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Risk-costing and optimization

Confidentialité : ☐ NON ☑ OUI (Durée : ☐ 1 an ☑ 2 ans)

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Abstract

For decades, reinsurance has been one of the most important vehicles for risk management. It is common that nowadays, some big insurance groups have it own internal reinsurer entity in order to reduce the level of ceded profit externally in the one hand, and to benefit the diversification effect in the other hand.

It exists two main missions for an internal reinsurer. The first one consists of the optimisation of reinsurance program giving the specific needs of the local insurance entity. The second one consists of a pricing model in order to challenge the external reinsurer's prices.

The objectives of this report are in the one hand, to develop pricing models for **non-proportional reinsurance** , and to assess the optimisation of reinsurance for an insurance company in the other hand.

Regarding the first objective, the two most common non-proportional reinsurance were taken into account in this report: **Excess of loss per life** and excess of loss per event **Excess of loss per event**.

With regard to the excess of loss per life, the first pricing model is a **frequency-severity** model which purely bases on the historical claim data and the size of portfolio. Our study was trying to introduce the **truncated distributions** in the calibration to see whether we can obtain better calibration of atypical risk than the use of usual statistic distributions can. The calibration using truncated distribution was largely applied in $P\&C$ reinsurance, it seems to be interesting to see its application in life reinsurance.

In case of very limited claim data, the internal reinsurer could have alternative pricing model: using the Best Estimate incident rate risk and using the portfolio model point by Sum At Risk and by age.

With regard to the excess of loss per event, the report aims to continue the road of the past CAT models in life insurance such as Strickler's (1960) and Erland Ekheden's (2008) by: extending the CAT events database - using terrorism database (GTD), simulating claim data using Sum At Risk model point and adding geographical deterministic scenarios in the simulation. In particular, we would like to test the application of other "fat tail" distributions than the traditional distribution for CAT claims: Generalized Pareto Distribution.

For the second objective, we will assume that the insurance company uses standard formula factors in its calculation of Solvency Capital Requirement. Then based on its specific needs in terms of volatility and required capital, we try to determine the optimized reinsurance structure.

Key words: *non-proportional reinsurance, excess of loss per life, excess of loss per event, truncated distribution, Generalized Pareto Distribution, Solvency Capital Requirement, optimisation.*

Résumé

Depuis des décennies, la réassurance est l'un des principaux outils de la gestion des risques. Il est fréquent que de nos jours, certains grands groupes d'assurance disposent d'un réassureur interne qui aide d'une part à réduire le profit cédé extérieurement et d'autre part à bénéficier de l'effet de diversification.

Il existe deux missions principales pour un réassureur interne. La première consiste à optimiser le programme de réassurance de l'entité locale en répondant à ses besoins spécifiques. La seconde consiste à établir un modèle de tarification afin de challenger les prix du réassureur externe.

Les objectifs de ce rapport sont d'une part d'élaborer des modèles de tarification pour la **réassurance non-proportionnelle** et d'autre part de proposer les indicateurs d'optimisation de la réassurance pour une entité locale.

En ce qui concerne le premier objectif, les deux réassurances non-proportionelles les plus courants sont considérées: **excédent de sinistre par tête** et **excédent de sinistre par événement**.
Pour la réassurance excédent de sinistre par tête, le premier modèle de tarification est un modèle de **fréquence-sévérité** qui s'appuie sur les données historiques de sinistres. Notre étude a tenté d'introduire les distributions tronqées dans la calibration pour voir si nous pouvons obtenir un meilleur qualité de fit du risque atypique que l'utilisation des distributions statistiques habituelles. La calibration utilisant les **distributions tronquées** ont été largement appliqué en réassurance on-vie. Il semble intéressant de voir son application dans la réassurance vie.
Au cas où les données de sinistres sont très limitées, le réassureur pourrait utiliser un autre modèle de tarification: modèle de taux d'incident Best Estimate en utilisant le modèle point du portefeuille par Somme At Risk et l'âge.

