation hypothesis must be cosidered.

We implemented both approches. First using a least square method the mortality curves where fitted by an exponential mortality. The squared errors where high and this methodology showed not to be appropriate from the beggining; in particular the exponential mortality fits poorly for high ages. The following table presents the lambdas fitted to a 15 year mortality curve interval for policyholders that had 30, 40, 50 and 60 years in 2009.

Age in 2009	30	40	50	60
lambda	0,038%	0,093%	0,252%	0,642%
error	0,106%	0,389%	0,933%	3,358%

From this approach no information could be implied to the mortality table, insted this approches results where very usefull to test the second approach. For the second approach two time steps where cosidered: a year and a month. In order to interpolate the mortality table to monthly steps we supposed the uniform distribution of mortality during the months of a given year, that is

$$S_{12t+u} = (1 - \tfrac{u}{12})S_{12t} + \tfrac{u}{12}S_{12(t+1)},$$

where $t \in \mathbb{N}$ and $u \in \{0, 1, ..., 11\}$.

In order to test the time discretization bias we fitted a λ for each age considered and calculated the insurance risk charge for the exponential model. Then the second approach was applied using the mortality table constructed by the exponential model. Since we used the implicit exponential model table, then the first approach results corresponded exactly to the continous time values (for the exponential table) and we could test the discretization of time in the second approach. That is, as the time step becomes smaller the second approach converges to the first. The following tables represent the insutance risk charges for the exponental model and the discrete approach models.

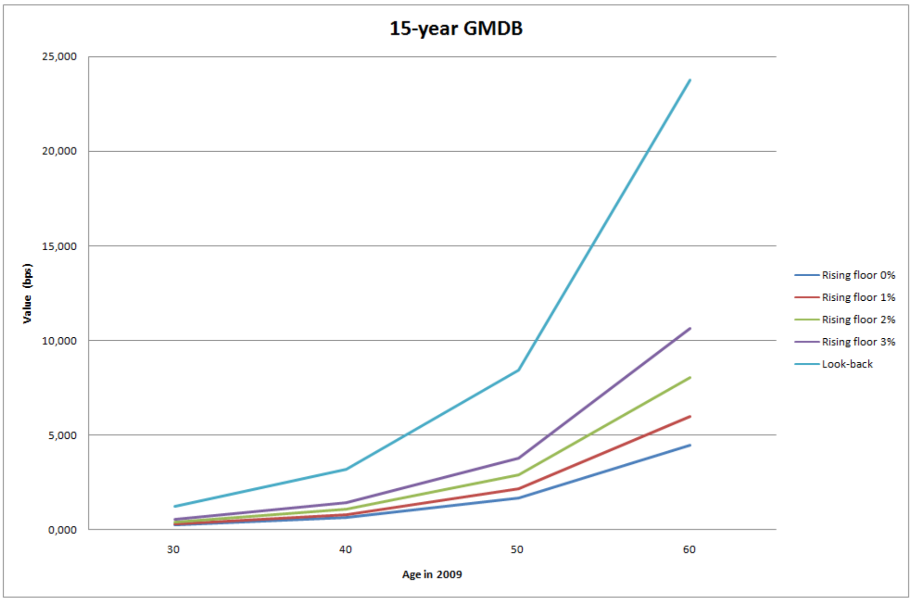
Age in 2009		30						
Guarantee		Exponential model value (bp)	year step (1/1) value (bp)	relative error	month step (1/12) value (bp)	relative error	week step (1/52) value (bp)	relative error
Rising floor	0%	0,227	0,229	0,86%	0,227	0,17%	0,227	0,04%
	1%	0,292	0,297	1,67%	0,293	0,22%	0,292	0,05%
	2%	0,376	0,385	2,44%	0,377	0,26%	0,376	0,06%
	3%	0,481	0,496	3,15%	0,482	0,31%	0,481	0,07%
Look-back		1,065	1,102	3,49%	1,068	0,33%	1,066	0,08%
Age in 2009		40						
Guarantee		Exponential model value (bp)	year step (1/1) value (bp)	relative error	month step (1/12) value (bp)	relative error	week step (1/52) value (bp)	relative error
Rising floor	0%	0,555	0,560	0,84%	0,556	0,16%	0,555	0,04%
	1%	0,715	0,727	1,66%	0,717	0,21%	0,716	0,05%
	2%	0,919	0,941	2,42%	0,921	0,26%	0,920	0,06%
	3%	1,176	1,213	3,14%	1,180	0,31%	1,177	0,07%
Look-back		2,605	2,696	3,47%	2,614	0,33%	2,607	0,08%
Age in 2009		50						
Guarantee		Exponential model value (bp)	year step (1/1) value (bp)	relative error	month step (1/12) value (bp)	relative error	week step (1/52) value (bp)	relative error
Rising floor	0%	1,511	1,523	0,80%	1,513	0,16%	1,511	0,04%
	1%	1,947	1,978	1,62%	1,951	0,21%	1,948	0,05%
	2%	2,502	2,562	2,39%	2,509	0,26%	2,504	0,06%
	3%	3,202	3,302	3,11%	3,212	0,31%	3,205	0,08%
Look-back		7,088	7,331	3,42%	7,112	0,33%	7,094	0,08%
Age in 2009		60						
Guarantee		Exponential model value (bp)	year step (1/1) value (bp)	relative error	month step (1/12) value (bp)	relative error	week step (1/52) value (bp)	relative error
Rising floor	0%	3,874	3,901	0,69%	3,880	0,16%	3,876	0,04%
	1%	4,992	5,067	1,52%	5,002	0,21%	4,994	0,05%
	2%	6,418	6,565	2,30%	6,435	0,26%	6,422	0,07%
	3%	8,222	8,471	3,03%	8,247	0,31%	8,228	0,08%
Look-back		18,161	18,762	3,31%	18,220	0,33%	18,175	0,08%

As axpected, relative error decreases as the time step is smaller. This is natural, since the discretemodel converges to the discrete model. A more interesting observation is that the discrete model gives always a higher insurance risk value, this is because the discretisation considers deaths to occur at the end of period while in the continous model deaths happen through out all the period this implies that in the discrete model guarantees are payed later that in the continous model and since these guarantees are put options and put options are more expensive at higher maturities, so guaranttes become more expensive. This means that discretisation of time over-values the guarantees. This could be considered as a conservative mesure in the GMDB valuation.

It must also be obseved that relative error increases as the guaranted interest rate is higher. This follows from the fact that the Put options value increases faster than linear with respect to guaranteed interest rate. In practical terms it means that a smaller time step should be taken for higher guaranteed interest rates. Observe as well that relative error is not very sensible to the age of the policyholder.

It is always difficult to define a relative error small enough to be accepted, in our consideration the week step is a good enough approximation to the "real" value. Therefore the next calculatons are done with a week-step.

In the next graph we represent the relation between the look-back and several roll-up GMDB insurance risk charges. The graph was done with $\sigma = 15\%$ and $r = 5\%$ using a week-step discretization and the "real" mortality table.



Observe that the charge is larges as the guaranteed interest rate is bigger. This follows from the fact that if a better rate is guaranteed then the guarantee should cost more. Another way to see this is that a Black-Sholz and Merton put option value increases if the g interest rate is increased.

Observe as well that the look-back is more expensive than the roll-ups given in the graph, this follows from the fact that the interest rate r taken is larger than each of the

g guaranteed, which is usually the case. The look-back guarantees the in expectancy the growth of the underlying and which in our case is r.

Chapter 5

GMAB valuation

By definition GMAB are guarantees that will take effect only if the policyholder is alive after a certain date K. In such a case she is paid $\max(S_0, S_K)$ if is a premium return guarantee or the $\max(S_0 e^{gK}, S_K)$ if is a roll-up guarantee, if she dies before K she will get S_T where T is her moment of death.
For the policyholder's engagement we will have the case $\tau = \min(T, K)$ which as demonstrated in the GMDB Section is

$$E_x[F_\tau] = 1 - e^{-\alpha K}(1 - F_x(K)) - \int_0^K e^{-\alpha t} f_x(t)dt,$$

where f_x is the mortality density and F_x its cumuative ditribution function.

5.1 The Premium return

In the premium return guarantee we have that the payment is

$$S_T \mathbf{1}_{T<K} + \max(S_0, S_K)\mathbf{1}_{T\geq K} = S_\tau + \max(S_0 - S_\tau, 0)\mathbf{1}_{T\geq K}.$$

Therefore the guarantee is

$$\begin{aligned} GMAB &= E_x[\max(S_0 - S_K, 0)\mathbf{1}_{T\geq K}] = E_x[Put(1,1,K)\mathbf{1}_{T\geq K}] \\ &= \int_K^\infty Put(1,1,t)dt = Put(1,1,K)\mathcal{S}_x(K), \end{aligned}$$

where $\mathcal{S}_x(K)$ is the probability that an individual which is has x years at t continues to be alife at K and Put coorresponds to the Black-Sholes/Merton price of a put option. That is

$$Put(1,1,K) = e^{-rK}\mathcal{N}(-d_2\sqrt{K}) - e^{-\alpha K}\mathcal{N}(-d_1\sqrt{K}),$$

where $\mathcal{N}$ is the gaussian normal cumulative distribution function, and

$$d_1 = \frac{r - \alpha + \frac{1}{2}\sigma^2}{\sigma} \qquad d_2 = d_1 - \sigma.$$

5.2 The Rising Floor

For the Rising floor with a similar argument we have that

$$\begin{aligned} GMAB = E_x[\max(e^{gK} - S_K, 0)\mathbf{1}_{T>K}] &= \int_K^\infty BSM(K, g|\sigma, r, \alpha) f(t)dt \\ &= BSM(K, g|\sigma, r, \alpha)\mathcal{S}_x(K), \end{aligned}$$

where BSM coorresponds to the strike modified Black-Sholes/Merton price of a put option. That is

$$BSM(K, g|\sigma, r, \alpha) = e^{(g-r)K}\mathcal{N}(-\xi_2\sqrt{K}) - e^{-\alpha K}\mathcal{N}(-\xi_1\sqrt{K}),$$

where $\mathcal{N}$ is the gaussian normal cumulative distribution function, and

$$\xi_1 = \frac{r - g - \alpha + \frac{1}{2}\sigma^2}{\sigma} \qquad \xi_2 = \xi_1 - \sigma.$$

5.3 The Greeks

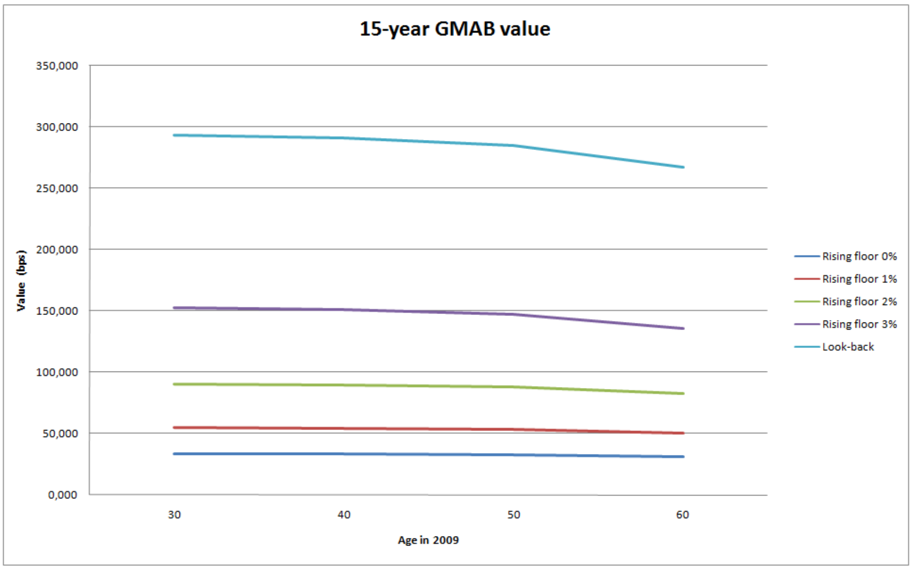
As with the GMDB guarantee, the greeks are usually easily valuated by:

$$\begin{aligned} \frac{\partial}{\partial y} GMAB(y, t) &= \int_K^\infty Greek_y(y, K) f(t)dt \\ &= Greek_y(y, K)\mathcal{S}_x(K), \end{aligned}$$

where y is any of the variables of which the option value depends. That is, the underlying value S, time to maturity $T - t$, volatility σ, market interest rate r or the guaranteed interest rate g. It is usual olso to consider the second derivate with respet to the underlying.

5.4 Numerical examples

In order to illustrate the values of the GMAB guarantee we considered policyholders that had 30, 40, 50 and 60 years in 2009. We took σ to be 20% and r the market interest rate to be 5%. A week-step approximation was used to calculate $E_x[F_\tau]$ while no approximation was required to calculate $E_x[V_\tau]$ since the guarantee can only take place in a precise time moment: K. The following graph was obtained:



First of all observe that the guarantee charge α decreases as the policyholder becomes older, this is because as she is older there is less probability that she will survive to the end of the contract, in this case 15 years, and so that the guarantee will take place. Notice that this trend is exactly the oposite to the GMDB guarantee which increases which age, this is easily explained since GMDB is a "in case of death" guarantee while the GMAB is a "in case of life" one.

Now observe that the as the guaranteed interest rate increases the charge value also does. The reason is that the put option value increases as the guaranteed interest rate does. We see that the look back option is more expensive than the rising-floor for the guaranteed interest rates given, this is because the look back grows in a similar speed as guaranteing the market interest rate, in our exemple the market interest rate is 5% which is higher to the guaranteed interest rates used. This is very similar to the GMWB guarantee logic.

Chapter 6

GMIB valuation

6.1 An asset composed of risk-free bonds

The Guaranteed Minimum Income Benefit guarantees minimal conditions to an annuity if the policyholder desires to annuitise her contract. This benefit depends on the asset's performance as in the interest rate level at the time of conversion if there is an interest rate guarantee. In order to reduce the complexity of the problem following Milevsky and Promislow's article [33] we will consider the asset to be composed of bonds, and therefore the only risk to be the interest rate risk. According to these authors "(...) this call option on annuity purchase factors can be viewed as the right, but not the obligation, to purchase a fixed immediate life annuity, for a deterministic strike price during the life of the contract. The company has essentially granted the policyholder an option on two underlying stochastic variables; future interest rates and future mortality rates". As it follows from this sentence, mortality rates are considered to be stochastic and play a predominant role in their model.

Denote r_t the instantaneous interest rate and h_t the hazard rate. We will suppose r_t to be independent of h_t. Define

$$\xi_t = r_t + h_t$$

the hazard-plus-interest rate process. The following two paragraphs show the interest of adding these two processes and the effect that this has on the model.

Consider on one hand the interest rate process. In absence of mortality the price of a zero coupon bond at time t with maturity at T, corresponds to

$$D_t(T) = E_Q\left[e^{-\int_t^T r_u du} \middle| \mathcal{F}_t \right],$$

where $E_Q[\cdot]$ corresponds to the risk neutral expectation. $D_t(T)$ can be seen as well as the discount factor and is the value seen from t of a monetary unit on T.

On the other hand we have the mortality process. The probability of survival to time T, conditional to being alive at time t, is

$$p_t(T) = E\left[e^{-\int_t^T h_u du} \middle| \mathcal{F}_t \right].$$

From the preceding to formulas it is clear that r_t plays in the stochastic process a very similar role to h_t in the mortality process.

Now consider a mortality contingent claim, that is a monetary claim that the policyholder can only make if she survives up to time T. The value of this claim seen from time T, with $t < T$, corresponds to

$$\begin{aligned}\Lambda_t(T) &= D_t(T)p_t(T) \\ &= E_Q\left[e^{-\int_t^T r_u du} \middle| \mathcal{F}_t \right] E\left[e^{-\int_t^T h_u du} \middle| \mathcal{F}_t \right] \\ &= E_Q\left[e^{-\int_t^T \xi_u du} \middle| \mathcal{F}_t \right].\end{aligned}$$

Note that the last equality uses the fact that the two processes are independent. Observe that a annuity set today corresponds to a set of mortality contingent claims.

Analogously to the forward price of a default-free zero-coupon bond, it can be defined a $\Lambda_t(s,T)$ pure endowment bought at time s, seen from time t and with maturity at T, with $t < s < T$. This corresponds to a mortality contingent claim that is bought in the future. A future annuity would just be a set of pure endowments. Notice as well that the value of a pure endowment

$$\Lambda_t(s,T) = E_Q\left[e^{-\int_s^T \xi_u du} \middle| \mathcal{F}_t \right] = \frac{\Lambda_t(T)}{\Lambda_s(T)}.$$

The GMIB guarantee with an asset composed of risk-free bonds corresponds to a set of call options on pure endowments. Let's denote $C_t(s,T,\Lambda)$ a call option on a pure endowment $\Lambda_t(s,T)$ with strike value Λ. We have therefore that

$$\begin{aligned}C_t(s,T,\Lambda) &= E_Q\left[e^{-\int_t^s \xi_u du} \max(\Lambda_s(T) - \Lambda, 0) \middle| \mathcal{F}_t \right] \\ &= E_Q\left[e^{-\int_t^s \xi_u du} \max\left(E_Q\left[e^{-\int_s^T \xi_u du} \middle| \mathcal{F}_s \right] - \Lambda, 0 \right) \middle| \mathcal{F}_t \right].\end{aligned}$$

In their article, Milevsky and Promislow, chose a CIR model for the interest rate and a Mean Reverting Brownian Gompertz for the hazart rate. Even though CIR produces well known closed-formed solutions, the Mean Reverting Brownian Gompertz process does not and therefore they are forced to apply a Monte Carlo simulation to obtain numerical examples.

A CIR model is one of the most popular interest rate models and satisfies the following stochastic differential equation

$$dr_t = \kappa(\theta - r_t)dt + \sigma_r \sqrt{r_t} dB_t.$$

It can be shown that

$$D_t(T) = C_1(t,T)e^{-r_t C_2(t,T)},$$

where

$$C_1(t,T) = \left[\frac{2\gamma e^{(\kappa+\gamma)(T-t)/2}}{(\gamma+\kappa)\left(e^{\gamma(T-t)}-1\right)+2\gamma}\right]^{2\kappa\theta/\sigma_r^2}$$

$$C_2(t,T) = \frac{2\left(e^{\gamma(T-t)}-1\right)}{(\gamma+\kappa)\left(e^{\gamma(T-t)}-1\right)+2\gamma}$$

and $\gamma = \sqrt{\kappa^2 + 2\sigma_r^2}$.

The Mean reverting Brownian Gompertz is introduced on Milevsky and Promislow's [33] article. It consists on process that expected to grow exponentially, that has a variance proportional to the value of the hazard rate, that will never hit zero and that exhibits mean reversion. It has the following form

$$h_t = h_0 e^{gt+\sigma Y_t} \quad \text{with} \quad g, \sigma, h_0 > 0 \quad \text{and} \quad dY_t = -bY_t dt + dB_t^h \quad \text{with} \quad b > 0.$$

When $b \to 0$ the expected hazard rate is equal to the standard Gompertz function. It can be shown that

$$dh_t = \left(g + \tfrac{1}{2}\sigma^2 + b\ln(h_0) + bgt + b\ln(h_t)\right)h_t dt + \sigma h_t dB_t^h,$$

which is very similar to the Black-Derman-Toy short rate model.

Unfortunately the Mean reverting Brownian Gompertz process does not have a closed-formula solution and requires some kind of approximation. Numerical examples are shown in Milevsky and Promislow's article. To conclude it is easy to see that

$$GMIB = \sum_{i=1}^{\infty} C_t(s, T_i, \Lambda),$$

where T_i are the dates the annuity is paid.

Part III

Valuation models of the GMWB and GMLB Variable Annuities

The Guaranteed Minimum Withdrawal Benefit (GMWB) is the most complex of the GMxBs. Not only the basic payment structure is strongly related to Asian options, but the guarantee value is very sensible to the policyholder's behavior. That is, in order to price this guarantee it is important to take into account the surrender behavior of the policyholder.

A variation of this product is the GMLB, Guaranteed Minimum Withdrawal Benefit for Life. It corresponds to the case where the guarantee is maintained for all the life of the policyholder. This variation implies some important differences in the way to model the product.

This GMWB/GLWB Part of the report will be divided in five Chapters. In Chapter 7 we will introduce the notations, assumptions and the general guidelines we will use for the GMWB/GLWB model. On Chapter 8 we will present the valuation of the GMWB in the case the policyholder restrains herself to withdraw on each period the contract established amount. That is she does not do withdrawals with penalization. The financial strategies used in this Section are those of the Asian Options. On Chapter 9 we will evaluate the GMWB in the case the policyholder optimizes her withdrawals. To proceed to this evaluation we will require stochastic control instruments to model the policyholders optimal behavior. On Chapter 10 we will consider a GMWB where the policyholder either withdrawals the contract established amount or she makes a total surrender of her contract. In such a case the only option she has is when to make the total surrender, or not to make it. The financial instruments related to this contract are the American Asian Options. At last, on Chapter 12 we will evaluate GLWB, in order to do so we will use the strategies already developed in the GMWB Section and will adapt them to the for Life case.

Chapter 7

Introduction to GMWB/GLWB

7.1 Financial Assumptions and Considerations

We will make the following assumptions on the financial market conditions:

- The financial market is complete.
- The financial market is free of arbitrage.
- There are no transaction costs.
- There is no restriction on short selling.

Since we are looking for an actuarial price, we will not consider acquisition or administration fees.

From a theoretical point of view it is usual to do some simplifying assumptions even though these assumptions are not valid in the real world market. In particular there exist market frictions such as transaction costs, taxes, bid-offer spread and liquidity. It is also not possible to make transactions in a continuous manner in a real number amount. Transactions are done in a discrete manner in amounts that are whole numbers. All these real world characteristics add risks into the products and therefore imply an increase in the product's value. We will not model them but some considerations are due:

- The discrete nature of real world market should not be in itself an important source of value change. In fact trading can be done several times a day, which for a long maturity product such as Variable Annuity becomes almost an instantaneous trading. As well, the huge asset value of Variable Annuities makes the discrete nature of the amount not a problem in itself.
- Variable Annuity assets are usually "blue chip" kind of assets which have high trading quality and therefore they are normally very liquid.

- Transaction costs and taxes on the contrary can increase the product's value. First because these are cost by themselves and second because transactions are avoided to reduce costs and so hedging becomes more imperfect, this implies that more risk is taken. Transaction costs can be easily modeled from a Monte-Carlo point of view.

7.1.1 On liquidity risk

The previous considerations suppose a tranquil market. If the market stresses, liquidity can become an issue. In fact, if the market agents become mistrustful of the market (or at least of the assets that compose the Variable Annuity Unit Link) the trading of the Variable Annuity's assets can become difficult. This can be expressed in two manners:

- The Bid-Offer spread can expand. That is, the price for buying instantly becomes considerably higher to the price of selling instantly. This means that the cost of changing the hedge position or of selling the asset in order to pay withdrawals, death outcomes, maturities and surrenders can become higher than expected.

- The selling or buying of certain assets becomes impossible. This is the extreme case where no negotiation is done and it is not possible to change the position. In such a case the asset manager can only expect for the liquidity to recover in order to readjust his hedge or to sell to pay for outcomes. In practical terms having a highly expanded Bid-Offer spread is almost equivalent to be in a not selling or buying case.

Liquidity modeling can also be included in a Monte-Carlo framework. In order to approach this risk a variable can be included to determine periods of non-trading. Another way to approach liquidity model directly the Bid-Offer spread. This model would require a trading rule that takes into account the cost obtained in the presence of a large Bid-Offer spread.

7.2 Model Assumptions and Notation

We will make the following assumptions on the underlying reference portfolio process:

- There exists a risk neutral probability measure Q such that the asset price processes are Q martingales.

- Under Q the asset follows a geometric Brownian motion:

  $$dS_t = rS_t dt + \sigma S_t dB_t$$

  where r is the risk free interest rate, σ is a constant called the volatility and B_t is a Q-Brownian motion.

- In the case mortality is taken into account, mortality is independent of the financial risk.

Let $\mathcal{F}_t$ be the filtration generated by B_t, that is $\mathcal{F}_t = \sigma(B_u : 0 \leq u \leq t)$ for $t \geq t$. For notation simplicity we will denote $E_t^Q[\cdot]$ for $E^Q[\cdot|\mathcal{F}_t]$. The probability measure where S_t is taken as a numeraire will be noted Q_S and so the corresponding expected value will be noted $E_t^{Q_S}[\cdot]$.

Let G be the amount established in the contract which is the limit before penalization. In a GMWB contract G is defined as the lump sum premium ω_0 divided by T the number of years of the guarantee. That is, if after T years the policyholder withdrawals G each year, then the contracts maturity T she would have withdrawn her premium w_0. To observe some basic relations we will observe what happens when she does only withdraws G each year.

Let W_t be the unit-link sub account. This sub account represents the amount of funds asset that is in her account after withdrawals and insurance fee. Under the assumption of constant withdrawals G, W_t follows a dynamic similar to the asset S_t except that a insurance fee proportional to the W_t amount is charged and that the W_t amount is reduced by the withdrawals G, that is:

$$\begin{cases} dW_t &= (r-\alpha)W_t dt - G dt + \sigma W_t dB_t \\ W_0 &= \omega_0 \end{cases} \tag{7.1}$$

for $W_t > 0$. This stochastic process could become negative as the $-Gdt$ element can make dW_t negative. This property, among others, shows that this process is far from being a geometric Brownian motion. In fact, if $W_t = 0$ then the unit-linked account is empty and it can no further change. Therefore if $W_t = 0$ then $dW_t = 0$. Stated in another manner, if $W_t = 0$ then for all $s > t$ we have that $W_s = 0$.

7.3 The basics on the GMWB model

Lets first consider the case $W_t > 0$ for $0 \leq t \leq T$. By using theorem .1.1 in the appendix we have that

$$W_t = e^{(r-\alpha-\frac{1}{2}\sigma^2)t+\sigma B_t}\left(\omega_0 - G\int_0^t e^{-(r-\alpha-\frac{1}{2}\sigma^2)s-\sigma B_s} ds\right). \tag{7.2}$$

Now, since $e^{-(r-\alpha-\frac{1}{2}\sigma^2)s-\sigma B_s} > 0$ then for each trajectory the random variable

$$\int_0^t e^{-(r-\alpha-\frac{1}{2}\sigma^2)s-\sigma B_s} ds$$

is increasing in t. Which means that once

$$e^{(r-\alpha-\frac{1}{2}\sigma^2)t+\sigma B_t}\left(\omega_0 - G\int_0^t e^{-(r-\alpha-\frac{1}{2}\sigma^2)s-\sigma B_s}ds\right)$$

has arrived to 0 it can only be 0 or negative. This implies that W_t can be expresed in the following manner

$$W_t = e^{(r-\alpha-\frac{1}{2}\sigma^2)t+\sigma B_t}\max\left(0, \omega_0 - G\int_0^t e^{-(r-\alpha-\frac{1}{2}\sigma^2)s-\sigma B_s}ds\right). \tag{7.3}$$

By definition $G = \frac{\omega_0}{T}$, which means that

$$W_t = \omega_0 e^{(r-\alpha-\frac{1}{2}\sigma^2)t+\sigma B_t}\left[1 - \frac{1}{T}\int_0^t e^{-(r-\alpha-\frac{1}{2}\sigma^2)s-\sigma B_s}ds\right]^+. \tag{7.4}$$

Note $Y_t = e^{-(r-\alpha-\frac{1}{2}\sigma^2)t-\sigma B_t}$. We have then that

$$W_t = \omega_0\frac{1}{Y_t}\left[1 - \frac{1}{t}\int_0^t Y_s ds\right]^+. \tag{7.5}$$

This equation has an interesting financial interpretation. A Quanto option is an option in which the payoff function is defined in a currency A but it is paid in another currency (say B). Let Y_t denote the currency exchange value. Then a Quanto option has the form $\frac{1}{Y_t}f(Y_t)$. We can see that W_t is a Quanto option.

An Asian option is an option where the mean value of an asset is considered in order to define the options pay off. There are two factors that should be considered to understand the four types of Asian options. First, the mean can be calculated as a geometric mean or as an arithmetic mean. This first case is usually simple to evaluate since the product of log-normal variables is a log-normal variable, on the contrary an arithmetic Asian option is much harder to value. Second, the mean can play the role of strike or of underlying in the payoff. When the mean plays the role of strike then the pay off compares this mean with the final value of the asset. We say that it is a variable strike Asian option. When the mean plays the role of underlying it is compared with a fixed strike. Therefore in the case where the pay-off is in the form $[K - \frac{1}{T}\int_0^T X_t dt]^+$ we speak of an arithmetic fixed strike Asian option. For the rest of this report when we speak of an Asian option we refer to an arithmetic fixed strike Asian option.

A Quanto Asian Put is a put on a currency A who's value depends on the average value of the currency A, but then is paid in a currency B. If follows that W_T is the payoff function of ω_0 unities of a Quanto Asian Put.

In a GMWB contract the policyholder receives the money she has withdrawn plus the amount in her account at the maturity of the contract. That is the value of the GMWB contract at time t is

$$\underbrace{e^{-r(T-t)}E_t^Q[W_T]}_{\text{UL account value}} + \underbrace{\int_t^T e^{-r(s-t)}Gds}_{\text{withdrawals}}.$$

Since the premium consists of a lump sum ω_0 at time 0, from an actuarial price perspective we have that

$$\omega_0 = e^{-rT}E_0^Q[W_T] + \int_0^T e^{-rt}Gdt \tag{7.6}$$

$$= e^{-rT}E_0^Q\left[\omega_0\frac{1}{Y_T}\left[1-\frac{1}{T}\int_0^T Y_t dt\right]^+\right] + \int_0^T e^{-rt}\omega_0 g dt \tag{7.7}$$

$$= \omega_0 e^{-rT}E_0^Q\left[\frac{1}{Y_T}\left[1-\frac{1}{T}\int_0^T Y_t dt\right]^+\right] + \frac{\omega_0 g}{r}(1-e^{-rT}), \tag{7.8}$$

where $g = \frac{1}{T}$ the amount withdrawn for a monetary unit of guarantee. We have just proved the following theorem.

Theorem 7.3.1 (Fundamental static GMWB relation)**.** Let W_t be the unit-link subaccount which follows the following dynamic

$$\begin{cases} dW_t = (r-\alpha)W_t dt - Gdt + \sigma W_t dB_t & \text{if } W_t > 0 \\ dW_t = 0 & \text{if } W_t = 0 \\ W_0 = \omega_0 \end{cases}$$

where α is the insurance charge and G the fixed amount the policyholder withdrawals each year. Let $Y_t = e^{-(r-\alpha-\frac{1}{2}\sigma^2)t-\sigma B_t}$. Then the following relation is due

$$e^{-rT}E_0^Q\left[\frac{1}{Y_T}\left[1-\frac{1}{T}\int_0^T Y_t dt\right]^+\right] + \frac{g}{r}(1-e^{-rT}) = 1, \tag{7.9}$$

where $g = \frac{1}{T}$.

One can notice that this relation decompose a GMWB contract into

1. A fixed annuity that pays G each year during T years,

2. A Quanto Asian Put.

In what follows we will note

$$H_t := e^{-r(T-t)}E_t^Q\left[\frac{1}{Y_T}\left[1-\frac{1}{T}\int_t^T Y_u du\right]^+\right] + \frac{g}{r}(1-e^{-r(T-t)})$$

the GMWB value at time t and

$$V_t := E_t^Q\left[\frac{1}{Y_T}\left[1-\frac{1}{T}\int_t^T Y_u du\right]^+\right]$$

the Quanto Asiatic option non-discounted expected pay-off.

Chapter 8

The static strategy for GMWB

8.1 The general case

In this Section we will suppose that the policyholder withdrawals the amount G which is the limit before penalization in the contact. This amount G is withdrawn every year and is equal to the principal amount ω_0 divided by T the number of years of the guarantee. This case corresponds exactly with the Theorem 7.3.1 case. What is left to do is to evaluate the Quanto Asiatic option. In order to so so we will use some of the strategies usual in the Asian Options literature.

To approach $E_0^Q\left[\frac{1}{Y_T}\left[1-\frac{1}{T}\int_0^T Y_s ds\right]^+\right]$ numerically different strategies have been proposed in the Asian options literature. We are going to observe three of them. First, we are going to make an approximation of the integral term by a log-normal distribution. Second, we are going to estimate a tight lower bound to the asian option term. And third, we are going to use a partial differential equation for the value of the Asian Option. Demonstrations to each strategy can be found in the appendix.

However, a simple problem remains to be treated, our option is Quanto Asiatic and not simply Asiatic. In order to profit of the rich literature on Asiatic options it would be an advantage to eliminate the Quanto term. A change of numeraire is required to solve this difference. Observe that

$$E_0^Q\left[\frac{1}{Y_T}\left[1-\frac{1}{T}\int_0^T Y_t dt\right]^+\right] = e^{(r-\alpha)T}E_0^{Q^S}\left[\left[1-\frac{1}{T}\int_0^T Y_t dt\right]^+\right],$$

where $Y_t = e^{-(r-\alpha+\frac{1}{2}\sigma^2)t-\sigma B_t^{Q^S}}$ and Q^S is the probability that takes S_t as numeraire. This second presentation of the problem will be used in the Log-Normal approximation Section and in the lower bound approximation Section. For the PDE it is easier to work with the Quanto Asiatic form.

Calculation by Log-Normal Approximation

The sum of log-normal variables produces a random variable that has a distribution hard to manage. This distribution con not expressed as a function of the usual functions. A standard way to approach this difficulty is to approximate the sum distribution by a log-normal distribution. However there are other ways to approach it, such as with gamma function (Milevsky and Posner [31]). To fit a log-normal distribution to a sum of log-normal distributions there is a plenty of possible criteria, we are going to use a classic one: we suppose that both have the same first and second moments. That is, the first two moments of the sum of log-normal variables is calculated and are used as parameters of the log-normal distribution that is an approximation of the sum. This approximation has proved to be quite good, however there is no guarantee that it is good enough for certain parameter values. That is, it should be carefully used. However it has a great advantage: it's implementation es very simple.
Closed formulas for the first and second moments of the arithmetic mean of a geometric brownian motion have been calculated by Turnbull and Wakeman (refer to Hull [25]). Adapting their formulae to our notation we find that if $Y_t = e^{-(r-\alpha+\frac{1}{2}\sigma^2)t-\sigma B_t^{Q^S}}$ and we define $\mu = -(r-\alpha)$ then

$$M1 := E_0^{Q^S}\left[\frac{1}{T}\int_0^T Y_t dt\right] = \frac{e^{\mu T}-1}{\mu T}$$

and

$$M2 := E_0^{Q^S}\left[\left(\frac{1}{T}\int_0^T Y_t dt\right)^2\right] = \frac{2e^{(2\mu+\sigma^2)T}}{(\mu+\sigma^2)(2\mu+\sigma^2)T^2} + \frac{2}{\mu T^2}\left(\frac{1}{2\mu+\sigma^2} - \frac{e^{\mu T}}{\mu+\sigma^2}\right).$$

Once the arithmetic mean of our problem was become a log-normal distribution the calculation of the price of the GMWB can be easily written with the help of the Black and Scholes put option valuation. That is, the GMWB product with static strategy is worth

$$V = e^{-\alpha T}\left(\mathcal{N}(-d_2) - F\mathcal{N}(-d_1)\right) + \frac{1}{rT}\left(1-e^{-rt}\right),$$

where

$$d_1 = \frac{\ln(F)+\frac{1}{2}\sigma_I^2 T}{\sigma_I\sqrt{T}}, \qquad d_2 = d_1 - \sigma_I\sqrt{T}, \qquad F = M_1, \qquad \text{and} \qquad \sigma_I^2 = \frac{1}{T}\ln\left(\frac{M_2}{M_1^2}\right).$$

The simplicity of this formula is astonishing in comparison to the approches that follow. Some numerical examples of the results obtained by the different approaches will be presented at the end of this Section.

Calculation by Upper and Lower Bounds

Next we will bound this guarantee's value by the Rogers and Shi methodology [41]. Let A and Z be two random variables, then by the tower property end the Jensen's inequality we have that

$$E[\max(A,0)] = E[E[\max(A,0)|Z]] \geq E[\max(E[A|Z],0)]$$

and

$$\begin{aligned} E[\max(A,0)] &= E[\max(E[A|Z],0)] + \tfrac{1}{2}\left(E[abs(A)|Z] - abs(E[A|Z])\right) \\ &\leq E[\max(E[A|Z],0)] + \tfrac{1}{2}E\left[abs(A - E[A|Z])|Z\right] \\ &\leq E[\max(E[A|Z],0)] + \tfrac{1}{2}E[\sqrt{Var[A|Z]}], \end{aligned}$$

so

$$E[\max(E[A|Z],0)] \leq E[\max(A,0)] \leq E[\max(E[A|Z],0)] + \tfrac{1}{2}E[\sqrt{Var[A|Z]}].$$

The tightness of the bounds depends on the choice of Z. Z should be chosen to make $Var[A|Z]$ small.

The difficulty to value Asian options is that the arithmetic mean of log-normal variables is not easily tracktable. On the contrary, the geometric mean of log-normal variables is easy to deal with and usually near enough to the arithmetic mean of log-normal variables. Geometric mean of log-normal variables is easy to deal with because it reduces to the arithmetic mean of normal variables wich is a normal variable. On the Rogers and Shi approximation [41] the information that is gathered from the arithmetic mean of normal variables is used to approach with conditional expectation the arithmetic mean of log-normal variables. That is, we find the expected value and variance of Y_t conditional to $\int_0^T B_u du$. Let's see how this is done.

Let $Z = \int\limits_0^T B_u^{Q^S} du$, as shown in lemma 5 in the appendix we have that $E^{Q^S}[Z] = 0$ and $Var^{Q^S}[Z] = \frac{T^3}{3}$. Observe that $Z \sim \mathcal{N}(0, \frac{T^3}{3})$. For the rest of this Section take $t \leq T$,

$$E^{Q^S}\left[B_t^{Q^S} Z\right] = E^{Q^S}\left[\int_0^T B_t^{Q^S} B_u^{Q^S} du\right] = \int_0^T \min(t,u)du = t\left(T - \frac{t}{2}\right).$$

By the Projection Theorem we have that

$$\begin{aligned} E^{Q^S}[B_t^{Q^S}|Z] &= 3\frac{t}{T^3}\left(T - \frac{t}{2}\right) Z \\ Var^{Q^S}[B_t^{Q^S}|Z] &= t - 3\frac{t^2}{T^3}\left(T - \frac{t}{2}\right)^2 = t - \frac{3}{T}\left(t - \frac{t^2}{2T}\right)^2. \end{aligned}$$

Let

$$m_t = 3\frac{t}{T^3}\left(T - \frac{t}{2}\right) \qquad \text{and} \qquad v_t^2 = t - \frac{3}{T}\left(t - \frac{t^2}{2T}\right)^2.$$

Since Y_t id log-normal we have that

$$\begin{aligned} E^{Q^S}[Y_t|Z] &= e^{-(r-\alpha+\sigma^2)t - \sigma m_t Z + \frac{1}{2}\sigma^2 v_t^2} \\ E^{Q^S}[Y_t^2|Z] &= e^{-2(r-\alpha+\sigma^2)t - 2\sigma m_t Z + 2\sigma^2 v_t^2}. \end{aligned}$$

Now consider the random variable Y_tY_T, we have that

$$\begin{aligned} E^{Q^S}\left[(B_t^{Q^S} + B_T^{Q^S})Z\right] &= t\left(T - \frac{t}{2}\right) + \frac{T^2}{2} \\ E^{Q^S}[B_t^{Q^S} + B_T^{Q^S}|Z] &= (m_t + m_T)Z \\ V_t^2 := Var^{Q^S}[B_t^{Q^S} + B_T^{Q^S}|Z] &= Var^{Q^S}[B_t^{Q^S} + B_T^{Q^S}] - \frac{3}{T^3}\left(t\left(T - \frac{t}{2}\right) + \frac{T^2}{2}\right)^2 \\ &= T + 3t - \frac{3}{T^3}\left(t\left(T - \frac{t}{2}\right) + \frac{T^2}{2}\right)^2 \end{aligned}$$

and since Y_tY_T is log-normal we have that

$$E^{Q^S}[Y_TY_t|Z] = e^{-(r-\alpha+\sigma^2)(T+t) - \sigma(m_t+m_T)Z + \frac{1}{2}\sigma^2 V_t^2}.$$

As shown on the appendix (lemma 5), we have that

$$\begin{aligned} E^{Q^S}\left[\int_0^T Y_t dt\middle| Z\right] &= \int_0^T E^{Q^S}[Y_t|Z]dt \\ Var^{Q^S}\left[\int_0^T Y_t dt\middle| Z\right] &= \frac{1-2\sigma}{\mu^2}(1 - 2E^{Q^S}[Y_T|Z] + E^{Q^S}[Y_T^2|Z]) + \frac{\sigma^2}{\mu^2}\int_0^T E^{Q^S}[Y_t^2|Z]dt \\ &\quad + 2\frac{\sigma}{\mu}\int_0^T E^{Q^S}[Y_t|Z]dt - 2\frac{\sigma}{\mu}\int_0^T E^{Q^S}[Y_TY_t|Z]dt - \left(\int_0^T E[Y_t|Z]dt\right)^2, \end{aligned}$$

for $\mu = r - \alpha + \sigma^2$. So by Rogers and Shi approximation we have that

$$E^{Q^S}\left[\left[1 - \frac{1}{T}\int_0^T E^{Q^S}[Y_t|Z]dt\right]^+\right] \leq E^{Q^S}\left[\left[1 - \frac{1}{T}\int_0^T Y_t dt\right]^+\right]$$

and

$$\begin{aligned} E^{Q^S}\left[\left[1 - \frac{1}{T}\int_0^T Y_t dt\right]^+\right] \leq & E^{Q^S}\left[\left[1 - \frac{1}{T}\int_0^T E^{Q^S}[Y_t|Z]dt\right]^+\right] \\ & + \frac{1}{2T}E^{Q^S}\left[\sqrt{Var^{Q^S}\left[\int_0^T Y_t dt\middle| Z\right]}\right]. \end{aligned}$$

Notice that since $Z \sim \mathcal{N}(0, \frac{T^3}{3})$ then

$$E^{Q^S}[f(Z)] = \int_{-\infty}^{\infty} f(z) \frac{3}{T^3\sqrt{2\Pi}} e^{-\frac{3z^2}{2T^3}} dz.$$

Observe that the option value has been under and upper bounded and that this bound only require a numerical integration to be calculated.

The Rogers and Shi lower bound is usually quite near to the real value. Unfortunately the upper bound can be too high, for a tighter approach to the upper bound of Asian options see Thompson [44]. In what follows it will be given more importance to the lower bound than to the upper bound. In fact, the lower bound tourns out to be a fair well approximation to the guarantees value.