Pour la réassurance excédent de sinistre par événement, le rapport vise à poursuivre des modèles de tarification du risque CAT en vie tels que Strickler (1960) et Erland Ekheden (2008) en élargissant la base de données CAT en utilisant la base pour les événements terrorists - GTD; en simulant les montants de sinistre à l'aide du modèle point par Sum At Risk et en ajoutant des scénarios déterministes. En particulier, nous aimerions tester l'application d'autres distributions de la "queue épaisse" que la distribution traditionnelle pour la caribration du risque CAT - distribution Pareto généralisée.

En ce qui concerne le deuxième objectif, nous supposerons que la compagnie d'assurance utilise les formules standards dans son calcul du capital de solvabilité requis . En suite, en fonction de ses besoins spécifiques en termes de volatilité ou de capital requis, nous tentons de déterminer la structure de réassurance optimisée.

Mots clés: *réassurance non proportionnelle, excédent de sinistre par tête, excédent de sinistre par événement, distribution tronquée, Pareto distribution généralisée, capital requis, réassurance optimisée.*

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Introduction to life reinsurance

1.1 Brief definition of life reinsurance

1.1.1 Definition

Reinsurance can be defined as a coverage purchased by an insurer to cover all or a part of risks from insurance policies issued by this company. This buying action is called a "cession", the insurance company is called "ceding company". The Reinsurer can then also cede the risk or a part of it to other Reinsurers. This action is called "retrocession" and the other Reinsurers are known as "retrocessionaires".

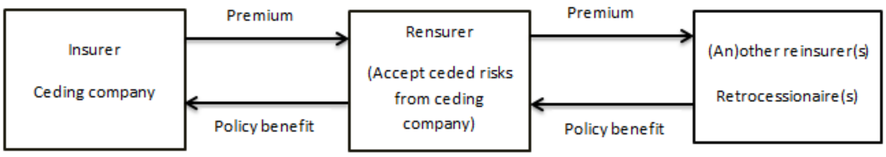


Figure 1.1: Risk retrocession

1.1.2 Covered risks

One or more categories of risks can be covered by a reinsurance contract. In particular, for life insurance business:

- Mortality and morbidity
- Disability
- Medical expense
- Critical illness
- Longevity
- Others

In a reinsurance treaty, we have often a list of exclusion. For example: participating in a criminal act, suicide or self-inflicted injury, etc. In general, they are risks that are difficult to handle by the reinsurer.

In addition, the "High Risk Occupation" (HRO) and the "High Risk Business" (HRB) are submitted as "Special Acceptances". The reinsurer could either decline or accept the risk with or without extra-loading.

High Risk Business: Individual Policies of the kinds particularized in the reinsurance treaty. For example: *Short-term travel accident policies.*

High Risk Occupation: The occupations of Original Insured considered to be of greater risk such as: *Professional sport-players, cabin crew and pilots of the airline companies, workers in mineral activities.*

1.1.3 Contractual elements

The contractual elements between the players mentioned in previous section are written in a legal agreement called: *Reinsurance treaty.* One can find following important technical notions in a life reinsurance treaty:

Reinsurer expense: a sum calculated as a percentage of the Reinsurance Premium. It represents the expenses issued from Reinsurer's activities linked to the covered treaty.

Premium commission: The commission payable by Reinsurer to Reinsured for its administrative activities of original insurance policies.

Profit sharing commission: The amount of profit from Reinsurer to be shared with Reinsured if the treaty makes profit for the Reinsurer.

Minimum Deposit Premium: The first and fixed amount payable by Reinsured to Reinsurer as premium. The remaining amount will be calculated in function of claims amount during the year.

Sum at risk: under a Policy or for a given period of insurance, the difference between:the insured benefit or the present value of the benefits payable in the event of a Claim and the corresponding technical reserve, if any, at the date of the Claim. From now on, this term will be noted as "SAR".

Deductible: in respect of any Claim, the amount of Ultimate Net Loss retained by the Reinsured for its own account.

Limit: the maximum amount covered by the Reinsurance Agreement in respect of each Covered Loss, in excess of the Deductible.

Annual Aggregate Limit: the maximum amount covered by the Reinsurance Agreement in one covered year.

Special Acceptances: agreement by the Reinsurer to include risks as Covered risks where, unless specifically agreed, Acceptances such risks would ordinarily not be accepted or would be excluded from cover or, if covered, would be subject to limitations.

Claim bordereau: The file sent by reinsured containing historical claim triggering the reinsurance treaty.