Calculation by Differential Equation

Another usual approach to value Asian Option is the use of differential equations[1]. Remember that we are interested in finding the value of

$$E_Q \left[\frac{1}{Y_T} \left[1 - \frac{1}{T} \int_0^T Y_t dt \right]^+ \right],$$

where $Y_t = e^{-(r-\alpha-\frac{1}{2}\sigma^2)t - \sigma B_t}$, that is

$$dY_t = -\mu Y_t dt - \sigma Y_t dB_t \qquad \text{with } Y_0 = 1 \qquad \text{and } \mu = r - \alpha - \sigma^2.$$

Let $X_t = \int_0^t g_s dY_s$, where $g_t = \left(\frac{t-T}{T} \right)$ so

$$X_t = \int_0^t g_s dY_s = 1 + g_t Y_t - \frac{1}{T} \int_0^t Y_s ds$$

and let $Z_t = \frac{X_t}{Y_t}$, so since $g_T = 0$ and $Y_T > 0$

$$E_Q \left[\frac{1}{Y_T} \left[1 - \frac{1}{T} \int_0^T Y_t dt \right]^+ \right] = E_Q \left[\frac{1}{Y_T} [X_T]^+ \right] = E_Q \left[[Z_T]^+ \right].$$

Now,

$$\begin{aligned} dZ_t &= \frac{1}{Y_t} dX_t + X_t d\left(\frac{1}{Y_t}\right) + dX_t d\left(\frac{1}{Y_t}\right) \\ &= \frac{1}{Y_t} g_t dY_t + X_t \left(\frac{1}{Y_t}(r-\alpha)dt + \frac{1}{Y_t}\sigma dB_t \right) - g_t \sigma^2 dt \\ &= (r-\alpha)(Z_t - g_t)dt + \sigma(Z_t - g_t)dB_t. \end{aligned}$$

[1] On this Section we will follow the methodology of Večeř [45] as presented by Lord [29].

Therefore by Feyman-Kac theorem we have that $V(t,z) = E_t^Q\left[[Z_T]^+\right]$ satisfies the following differential equation

$$\frac{\partial V}{\partial t} + (r-\alpha)(Z_t - g_t)\frac{\partial V}{\partial Z} + \frac{1}{2}\sigma^2(Z_t - g_t)^2\frac{\partial^2 V}{\partial Z^2} = 0,$$

with teminal condition $V(T,Z) = [Z]^+$.

Let $A(t,Z) = (r-\alpha)(Z_t - g_t)$ and $B(t,Z) = \frac{1}{2}\sigma^2(Z_t - g_t)^2$, then, we propose the implicit difference scheme

$$\frac{V_j^{n+1} - V_j^n}{\Delta t} + A(t_n, Z_j)\frac{V_{j+1}^n - V_{j-1}^n}{2\Delta Z} + B(t_n, Z_j)\frac{V_{j+1}^n - 2V_j^n + V_{j-1}^n}{\Delta Z^2} = 0,$$

for $n = 0, ..., N$ and $j = 1, ..., J$. If we consider $V_0^n = 0$ and $V_{J+1}^n = V_{J+1}^{n+1}$ the extreme values can be treated. Observe that if $t_N = T$ then $V_j^N = [Z_j]^+$ and if $t_n < T$ then the vector V^n satisfies

$$V^n = (I - \Delta t L^n)^{-1}(V^{n+1} + \Delta t F^n),$$

where I is the identity matrix, L^n is the tridiagonal square matrix such that $L_{j,j-1}^n = \dfrac{B(t_n, Z_j)}{\Delta Z^2} - \dfrac{A(t_n, Z_j)}{2\Delta Z}$, $L_{j,j}^n = -2\dfrac{B(t_n, Z_j)}{\Delta Z^2}$ and $L_{j,j+1}^n = \dfrac{B(t_n, Z_j)}{\Delta Z^2} + \dfrac{A(t_n, Z_j)}{2\Delta Z}$ and F^n is a zero vector except for the position J where it's value is equal to $L_{J,J+1}^n V_{J+1}^{n+1}$. A simple way to obtain V_{J+1}^{n+1} in each step is to reduce J by one each step.

Observe that $(I - \Delta t L^n)$ becomes then a tridiagonal matrix. There exist fast algorithms to invert these type of matrices, for more information please refer to the appendix.

8.2 Inclusion of stochastic interet rate

Following Peng, Leung and Kwonk [38] methodology we will consider the Vasicek interest rate model, which with the asset models gives the following system

$$dS_t = r_t S_t dt + \sqrt{1-\rho^2}\sigma_S S_t dB_{1,t} + \rho\sigma_S S_t dB_{2,t}$$
$$dr_t = k(\theta - r_t)dt + \sigma_r dB_{2,t}$$

where $B_{1,t}$ and $B_{2,t}$ are independent standard Q-Bronwnian processes, ρ is a constant called correlation, θ, k and σ_r are the Vasicek model parameters and σ_S is the asset model volatility. As shown on the appendix, the value of a zero-coupon Bond under the Vasicek model is

$$D(t,T) = e^{m(t,T) - n(t,T)r_t},$$

where $b(t,T) = \frac{1}{k}\left(1 - e^{-k(T-t)}\right)$ and $m(t,T) = \left(\theta - \dfrac{\sigma^2}{2k^2}\right)(b(t,T) - (T-t)) - \dfrac{\sigma^2}{4k} b^2(t,T)$ and the following relation is respected

$$\frac{dD(t,T)}{D(t,T)} = r_t dt - \sigma_r b(t,T) dB_{2,t}$$

therefore the financial model can be expressed as

$$\begin{aligned}\frac{dS_t}{S_t} &= r_t dt + \boldsymbol{\sigma}_S d\mathbf{B}_t \\ \frac{dD_t}{D_t} &= r_t dt + \boldsymbol{\sigma}_D d\mathbf{B}_t,\end{aligned}$$

where $\mathbf{B}_t = \begin{pmatrix} B_{1,t} \\ B_{2,t} \end{pmatrix}$ $\boldsymbol{\sigma}_S = (\sqrt{1-\rho^2}\sigma_S \quad \rho\sigma_S)$ and $\boldsymbol{\sigma}_D = (0 \quad -\sigma_r b(t,T))$. The sub-account process will have the following partial fifferential equation
$W_0 = \omega_0$, if $W_t > 0$ then $dW_t = (r_t - \alpha)W_t dt - G dt + \boldsymbol{\sigma}_S W_t d\mathbf{B}_t$ and if $W_t = 0$ then $dW_t = 0$. By the application of theorem .1.1 we have that for $W_t > 0$

$$W_t = Z_t \left(w_0 - G \int_0^t \frac{1}{Z_u} du\right)$$

for

$$Z_t = e^{\int_0^t \left(r_u - \alpha - \frac{1}{2}\boldsymbol{\sigma}_S \boldsymbol{\sigma}_S^T\right) du + \int_0^t \boldsymbol{\sigma}_S d\mathbf{B}_u}$$

and so

$$W_t = Z_t \left(\max\left(w_0 - G \int_0^t \frac{1}{Z_u} du, 0\right)\right).$$

The value of the GMWB contract with stochastic interest rate would then be

$$\begin{aligned}V(W,r,0) &= E_0^Q\left[e^{-\int_0^T r_u du} W_T + \int_0^T e^{-\int_0^u r_s ds} G du\right] \\ &= E_0^Q\left[e^{-\int_0^T r_u du} W_T\right] + G\int_0^T D(0,u) du.\end{aligned}$$

In order to calcul $E_0^Q\left[e^{-\int_0^T r_u du} W_T\right]$ it is required to change the numeraire. Consider a new mesure Q^S with S_t as numeraire. If M_t is the money market account process, numeraire de Q, then

$$\left.\frac{dQ^S}{dQ}\right|_{\mathcal{F}_T} = \frac{S_T/S_0}{M_T/M_0}$$

and so we have that

$$
\begin{aligned}
E_0^Q\left[e^{-\int_0^T r_u du} W_T\right] &= E_0^{Q^S}\left[e^{-\int_0^T r_u du}\frac{S_T/S_0}{M_T/M_0} W_T\right] \\
&= E_0^{Q^S}\left[\frac{S_T}{S_0} Z_T \max\left(w_0 - G\int_0^T \frac{1}{Z_u} du, 0\right)\right] \\
&= e^{-\alpha T} E_0^{Q^S}\left[\max\left(w_0 - G\int_0^T \frac{1}{Z_u} du, 0\right)\right].
\end{aligned}
$$

Observe that is equation es very similar to the one we use without stochastic interest rate.

Now, by Girsanov Theorem the Q^S-Brownian motion $\mathbf{B}_t^{Q^S}$ satisfies the relation $d\mathbf{B}_t^{Q^S} = d\mathbf{B}_t - \boldsymbol{\sigma}_S^T dt$. Therefore

$$
Z_t = e^{\int_0^t \left(r_u - \alpha + \frac{1}{2}\boldsymbol{\sigma}_S \boldsymbol{\sigma}_S^T\right)du + \int_0^t \boldsymbol{\sigma}_S d\mathbf{B}_u^{Q^S}},
$$

where r_t satisfies $dr_t = k(\theta + \frac{\rho}{k}\sigma_S - r_t)dt + \sigma_r dB_{2,t}^{Q^S}$.

Different numerical applications are possible to solve this equation. The methodology followed by Peng, Leung and Kwonk [38] corresponds to a calculation by upper and lower bounds, they use Rogers and Shi [41] lower bound, as we do on the previous Section, and for un upper bound they use Thompson's [44] methodology.

8.3 Numerical Examples

Three methodologies where implemented: the log-normal approximation, the calculation by lower bounds and the calculation by partial difference equation. The upper bound is usually too high and was not considered for this table. No implementation of the stochastic interest rate for the static strategy was done. However it was implemented for the dynamic strategy and can be observed on the following Section.

We considered $r = 5\%$ and took two possible values for σ, $\sigma = 20\%$ and $\sigma = 30\%$. The results are resumed in the following table:

		$\sigma = 20\%$			$\sigma = 30\%$		
Contractual rate, g	Maturity T = 1/g	Log-Normal Approx.	Lower Bound	PDE	Log-Normal Approx.	Lower Bound	PDE
4%	25,00	17,11	17,59	27,71	58,55	50,40	54,38
5%	20,00	27,87	28,32	38,60	85,09	75,61	80,00
6%	16,67	40,09	40,45	48,92	113,46	102,80	107,05
7%	14,29	53,35	53,60	61,11	143,04	131,31	135,33
8%	12,50	67,40	67,49	74,57	173,45	160,70	164,48
9%	11,11	82,00	81,94	88,88	204,38	190,70	194,24
10%	10,00	97,06	96,80	103,79	235,66	221,08	224,45

As we can see, as the contractual rate is higher, the option charge also increases. It seems reasonable that one would pay more if she is to obtain a higher guaranteed value. As is usual in the option realm, with a higher volatility the price (in this case the charge) increases. In fact one can see that in the passage from $\sigma = 20\%$ to $\sigma = 30\%$, the charge doubles. As expected we can also see that the lower bound is a bit lower than the PDE, but we can see that it is not at many basic points of distance.

The log-normal approximation is for $\sigma = 20\%$ in almost all values of g lower than the lower bound. In fact it is almost the lower bound. On the contrary for $\sigma = 30\%$ the log-normal approximation is some basic points too high.

If a precise value is desired the PDE approach is the most accurate. However is the hardest to implement and a tiny differential step is required to obtain a good precision. The lower bound is a good approximation that is known to be beneath the real value, its implementation is quite easy, eventhough a numerical integral is required. The log-normal approximation is quite easy to implement but one should be mistrustful to the values obtained.

Chapter 9

The dynamic strategy for GMWB

9.1 The basic case

Let S_t be the asset value, which is in fact the guarantee's underlying. We suppose that S_t follows a geometric Brownian motion

$$dS_t = rS_t dt + \sigma S_t dB_t,$$

where B_t is a standard Brownian motion, σ the asset volatility and r the risk-free interest rate. Let α be the insurance risk charge. Suppose $\mathcal{F}_t$ is the filtration natural to B_t. If not stated otherwise, expectations will be taken under the risk-neutral probability $E^Q[\cdot]$.

Les A_t be the value of the GMWB account. We take A_t to be a right-continuous $\mathcal{F}_t$ adapted process. At the beginning of the contract A_t is equal to ω_0 which is the product's premium. During the contracts life, A_t will decrease with the withdrawals until it arrives to a 0 value. If at T, maturity of the contract, the value is A is still positive all the account value is withdrawn by the policyholder and A_T becomes zero.

We will take γ_s the withdrawal rate such that

$$A_t = A_0 - \int_0^t \gamma_s ds, \qquad 0 \leq \gamma_0 \leq \lambda$$

where λ is the maximal withdrawal rate, suppose that λ is large enough.

When the policyholder withdrawal more than the quantity G established in the contract, a penalty charge is due. Let k be this penalty charge. Let $\lambda > G$. Therefore, if the policyholder withdraws at a rate γ_s less than G she receives the same rate γ_s, but if she withdrawals more than G then she will receive a rate of $G + (1-k)(\gamma - G)$. We define function h to represent this relation

$$h(\gamma_s) = \begin{cases} \gamma_s & \text{if } 0 \leq \gamma_s \leq G \\ G + (1-k)(\gamma_s - G) & \text{if } \gamma_s > G \end{cases}.$$

Let W_t be the unit-liked account value. That is W_t is an account that varies with S_t but that is reduced by the withdrawals. W_t follows then the following relation

$$dW_t = (r-\alpha)W_t dt + \sigma W_t dB_t + dA_t, \qquad \text{if } W_t > 0,$$

when W_t becomes zeros, it maintains its value in zero. That is, if $W_s = 0$ then for all $t > s$ we have that $W_t = 0$. $W_0 = \omega_0$ which is the premium paid by the policyholder. If at the contract's maturity T the asset account W_T is more than zero, then the policyholder will receive W_T.

The contracts value at moment t is the expected value of all possible future flows. Therefore

$$V(W,A,t) = \max_{(\gamma_s)_{s\in]0,T[}\in\mathcal{A}} E_t\left[\underbrace{e^{-r(T-t)}\max(W_T,0)}_{\text{what is left in the account}} + \underbrace{\int_t^T e^{-r(u-t)}h(\gamma_u)du}_{\text{what will be withdrawn}}\right].$$

We are going to use the Hamilton-Jacobi-Bellman (HJB) relation to find an stochastic differential equation that is satisfied by the value function V. Using Section .2 notation we have that $\mathcal{O} =]0,+\infty[\times]0,+\infty[$ and $\mathcal{A} = [0,\lambda]^{]0,T[}$, we will have a two dimensional state process X_t defined in the following manner

$$X_t = \begin{pmatrix} A_t \\ W_t \end{pmatrix}$$

$$\begin{aligned} dA_t &= -\gamma_t dt \\ dW_t &= ((r-\alpha)W_t - \gamma_t)\, dt + \sigma W_t dB_t \end{aligned}$$

$$b(X_t,\gamma_t) = \begin{pmatrix} -\gamma_t \\ (r-\alpha)W_t - \gamma_t \end{pmatrix} \qquad \sigma(X_t,\gamma_t) = \begin{pmatrix} 0 \\ \sigma W_t \end{pmatrix}.$$

And the value function as defined in Section .2 are

$$f(t,X_t,\gamma_t) = h(\gamma_t) \qquad \Psi(\tau,X_\tau) = \max(W_\tau,0) \qquad r(X_t,t,\gamma_t) = r.$$

Let's first verify if Section .2.4 conditions are fullfilled. The Lipschitz conditions are easily verified

$$|b(X_{1,t},\gamma_t) - b(X_{2,t},\gamma_t)| = \left|\begin{pmatrix} 0 \\ (r-\alpha)(W_{1,t} - W_{2,t}) \end{pmatrix}\right| \leq (r-\alpha)|X_{1,t} - X_{2,t}|$$

$$|\sigma(X_{1,t},\gamma_t)-\sigma(X_{2,t},\gamma_t)| = \left|\begin{pmatrix} 0 \\ \sigma(W_{1,t}-W_{2,t}) \end{pmatrix}\right| \le \sigma|X_{1,t}-X_{2,t}|$$

so it is enoght to take $L=\max(r-\alpha,\sigma)$.

The good definition of V is given by the fact that $f(X_t,\gamma_t)\le\lambda(1+|X_t|)$ and $\Psi(X_t)\le\lambda(1+|X_t|)$.

The continuity of the stopping time one can take $\mathcal{O}_1=\mathbb{R}\times]0,+\infty[\times\mathbb{R}$ and $\mathcal{O}_2=\mathbb{R}^2\times]0,+\infty[$ to accomplish $H'2$. Then $\delta_1(t,w,a)=w$ and $\delta_2(t,w,a)=a$ which are clearly $\mathcal{C}^2$ functions and if $(w,a)\in\Gamma_{transversal}$ we have that $w=0$ or $a=0$ and so either $\delta_1(w,a)=0$ or $\delta_2(w,a)=0$ and we have that

$$\frac{\partial\delta_i}{\partial t}+\mathcal{L}^\alpha\delta_i=-\gamma_t\le 0 \text{ for } i=\{1,2\}$$

which means that the Hamilton-Jacobi-Bellman equation is satisfied, that is

$$V(W,A,t)=\begin{cases}\sup\limits_{\gamma\in\mathcal{A}}\left(\dfrac{\partial V}{\partial t}+\mathcal{L}^\alpha V+h(\gamma_t)-rV\right) & =0 \text{ if inside } \mathcal{D}\\ V & =\Psi \text{ if } t=\tau\end{cases}$$

which for $(W,A,t)\in\mathcal{D}$ is

$$\frac{\partial V}{\partial t}+\max_{\gamma_t}\left(h(\gamma_t)-\gamma_t\left(\frac{\partial V}{\partial A}+\frac{\partial V}{\partial W}\right)\right)+(r-\alpha)W\frac{\partial V}{\partial W}+\frac{1}{2}\sigma^2W^2\frac{\partial^2 V}{\partial W^2}-rV=0.$$

Now, let $C=\dfrac{\partial V}{\partial A}+\dfrac{\partial V}{\partial W}$. It is clear that for any $\delta>0$

$$V(W+\delta,V+\delta,t)\le V(W,A,t)+\delta,$$

which implies that $C\le 1$. Let $g(C)=\max\limits_{\gamma_t}\left(h(\gamma_t)-\gamma_t C\right)$ which is

$$\begin{aligned} g(C) &= \begin{cases}(1-C)G & \text{si } (1-k)<C\le 1\\ kG+\lambda(1-k-C) & \text{si } 0\le C\le(1-k)\end{cases}\\ &= G\min(1-C,k)+\lambda\max(1-k-C,0)\end{aligned}$$

and so inside $\mathcal{D}$ we have that

$$\frac{\partial V}{\partial t}+g\left(\frac{\partial V}{\partial A}+\frac{\partial V}{\partial W}\right)+(r-\alpha)W\frac{\partial V}{\partial W}+\frac{1}{2}\sigma^2W^2\frac{\partial^2 V}{\partial W^2}-rV=0.$$

Which is equivalent to saying that the following system is true and at least one of the inqualities is satisfied in the frontier

$$\frac{\partial V}{\partial t}+\left(1-\frac{\partial V}{\partial A}+\frac{\partial V}{\partial W}\right)+(r-\alpha)W\frac{\partial V}{\partial W}+\frac{1}{2}\sigma^2W^2\frac{\partial^2 V}{\partial W^2}-rV\geq 0$$
$$\frac{\partial V}{\partial t}+kG+\lambda\left(1-k-\frac{\partial V}{\partial A}+\frac{\partial V}{\partial W}\right)+(r-\alpha)W\frac{\partial V}{\partial W}+\frac{1}{2}\sigma^2W^2\frac{\partial^2 V}{\partial W^2}-rV\geq 0$$

when $\lambda\to\infty$ the system becomes equivalent to saying that the following system is true and at least one of the inqualities is satisfied in the frontier

$$\frac{\partial V}{\partial t}+\left(1-\frac{\partial V}{\partial A}+\frac{\partial V}{\partial W}\right)+(r-\alpha)W\frac{\partial V}{\partial W}+\frac{1}{2}\sigma^2W^2\frac{\partial^2 V}{\partial W^2}-rV\geq 0$$
$$1-k-\frac{\partial V}{\partial A}+\frac{\partial V}{\partial W}\geq 0$$

and there fore the system is equivalent to

$$\min\left(\frac{\partial V}{\partial t}+\left(1-\frac{\partial V}{\partial A}+\frac{\partial V}{\partial W}\right)+(r-\alpha)W\frac{\partial V}{\partial W}+\frac{1}{2}\sigma^2W^2\frac{\partial^2 V}{\partial W^2}-rV,\right.$$
$$\left.1-k-\frac{\partial V}{\partial A}+\frac{\partial V}{\partial W}\right)=0.$$

In order to apply a numerical approach to the differential equation, boundary conditions are required. The payout at maturity T is $V(W,A,T)=\max(X,(1-k)A)$ and if $A=0$ then the value function becomes the actual value of the underlying : $V(W,0,t)=We^{-\alpha(T-t)}$. In fact as W is large in relation to A this relation tends to be satisfied, that is $\lim_{W\to\infty}V(W,A,t)=We^{-\alpha(T-t)}$. Now, since in our stochastic differential equation A has only first order derivates, we can numerically avoid to know the value of function when A is large. For $V(0,A,t)$ in the appendix (lemma 6) is prorfed that

$$V(0,A,t)=(1-k)\max(A-G\tau^*,0)+\frac{G}{r}\left(1-e^{-r\min(\frac{A}{G},\tau^*)}\right),$$

where $\tau^*=\min\left(-\frac{\ln(1-k)}{r},T-t\right)$.

Following Dai, Kwong and Zwong [17] discretisation methodolgy a two-level implicit finite difference scheme was implemented. Let

$$R^n_{j,k}=\frac{\sigma^2}{2}W_j^2\frac{V^n_{j+1,k}-2V^n_{j,k}+V^n_{j-1,k}}{\Delta W^2}+(r-\alpha)W_j\frac{V^n_{j+1,k}-V^n_{j-1,k}}{2\Delta W}-rV^n_{j,k}$$
$$R^{n+1}_{j,k}=\frac{\sigma^2}{2}W_j^2\frac{V^{n+1}_{j+1,k}-2V^{n+1}_{j,k}+V^{n+1}_{j-1,k}}{\Delta W^2}+(r-\alpha)W_j\frac{V^{n+1}_{j+1,k}-V^{n+1}_{j-1,k}}{2\Delta W}-rV^{n+1}_{j,k}.$$

For the linear R part a Crank-Nicholson scheme was applied while the non-linear part was completely implicit, that is

$$\frac{V_{j,k}^{n+1} - V_{j,k}^{n}}{\Delta\tau} = \frac{1}{2}R_{j,k}^{n} + \frac{1}{2}R_{j,k}^{n+1} + g(C_{j,k}^{n+1})$$

where $C_{j,k}^{n+1} = \dfrac{V_{j,k}^{n+1} - V_{j-1,k}^{n+1}}{\Delta W} + \dfrac{V_{j,k}^{n+1} - V_{j,k-1}^{n+1}}{\Delta A}$ and $\tau = T - t$ the time to maturity.

The discrete withdrawal model

The previous approach supposes that withdrawals are made continuously. In fact in real a contract that is not the case. To correct this inconsistency Dai Kwonk and Zong [17] (see also Chen and Forsyth [13]) propose to use the following model. Withdrawals are made on dates t_i for $i = 1, .., N$. Then the time reverse scheme would be divided into two time sets

1. The time intervals between withdrawals $]t_i t_{i+1}[$ where the GMWB account does not change and so the option value does not change accordingly, that is $\frac{\partial V}{\partial A} = 0$, so the PDE becomes
$$\frac{\partial V}{\partial t} + \mathcal{L}V = 0.$$

2. At withdrawal dates $t_1, ..., t_N$ the policyholder withdrawals in such a manner that the GMWB value plus the amount received from the withdrawal is maximized, that is
$$V(W, A, t_i^-) = \max_{0 \le \gamma \le A} \left\{ V(\max(W - \gamma, 0), A - \gamma, t_i^+) + f(\gamma) \right\},$$
where
$$f(\gamma) = \begin{cases} \gamma & \text{if } 0 \le \gamma \le G \\ G + (1-k)(\gamma - G) & \text{if } \gamma > G \end{cases}.$$

Boundary conditions should be established for these equations. Define $V_0(A, t) = V(0, A, t)$, the value function where no money is left in the unit liked account. In such a case the value variation in the non-withdrawal periods only varies with the discount rate, that is

$$\frac{\partial V_0}{\partial t} = rV \qquad \text{if } t \notin \{t_1, ..., t_N\}.$$

If it is a withdrawal period withdrawals can only be made on the GMWB account, that is

$$V_0(A, t-) = \max_{0 \le \gamma \le A} \left\{ V_0(A - \gamma, t_i^+) + f(\gamma) \right\}.$$

The discrete time formulation of the problem is more realistic and simpler to implement. The PDE depends on a single variable and is linear. That is not the case of the continuous formulation. In fact the non linearity of the continuous formulation makes the differential scheme extremely unstable. Chen and Forsyth explore alternatives to stabilize such difference scheme. In our implementation we considered simpler and nearer to reality to apply the discrete version. In Dai Kwonk and Zong it is suggested that there are time steps that are of type 1 (time between withdrawals) and others are type 2 (withdrawal moments). In our implementation we considered more accurate to apply at each time step the difference equation but in the appropriate gaps between time steps apply the maximization equation (optimized withdrawals). These small algorithm differences explain the slight difference between our results.

Some numerical examples following the discrete withdrawal approach will be presented at the end of this Chapter.

9.2 The stochastic interest rate case

To include into our model an stochastic interest rate, we propose the following system:

$$\begin{aligned}
dA_t &= -\gamma_t dt \\
dW_t &= ((r_t - \alpha)W_t - \gamma_t)\, dt + \sqrt{1-\rho^2}\sigma W_t dB_t^1 + \rho\sigma W_t dB_t^2 \\
dr_t &= p(r_t, t)dt + q(r_t, t)dB_t^2
\end{aligned}$$

where B_t^1 and B_t^2 are independenr brownian motions. Observe that W_t's brownien motion $\sqrt{1-\rho^2}B_t^1 + \rho B_t^2$ is correlated to r_t's brownien motion B_t^2 by a factor of ρ. To begin, we will let $p(r, t)$ and $q(r, t)$ be any mesurable function, later on this Chapter we will impose some conditions in order to make the process converge.

Let $\mathcal{O} =]0, \omega_0[\times]0, +\infty[\times]0, +\infty[$ be the state space and $\mathcal{A} = [0, \lambda]^{]0,T[}$ the strategy space.

The system can be written in the matrix form

$$X(t) = \begin{pmatrix} A_t \\ W_t \\ r_t \end{pmatrix} \qquad b(X_t, \gamma_t) = \begin{pmatrix} -\gamma_t \\ (r_t - \alpha)W_t - \gamma_t \\ p(r_t, t) \end{pmatrix}$$

$$\sigma(X_t, \gamma_t) = \begin{pmatrix} 0 & 0 \\ \sqrt{1-\rho^2}\sigma W_t & \rho\sigma W_t \\ 0 & q(r_t, t) \end{pmatrix}$$

$$\frac{1}{2}\sigma \cdot \sigma^t = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2}\sigma^2 W_t^2 & \frac{q(r_t,t)\sigma W_t}{2} \\ 0 & \frac{q(r_t,t)\sigma W_t}{2} & \frac{q^2(r_t,t)}{2} \end{pmatrix}$$

with the same target-functions as the non-interest rate case

$$f(t, X_t, \gamma_t) = h(\gamma_t) \qquad \Psi(\tau, X_\tau) = \max(W_\tau, 0).$$

The good definition of V is proofed by the same argument that for the non-stochastic interest rate case.

For the continuity of the stopping time take $\mathcal{O}_1 = \mathbb{R}\times]0, +\infty[\times\mathbb{R}^2$ and $\mathcal{O}_2 = \mathbb{R}^2\times]0, +\infty[\times\mathbb{R}$ and $\mathcal{O}_3 = \mathbb{R}^3\times]0, +\infty[$. Then $\delta_1(t, w, a, r) = w$, $\delta_2(t, w, a, r) = a$ and $\delta_3(t, w, a, r) = r$ which are clearly $\mathcal{C}^2$ functions and if $(w, a, r) \in \Gamma_{transversal}$ we have that $w = 0$, $a = 0$ or $r = 0$ and so either $\delta_1(w, a, r) = 0$, $\delta_2(w, a, r) = 0$ or $\delta_3(w, a, r) = 0$ and we have that: if $w = 0$ then $\frac{\partial\delta_1}{\partial 1} + \mathcal{L}^\alpha\delta_1 = -\gamma_t \leq 0$, if $a = 0$ then $\frac{\partial\delta_2}{\partial 2} + \mathcal{L}^\alpha\delta_2 = -\gamma_t \leq 0$ and if $r = 0$ then $S * D\delta_3 = q(0, t)$ and $q(0, t) \neq 0$ for the Vasicek and the Hull and White Models.

Followung the same line of argument of the previous Chapter, the Hamilton-Jacobi-Bellman equation is:

$$V(W, A, t) = \begin{cases} \sup\limits_{\gamma\in\mathcal{A}} \left(\dfrac{\partial V}{\partial t} + \mathcal{L}^\alpha V + h(\gamma_t) - rV\right) & = 0 \text{ if inside } \mathcal{D} \\ V & = \Psi \text{ if } t = \tau \end{cases}$$

which for $(W, A, r, t) \in \mathcal{D}$. By the HJB theorem the value function V satisfies the partial differential equation

$$\begin{aligned}\frac{\partial V}{\partial t} &+ g\left(\frac{\partial V}{\partial A} + \frac{\partial V}{\partial W}\right) + (r-\alpha)W\frac{\partial V}{\partial W} + \frac{1}{2}\sigma^2 W^2\frac{\partial^2 V}{\partial W^2} - rV \\ &+ \frac{q^2(r_t, t)}{2}\frac{\partial^2 V}{\partial r^2} + q(r_t, t)\sigma W_t\frac{\partial^2 V}{\partial W\partial r} + p(r_t, t)\frac{\partial V}{\partial r} = 0.\end{aligned}$$

In the following we will consider a Vasicek interest rate model. This simple model allows us to see the interest rate model inclusion consequences on the GMWB guarantee. The Vasicek model is

$$dr_t = (a - br_t)dt + \sigma_r dB_t^2$$

where a, b and σ are positive constants.

The GMWB partial differential equation with the Vasicek interest rate model becomes

$$\begin{aligned}\frac{\partial V}{\partial t} &+ g\left(\frac{\partial V}{\partial A} + \frac{\partial V}{\partial W}\right) + (r-\alpha)W\frac{\partial V}{\partial W} + \frac{1}{2}\sigma^2 W^2\frac{\partial^2 V}{\partial W^2} - rV \\ &+ \frac{\sigma_r^2}{2}\frac{\partial^2 V}{\partial r^2} + \sigma_r W_t\frac{\partial^2 V}{\partial W\partial r} + (a - br_t)\frac{\partial V}{\partial r} = 0,\end{aligned}$$

which from a discrete withdrawal perspective means that on the non-withdrawal time set, V satisfies the following PDE

$$\frac{\partial V}{\partial t} + \mathcal{L}V = 0$$

and that on the withdrawal days

$$V(W, A, r, t) = \max_{0<\gamma<A} \left\{ V([W-\gamma]^+, A-\gamma, r, t_+) + f(\gamma) \right\}$$

where

$$\begin{aligned}\mathcal{L}V = (r-\alpha)W\frac{\partial V}{\partial W} + \frac{1}{2}\sigma^2 W^2 \frac{\partial^2 V}{\partial W^2} - rV \\ + \frac{\sigma_r^2}{2}\frac{\partial^2 V}{\partial r^2} + \sigma_r W_t \frac{\partial^2 V}{\partial W \partial r} + (a - br_t)\frac{\partial V}{\partial r}.\end{aligned}$$

The inclusion of stochastic interest rate adds a dimension into the partial difference equation. From a discrete withdrawals perspective this means that not counting time t and the GMWB account A_t, the value process V_t is composed of two dimensions: the unit-link account and the stochastic interest rate process. In order to resolve numerically the PDE a Crank-Nicholson scheme was applied. We used the Kronecker product to deal with the two dimensionality of the matrices. In what follows we are going to present in more detail the numerical approach. For a fixed moment t and a fixed value of A_t the value process V_t can be represented by a $N \times M$ matrix V where the first dimension represents the unit-link value W_t and the second dimension represents the interest rate value r_t.

We will represent the unit-link value as a diagonal matrix W with the set of possible values $W_1, ..., W_N$ in the diagonal, homogeneously separated ones to each other with a step of ΔW. We will represent the interest rate with a matrix R that is diagonal, with values $r_1, ..., r_M$ on the diagonal, separated in with a step Δr. As a general rule W will multiply on the left of the value matrix V and R on the right. This will allow as to use the correct value of W_i and r_j for the corresponding value cell $V_{i,j}$.
In order to compute the derivates we use the matrix D_x^1 and D_x^2:

$$D_x^1 = \frac{1}{2\Delta x}\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ 0 & -1 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad D_x^2 = \frac{1}{\Delta x^2}\begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -2 \end{pmatrix}.$$

Observe that D_W^1 and D_W^2 are $N \times N$ matrices while D_r^1 and D_r^2 are $M \times M$ matrices. To avoid to overcharge the notation D_r^1 will correspond to the matrix that has -1 on the upper diagonal and 1 on the lower diagonal. Notice also that the simple application of these matrices losses the border values, in order to correct this problem we introduce the following matrices:

$$H_W^1 = \frac{1}{2\Delta W}\begin{pmatrix} -V_{0,1} & -V_{0,2} & \cdots & -V_{0,M} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \\ V_{N+1,1} & V_{N+1,2} & \cdots & V_{N+1,M} \end{pmatrix},$$

$$H_W^2 = \frac{1}{\Delta W^2}\begin{pmatrix} V_{0,1} & V_{0,2} & \cdots & V_{0,M} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \\ V_{N+1,1} & V_{N+1,2} & \cdots & V_{N+1,M} \end{pmatrix},$$

$$H_r^1 = \frac{1}{2\Delta r}\begin{pmatrix} -V_{1,0} & 0 & \cdots & 0 & V_{1,M+1} \\ -V_{2,0} & 0 & \cdots & 0 & V_{2,M+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -V_{N,0} & 0 & \cdots & 0 & V_{N,M+1} \end{pmatrix} \quad \text{and} \quad H_r^2 = \frac{1}{\Delta r^2}\begin{pmatrix} V_{1,0} & 0 & \cdots & 0 & V_{1,M+1} \\ V_{2,0} & 0 & \cdots & 0 & V_{2,M+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ V_{N,0} & 0 & \cdots & 0 & V_{N,M+1} \end{pmatrix},$$

where H_W^1,H_W^2, H_r^1 and H_r^2 are all $N \times M$ matrices. The values of these matrices are the border values and are supposed to be known, that is for a given time and value A of the GMWB account these matrices are constant matrices.

This matrix notation allows us to write numerically

$$\begin{aligned} &\frac{\partial V}{\partial t} + (r-\alpha)W\frac{\partial V}{\partial W} + \tfrac{1}{2}\sigma^2 W^2 \frac{\partial^2 V}{\partial W^2} - rV + \tfrac{1}{2}\sigma_r^2\frac{\partial^2 V}{\partial r^2} + \rho\sigma\sigma_r W\frac{\partial^2 V}{\partial W \partial r} + (a-br)\frac{\partial V}{\partial r} \\ \approx\ & \frac{\partial V}{\partial t} + W(D_W^1 V + H_W^1)(R - \alpha I_M) + \tfrac{1}{2}\sigma^2 W^2 (D_W^2 V + H_W^2) - VR \\ & + \tfrac{1}{2}\sigma_r^2(VD_r^2 + H_r^2) + \rho\sigma\sigma_r W((D_W^1 V + H_W^1)D_r^1 + H_r^1) + (VD_r^1 + H_r^1)(aI_M - bR) \\ =\ & \frac{\partial V}{\partial t} + WD_W^1 V(R - \alpha I_M) + \tfrac{1}{2}\sigma^2 W^2 D_W^2 V + V(\tfrac{1}{2}\sigma_r^2 D_r^2 - R + D_r^1(aI_M - bR)) \\ & + \rho\sigma\sigma_r W D_W^1 V D_r^1 + H, \end{aligned}$$

where

$$H = \tfrac{1}{2}\sigma^2 W^2 H_W^2 + \tfrac{1}{2}\sigma_r^2 H_r^2 + H_r^1(aI_M - bR) + WH_W^1(R - \alpha I_M) + \rho\sigma\sigma_r W(H_W^1 D_r^1 + H_r^1)$$

is a constant matrix.

We will like to write this expression in the form $\mathcal{I}v + h$ where v and h are vector representations of V and H. To be more precise v corresponds to the column vector that has as first N lines the first column of V, as second N lines the second column of V and so on. If A is a $N \times N$ matrix and B is a $M \times M$ matrix then

$$AVB = Y \Leftrightarrow (B^T \otimes A)v = y,$$

where $\otimes$ is the Kronecker product and y is the column vector that represents Y. Using this relation we have that our differential equation becomes

$$\frac{\partial v}{\partial t} + \mathcal{I}v + h,$$

where

$$\begin{aligned}\mathcal{I} =& \frac{1}{2}\sigma^2(I_M \otimes W^2 D_W^2) + \left((\tfrac{1}{2}\sigma_r^2 D_r^2 - R + D_r^1(aI_M - bR))^T \otimes I_M\right) \\ &+ \left((R - \alpha I_M)^T \otimes (WD_W^1)\right) + \rho\sigma\sigma_r \left((D_r^1)^T \otimes WD_W^1\right)\end{aligned}$$

and h is the vector representation of H.

The Crank-Nicholson approach gives us

$$\frac{v^{n+1} - v^n}{\Delta t} = \frac{1}{2}\mathcal{I}v^n + \frac{1}{2}\mathcal{I}vn + 1 + h,$$

that is

$$v^n = \left(I_{NM} - \tfrac{1}{2}\Delta t\mathcal{I}\right)^{-1} \left((I_{NM} + \tfrac{1}{2}\Delta t\mathcal{I})v^{n+1} + \Delta th\right).$$

Observe that $I_{NM} - \frac{1}{2}\Delta t\mathcal{I}$ is a $NM \times NM$ matrix, that should be inversed. However since $\mathcal{I}$ does not depend on the time moment t or of the GMWB account A, this inversion should only be done once, that is, before all loops.

9.3 Numerical Examples

Discrete schemas where applied for the three variations for the dynamic strategy GMWB. In the three cases the discrete withdrawal model was considered. In all our calculations we took the following parameters : $r = 5\%$ and $\sigma = 20\%$ or $\sigma = 30\%$. Present short-period interest rate is lower than 5% but 5% is a standard amount to be used in Variable Annuities literature for numerical examples. The fact of using this amount allows us to compare our results to the ones of other researchers. This is in accordance with the spirit of a report that is mostly theorical and proposes instruments that could be used in any financial market environment. Volatilities of $\sigma = 20\%$ or $\sigma = 30\%$ are usual in the financial market. A $\sigma = 30\%$ can be expected in perturbed agitated financial markets like the present one. Let's see the results.

The basic case

For the basic case we constructed a table with the same structure as the one done by Dai Kwonk and Zong [17].

Contractual	Maturity	k = 5%		k = 10%	
rate, g	T = 1/g	$\sigma = 20\%$	$\sigma = 30\%$	$\sigma = 20\%$	$\sigma = 30\%$
4%	25,00	0,98%	2,03%	0,52%	1,25%
5%	20,00	1,19%	2,49%	0,66%	1,56%
6%	16,67	1,40%	2,96%	0,81%	1,88%
7%	14,29	1,58%	3,37%	0,94%	2,15%
8%	12,50	1,75%	3,77%	1,08%	2,45%
9%	11,11	1,94%	4,19%	1,22%	2,76%
10%	10,00	2,09%	4,54%	1,35%	3,02%

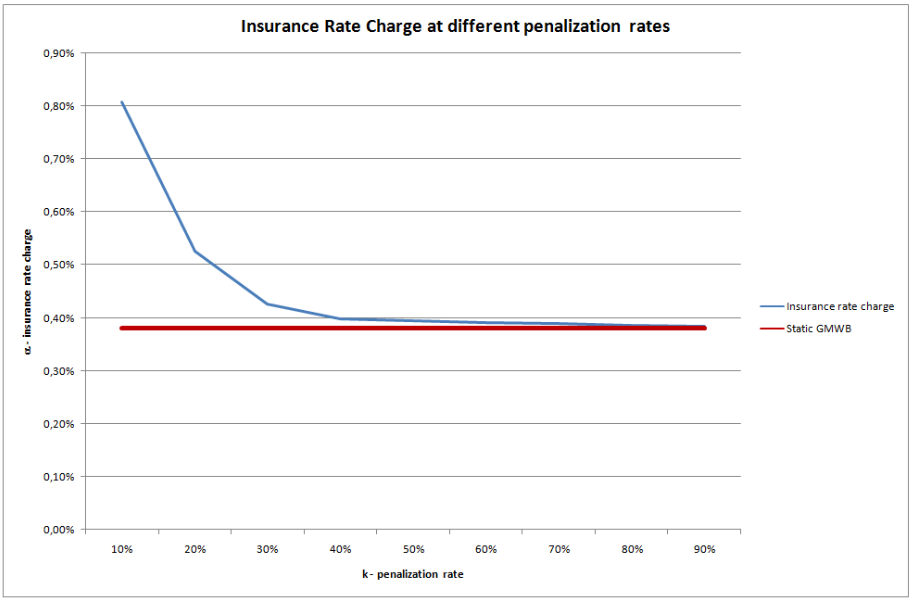
Our results are very similar to those of Dai Knok and Zong. The main difference withe their results is that we considered the discretized withdrawals while in their correspononding table the withdrawals are continous. Other difference are due to possible discretitation difference such as the grill construction. In our discretization the policyholder withdraws each 6 months, which in terms of a guarantee that lastes for several years gives values near to those the continous model, however a discrete withdrawal gives a guarantee that is less expensive. Comparing both table one can see that this differences costs between 2 and 10 basic points.

In the table we can observe that the insurance rate charge is larger in a more volatile situation. This result is in accordance with the fact that volatility affects possitively the price of financial options. We can observe that the step from $\sigma = 20\%$ to $\sigma = 30\%$ can double the insurance rate charge.

We observe as well that with a higher penalization rate k the GMWB value decreases, that is the required insurance rate charge is smaller. This is natural since with higher penalization the policyholder is restrained to use her surrender option. That is, since depasing the contractual withdrawal rate g becomes more expensive optimal situations to do so will be lesser and the option becomes less interesting. The optimal value bemcomes lesser and so the insruance rate charge α is also reduced.

9.4 The static-dynamic strategies relation

If the policyholder follows a static strategy the value of the GMWB is always less than that of the GMWB of a policyholder that follows a dynamic strategy, given that all other conditions are equal. This follows directly from the fact that the dynamic strategy is the optimal between all possible strategies in the sense of increasing the option value. Now according the the penalty value, the dynamic strategy value varies as seen in the following graph:



which was calculated for $G = 6\%$, $\sigma = 20\%$ and $r = 5\%$.