Premium bordereau: The file containing all information about reinsurance premium, especially using for reinsurance per insured life.

1.2 Traditional life reinsurance and structured life reinsurance

We distinguish two main kinds of life reinsurance which are **traditional reinsurance** and **structured reinsurance**. Traditional reinsurance can be defined as a vehicle of ***transfer of risks*** for example: mortality, morbidity, disability and medical expense. The current thesis will only focus on the traditional reinsurance.

It's also interesting to notice that in the more recent period, structured reinsurance by definition, is used as important tools for insurers to ***achieve some objectives***, for example: to optimize the diversification of risks, to reduce the taxes, volume of reserve or cost of capital etc. When structured reinsurance involves risk transfer, the risk transfer purpose is only secondary.

1.3 Focus on traditional life reinsurance

1.3.1 Different types of traditional life reinsurance

There are two main kinds of traditional reinsurance: proportional and non-proportional.

In proportional basis, the reinsurance premium is indicated proportionally to the insurance premium or to the ceded sum at risk of each insured. In contrast, in non-proportional basis, the reinsurance premium is indicated globally for the total ceded portfolio.

Technical notions: If the insurance portfolio contains n policies with the corresponding sum at risks $X_{i,i=1,..,n}$ and the corresponding insurance premiums: $P_{i,i=1,..,n}$.

The reinsurance premium is noted as P^{re}. If the reinsurance premiums are given per life, we note $P^{re}_{i,i=1,..,n}$

1.3.1.1 Proportional reinsurance

The main types of proportional reinsurance in the market are: quota-share and surplus.

a) Quota-share - QS

Reinsurer and insurer share the premium and the amount of claim in quota basis. The quota-share ratio, i.e. *"cession rate"* is noted q, then 1 -q would be called *"retention rate"*. We have reinsurance premium:

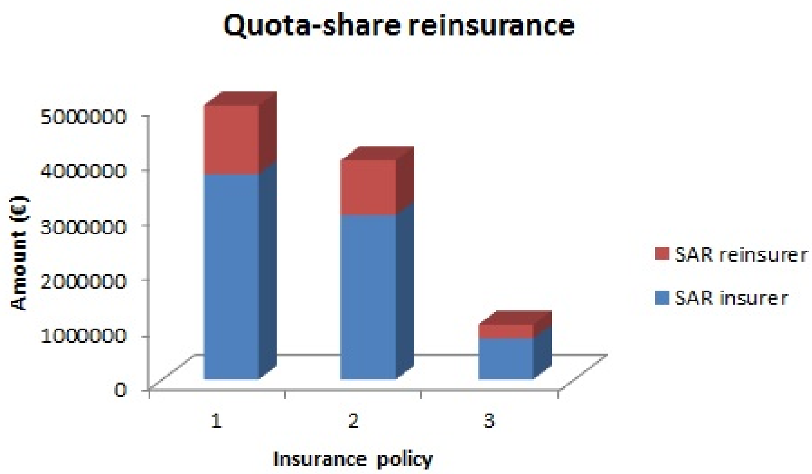
$$P^{re} = q.\sum_{i=1}^{n} P_i$$

If there is a claim occurred for policy j, then, the amount of claim paid by Reinsurer called "claim recovery" is equal to:

$$C_j = q.X_j \quad and \quad C = \sum_j C_j$$

where C is total amount of claims during the covered period In the treaty wording, there would be a clause which limits the amount of sum at risk accepted by Reinsurer. This limit is often called *"Underwriting limit"*. Above this level, the policies could be classed as "Special acceptances".

The following graph shows an example of quota-share reinsurance with the ceded quota-share ratio being equal to 25%.



Policy	SAR	Cession rate	SAR insurer	SAR reinsurer
1	5 000 000	25%	3 750 000	1 250 000
2	4 000 000	25%	3 000 000	1 000 000
3	1 000 000	25%	750 000	250 000

Figure 1.2: Quota-share reinsurance

The treaty could introduce a clause of profit sharing commission. Thus, the reinsurance result is the combination of reinsurance premium and claim, expense and commission, and profit sharing, i.e:

$$Reins.re = P_{re}(1 - RC) - C - PS.(P_{re}(1 - RC - \alpha) - C)$$

Where :

- Preins.re: Reinsurance result
- RC : Reinsurance premium commission
- C: Claims amount
- PS: Profit sharing ratio
- α: Reinsurer expense

b) Surplus per life - XP

The Surplus treaty is another form of proportional reinsurance but in practice, it introduces following notions as in non-proportional treaty:

- R: Retention
- L: Limit

The treaty Surplus is noted in this case: L XP R.