Observe that the value of the dynamic strategy converges to the static strategy value as $K \to 1$. Which is due to the fact that if the penalty $K = 100\%$ there is no interest in withdrawing more than the value specified in the contract. Since we have a discrete withdrawel, the static strategy for this withdrawl is just a few basic point beneath the dynamic strategy one.

Chapter 10

The Total Surrender GMWB

In the Dai Kwok Zong model the surrender behaviour is modelled as a result of partial surrender. However nowadays most of the GMWB products contain clauses that neutralise the partial surrender behaviour. That is, in many GMWB products when the policyholder surrender an amount over the withdrawal guaranteed rate, the policy characteristics (such as withdrawal amount guaranteed or insurance fee) change in such a manner that it this becomes equivalent to making a total surrender of the contract and buying a new contract with these new conditions.

These kinds of clauses are also present in the step-up guarantee. This guarantee changes the policy characteristics when the unit link is too favourable in relation to the GMWB product in order to re-establish the equilibrium. In fact if the unit link price rises fast and it becomes interesting to the policyholder to surrender, the step-up guarantees makes a re-adjustment of the policy terms in order to make the GMWB as interesting as the unit link and therefore to discourage the policyholder to surrender.

Under this logic it becomes relevant to model the GMWB product as a product where only a total surrender is possible. Implicitly it is acknowledge that partial surrender is possible but in it is supposed that the product is re-adjusted in such a manner that in terms of modelling it is enough to consider only the total surrender case.

The Dai Kwok Zong strategy to model partial surrender is to use the Hamilton-Jacobi-Bellman machinery. We will adapt their instruments to consider the total surrender case using an American option strategy.

In order to keep our results the best adjusted to the present market conditions we will use the following information as input parameters to the closed-formula models:

- Swap yield curve as at 31 December 2008.
- Asset mixes of 30% / 70%, 50% / 50% and 70% / 30%. On the 31^{st} may 2009

NAVA's repport one can see that in average the US market asset composition has 42% equity and 58% of other assets of fixed accoutns, allocation, bonds and money market. Variable Annuities products are sold on asset mixes, that is, they are not only composed of equity but also they are composed of bonds and therefore their volatility is less than the market volatility, which is now a day above 30%.

- The contractual rate in the GMWB guarantee will be taken from 4% to 10% which is the range of values used in the present market.
- We will use a mortality table by generation TGF05. We wuppose our policyholder is a 60 year old woman.

10.1 The basic model

Consider the value of a static GMWB at time t

$$H_t = E_t^Q\left[e^{-\int_t^T r_v dv} W_T \frac{S_x(T)}{S_x(t)} + \int_t^T e^{-\int_t^u r_v dv}\left(g\frac{S_x(u)}{S_x(t)} + W_u \frac{f_x(u)}{S_x(t)}\right) du\right],$$

where S_x is the survival function of a policyholder that has x year at $t = 0$. We will suppose that S_x is differentiable. Let f_x be the density function of mortality, W_t the value of the unit-link and g the fixed withdrawal rate. Let

$$V_t = H_t S_x(t) = E_t^Q\left[e^{-\int_t^T r_v dv} W_T S_x(T) + \int_t^T e^{-\int_t^u r_v dv}\left(g S_x(u) + W_u f_x(u)\right) du\right].$$

If the Hamilton-Jacobi-Bellman theorem is used with a singleton control space $\mathcal{A} = \{g\}$ the theorem becomes a generalized version of the Feymen-Kac theorem. In such a case we have that V_t satisffes the following PDE

$$\frac{\partial V}{\partial t} + \mathcal{L}V + gS_x(t) + W_t f_x(t) - rtV = 0$$

with final condition $V_T = W_T S_x(T)$. We have also that

$$\frac{\partial V}{\partial t} = S_x(t)\frac{\partial H}{\partial t} - H_t f_x(t) \qquad \text{and} \qquad \mathcal{L}V = S_x(t)\mathcal{L}H.$$

Therefore

$$S_x(t)\frac{\partial H}{\partial t} - H_t f_x(t) + S_x(t)\mathcal{L}H + gS_x(t) + W_t f_x(t) - r_t H_t S_x(t) = 0$$

or the equivalent

$$\frac{\partial H}{\partial t} + \mathcal{L}H + g - (H_t - W_t)\mu_x(t) - r_t H_t = 0,$$

where $\mu_x(t)$ is the force of mortality $\dfrac{f_x(t)}{S_x(t)}$ and with final condition $H_T = W_T$.

Please remark that this PDE is valid if r_t is a deterministic function but it is also valid if r_t is an stochastic process. The difference between this two cases is the form of the operator $\mathcal{L}$.

10.1.1 The In-the-moneyness of the GMWB

At every moment t, the policyholder might be interest in finishing her contract. H_t represents the value the contract has including all the information due to future payments of annuity g and to the future withdrawal of the unit-link W_s (with $s > t$) due to surrender, to death or to arrival to maturity. This is the value she gets if she keeps the contract. While W_t is what she gets if the contract is finished due to surrender or death.

We will say that the GMWB is in-the-money if $H_t > W_t$, at-the-money of $H_t = W_t$ and out-of-the-money if $H_t < W_t$. It only makes sense to surrender if the contract is out-of-the-money. In fact we will use this relation when we include a surrender behavior into the model. If surrender is included to the model the contact can never be out-of-the-money because in such a case is better to surrender. We can keep the PDE but at each time step establish $H_{t^-} = \max(W_t, H_{t^+})$. Eventhough this approach by lower bound is not necessarily optimal, this form of modeling is usual in American type options.

The effect of mortality on the contract is also tightly related to the in-the-moneyness of the GMWB. From the PDE we can observe that an increment of mortality in-creases the value of the GMWB if the contract is out-of-the-money while it reduces the value of the GMWB if the contract is in-the-money. This is a clear consequence of the fact that is better to keep the contract if it is in-the-money and mortality is a way out of the contract.

10.1.2 Including Surrender Charges

A surrender charge can easily be included in the total surrender GMWB model. The PDE presented previously is maintained. Buy the optimal surrender is done taking into account the price

$$H_{t^-} = \max(H_{t^+}, (1 - k)W_t)$$

where k is the surrender charge. Observe that no fee will be charged in case of death. Only the surrender behaviour is charged.

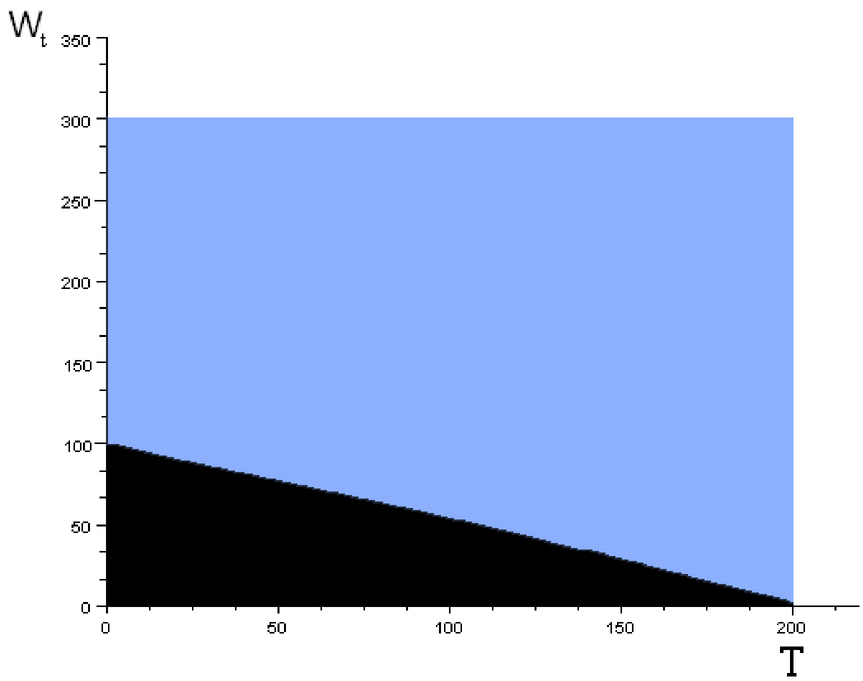
Our model supposes that surrender corresponds to the policyholder decision of partial surrender but it can also correspond to application of a step-up guarantee. Therefore this surrender charges should also be considered as a step-up fee.

10.2 Numerical Implementation

Using the previous model we calculated the insurance fee in basic points from an acturial pricing perspective. The fees obtained are the following.

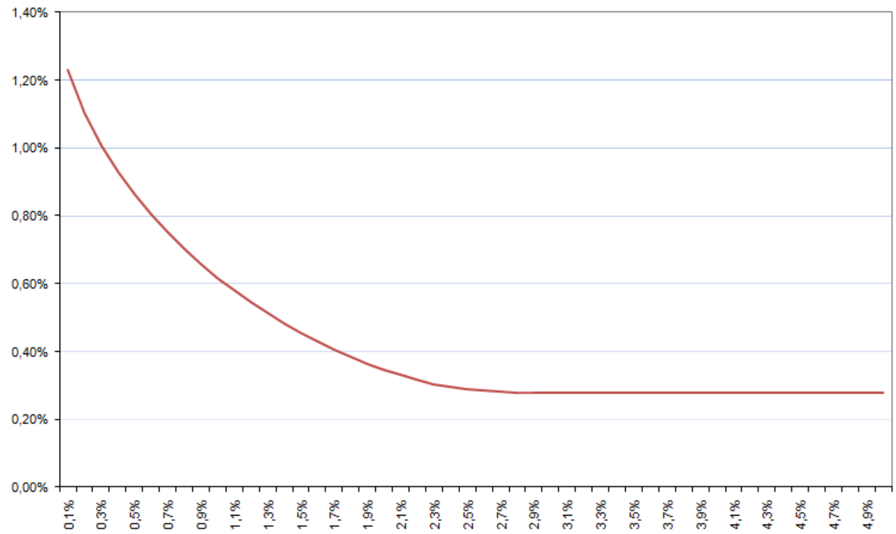
k = 0 bps	Equity / Bond composition			
Contractual rate (g)	30%	50%	70%	100%
0,04	5,808	59,699	171,845	437,657
0,05	11,184	91,224	246,986	607,428
0,06	18,948	128,815	330,958	787,832
0,07	28,228	168,555	416,319	964,894
0,08	40,624	215,955	513,717	1159,971
0,09	52,546	259,126	600,959	1331,738
0,10	68,404	312,304	705,003	1532,186

The following graph represent the optimal surrender behaviour for a GMWB guarantee without surrender charge.



In this example $W_0 = 100$. Observe that the optimal behaviour is almost a straight line from $(0, 100)$ to $(T, 0)$, that is, the policyholder optimal behaviour consists basically in surrender each time the W_t overpasses A_t where A_t is the GMWB account.

We also considered the case when a fee is charged to surrender. In the following graph we observe the effect of the surrender charge on the insurance fee from an actuarial pricing perspective.



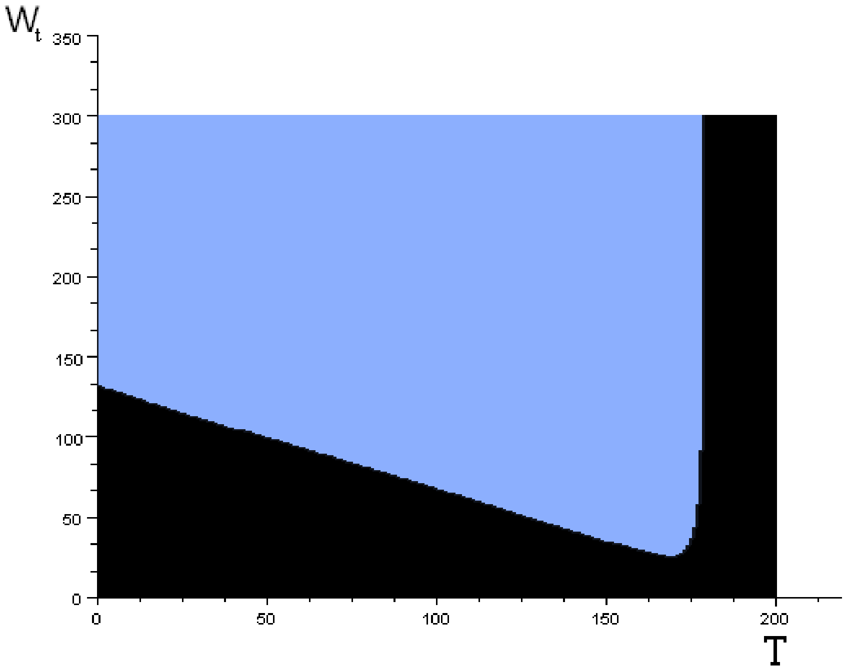
It is clear that a higher surrender charge will reduce the insurance fee. This is a quite logical relation. Now, it is important to compare this graph to the one we had calculated for the partial surrender in the Dai Kwonk Wong model. If graphs are compared it is quite clear that the surrender charge effect is much stronger in the total surrender case. While surrender charges that strongly reduce the behavioural range of the policyholder are too high in the partial surrender context, they are commercially attainable in the total surrender model.

On the next table you can observe the insurance fee in basic points for a total surrender GMWB with a 50 bps surrender charge.

k = 50 bps	Equity / Bond composition			
Contractual rate (g)	30%	50%	70%	100%
0,04	0,77	27,07	117,22	349,96
0,05	1,72	48,19	177,66	499,16
0,06	3,29	75,25	247,22	660,73
0,07	5,40	104,99	319,28	821,73
0,08	9,10	142,00	404,10	1000,12
0,09	14,49	176,12	477,82	1157,17
0,10	23,41	219,31	573,76	1353,01

These results are in basic points. It is clear that a higher level of risk in the equity / bond composition implies a higher price. Higher volatility usually increases the price of derivate products.

The following graph represent the optimal surrender behaviour for a GMWB guarantee without surrender charge.



Observe that at $t = 0$ the optimal surrender fronteir is stablished at a value higher than $W_0 = 100$ this is asily explained by the fact that by definition $H_0 = 100$ and the $\max(H_0, (1-k)W_0)$ has to be H_0 in a neighbourhood of this value. However the frontier descends as the time passes, this is explainable by the fact that there is a continous constant withdrawal and there is a lesse amount to be taken into account. At the end a huge change appears in the frontier, all of a sudden the frontier becomes quite high. To explain this sudden change it migh be better to consider the fontier at T. At T we have that $H_T = W_T$ so the $\max(H_T, (1-k)W_T)$ has to be H_T, that is it does not make sense to surrender. As k is higher there should be a neighbourhood to this phenomene, this neighbourhood is the sudden change that we observe in the frontier. The higher k is the earlier the jump will appear.

From this graph we can assert that for $k > 0$ there is a moment of time when it makes no more sense to surrender and the optimal strategy is to carry the guarantee to maturity.

Chapter 11

GMWB and QIS 4

On this Section we will apply some of the QIS4 Standard Methodology instruments to measure some of the risks of the GMWB product. The following risks will be considered:

- Interest rate risk
- Equity risk
- Mortality risk
- Longevity risk

Following the QIS4 standard methodology we will calculate the GMWB net present value after stressing each of these variables. In fact the methodology was the following:

1. The central scenario was used to calculate the insurance fee. Since we are working at an actuarial price level, the final NAV of this exercise was 0.
2. Using the insurance fee calculated at 1) we calculated the NAV of the GMWB product within each stress scenario. As the NAV in 1) is 0 then the NAV change is equal to the NAV in 2).

On each of our results we will take 100 € to be the initial lump sum paid by the policyholder, that is our nominal will be of a hundred.

11.1 Interest rate risk

QIS 4 standard methodology proposes a set of impacts to be applied to the interest rate curve. The relative movement changes considered are the following:

Maturity (years)	1	2	3	4	5	6	7	8	9	10
relative change UP	0,94	0,77	0,69	0,62	0,56	0,52	0,49	0,46	0,44	0,42
realtive change DOWN	-0,51	-0,47	-0,44	-0,42	-0,40	-0,38	-0,37	-0,35	-0,34	-0,34

Maturity (years)	11	12	13	14	15	16	17	18	19	20+
relative change UP	0,42	0,42	0,42	0,42	0,42	0,41	0,4	0,39	0,38	0,37
realtive change DOWN	-0,34	-0,34	-0,34	-0,34	-0,34	-0,33	-0,33	-0,32	-0,31	-0,31

These impacts affect the interest rate curve used in the model and it also affects the value of the bond part of the portfolio. The bond part of all the variable annuity net assets sub account in 2009 in the USA was composed as follows [26]:

	Billions of Dollars
Long-Term Corporate	1,30
Long-Term Government	1,00
Intermediate-Term Corporate	66,60
Intermediate-Term Government	27,10
Short-Term Corporate	5,50
Short-Term Government	0,90

Which means that more than 90% of the bond portfolio is intermediate-term. Accordingly we will suppose that the bond part of the portfolio of our example has a modified duration is of 3 years. Using this duration, the interest rate impact and the Equity/Bond composition we can calculate the initial value of the portfolio for each stress scenario.

The changes in the interest rate curve produce two stress scenarios: an interest rate UP scenario and an interest rate DOWN scenario. Let's first consider the effect of a plummet in the interest rate.

k = 0 pbs	Equity / Bond composition			
Contractual rate (g)	30%	50%	70%	100%
0,04	-0,28	-0,94	-1,55	-2,16
0,05	-0,33	-0,96	-1,51	-2,06
0,06	-0,35	-0,93	-1,42	-2,00
0,07	-0,34	-0,87	-1,33	-1,96
0,08	-0,34	-0,83	-1,32	-1,94
0,09	-0,32	-0,79	-1,28	-1,88
0,10	-0,32	-0,77	-1,29	-1,89

k = 50 bps	Equity / Bond composition			
Contractual rate (g)	30%	50%	70%	100%
0,04	-0,45	-1,60	-2,31	-2,92
0,05	-0,55	-1,58	-2,19	-2,70
0,06	-0,64	-1,50	-2,02	-2,53
0,07	-0,67	-1,37	-1,85	-2,39
0,08	-0,68	-1,29	-1,76	-2,34
0,09	-0,64	-1,19	-1,67	-2,23
0,10	-0,61	-1,14	-1,62	-2,19

It must first be observed that a decrease in the value of the interest rate curve implies a loss to the insurer. Insurance fee was calculated on the hypothesis of a certain interest rate curve, that of 31 December 2008. The GMWB contract guarantees a minimum set of payments, when the interest rate drops these payments become more expensive for the insurer to pay.

It is quite clear that this negative impact of a drop in the interest rate curve is stronger if the unit link has a larger part in equity. Both risk-free and equity parts of the portfolio are sensible to interest rate movements but the equity part has volatility.

More interesting is the relation between the contractual rate (g) and the drop in interest rate. Observe that the effect is opposite if the unit link is composed entirely of equity or not. A high volatility the GMWBs will be more affected by an interest rate drop if it has a small contractual rate, while if it has a small volatility it is the contracts that have a small contract rate that are less affected. The intricate relations between these variables are difficult to grasp but at least one observation can be made. The contracts with smaller g are the ones which have a longer maturity, remember that the maturity T is equal to $\frac{1}{g}$. Longer periods of time permit high volatilities to make an important effect. This might explain the cross relation between these variables.

Observe now that the impact of the interest rate drop is stronger when a surrender charge is included. In fact what is happening is that it becomes in itself less interesting for the policyholder to surrender her contract when the interest rate has dropped but the insurance fee α is higher on the no surrender charge contract. In other words the contract that has a surrender charge becomes penalized as there is less surrender while having a lower insurance fee.

Now we are going to consider the UP stress scenario. Consider first the case when there is no surrender charge:

k = 0 bps	Equity / Bond composition			
Contractual rate (g)	30%	50%	70%	100%
0,04	0,00	0,00	0,00	0,00
0,05	0,00	0,00	0,00	0,00
0,06	0,00	0,00	0,00	0,00
0,07	0,00	0,00	0,00	0,00
0,08	0,00	0,00	0,00	0,00
0,09	0,00	0,00	0,00	0,00
0,10	0,00	0,00	0,00	0,00

In the table for a no surrender charge contract we optained a zero result no matter the policy characteristics. This result might seem very strange at first glance. But the reason for this is very simple and quite enlightening.

In fact if what happens is that the optimal surrender of the policyholder profits from all the possible stress gains the insurer could get from the GMWB contract. Surrender is done at each gain opportunity since there is no surrender charge.

Even more enlightening is what happens when there is in fact a surrender charge.

k = 50 bps	Equity / Bond composition			
Contractual rate (g)	30%	50%	70%	100%
0,04	0,104	0,478	0,490	0,500
0,05	0,167	0,480	0,490	0,500
0,06	0,238	0,480	0,490	0,500
0,07	0,294	0,480	0,490	0,500
0,08	0,326	0,480	0,490	0,500
0,09	0,325	0,480	0,490	0,500
0,10	0,329	0,480	0,490	0,500

The NAV change is upper bounded by the surrender charge k times the shocked initial lump sum. In our examples le lump sum is 100 €. Once shocked, the inital lump sums are 95 €, 96 €, 98€ and 100€. The NAVs obtained correspond to these initial lump sums times the surrender charge of 50 bps. The following assertion can be stated:

No matter the stress scenario, if the policyholder behaviour is optimal the NAV change is bounded by the surrender charge times the initial shocked lump sum.

11.2 Equity Risk

In what follows and following the QIS 4 standard model we study the NAV of the GMWB given a drop of 32% of the equity value. Observe that if the equity decreases 32% this does not mean that the unit link portfolio will do the same, in fact only the risky part of this portfolio will shut. The unit link shocks are the following for the different equity/bond compositions

Equity Part	UL shock
100%	32%
70%	22%
50%	16%
30%	10%

The following table represents the NAV change for a lump sum of 100 €:

k = 0 bps	Equity / Bond composition			
Contractual rate (g)	30%	50%	70%	100%
0,04	-0,1321	-1,0550	-2,7211	-6,7434
0,05	-0,2506	-1,5765	-3,7812	-8,8664
0,06	-0,4158	-2,1617	-4,8696	-10,8578
0,07	-0,6077	-2,7497	-5,8950	-12,6019
0,08	-0,8521	-3,4044	-6,9656	-14,3070
0,09	-1,0829	-3,9742	-7,8580	-15,6559
0,10	-1,3721	-4,6205	-8,8210	-17,0499

k = 50 bps	Equity / Bond composition			
Contractual rate (g)	30%	50%	70%	100%
0,04	-0,2379	-1,6262	-3,4772	-7,5291
0,05	-0,4300	-2,1997	-4,5268	-9,5546
0,06	-0,6828	-2,8060	-5,5747	-11,4370
0,07	-0,9565	-3,3880	-6,5425	-13,0798
0,08	-1,2692	-4,0205	-7,5393	-14,6945
0,09	-1,5225	-4,5564	-8,3814	-15,9773
0,10	-1,8275	-5,1627	-9,2591	-17,2874

It must first be notice the high impact a equity plummet can have on an GMWB guarantee. On our model the insurer uses a percentage of the equity in order to hedge the guarantee's risk. However if the equity drops the amount collected in order to produce the hedge is reduced and the probability of having an exercise of the guarantee increases. To avoid this pitfall some insurers charge an amount that do not depend on the equity level.

The impact of equity drop is higher on higher volatility contracts. To elements should be considered. First, as the unit link is composed of a high percentage of equity, the equity drop has a higher impact. Second, as with the drop of interest rates, higher volatility contracts get higher impacts on variable changes.

When the equity losses value the contracts with higher contractual rate (g) are the most affected. This can be explained by the fact that the probability of having to exercise the guarantee increases. The higher the contractual rate the more the insurer will have to take in charge in case of a diminish of the unit link.

At last it is important to observe that the contracts with surrender charge where more affected by the equity drop. As with the interest rate drop case the explanation is that this change in the variable implies a reduction in the probability to surrender. That is policyholder behaviour between surrender and no surrender charge contracts becomes more similar while the insurance charges α are smaller for the surrender charge contract.

11.3 Mortality/Longevity risk

Following the QIS4 Standard model methodology we stressed the mortality in by increasing the mortality rates by 10% for each age and we constructed a longevity scenario by decreasing the mortality rates by 25% for each age.

The following are the results for the longevity case:

k = 0 bps	Equity / Bond composition			
Contractual rate (g)	30%	50%	70%	100%
0,04	0,0000	-0,0004	-0,0009	-0,0015
0,05	-0,0001	-0,0004	-0,0009	-0,0015
0,06	-0,0001	-0,0005	-0,0010	-0,0015
0,07	-0,0001	-0,0005	-0,0011	-0,0017
0,08	-0,0001	-0,0006	-0,0011	-0,0018
0,09	-0,0001	-0,0006	-0,0011	-0,0017
0,10	-0,0001	-0,0006	-0,0012	-0,0018

k = 50 bps	Equity / Bond composition			
Contractual rate (g)	30%	50%	70%	100%
0,04	-0,0001	-0,0011	-0,0016	-0,0015
0,05	-0,0001	-0,0010	-0,0015	-0,0017
0,06	-0,0001	-0,0009	-0,0013	-0,0019
0,07	-0,0002	-0,0008	-0,0011	-0,0014
0,08	-0,0002	-0,0008	-0,0010	-0,0007
0,09	-0,0003	-0,0007	-0,0009	-0,0022
0,10	-0,0003	-0,0008	-0,0013	-0,0005

The following are the results for the mortality case:

k = 0 bps	Equity / Bond composition			
Contractual rate (g)	30%	50%	70%	100%
0,04	0,0000	0,0000	0,0000	0,0000
0,05	0,0000	0,0000	0,0000	0,0000
0,06	0,0000	0,0000	0,0000	0,0000
0,07	0,0000	0,0000	0,0000	0,0000
0,08	0,0000	0,0000	0,0000	0,0000
0,09	0,0000	0,0000	0,0000	0,0000
0,10	0,0000	0,0000	0,0000	0,0000

k = 50 bps	Equity / Bond composition			
Contractual rate (g)	30%	50%	70%	100%
0,04	0,0000	0,0004	0,0007	0,0006
0,05	0,0000	0,0004	0,0006	0,0007
0,06	0,0001	0,0004	0,0005	0,0007
0,07	0,0001	0,0003	0,0004	0,0006
0,08	0,0001	0,0003	0,0004	0,0004
0,09	0,0001	0,0003	0,0004	0,0009
0,10	0,0001	0,0003	0,0005	0,0002

Observe that NAV change is negative in the longevity risk and positive for the mortality risk. A simple explanation is due: as more policyholders survive up to the maturity of the contract more guarantees must be paid by the insurer.

However these values are immaterial. One can venture to say that the surrender behavior, in our model, takes most of the policyholder exists and not much involuntary exists remain.

As results are very small, some value oscilation is due to numerical implementation and approximation. We will restrain ourselves of further analysis.

11.4 Equity and Longevity Cross Effect

Even though the QIS4 do not propose to consider the effect of simultaneously having an equity stress and a mortality stress we will consider this case as well. The following results are the change in NAV when both equity and longevity are shocked:

k = 0 bps	Equity / Bond composition			
Contractual rate (g)	30%	50%	70%	100%
0,04	-0,1324	-1,0573	-2,7254	-6,7496
0,05	-0,2511	-1,5790	-3,7856	-8,8726
0,06	-0,4164	-2,1643	-4,8740	-10,8638
0,07	-0,6084	-2,7524	-5,8994	-12,6077
0,08	-0,8529	-3,4072	-6,9699	-14,3126
0,09	-1,0838	-3,9770	-7,8622	-15,6612
0,10	-1,3731	-4,6233	-8,8252	-17,0549

k = 50 bps	Equity / Bond composition			
Contractual rate (g)	30%	50%	70%	100%
0,04	-0,2383	-1,6289	-3,4819	-7,5356
0,05	-0,4305	-2,2024	-4,5315	-9,5610
0,06	-0,6834	-2,8087	-5,5793	-11,4432
0,07	-0,9572	-3,3908	-6,5469	-13,0856
0,08	-1,2701	-4,0233	-7,5436	-14,7002
0,09	-1,5234	-4,5592	-8,3856	-15,9827
0,10	-1,8284	-5,1654	-9,2632	-17,2925

These results do not seem very different to those obtained when only the equity was shocked. As longevity is included the value decreases. However observe what happens if we subtract the equity shocked tables from the previous tables, we obtain:

k = 0 bps	Equity / Bond composition			
Contractual rate (g)	30%	50%	70%	100%
0,04	-0,0004	-0,0023	-0,0043	-0,0061
0,05	-0,0005	-0,0024	-0,0044	-0,0061
0,06	-0,0006	-0,0026	-0,0044	-0,0060
0,07	-0,0007	-0,0027	-0,0044	-0,0058
0,08	-0,0008	-0,0028	-0,0044	-0,0056
0,09	-0,0009	-0,0027	-0,0042	-0,0053
0,10	-0,0010	-0,0028	-0,0041	-0,0051

k = 50 bps	Equity / Bond composition			
Contractual rate (g)	30%	50%	70%	100%
0,04	-0,0004	-0,0027	-0,0047	-0,0065
0,05	-0,0005	-0,0027	-0,0046	-0,0063
0,06	-0,0006	-0,0028	-0,0046	-0,0061
0,07	-0,0007	-0,0028	-0,0044	-0,0059
0,08	-0,0008	-0,0028	-0,0044	-0,0056
0,09	-0,0009	-0,0027	-0,0042	-0,0053
0,10	-0,0010	-0,0027	-0,0041	-0,0051

These values are four or five times more important that those obtained when only longevity was considered. The risk of longevity increases in an important manner in the presence of equity risk. It seems clear that the addition of both risks is far from representing the effect of the mutual risk presence.

Chapter 12

GLWB valuation

GMWBs for-life contracts can be approached from a Monte-Carlo perspective as is done by Holz, Kling and Russ [24]. On this Section we will adapt the GMWB guarentee develepments of the previous Section to the "for-life" case.

The GLWBs can be considered as very long maturity GMWBs. One important difference must be high-lighted with respect to the way GMWBs have been modeled in this report: in GMWBs we have taken the maturity T to be equal to $\frac{\omega_0}{G}$ where G is the amount the policyholder is allowed to withdrawal without a penalization, this can not longer be the case for GLWBs.

In this Section we will construct a static strategy GLWB as a portfolio of GMWBs. In order to do so we must consider a GMWB where T is unrelated to G. Thes guarantee will be called the general-GMWB. It follows from equation 7.3 that

$$W_T = e^{(r-\alpha-\frac{1}{2}\sigma^2)T+\sigma B_T} \max\left(0, \omega_0 - G\int_0^T e^{-(r-\alpha-\frac{1}{2}\sigma^2)t-\sigma B_t}dt\right) \tag{12.1}$$

and following the same argument that is followed in Section 8.1 we have that the value of the general-GMWB is

$$V[\text{general-GMWB}] = e^{-rT}E_Q\left[\frac{1}{Y_T}\left[1 - g\int_0^T Y_t dt\right]^+\right] + \frac{g}{r}(1-e^{-rT}).$$

Observe that the only difference with equation 7.9 is that we no longer have that $g = \frac{1}{T}$. Now we will concentrate in finding the value of

$$E_Q\left[\frac{1}{Y_T}\left[1 - g\int_0^T Y_t dt\right]^+\right].$$

This option turns out to be also Quantum Asiatic but with another strike if written in the following manner

$$gT\ E_Q\left[\frac{1}{Y_T}\left[\frac{1}{gT}-\frac{1}{T}\int_0^T Y_t dt\right]^+\right],$$

where

$$Y_t = e^{-(r-\alpha-\frac{1}{2}\sigma^2)t-\sigma B_t}.$$

Now a GLWB guarantee is equivalent to a portfolio of general-GMWB weighted by the percentage of policyholders that dies at each maturity year. That is,

$$GLWB = \int_0^\infty \text{general-GMWB}(t,\sigma,r,g)dF_x(t),$$

where $F_x()$ is the cumulative distribution function of a policyholder of age x at the time of valuation and t is the maturity of the general-GMWB guarantee.

Calculation by Differential Equation

This option can be valued with a partial differential equations approach very similar to the one used for GMWBs with static strategy. Let

$$X_t = \frac{1}{gT} - 1 + \int_0^t g_s dY_s = \frac{1}{gT} + g_t Y_t - \frac{1}{T}\int_0^t Y_s ds$$

with $g_t = \left(\frac{t-T}{T}\right)$. Let $Z_t = \frac{X_t}{Y_t}$, then

$$E_Q\left[\frac{1}{Y_T}\left[\frac{1}{gT}-\frac{1}{T}\int_0^T Y_t dt\right]^+\right] = E_Q\left[[Z_T]^+\right]$$

and

$$\begin{aligned}
dZ_t &= \frac{1}{Y_t}dX_t + X_t d\left(\frac{1}{Y_t}\right) + dX_t d\left(\frac{1}{Y_t}\right) \\
&= \frac{1}{Y_t}g_t dY_t + X_t\left(\frac{1}{Y_t}(r-\alpha)dt + \frac{1}{Y_t}\sigma dB_t\right) - g_t\sigma^2 dt \\
&= (r-\alpha)(Z_t - g_t)dt + \sigma(Z_t - g_t)dB_t.
\end{aligned}$$

Therefore by the thorem of Feyman-Kac theorem we have that $V(t,z) = E_t^Q\left[[Z_T]^+\right]$ satisfies the following differential equation

$$\frac{\partial V}{\partial t} + (r-\alpha)(Z_t - g_t)\frac{\partial V}{\partial Z} + \frac{1}{2}\sigma^2(Z_t - g_t)^2\frac{\partial^2 V}{\partial Z^2} = 0,$$

with teminal condition $V(T, Z) = [Z]^+$. Observe that the general-GMWB EDP is exactly the same EDP that for the $T = \frac{1}{g}$ case. The only difference between the general-GMWB system of equations and the GMWB system of equations is the value of Z_t at 0, that is, in the general case we have that

$$Z_0 = \frac{1}{gT} - 1,$$

while on the GWBDs we have $Z_0 = 0$. This implies that the same difference scheme that was used for the GMWB might be applied.

However the calculation should be done for as many periods as those in xhich the policyholder has a non-negative probability of being alife, this in addition to the fact that a precise EDP requires a fine grid implies that the algorithm is very slow. We have already shown that the lower bound approximation is in fact very near to the real value so we suggest to use the approximation, which is quite fast in computational terms to approach the value of the general-GMWBs. On the next Section we will show how.

Approximation by Lower Bounds

From the calculations of lower bonds made in Section 8.1 it follows that if

$$m_t = 3\frac{t}{T^3}\left(T - \frac{t}{2}\right) \qquad v_t^2 = t - \frac{3}{T}\left(t - \frac{t^2}{2T}\right)^2$$

and

$$E^{Q_S}[Y_t|Z] = e^{-(r-\alpha+\sigma^2)t - \sigma m_t Z + \frac{1}{2}\sigma^2 v_t^2},$$

then

$$E^{Q_S}\left[\left[1 - g\int_0^T E^{Q_S}[Y_t|Z]dt\right]^+\right] \leq E^{Q_S}\left[\left[1 - g\int_0^T Y_t dt\right]^+\right],$$

where Q_S is the probability that take S as numeraire. This lower bound is a good approximation to the function value. And so the general-GMWB is worth

$$\begin{aligned}
V[\text{general-GMWB}] =& e^{-rT} E_Q\left[\frac{1}{Y_T}\left[1-g\int_0^T Y_t dt\right]^+\right] + \frac{g}{r}(1-e^{-rT}) \\
\approx& e^{-\alpha T} E^{Q_S}\left[\left[1-g\int_0^T E^{Q_S}[Y_t|Z]dt\right]^+\right] + \frac{g}{r}(1-e^{-rT}) \\
=& \frac{e^{-\alpha T}}{\sqrt{\frac{2}{3}\Pi T^3}} \int_{-\infty}^{\infty}\left[1-g\int_0^T e^{-(r-\alpha+\sigma^2)t-\sigma m_t z+\frac{1}{2}\sigma^2 v_t^2} dt\right]^+ e^{-\frac{3}{2}\frac{z^2}{T^3}} dz \\
&+ \frac{g}{r}(1-e^{-rT}),
\end{aligned}$$

where only numerical integrations are required.

Conclusion

On the present report we have presented a general glance of the variable annuities. We have introduced the product and its standard guarantees. As well we presented some general figures on the markets and on the possible future development of these products. Our main focus of attention was the GMWBs guarantee. In particular we have focused on the study of the effect of optimal surrender on this guarantee. However, the other guarantees where also briefly treated.

We have chosen to approach the variable annuities from a non-simulation point of view, that is from an "analytical" point of view. From a practical point of view it is normally easier to follow a Monte-Carlo approach. In fact not to follow a Monte-Carlo approach usually implies several simplifications in the modeling. However our choice of an analytical approach is well grounded. Not only it allowed us to examine the effects of passing from a continuous model to a discrete one in GMDBs but mainly it allowed us to explore the effects of optimal surrender in GMWBs. The problem of optimal behavior is better suited from an analytical approach. In fact the Monte-Carlo approach follows a past-to-future path but optimality supposes in fact to choose the best moment and amount by taking into account the future possible outcomes. That is, it is better fitted by a future-to-past perspective. Amongst the analytical approaches there are some that follow this direction. In particular the PDE approaches are very well suited for optimality problems.

The first Part of the report was dedicated to the GMDB, GMAB and GMIB guarantees. Our presentation was general the basics on value calculation of these products where presented and some numerical examples.

A second Part was dedicated to GMWBs and in a lesser manner to GLWB. Three general approaches where taken with respect to the GMWBs. First it was considered the case where no surrender was done by the policyholder. This case leaded us to a very keen relation between GMWBs and financial derivates: GMWBs are strongly related to Asian options. The no-surrender case decomposes the GMWBs into a quanto-asiatic option and a fixed annuity. Therefore the valuation of the no-surrender GMWB was inspired on Asiatic option techniques. We presented three of them: by approximation of the sum of log-normal variables, by a lower bound and by PDEs. The approximation of the sum of -log-normal variables produces a solution very easy to implement and that showed in our examples not to be too misleading, however as being an approximation it must be treated with caution, in particular in extreme parameter configurations. The lower bound showed to be a very good approximation, however from a prudent pricing perspective it would be better to have an upper bound. The PDE is without doubt the most precise approach, however it is as well the harder to implement.

The second general approach applied to the GMWBs was to consider optimal partial surrender. This approach is treated with the help of the Hamilton-Jacobi-Bellman machinery. Two surrender charges were taken into account 5% and 10%. We observed that the insurance charge was larger for the smaller surrender charge. As well we observed that for high volatility the insurance charge is greater. We compared the insurance charges with different values of surrender charge and observed that the insurance charge dropped vividly as the surrender charge increased up to joining the value of those obtained for a no-surrender model.

The third general approach consisted in allowing only total surrender. This approach is related to the family of present contracts in which the contract characteristics are changed if the policyholder makes a partial surrender, this changes usually are equivalent to making a total surrender and then buying a new contract under new characteristic of less value. The approach is also related to the step-up type of guarantee. In order to model this approach we used instruments borrowed from the American-Asian options. As a result we established a PDE who's values where adjusted from the future to the past, according to the optimal surrender behavior. Two surrender charges were considered: a surrender charge of 0 bps and a surrender charge of 50 bps. Observe that the surrender charges are much smaller than those considered for the partial surrender model. We observed that the insurance fees were higher if there was a higher volatility and smaller for the higher surrender charge. We also observed that the insurance fee drop rapidly if we increase the surrender charge up to establishing in a no-surrender value. For this Part of the model we incorporated an interest rate curve, while up to this moment we had supposed a flat forward curve, and we also incorporated a mortality curve. We observed the optimal surrender behavior. This behavior is almost linear when there is no surrender charge while a more sophisticated form appears in the presence of a surrender charge. This form implies a non-surrender zone around the lump value at the beginning of the contract and also a no-surrender zone for all contract values at the end of the contract. This to zones correspond to the fact that at the beginning the contract is at a fair price and surrender would imply losing the surrender charge and that at the end of the contract there is always a payment of what is left in the unit-link and so it makes no sense to surrender just before.

Once the three general approaches were presented we considered some of the risks associated to this product. To do so we took the third approach as the basic approach and we applied the QIS4 standard methodology to measure the interest rate risk, the equity risk, the mortality risk and the longevity risk. Some general principles where observed. In fact we observed that the NAV change was bounded by the surrender charge times the initial shocked lump sum. This is a consequence that once this limit is crossed the policyholder surrenders taking the remaining gain. We also observed that the NAV change was negative when there was a drop in the interest rate, a drop in the equity or a reduction of mortality. It is clear that the first reduces the value of the unit-link, the second has an impact on the feasibility of paying the withdrawals while the third permits more policyholders to arrive to the final of the contract where the guarantee takes place. In general we observed that the effect of volatility was to increase the NAV. Now, the presence of a surrender charge has an affect similar to that of volatility: if the NAV change is positive there is a higher gain, if the NAV change is negative there is a higher loss. The presence of a surrender charge permits the insurer to charge a lower insurance charge, if the market goes down then there is not much need of a the surrender charge while there is a lower insurance charge and if the market go up then the presence of the surrender charge discourages the policyholder to surrender leaving a gain for the insurer. At last we considerd the case of simial longevity and equity shock and we observed that the longevity equity is aplified by the presence of the equity shoch.

In the last Section we presented some ideas on the application of the GMWB techniques into the GLWB guarantee. Mainly the point is to disaggregate the relation between contractual rate and maturity while maintaining the advantages of the techniques just developed for the GMWB. However our main focus of attention was the GMWB and we did not pursue a further analysis of the GLWB guarantee.

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Appendix

In this Section the mathematical background to follow most of the demonstrations will be presented. A brief introduction to stochastic calculus is due and the principles of stochastic control and in paticular the Hamilton-Jacobi-Bellman equation are presented.

.1 Basic Definitions

The present Subsection will only present the basic stochastic definitions and theorems required to introduce the stochastic control. A good introduction to the stochastic calculus is Karatzas and Shreve's book [28]. A good short introduction can be found in Karatzas' notes [27]. For a more financial approach a good source is Shreve's book [42].

.1.1 Some Definitions

A σ-albegra on a set Ω is a collection of subsets of Ω such that the empty set is included, and is stable under complementation and enumerable union. It can be shown that for every collection of subsets $\mathcal{A}$ of Ω there exists a unique σ-algebra that is the smallest σ-algebra that contains $\mathcal{A}$, we denote this σ-algebra as $\sigma(\mathcal{A})$. We will denote $\mathbb{R}$ the set of the real numbers with σ-algebra σ(all the open subsets of the real numbers). A mesurable function $f : A \to B$ is a function from the set A with σ-algebra $\mathcal{A}$ to the set B with σ-algebra $\mathcal{B}$ such that if $C \in \mathcal{B}$ then $f^{-1}(C) \in \mathcal{A}$. We denote $\sigma(f)$ the smallest σ-algebra in A such that f is a mesurable function for a given $(B, \mathcal{B})$. A random variable is a mesurable function $Y(\omega) : \Omega \to \mathbb{R}$, defined on a probability space $(\Omega, \mathcal{F}, P)$. A stochastic process is a family of random variables $X = \{X_t; 0 \leq t \leq \infty\}$. For every $\omega \in \Omega$, the function $t \to X_t(\omega)$ is called the *sample path* or trajectory of the process. We will denote $\mathcal{F}_t^X = \sigma(X_s, 0 \leq s \leq t)$ the smallest σ-algebra such that all functions in the family $\{X_s, 0 \leq s \leq t\}$ are measurable. A set of σ-algebras $\{\mathcal{F}_t; 0 \leq t < \infty\}$ such that $\mathcal{F}_s \subset \mathcal{F}_t$ for all $0 \leq s \leq t < \infty$ is called a filtration. If $\mathcal{F}_t^X \subseteq \mathcal{F}_t$ for all $t \geq 0$, we say that X is adapted to the filtration $\mathcal{F}_t$.

.1.2 Brownian Motion

Brownian motion is a strochastic process $\{B_t; 0 \leq t \leq \infty\}$ that satisfies

1. $B_0 = 0$
2. $B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}}$ are independent, for every $n \geq 1$ and $0 = t_0 < t_1 < ... < t_n < \infty$.
3. $B_t - B_s \sim \mathcal{N}(0, \sigma^2(t-s))$, for every $0 < s < t < \infty$.
4. B_t has continous sample paths.

It can be shown that this process exists. In fact is one of the most important processes in financial mathematics. When $\sigma = 1$ we say that it is a *standard* Brownian Motion.

If $B^{(1)}, ..B^{(d)}$ are d independent, standard Brownian motions, the vector-valued process $B = (B^{(1)}, ..B^{(d)})$ is called a standard Brownian motion in $\mathbb{R}^d$.

.1.3 Stochastic Integration

We will now brefly define the stochastic integral with respect to a Brownian motion. Consider a Brownian motion $\{B_t\}$ adapted to a given filtration $\{\mathcal{F}_t\}$; for an adapted process X we would like to define the stochastic integral

$$\int_0^t X_s dB_s.$$

We will first define this integral with respect to a *simple* process. A process X is called *simple* if there exists a partition $0 = t_0 < t_1 < ... < t_n < t_{n+1} = T$ such that $X_t(\omega)$ is constant in t for each subinterval $[t_j, t_{j+1})$ and all $\omega \in \Omega$. We define

$$\int_0^t X_s dB_s := \sum_{j=0}^{m} X_{t_j}(B_{t_{j+1}\wedge t} - B_{t_j}) \qquad t_m < t \leq t_{m+1}.$$

Now we will like to define this integral for a more general set of processes. Let Γ be the set of the $\mathcal{F}_t$-adapted right-continous with left limit processes θ_t such that

$$E\left[\int_0^\infty \theta_t^2 dt\right] < \infty.$$

Define $||\theta||^2 = E\left[\int_0^\infty \theta_t^2 dt\right]$. Then the set Γ with $||\cdot||$ as norm is a complete space and we can define our integral for all elements of this space. It is easy to see that all simple functions are in Γ and it can be shown that for every function $\theta \in \Gamma$ there exists a series of simple functions $\theta^{(n)}$ such that $\theta^{(n)} \xrightarrow[L^2([0,+\infty))]{n\to\infty} \theta$. Then

$$\int_0^t \theta_s^{(n)} dB_s \xrightarrow[L^2(\Omega)]{n\to\infty} \int_0^t \theta_s dB_s$$

is a good definition.

This definition can be generalized to an integral with respect to other stochastic processes. We will not require this generalization for this report.

.1.4 Itô's lemma

An Itô process is a stochastic process X_t that can be represented as

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dB_s,$$

where X_0 is nonrandom and b_t and σ_t are adapted stochastic processes.
Let $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^{1\times 2}$ function with bounded derivates. Then the following relation is true for all itô processes and is called Itô's lemma.

$$\begin{aligned} f(t, X_t) = f(0, X_0) + \int_0^t f_t(s, X_s)ds + \int_0^t f_x(s, X_s)b_s dB_s + \int_0^t f_x(s, X_s)\sigma_s ds \\ + \frac{1}{2}\int_0^t f_{xx}(s, X_s)\sigma_s^2 ds. \end{aligned}$$

.1.5 An itô's lemma application

The following theorem corresponds to Karatzas et Shreve's [28] page 361 exercice. It is a good illustration of the Itô's lemma and it we will us this result for the static strategy of the GMWBs.

Theorem .1.1. Let $X(t)$ be the stochastique process that is the solution to

$$dX(u) = (a(u) + b(u)X(u))du + (\gamma(u) + \sigma(u)X(u))dB(u),$$

where $B(u)$ is a Brownian motion with respect to a filtration $\mathcal{F}(u)$, $u \geq 0$, and $a(u), b(u), \gamma(u), \sigma(u)$ are processes adapted to this filtration. Let $t \geq 0$ and $x \in \mathbb{R}$ the time and initial value of the process. Let

$$\begin{aligned} Z(u) &= exp\left\{\int_t^u \sigma(s)dB(s) + \int_t^u \left(b(s) - \frac{1}{2}\sigma^2(s)\right) ds\right\} \\ Y(u) &= x + \int_t^u \frac{a(s) - \sigma(s)\gamma(s)}{Z(s)} ds + \int_t^u \frac{\gamma(s)}{Z(s)} dB(s) \end{aligned}$$

then $X(u) = Y(u)Z(u)$.

Proof. First, $Y(0)Z(0) = x \cdot 1 = x$ then the initial value is fulfilled. Also,

$$\begin{aligned} dZ(u) &= d\left(exp\left\{ \int_t^u \sigma(s)dB(s) + \int_t^u \left(b(s) - \frac{1}{2}\sigma^2(s) \right) ds \right\} \right) \\ &= Z(u)\left(\left(b(u) - \frac{1}{2}\sigma^2(u) \right) du + \sigma(u)dB(u) + \frac{1}{2}\sigma^2(u)du \right) \\ &= \sigma(u)Z(u)dB(u) + b(u)Z(u)du, \qquad u \geq t, \end{aligned}$$

$$\begin{aligned} dY(u) &= d\left(x + \int_t^u \frac{a(s) - \sigma(s)\gamma(s)}{Z(s)} ds + \int_t^u \frac{\gamma(s)}{Z(s)} dB(s) \right) \\ &= \frac{a(u) - \sigma(u)\gamma(u)}{Z(u)} du + \frac{\gamma(u)}{Z(u)} dB(u), \qquad u \geq t. \end{aligned}$$

An so

$$\begin{aligned} dX(u) =& d(Y(u)Z(u)) \\ =& Y(u)dZ(u) + Z(u)dY(u) + dZ(u)dY(u) \\ =& Y(u)Z(u)(\sigma(u)dB(u) + b(u)du) + Z(u)\left(\frac{a(u) - \sigma(u)\gamma(u)}{Z(u)} du + \frac{\gamma(u)}{Z(u)} dB(u) \right) \\ &+ Z(u)(\sigma(u)dB(u) + b(u)du)\left(\frac{a(u) - \sigma(u)\gamma(u)}{Z(u)} du + \frac{\gamma(u)}{Z(u)} dB(u) \right) \\ =& \sigma(u)X(u)dB(u) + b(u)X(u)du + (a(u) - \sigma(u)\gamma(u))du + \gamma(u)dB(u) \\ &+ (\sigma(u)dB(u) + b(u)du)\left((a(u) - \sigma(u)\gamma(u))du + \gamma(u)dB(u)\right) \\ =& (b(u)X(u) + a(u) - \sigma(u)\gamma(u))du + (\sigma(u)X(u) + \gamma(u))dB(u) + \sigma(u)\gamma(u)du \\ =& (a(u) + b(u)X(u))du + (\gamma(u) + \sigma(u)X(u))dB(u). \end{aligned}$$

.1.6 Feynman-Kac theorem

Consider the differential stochastic equation:

$$dX(u) = \beta(u, X(u))du + \gamma(u, X(u))dW(u).$$

Let $h(y)$ be a Borel-mesurable function. Let $F > 0$ fixed, and $t \in [0, T]$. Let

$$g(t, x) = E^{t,x}[h(X(T))]$$

then $E^{t,x}[|h(X(T))|] < +\infty$. Therefore $g(t, x)$ is solution to the partial differential equation

$$g_t(t, x) + \beta(t, x)g_x(t, x) + \frac{1}{2}\gamma^2(t, x)g_{xx}(t, x) = 0,$$

with the terminal condition $g(T, x) = h(x), \forall x$.

.2 Stochastic control

In this Section the Stochastic optimal control theory will be presented. This Section is based on [11], [10] and [27]. We will present the parabolic Hamilton-Jacobi-Bellman theorem. We have chosen the parabolic case since we will use it for a GMWB product, which has a bounded term. That is, the product arrives to an end, either because there is no more money left in the GMWB account or because the product has arrived to maturity (see the GMWB dynamic surrender section for more details). We will not use Hamilton-Jacobi-Bellman theorem in for-life GMWB.

.2.1 The control process

Let $(\Omega, \mathcal{F}, P)$ be a probability complete space. Let B be a p-dimensional Brownian motion with $\mathcal{F}_t$ its natural filtration. The decision variables, or control variables, are stochastic processes whose value can be decided at any time. That is, the value of these variables depends only on the information available up to the time of the decision t. To be more precise, a **control process** $(\alpha_t)_{0\leq t<T}$ is an stochastic process adapted to the natural filtration of the Brownian motion considered. We will denote A the image space of the control process and will demand A to be compact in a separable space. Denote $\mathcal{A}$ the set of all control processes, that is

$$\mathcal{A} = \{\alpha_t : 0 \leq t < T, \alpha_t \text{ is } \mathcal{F}_t - \text{mesurable}\}.$$

.2.2 The stochastic controlled process

Now suppose that X_t is a stochastic process (**state variable**) governed by the stochastic differential equation:

$$\begin{cases} dX_t &= b(X_t, \alpha_t)dt + \sigma(X_t, \alpha_t)dB_t \\ X_0 &= x \in \mathbb{R}^n \end{cases},$$

where B_t is the p-dimensional Brownian motion, the drift term $b(X_t, \alpha_t) : \mathbb{R}^n \times A \to \mathbb{R}^n$ is a continous function and the volatility term $\sigma(X_t, \alpha_t) : \mathbb{R}^n \times A \to \mathbb{R}^{n\times p}$ is also continous. A Lipschitz contidion[1] on the fonctions b et σ guarantees that the stochastic differential equation has a unique solution and therefore that X_t is well defined. Such an X_t is called a **stochastic controlled process** .

The image space of the stochastic control process is normally $\mathbb{R}^n$, but for the interest of GMWBs $\mathbb{R}^n$ is too large and we will rather take an open non-bounded subset of $\mathbb{R}^n$, let's denote $\mathcal{O}$ this open non-bounded set. We will suppose as well that $0 < t < T$, that is t is also inside an open interval of $\mathbb{R}^n$.

[1] see Section .2.4.

Now we will redefine our stocastic controlled process in order to include the time information into the process and also into it's image. Take

$$\begin{cases} dY_t & = B(t, Y_t, \alpha_t)dt + S(t, Y_t, \alpha_t)dB_t \\ Y_0 & = y \in \mathbb{R}^n \end{cases}$$

to be the stochastic controlled process of our interest. Define $X_0 = x = (0, y)$ and $X_t = (t, Y_t)$ the new stochastic controlled process. Take

$$b = \begin{pmatrix} 1 \\ B \end{pmatrix} \text{ and } \sigma = \begin{pmatrix} 0 \cdots 0 \\ S \end{pmatrix}$$

then

$$\begin{cases} dX_t & = b(X_t, \alpha_t)dt + \sigma(X_t, \alpha_t)dB_t \\ X_0 & = x = (0, y) \in (0, T) \times \mathcal{O}, \end{cases}$$

with this notation we can consider that our process stops when X_t goes out of $\mathcal{D} = (0, T) \times \mathcal{O}$. There fore, we can define a stop time $\tau = \inf\{t > 0; X_t \notin \mathcal{D}\}$.

.2.3 The value function

The general problem of the optimal control theory is to find an stochastic control process that will minimize a certain function of the trajectory of X. We will say that the expected cost function for a given strategy $\alpha \in \mathcal{A}$ is

$$V^\alpha(s, y) = E^\alpha_{s,y} \left[\int_s^\tau f(t, Y_t, \alpha_t) e^{-\int_s^t r(u, Y_u, \alpha_u)du} dt + \Psi(\tau, Y_\tau) e^{-\int_s^\tau r(u, Y_u, \alpha_u)du} \right],$$

where f and Ψ are continous and $r(u, Y_u, \alpha_u)$ is the interest rate process.
What we are looking for is the strategie $\hat{\alpha}$ that will minimze V^α. If we can not find $\hat{\alpha}$ we will like at least to have the value of $V^{\hat{\alpha}}$. Let

$$V(s, y) = \inf_{\alpha \in \mathcal{A}} V^\alpha(s, y)$$

the **value function**.

.2.4 Sufficient conditions

The good definition of X_t

The functions b and σ must be globaly Lipschitziens in $\overline{\mathcal{D}}$ uniformilly in $\mathcal{A}$ to assure the good definition of the controlled process. That is, there exists a constant L such that

$$
\begin{aligned}
|b(x,\alpha) - b(x_0,\alpha)| &\leq L|x - x_0| \\
|\sigma(x,\alpha) - \sigma(x_0,\alpha)| &\leq L|x - x_0|
\end{aligned}
$$

for all $(x, x_0) \in \overline{\mathcal{D}} \times \overline{\mathcal{D}}$ and $\alpha \in A$.

The good definition of V

One of the three following conditions is enought to assure the good definiton of V

(H1) f and Ψ are bounded in ther definition domain

(H2) f and Ψ increase like a polynom and X_t admits polynomial moments, that is there exists a $K > 0$ and an $m_0 \in \mathbb{N}$ such that

$$f(x,\alpha) \leq K(1 + |x|^{m_0}) \quad \text{and} \quad \Psi(x) \leq K(1 + |x|^{m_0})$$

and

$$\forall m \leq m_0, \qquad \sup_{\alpha \in \mathcal{A}} E_y^\alpha \left[\sup_{s \leq t \leq T} |X_t|^m \right] < \infty$$

(H3) f and Ψ increase exponentially and X_t admits exponential moments, that is there exists a $K > 0$ and an $\lambda_0 > 0$ such that

$$f(x,\alpha) \leq K e^{\lambda_0 |x|} \quad \text{and} \quad \Psi(x) \leq K e^{\lambda_0 |x|}$$

and

$$\forall \lambda \leq \lambda_0, \qquad \sup_{\alpha \in \mathcal{A}} E_y^\alpha \left[\sup_{s \leq t \leq T} e^{\lambda |X_t|} \right] < \infty.$$

The continuity of the stopping time τ

Let δ be the distance to the border of an open space O, that is

$$\delta(x) = \begin{cases} + \inf_{y \in \partial O} |y - x| & \text{if } x \in O \\ - \inf_{y \in \partial O} |y - x| & \text{if } x \notin O \end{cases}.$$

Let $\mathcal{L}_t^\alpha$ be the linear, second-order differential opperator

$$\mathcal{L}^\alpha V(x) = \sum_{i=1}^n b_i(x,\alpha) \frac{\partial V(x)}{\partial x_i} + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n s_{ij}(x,\alpha) \frac{\partial^2 V(x)}{\partial x_i \partial x_j}.$$

One of the following two hypothesis assures the border's non-degeneration. This assures the contiunuity of the stopping time.

(H'1) δ is $\mathcal{C}^2$ and $\forall x \in \delta\mathcal{O}, \forall \alpha \in \mathcal{A}, \sigma^*(x,\alpha)D\delta(x) \neq 0$ or $L^\alpha\delta(x) < 0$.

(H'2) We suppose that the open space $\mathcal{D}$ can be written as

$$\mathcal{D} = ((0,T) \times \mathbb{R}^n) \cap (\cap_{i=1}^q \mathcal{O}_i)$$

where the $\mathcal{O}_i$ are open spaces in $\mathbb{R}^n$. We denote δ_i the distance to the border of an open space. For all $i = 1,..,q$ we have that δ_i is $\mathcal{C}^2$ in a neighborhood of $\delta\mathcal{O}_i$ and for every $(s,y) \in \Gamma_{transversal}$ $\exists i \in \{1,...,q\}$ such that $\delta_i(s,y) = 0$ and $\forall \alpha \in \mathcal{A}, S^*(s,y,\alpha)D\delta_i(s,y) \neq 0$ or $\frac{\partial \delta_i}{\delta t}(s,y) + \mathcal{L}^\alpha\delta_i(s,y) < 0$.

.2.5 The principle of dynamic programming

Under the preceding conditions, V satisfies the following equation, for all $x \in \mathcal{D}$ and for any t such that $s \leq t \leq T$

$$V(s,x) = \inf_{\alpha\in\mathcal{A}} E^\alpha_{s,y}\left[V(t\wedge\tau, Y_{t\wedge\tau})e^{-\int_s^{t\wedge\tau} r(s,Y_s,\alpha_s)ds} + \int_s^{t\wedge\tau} f(u,Y_u,\alpha_u)e^{-\int_s^u r(s,Y_s,\alpha_s)ds}du\right]$$

we can interpret this fundamental principle as a principle that states that the optimal V can be "decomposed" into a the integral of f up to t and then to V starting at t. The optimal value function after t for an optimal strategy from s is in fact just the optimal value function after t.

.2.6 The Hamilton-Jacobi-Bellman equation

If V is $\mathcal{C}^2(\mathcal{D}) \cup \mathcal{C}(\bar{\mathcal{D}})$ and the sufficient conditions of Section .2.4 are fulfilled then V satisfies the classical Hamilton-Jacobi-Bellman equation:

$$\begin{cases} \inf\limits_{\alpha\in\mathcal{A}}\left(\dfrac{\partial V}{\partial t}(\cdot) + \mathcal{L}^\alpha V(\cdot) + f(\cdot,\alpha) - r(\cdot,\alpha)V(\cdot)\right) & = 0 \text{ if inside } \mathcal{D} \\ V & = \Psi \text{ if } t = \tau \end{cases}$$

where $\mathcal{L}^\alpha_t$ is the linear, second-order differential opperator:

$$\mathcal{L}^\alpha V(x) = \sum_{i=1}^n b_i(x,\alpha)\frac{\partial V(x)}{\partial x_i} + \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n s_{ij}(x,\alpha)\frac{\partial^2 V(x)}{\partial x_i \partial x_j}$$

and $B = (b_i)_{i=1,..,n}$ and $S = (s_ij)_{i,j=1,...,n}$.

Observe that if we are faced with a maximization problem $V(s,y) = \sup\limits_{\alpha\in\mathcal{A}} V^\alpha(s,y)$ the Hamilton-Jacobi-Bellman equation becomes

$$\begin{cases} \sup\limits_{\alpha\in\mathcal{A}}\left(\dfrac{\partial V}{\partial t}(\cdot) + \mathcal{L}^\alpha V(\cdot) + f(\cdot,\alpha) - r(\cdot,\alpha)V(\cdot)\right) & = 0 \text{ if inside } \mathcal{D} \\ V & = \Psi \text{ if } t = \tau \end{cases}.$$

.3 Vasicek's Interest Rate Model

The Vasicek's Interest Rate Model follows the following stochastic differential equation: $dr_t = (a - br_t)dt + \sigma dB_t$ where B_t is a Q-brownian motion.

Lemma 1. The solution to the Vasicek SDE is given by

$$r_t = r_s e^{-b(t-s)} + \frac{a}{b}\left(1 - e^{-b(t-s)}\right) + \sigma \int_s^t e^{-b(t-u)} dB_u.$$

Proof. Let $Y_t = r_t e^{b(t_s)}$ with $t \geq s$. Then by Ito's lemma we have that

$$\begin{aligned} dY_t &= e^{b(t-s)} dr_t + r_t e^{b(t-s)} dt \\ &= e^{b(t-s)}((a - br_t)dt + \sigma dB_t) + r_t b e^{b(t-s)} dt \\ &= e^{b(t-s)}(a dt + \sigma dB_t). \end{aligned}$$

Now,

$$r_t e^{b(t-s)} - r_s = \int_s^t dY_u = \int_s^t e^{b(t-u)} a du + \int_s^t e^{b(t-u)} \sigma dB_u$$

and so

$$r_t = r_s e^{-b(t-s)} + \frac{a}{b}\left(1 - e^{-b(t-s)}\right) + \sigma \int_s^t e^{-b(t-u)} dB_u.$$

Lemma 2. The price at time t of a zero-coupcon bond in Vasicek's model is:

$$B(t,T) = e^{m(t,T) - n(t,T) r_t},$$

where
$n(t,T) = \frac{1}{b}\left(1 - e^{-b(T-t)}\right)$ and $m(t,T) = \left(\frac{a}{b} - \frac{\sigma^2}{2b^2}\right)(n(t,T) - (T-t)) - \frac{\sigma^2}{4b} n^2(t,T)$.

Proof. For Q the is risk-neutral probability we have that

$$B(t,T) = E_t^Q\left[e^{-\int_t^T r_u du}\right].$$

Lets calculate $\int_t^T r_u du$, this is,

$$\begin{aligned} \int_t^T r_u du &= \int_t^T r_t e^{-b(u-t)} du + \int_t^T \frac{a}{b}\left(1 - e^{-b(u-t)}\right) du + \sigma \int_t^T \int_t^u e^{-b(u-s)} dB_s du \\ &= r_t n(t,T) + a \int_t^T n(u,T) du + \sigma \int_t^T \int_u^T e^{-b(u-s)} du dB_s \\ &= r_t n(t,T) + a \int_t^T n(u,T) du + \sigma \int_t^T n(u,T) dB_u. \end{aligned}$$

Observe that $-\int_t^T r_u du$ is a gaussian variable, and that,

$$E_t^Q\left[-\int_t^T r_u du\right] = -r_t n(t,T) - a\int_t^T n(u,T)du = -r_t n(t,T) - \frac{a}{b}(T-t-n(t,T))$$

$$V_t^Q\left[-\int_t^T r_u du\right] = \sigma^2\int_t^T n^2(u,T)du = \frac{\sigma^2}{b^2}(T-t-n(t,T)) - \frac{\sigma^2}{2b}n^2(t,T).$$

And so $e^{-\int_t^T r_u du}$ is a log-normal variable. Therefore

$$\begin{aligned} B(t,T) = E_t^Q\left[e^{-\int_t^T r_u du}\right] &= e^{-r_t n(t,T) - \frac{a}{b}(T-t-n(t,T)) + \frac{\sigma^2}{2b^2}(T-t-n(t,T)) - \frac{\sigma^2}{4b}n^2(t,T)} \\ &= e^{-r_t n(t,T) + \left(\frac{a}{b} - \frac{\sigma^2}{2b^2}\right)(n(t,T)-(T-t)) - \frac{\sigma^2}{4b}n^2(t,T)}, \end{aligned}$$

which proofs the lemma.

Observation Sometimes the Vasicek model is expressed as $dr_t = b(\theta - r_t)dt + \sigma dB_t$ in such a case

$$B(t,T) = e^{-r_t n(t,T) + \left(\theta - \frac{\sigma^2}{2b^2}\right)(n(t,T)-(T-t)) - \frac{\sigma^2}{4b}n^2(t,T)}.$$

Lemma 3. The dynamics for the bond price of the Vasicek model are

$$dB(t,T) = B(t,T)(r_t dt - \sigma n(t,T)dB_t).$$

Following Itô's lemma

$$\begin{aligned} dB(t,T) = &\frac{\partial B(t,T)}{\partial t}dt + \frac{\partial B(t,T)}{\partial r_t}dr_t + \frac{\sigma^2}{2}\frac{\partial^2 B(t,T)}{\partial r_t^2}dt \\ =&B(t,T)\frac{\partial}{\partial t}(m(t,T) - n(t,T)r_t)dt - B(t,T)n(t,T)dr_t + \frac{\sigma^2}{2}B(t,T)n^2(t,T)dt \\ =&B(t,T)\left(\frac{\partial}{\partial t}(m(t,T) - n(t,T)r_t) - n(t,T)(a - br_t) + \frac{\sigma^2}{2}n^2(t,T)\right)dt \\ &- B(t,T)n(t,T)\sigma dB_t. \end{aligned}$$

Now, $\frac{\partial}{\partial t}m(t,T) = \left(\frac{a}{b} - \frac{\sigma^2}{2b^2}\right)\left(\frac{\partial}{\partial t}n(t,T) + 1\right) - \frac{\sigma^2}{2b}n(t,T)\frac{\partial}{\partial t}n(t,T)$
and $\frac{\partial}{\partial t}n(t,T) = -e^{-b(T-t)}$.

So

$$
\begin{aligned}
&\frac{\partial}{\partial t}(m(t,T) - n(t,T)r_t) - n(t,T)(a - br_t) + \frac{\sigma^2}{2}n^2(t,T) \\
&= \left(\frac{a}{b} - \frac{\sigma^2}{2b^2}\right)\left(1 - e^{-b(T-t)}\right) + \frac{\sigma^2}{2b}n(t,T)e^{-b(T-t)} + e^{-b(T-t)}r_t - n(t,T)(a - br_t) + \frac{\sigma^2}{2}n^2(t,T) \\
&= an(t,T) - \frac{\sigma^2}{2b}n(t,T) + \frac{\sigma^2}{2b}n(t,T)e^{-b(T-t)} + r_t\left(e^{-b(T-t)} + bn(t,T)\right) - an(t,T) + \frac{\sigma^2}{2}n^2(t,T) \\
&= -\frac{\sigma^2}{2b}n(t,T)\left(1 - e^{-b(T-t)}\right) + r_t + \frac{\sigma^2}{2}n^2(t,T) \\
&= r_t,
\end{aligned}
$$

therefore $dB(t,T) = B(t,T)(r_t dt - \sigma n(t,T)dB_t)$.

.4 Central moments for Asian options bounds calculations

Lemma 4. Let $Z = \int_0^t B_u du$ therefore

$$E[B_u|Z] = m_u Z$$
$$Var[B_u|Z] = u - \frac{t^3}{3} m_u^2$$
$$Var[B_u + B_s|Z] = u + s + v_{u,s},$$

where $m_u = 3\frac{u}{t^3}\left(t - \frac{1}{2}u\right)$ and $v_{u,s} = 2\min(u,s) - \frac{t^3}{3}(m_u + m_s)^2$.

Proof. Consider fist Z and B_u moments $E[B_u] = 0$, $Var[B_u] = u$, $E[Z] = 0$,

$$\begin{aligned}
Var[Z] &= E\left[\left(tB_t - \int_0^t udB_u\right)^2\right] \\
&= E[t^2B_t^2] - 2tE\left[B_t\left(\int_0^t udB_u\right)\right] + E\left[\int_0^t u^2du\right] \\
&= E[t^2B_t^2] - 2tE\left[B_t\left(tB_t - \int_0^t B_udu\right)\right] + E\left[\int_0^t u^2du\right] \\
&= t^3 - 2t^3 + 2t\int_0^t udu + \int_0^t u^2du \\
&= \frac{t^3}{3} \\
Cov(B_u, Z) &= E\left[B_u\int_0^t B_sds\right] = \int_0^t \min(u,s)ds \\
&= u\left(t - \frac{1}{2}u\right).
\end{aligned}$$

Observe that Z and B_u are normal distributed. Now by the Projection Theorem we have that

$$
\begin{aligned}
E[B_u|Z] &= E[B_u] + \frac{Cov(B_u, Z)}{Var[Z]}(Z - E[Z]) \\
&= 3\frac{u}{t^3}\left(t - \frac{1}{2}u\right) Z \\
&= m_u Z \\
Var[B_u|Z] &= Var[B_u] - \frac{Cov(B_u, Z)^2}{Var[Z]} \\
&= u - 3\frac{u^2}{t^3}\left(t - \frac{1}{2}u\right)^2 \\
&= u - \frac{t^3}{3}m_u^2 \\
Cov[B_u, B_s|Z] &= Cov[B_u, B_s] - \frac{Cov(B_u, Z)Cov(B_s, Z)}{Var[Z]} \\
&= \min(u, s) - \frac{3}{t^3}\left(t - \frac{1}{2}u\right)\left(t - \frac{1}{2}s\right) \\
&= \min(u, s) - \frac{t^3}{3}m_u m_s \\
Var[B_u + B_s|Z] &= u - \frac{t^3}{3}m_u^2 + s - \frac{t^3}{3}m_s^2 + 2\min(u, s) - 2\frac{t^3}{3}m_u m_s \\
&= 2\min(u, s) + (u + s) - \frac{t^3}{3}(m_u + m_s)^2.
\end{aligned}
$$

Lemma 5. Let $Y_t = e^{-(\mu+\frac{1}{2}\sigma^2)t-\sigma B_t}$ and $Z = \int_0^t B_s ds$ therefore

$$
\begin{aligned}
E\left[\int_0^t Y_u du \middle| Z\right] &= \int_0^t e^{-\mu u - \sigma m_u Z + \frac{1}{2}\sigma^2 \frac{t^3}{3} m_u^2} du \\
Var\left[\int_0^t Y_u du \middle| Z\right] &= \int_0^t \int_0^t e^{-\mu(u+s) - \sigma(m_u+m_s)Z + \frac{1}{2}\sigma^2 v_{u,s}} duds.
\end{aligned}
$$

Proof.

$$
\begin{aligned}
E\left[\int_0^t Y_u du \middle| Z\right] &= \int_0^t E[Y_u|Z] du \\
&= \int_0^t e^{-(\mu+\frac{1}{2}\sigma^2)u - \sigma E[B_u|Z] + \frac{1}{2}\sigma^2 Var[B_u|Z]} du \\
&= \int_0^t e^{-(\mu+\frac{1}{2}\sigma^2)u - \sigma m_u Z + \frac{1}{2}\sigma^2\left(u - \frac{t^3}{3}m_u^2\right)} du \\
&= \int_0^t e^{-\mu u - \sigma m_u Z + \frac{1}{2}\sigma^2 \frac{t^3}{3} m_u^2} du
\end{aligned}
$$

and

$$
\begin{aligned}
E\left[\left(\int_0^t Y_u du\right)^2 \middle| Z\right] &= \int_0^t \int_0^t E[Y_u Y_s] duds \\
&= \int_0^t \int_0^t e^{-(\mu+\frac{1}{2}\sigma^2)(u+s)-\sigma E[B_u+B_s|Z]+\frac{1}{2}\sigma^2 Var[B_u+B_s|Z]} duds \\
&= \int_0^t \int_0^t e^{-\mu(u+s)-\sigma(m_u+m_s)Z+\frac{1}{2}\sigma^2 v_{u,s}} duds.
\end{aligned}
$$

.5 On the Inversion of a tridiagonal matrix

A Tridiagonal matrix is a square matrix such that that all entries that are not the principal diagonal or on the diagonals above and below the principal diagonal, are zero. That is, it is a matrix that has the following form:

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & 0 & 0 & 0 & \cdots & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} & 0 & 0 & \cdots & 0 \\ 0 & a_{3,2} & a_{3,3} & a_{3,4} & 0 & \cdots & 0 \\ 0 & 0 & a_{4,3} & a_{4,4} & a_{4,5} & & 0 \\ 0 & 0 & 0 & a_{5,4} & a_{5,5} & \ddots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & 0 & \ddots & a_{n,n} \end{pmatrix}.$$

Define $\mathcal{B}_k$ with $k = 1, ..., n$ the square submatrix if A composed of those elements $a_{i,j}$ such that $i \leq k$ and $j \leq k$. That is

$$\mathcal{B}_1 = \begin{pmatrix} a_{1,1} \end{pmatrix}, \mathcal{B}_2 = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}, \mathcal{B}_3 = \begin{pmatrix} a_{1,1} & a_{1,2} & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} \\ 0 & a_{3,2} & a_{3,3} \end{pmatrix}, ..., \mathcal{B}_n = A.$$

Let $B_k = \det(\mathcal{B}_k)$ for $k = 1, .., n$. By definition $B_1 = a_{1,1}$, $B_2 = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$ and it is easy to see that

$$B_k = a_{k,k}B_{k-1} - a_{k-1,k}a_{k,k-1}B_{k-2}$$

for $k = 3, ..., n$. This recursive formula allows to obtain the value of $\det(A)$ since $\det(A) = B_n$.

Now define $\mathcal{C}_k$ the square submatrix if A composed of those elements $a_{i,j}$ such that $i \geq k$ and $j \geq k$. That is

$$\mathcal{C}_1 = A, \qquad \mathcal{C}_2 = \begin{pmatrix} a_{2,2} & a_{2,3} & 0 & 0 & \cdots & 0 \\ a_{3,2} & a_{3,3} & a_{3,4} & 0 & \cdots & 0 \\ 0 & a_{4,3} & a_{4,4} & a_{4,5} & & 0 \\ 0 & 0 & a_{5,4} & a_{5,5} & \ddots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & \ddots & a_{n,n} \end{pmatrix},$$

$$\mathcal{C}_3 = \begin{pmatrix} a_{3,3} & a_{3,4} & 0 & \cdots & 0 \\ a_{4,3} & a_{4,4} & a_{4,5} & & 0 \\ 0 & a_{5,4} & a_{5,5} & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \ddots & a_{n,n} \end{pmatrix}, ..., \mathcal{C}_n = \begin{pmatrix} a_{n,n} \end{pmatrix}.$$

Let $C_k = \det(\mathcal{C}_k)$ for $k = 1, .., n$. Then we have that $C_n = a_{n,n}$, $C_{n-1} = a_{n-1,n-1}a_{n,n} - a_{n-1,n}a_{n,n-1}$ and

$$C_k = a_{k,k}C_{k+1} - a_{k+1,k}a_{k,k+1}C_{k+2}$$

for $k = 1, ..., n-2$. Let $D_k = \prod_{j=1}^{k} a_{j+1,j}$ for $k = 1, ..., n-1$ and $E_k = \prod_{j=1}^{k} a_{j,j+1}$ for $k = 1, ..., n-1$ the second diagonal products.

Each element of A^{-1} is the product or division of some of the elements B, C, D or E, lets see how. Consider first what happens to A if we delete colomn i and line i. Take for exemple $i = 3$

$$\left(\begin{array}{cc|cccc}
a_{1,1} & a_{1,2} & 0 & 0 & \cdots & 0 \\
a_{2,1} & a_{2,2} & 0 & 0 & \cdots & 0 \\
\hline
0 & 0 & a_{4,4} & a_{4,5} & & 0 \\
0 & 0 & a_{5,4} & a_{5,5} & \ddots & 0 \\
\vdots & \vdots & & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \ddots & a_{n,n}
\end{array}\right).$$

The resulting matrix has 2 non-zero square blocs in diagonal: $\mathcal{B}_{i-1}$ and $\mathcal{C}_{i+1}$ and so it's determinant is $B_{i-1}C_{i+1}$, which means that the diagonal element $\alpha_{i,i}$ of A^{-1} is

$$\alpha_{i,i} = \frac{B_{i-1}C_{i+1}}{B_n}.$$

Consider what happens to A^{-1} if we delete line i and colomn j with $i < j$. For exemple $i = 2$ and $j = 4$,

$$\left(\begin{array}{c|cc|ccc}
a_{1,1} & a_{1,2} & 0 & 0 & \cdots & 0 \\
\hline
0 & a_{3,2} & a_{3,3} & 0 & \cdots & 0 \\
0 & 0 & a_{4,3} & a_{4,5} & & 0 \\
\hline
0 & 0 & 0 & a_{5,5} & \ddots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \ddots & a_{n,n}
\end{array}\right).$$

The resulting matrix is an upper triangular by-blocs matrix, this means that it's determinant is the product of the determinants of the blocs in the diagonal. The firts bloc is $\mathcal{B}_{i-1}$, the second bloc is an upper triangular matrix and the third bloc is $\mathcal{C}_{j+1}$. Then

$$\alpha_{i,j} = (-1)^{i+j}\frac{B_{i-1}D_{j-1}C_{j+1}}{B_nD_{i-1}}$$

At last consider what happens to A^{-1} if we delete line i and colomn j with $i > j$. For exemple $i = 4$ and $j = 2$,

$$\left(\begin{array}{c|cc|ccc}
a_{1,1} & 0 & 0 & 0 & \cdots & 0 \\
\hline
a_{2,1} & a_{2,3} & 0 & 0 & \cdots & 0 \\
0 & a_{3,3} & a_{3,4} & 0 & \cdots & 0 \\
\hline
0 & 0 & a_{5,4} & a_{5,5} & \ddots & 0 \\
\vdots & \vdots & & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \ddots & a_{n,n}
\end{array}\right).$$

This is a lower triangular by-blocs matrix. In this case

$$\alpha_{i,j} = (-1)^{i+j} \frac{B_{j-1}E_{i-1}C_{i+1}}{B_n E_{j-1}}$$

Putting all together we have the following theorem.

Theorem .5.1. Let A be a tridiagonal matrix with elements $a_{i,j}$. Let $B_1 = a_{1,1}$, $B_2 = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$ and $B_k = a_{k,k}B_{k-1} - a_{k-1,k}a_{k,k-1}B_{k-2}$ for $k = 3, ..., n$. Let $C_n = a_{n,n}$, $C_{n-1} = a_{n-1,n-1}a_{n,n} - a_{n-1,n}a_{n,n-1}$ and $C_k = a_{k,k}C_{k+1} - a_{k+1,k}a_{k,k+1}C_{k+2}$. Define $D_k = \prod_{j=1}^{k} a_{j+1,j}$ for $k = 1, ..., n-1$ and $E_k = \prod_{j=1}^{k} a_{j,j+1}$ for $k = 1, ..., n-1$. Let $A^{-1} = [\alpha_{i,j}]$ then
for $i = j$

$$\alpha_{i,i} = \frac{B_{i-1}C_{i+1}}{B_n}$$

for $i < j$

$$\alpha_{i,j} = (-1)^{i+j} \frac{B_{i-1}D_{j-1}C_{j+1}}{B_n D_{i-1}}$$

for $i > j$

$$\alpha_{i,j} = (-1)^{i+j} \frac{B_{j-1}E_{i-1}C_{i+1}}{B_n E_{j-1}}.$$

.6 Derivation of a closed form formula for $V_0(A,t)$

Lemma 6. This lemma is taken from Dai, Kwonk and Zong [17]. Suppose that $V(W,A,t)$ is governed by the following differential stochastic equation

$$\min\left(-\frac{\partial V}{\partial t} - \frac{\sigma^2}{2}W^2\frac{\partial^2 V}{\partial W^2} - (r-\alpha)W\frac{\partial V}{\partial W} + rV - \max\left(1 - \frac{\partial V}{\partial W} - \frac{\partial V}{\partial A}, 0\right)G \right.$$
$$\left., \frac{\partial V}{\partial W} + \frac{\partial V}{\partial A} - (1-k)\right) = 0,$$

with $V(0,A,T) = A(1-k)$ and $V(0,0,t) = 0$. Then

$$V_0(A,t) = (1-k)\max(A - G\tau^*, 0) + \frac{G}{r}\left(1 - e^{-r\min(\tau^*, \frac{A}{G})}\right)$$

with

$$\tau* = \min\left(-\frac{\ln(1-k)}{r}, T-t\right).$$

Proof. Observe that if $W = 0$ then we have that $\dfrac{\partial V}{\partial W} = 0$ so the equation becomes

$$\min\left(-\frac{\partial V_0}{\partial t} + rV_0 - \max\left(1 - \frac{\partial V_0}{\partial A}, 0\right)G, \frac{\partial V_0}{\partial A} - (1-k)\right) = 0$$

with $V_0(A,T) = A(1-k)$ and $V_0(0,t) = 0$.

Case 1 If $1 \geq \dfrac{\partial V_0}{\partial A} > (1-k)$ so $-\dfrac{\partial V_0}{\partial t} + rV_0 - G + G\dfrac{\partial V_0}{\partial A} = 0$.

Let $U_0(A,t) = V_0(A,t)e^{r(T-t)} - G\int\limits_t^T e^{r(T-u)}du$ then

$$\frac{\partial U_0}{\partial t} - G\frac{\partial U_0}{\partial A} = 0,$$

which is an hyperbolic ecuation with auxiliary conditions $U_0(A,T) = A(1-k)$ and $U_0(0,t) = -G\int\limits_t^T e^{r(T-u)}du$. The solution of this kind of equations has the form

$$U_0(A,t) = F\left(t + \frac{A}{G}\right),$$

where F must be determined from the auxiliary conditions. This equations has a unique characteristic which is the line $T = t + \frac{A}{G}$, this line divides the function's domain into two solution regions.

At $t = T$ we have that $V_0(A,T) = A(1-k)$ so $\frac{\partial V_0}{\partial A}(A,T) = (1-k)$ which do not correspond to the case 1. So we will focus on $A = 0$ that is $U_0(0,t) = -G\int\limits_t^T e^{r(T-u)}du$ which means that

$$F(t) = -G\int\limits_t^T e^{r(T-u)}du$$

and so

$$U_0(A,t) = F\left(t+\frac{A}{G}\right) = -G\int\limits_{t+\frac{A}{G}}^T e^{r(T-u)}du.$$

Which implies that

$$V_0(A,t) = e^{-r(T-t)}\left(-G\int\limits_{t+\frac{A}{G}}^T e^{r(T-u)}du + G\int\limits_t^T e^{r(T-u)}du\right) = G\int\limits_t^{t+\frac{A}{G}} e^{-r(u-t)}du.$$

Observe that

$$\frac{\partial}{\partial A}\left(G\int\limits_t^{t+\frac{A}{G}} e^{-r(u-t)}du\right) = e^{-r\frac{A}{G}} < 1$$

and therefore $1 \geq \dfrac{\partial V_0}{\partial A}$ for $A \geq 0$ and $r, G > 0$. To satisfy case 1 conditions we would only require that $A < -\frac{G}{r}\ln(1-k)$. This region combined with the region determined by the characterisic gives

$$\left\{(A,t) : A < \min\left(-\frac{G}{r}\ln(1-k), G(T-t)\right)\right\}.$$

Let

$$\tau^* = \min\left(-\frac{1}{r}\ln(1-k), T-t\right).$$

Case 2 If $\frac{\partial V_0}{\partial A} = 1-k$

Then $V_0(A,t) = A(1-k) + C(t)$ where $C(t)$ is a function of t. To maintain continuity one would expect to have for $A \in [0, A_0]$ and $t \in [0,T]$ we have

$$G\tau^*(1-k) + C(t) = V_0(G\tau^*, t) = G\int\limits_t^{t+\tau^*} e^{-r(u-t)}du,$$

and so for $A \geq GB(A,t)$

$$V_0(A,t) = A(1-k) + G \int_t^{t+\tau^*} e^{-r(u-t)} du - G\tau^*(1-k).$$

Putting all together

Putting case 1 and case 2 solutions together we find that

$$V_0(A,t) = (1-k)\max(A - G\tau^*, 0) + \frac{G}{r}\left(1 - e^{-r\min(\tau^*, \frac{A}{G})}\right),$$

with

$$\tau* = \min\left(-\frac{\ln(1-k)}{r}, T - t\right).$$