The retention and limit are applied in per head or per policy level. A sum at risk X_i is assigned to each insured or each policy. From the following calculation, the treaty defines the cession rate r for each insured or each policy:

$$r = min\Big(\frac{L}{X_i}, max(1 - \frac{R}{X_i}, 0)\Big)$$

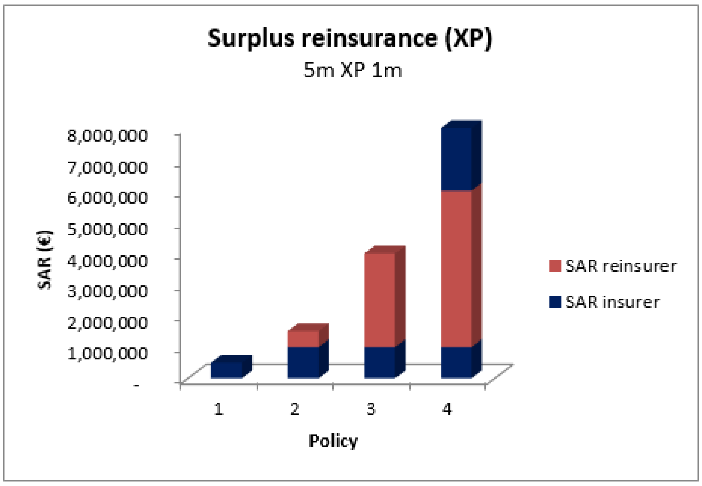
For mortality coverage with lump-sum payment in case of claim, the amount of claim is generally fixed by the sum at risk X_i and the treaty reimbursement mechanism works exactly as XL per Life non-proportional treaty: reinsurer engages to pay any amount of claim exceeding the retention and limited to the limit of the treaty, i.e. shown by following formulas:

$$X_i^{re} = min(L, max(X_i - R, 0))$$

However, for disability coverage, the amount of claim sometimes depends on the level of disability and therefore isn't fixed at the sum at risk level. The cession rate is used in this case:

$$X_i^{re} = X_i * r$$

The graph below shows the part of Reinsurer and insurer in sum at risk for a Surplus reinsurance 5m XP 1m with mortality coverage:



Policy	SAR	SAR insurer	SAR reinsurer
1	500 000	500 000	-
2	1 500 000	1 000 000	500 000
3	4 000 000	1 000 000	3 000 000
4	8 000 000	3 000 000	5 000 000

Figure 1.3: Surplus reinsurance

As precised from the start, the reinsurance premium is calculated in proportional basis, per policy by 2 kinds of basis:

- Quota-share basis:

$$P_{re}(i) = \frac{X_i^{re}}{X_i}.P_i$$

  Reinsurance premium is proportional to the original premium.

- Risk premium basis:

$$P_{re}(i) = X_i^{re}.q_{x_i}$$

  Reinsurer gives the specific rates applied to the ceded sum at risk which are generally different from insurance premium rate.

In practice, risk premium basis is more frequently used than quota-share basis. The reinsurance premium rate could be percentage of a reference table (e.g. regulation's mortality table) or it could be a proper premium table calibrated by Reinsurer.
An example of reinsurance premium table for policies with death benefit:

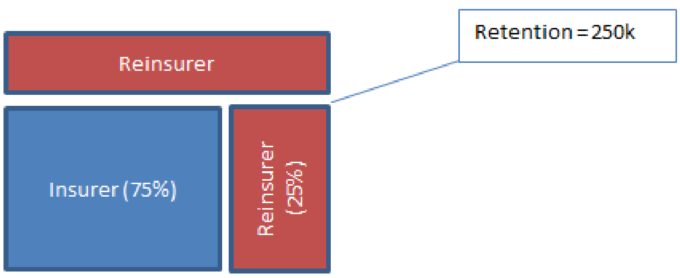
Age	qx			
	Male		Female	
	Smoker	Non smoker	Smoker	Non smoker
..	..	..	..	..
25	0,0025	0,0023	0,0021	0,0019
26	0,0027	0,0025	0,0023	0,0021
27	0,0029	0,0027	0,0025	0,0023
28	0,0031	0,0029	0,0027	0,0025
29	0,0033	0,0031	0,0029	0,0027
30	0,0035	0,0033	0,0031	0,0029
31	0,0037	0,0035	0,0033	0,0031
32	0,0039	0,0037	0,0035	0,0033
33	0,0041	0,0039	0,0037	0,0035
34	0,0043	0,0041	0,0039	0,0037
35	0,0045	0,0043	0,0041	0,0039
36	0,0047	0,0045	0,0043	0,0041
37	0,0049	0,0047	0,0045	0,0043
38	0,0051	0,0049	0,0047	0,0045
39	0,0053	0,0051	0,0049	0,0047
..	..	..	..	..

The rate could be different for male and female insured, smoking and non smoking insured.
The reinsurance result is calculated in the same manner as for quota-share reinsurance:

$$Reins.re = P_{re}(1 - RC) - C - PS.(P_{re}(1 - RC - \alpha) - C)$$

b) Surplus combined quota-share per life

It exists also a mixed form between surplus and quota-share. Under the surplus retention, insurer and Reinsurer shares the risk in a proportional basis, i.e. in a quota-share basis.
Example: Surplus with retention 250,000 combined with a quota-share 75% : 25%. The Reinsured retains 75% under the retention.



Quota-share 75:25 with retention 250K

If we note $1 - \alpha$ the cession rate below the surplus retention then the following formula shows the calculation of the reinsured sum at risk: Reinsured sum at risk:

$$X_i^{re} = min(L, max(X_i - R, 0)) + min(X_i, R) * (1 - \alpha)$$

Insurer's retained sum at risk: $X_i - X_i^{re}$

The reinsurance premium is normally defined in a risk premium basis (q_x applied in ceded sum at risk).

The reinsurance result is calculated in the same manner as for proportionally surplus treaty.

1.3.1.2 Non-proportional reinsurance

Unlike proportional treaties, the reinsurance premium for non-proportional treaties normally doesn't base on a per life basis. The premium is calculated in the portfolio level.

In general, it exists 3 popular kinds of non-proportional forms: *Excess of Loss (noted as "XL") per Life, XL per event (CAT)* and *Stop-loss*. In the scope of this document, we will focus only on XL per Life and XL CAT treaties for the pricing and the optimization of reinsurance.

a) XL per life

Reinsurer engages to pay any amount of claim exceeding the retention and limited to the limit of the treaty, i.e. shown by following formulas:

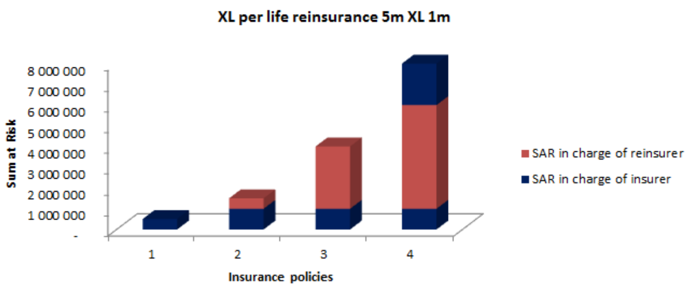
Reinsured sum at risk:

$$X_i^{re} = min(L, max(X_i - R, 0))$$

Insured sum at risk:

$$X_i = min(L, max(X_i - R, 0))$$

Example: 5m XL 1m



Policy	SAR	SAR insurer	SAR reinsurer
1	500 000	500 000	-
2	1 500 000	1 000 000	500 000
3	4 000 000	1 000 000	3 000 000
4	8 000 000	3 000 000	5 000 000

Figure 1.4: XL per Life reinsurance

In addition, it's common for non-proportional treaties to have an AAL clause (aggregate annual loss) per year which is not likely to be included in proportional basis.

$$C = min(\sum_j C_j, AAL)$$

Covered period:
The covered period of XL per life treaty is normally one year (yearly renewable). This treaty covers all claims exceeding the retention of the treaty incurred during the covered period. The claim could be reported after the covered period (IBNyR - Chapter 2 - 2.1.1), or not enough reported during the covered period (IBNeR - Chapter 2 - 2.1.1)
Treaty conditions:

- R - Retention (deductible): The retention of the treaty. The claim amount has to exceed this level in order to activate the treaty.
- L - Limit (Capacity): The maximum obligation of Reinsurer per claim.
- n - Number of reinstatement: The maximum number of amounts of limit could be payable by reinsurer per year after the first used limit.
- AAL - Annual Aggregate Limit - The maximum total amount of reinsured claim payable by Reinsurer per year: AAL = (n+1).L

Reinsurance premium: The reinsurance premium isn't calculated by a proportional basis per insured life. The reinsurance premium is given in form of rate of a **premium basis**. The premium basis could be EPI (Earned Premium Income) or ceded SAR. As the cover lasts during 1 year period, the rate is normally fixed at the beginning of the contract and re-adjusted at the end of the contract in function of the evolution of the premium basis. For example, if the premium basis stated in the treaty is EPI, then at the time that the reinsurance treaty is issued (begining of year n), the Reinsurer pays normally an amount called "Minimum Deposit Premium" (MDP).

$$MDP = q.EPI_{01/01/n}$$

At the end of the year, the Reinsurer and insurer readjust the reinsurance premium as following:

$$P_re = min(MDP, q.\frac{EPI_{01/01/n} + EPI_{31/12/n}}{2})$$

In general, the governance of reinsurance premium of XL per life treaty is much more simple than surplus reinsurance treaty as required no calculation per head. It's often used when there is limited data in per life basis (in some countries, insurer can not obtain all per life information in group business).

b) XL per event

The XL per event treaty (or also called in other words "CAT treaty") is placed by insurer after almost all other reinsurances such as per life and quota-share reinsurance in order to protect the portfolio against extreme losses caused by natural catastrophic events or man-made catastrophic events (industrial hazard, terrorism attacks, etc.).

Covered period:

As XL per life treaty, XL per event treaty has also normally one year covered period (yearly renewable). This treaty covers also all claims exceeding the retention of the treaty incurred during the covered period.

Treaty conditions:

- R - Retention (deductible): The retention of the treaty. The claim amount per event has to exceed this level in order to activate the treaty.
- L - Limit (Capacity): The maximum obligation of Reinsurer per event.
- n - Number of reinstatement: The maximum number of amounts of limit could be payable by reinsurer per year after the first used limit.
- AAL - Annual Aggregate Limit - The maximum total amount of reinsured claim payable by Reinsurer per year: AAL = (n+1).L
- Minimum number of victims: M - The minimum number of victims in the event in order to activate the treaty.

c) XL Stop loss

As CAT treaty, Stop-loss is also a reinsurance after the placement of per life reinsurances.

Stop-loss reinsurance will help to protect company result against the bad annual loss ratio due to either number or size of claims.

As already mentioned, a CAT cover will help the company be protected against a big catastrophic event. However, the CAT cover generally excludes the risk of epidemic and pandemic. The stop-loss cover, in complement to the CAT cover, can cover this risk and then can protect the final result of the insurance company. Stop-loss reinsuance, therefore, not only protect the insurer against large claims but also large number of small claims during the year.

Example: Stop-loss reinsurance with retention 80% and capacity 120%. If the annual claims ratio (total claim/ total premium) exceeds 80%, the treaty will pay the exceeding part limited by the capacity 120%.

Figure 1.5: Stop-loss reinsurance

1.3.2 Data available to Reinsurer

1.3.2.1 Portfolio data

The most expected data from Reinsurer side is in a per life basis, i.e:

Insured ID	Age	Gender	Smoking status	Type of benefit 1	Sum insured 1	Type of payment 1	Type of benefit 2	Sum insured 2	Type of payment 2
1988AND802	28	M	Yes	Death	100 000	Lum-sump	Disability	5 000	Annuity
1976BAG703	40	F	No	Critical Illness	200 000	Lum-sump	Disability	5 000	Annuity

This detailed information will help the Reinsurer project per life cash-flows scenario for future period. However, there are several constraints in terms of availability of data for a Reinsurer:

- The insurer provides only information about insureds protected by reinsurance. Therefore, it exists a basic risk: biased evaluation of risk for Reinsured portfolio. Indeed, the Reinsured portfolio doesn't take the same level of risk as the whole insured portfolio does.
- Per life information is not available for group business in some countries due to regulatory constraints.
- Per life information is not possible to obtain, only per policy information is, due to lack of insured ID in a non automatic underwriting process (happening more for old policies).
- The quality of data is not always as good as expected. Sometimes, the quality of data is very poor and Reinsurer has to make a lot of assumptions in its analysis.

a) Proportional reinsurance
In proportional reinsurance, insurer has to establish premium bordereau and claim bordereau for each policy. Therefore, Reinsurer has more chance to obtain in this case data per policy as needed above or even better, it can obtain also the accumulation per life information, i.e, the insured ID.
b) Non proportional reinsurance
In per life basis, as described in the section above, non proportional reinsurance is in general an agreement to solve the problem of per head data. The premium is paid in total of portfolio and not

per life. The information per life is not provided in detail but at some level, a model point is provided as in the following example:

XL per Life:

Example: A mortality portfolio (only mortality risk is covered) is ceded by a XL per life treaty 5,000,000 XL 1,000,000. The maximum SAR in this portfolio is 6,000,000 EUR. The table below shows the number of insureds, the average age per range of SAR:

Min SAR	Max SAR	Average SAR	Nb of insured lives	Average Age	Total SAR	Total SAR Retained by insurer	Total SAR Ceded to XL per Life
-	500 000	250 000	70 000	35	17 500 000 000	17 500 000 000	-
500 000	1 000 000	750 000	12 000	45	9 000 000 000	9 000 000 000	-
1 000 000	1 500 000	1 250 000	3 000	47	3 750 000 000	3 000 000 000	750 000 000
1 500 000	2 000 000	1 750 000	1 000	50	1 750 000 000	1 000 000 000	750 000 000
2 000 000	2 500 000	2 250 000	600	55	1 350 000 000	600 000 000	750 000 000
2 500 000	3 000 000	2 750 000	300	57	825 000 000	300 000 000	525 000 000
3 000 000	3 500 000	3 250 000	120	60	390 000 000	120 000 000	270 000 000
3 500 000	4 000 000	3 750 000	100	59	375 000 000	100 000 000	275 000 000
4 000 000	4 500 000	4 250 000	58	58	246 500 000	58 000 000	188 500 000
4 500 000	5 000 000	4 750 000	7	60	33 250 000	7 000 000	26 250 000
5 000 000	5 500 000	5 250 000	6	61	31 500 000	6 000 000	25 500 000
5 500 000	6 000 000	5 750 000	2	60	11 500 000	2 000 000	9 500 000
6 000 000	++++	-	-	-	-	-	-
Total			87 193	42	35 262 750 000	31 693 000 000	3 569 750 000

Figure 1.6: Model point

Based on this table, the Reinsurer could have a brief view on the portfolio without going into detail of each insured life.

XL per event (XL CAT):

For example, if the life portfolio is reinsured firstly by a XL per life and then in a second state by a CAT treaty, then the CAT treaty will cover the total SAR retained by insurer after the XL per life treaty.

By consequence:

- The SAR covered by CAT treaty: EUR 31,693,000,000
- The number of heads covered by CAT treaty: 87,193
- The average SAR per head covered by CAT treaty: EUR 363,481. Note that before the XL per Life treaty, the average SAR per head is higher: EUR 404,422

In practice, it's more important to have the information splitted by line of business, between group and individual policies since the catastrophic risk is much higher for group policies with high concentration. In this example:

	Group			Individual		
Risk class	*SAR*	*Nb of insureds*	*Average SAR*	*SAR*	*Nb of insureds*	*Average SAR*
Mortality	20 502 000 000	51 000	402 000	11 191 000 000	36 193	309 203

The risks covered by a reinsurance treaty should be relevant to the risks covered in the original policies (insurance policies), i.e., the CAT treaty shouldn't cover the risks which are not covered by the original policies. Therefore, the risks covered by insurance policies are provided as well.
In this example:

Class of risk	Typology	Excluded in the original policies (Y / N) Individual	Group
Political instability	Increase of riots	N	N
	Emergency state	N	N
	Civil war	N	N
	Active war	Y	Y
	Passive war	N	N
Terrorism	Ambushes	N	N
	Bomb attacks	N	N
	Chemical attacks	Y	Y
	Biological attacks	Y	Y
	Nuclear attacks	Y	Y
Industrial hazard	Nuclear risks	Y	Y
	Chemical risks	N	Y
	Biological risks	N	Y
	Contamination	N	N
Epidemic or pandemic	Air transmission	N	N
	Human contact transmission	N	N
	Water transmission	N	N
	Animal transmission	N	N
High Risk Occupations	Professional sports player	N	N
	Armed forces	N	N
	Ships crew	N	N
	Miners	N	N
	Rescuer (firemen, etc ...)	N	N
	Road conveyor	N	N
	Cabin crew	N	N
	Atomic energy workers	N	Y
	Ammunition workers	N	N
	Polluted workplace	N	N
Natural hazard	Earthquake	N	N
	Climatic catastrophes	N	N
	Volcanic activities	N	N
	Landslides	N	N
	Flood	N	N
Miscellaneous	Others		

Figure 1.7: Exclusion XL per event (CAT treaty)

In practice, the original policies always covers natural catastrophe risk and epidemic or pandemic risk. However, the epidemic and pandemic are not in the scope of this document. Also, the original policies may cover or not terrorism risk (bomb attacks, ambushes, etc), NBC (Nuclear, Biological, Chemical) and NBC terrorism risk. Political instability causes are normally covered except passive participation in war. High Risk Occupation is by default excluded in the reinsurance treaty. If Insurer wants to cover a group of high risk occupations (pilots, sport men, etc.), it has to request Reinsurer to put the group in a Special Acceptance.

1.3.2.2 Public catastrophes data

Natural CAT and man-made CAT except terrorism:

For CAT treaties, in most cases, Reinsurer doesn't possess enough claim data in its underlying portfolio. For the risk assessment, it has to use the data coming from general population in order to do the calibration. A transformation step will be done in order to deduce the risk on the reinsured portfolio.

One of the most popular source of CAT data is EMDAT database (http://emdat.be/). An example of available information can be shown in the following table:

Country	Location	Type	Sub Type	Name	Total_Killed	Year
United States	Mobile, Bayou La Batre, D ...	Storm	Tropical cyclone	Katrina	1833	2005
United States	Colorado	Storm	Local storm		600	1984
United States	West Virginia	Mass movement wet	Landslide		400	1972
United States	Alabama, Akansas, Kentuck ...	Storm	Local storm		354	2011

Figure 1.8: EMDAT database extraction

In order to understand better the database, we did some following descriptive statistics of the database. The using period is: 1900-2014.

Worst events in terms of number of deaths from 1970:

Worst events from 1970	Number of deaths
Indonesia , 2004 , Earthquake_(seismic_activity)	1 833
United_States , 2005 , Storm Katrina	17 127
Papua_New_Guinea , 1998 , Earthquake_(seismic_activity)	14 600
Bangladesh , 1991 , Storm	2 182
Soviet_Union , 1988 , Earthquake_(seismic_activity)	329
Somalia , 1997 , Flood	35 399
Ireland , 1985 , Transport_accident	183
Iran_Islam_Rep , 1990 , Earthquake_(seismic_activity)	25 000
South_Africa , 1987 , Flood	2 633
Haiti , 2010 , Earthquake_(seismic_activity)	165 708
Japan , 2011 , Earthquake_(seismic_activity)	40 000
Sri_Lanka , 2004 , Earthquake_(seismic_activity)	2 311
Honduras , 1998 , Storm	222 570
Turkey , 1999 , Earthquake_(seismic_activity)	506
Algeria , 1980 , Earthquake_(seismic_activity)	19 846
Poland , 1987 , Transport_accident	87 476
China_P_Rep , 2008 , Earthquake_(seismic_activity)	138 866

Statistics by type of disaster:

- Severity: 1900-2014

Type	Mean	Median	Mode	Maximum	Count	Cause of worst disaster
Earthquake (seismic activity)	2 854	21	1	242 000	899	Earthquake (ground shaking), China, 1976
Flood	488	11	3	500 000	3 053	Flood, China, Honan Province, 1939
Industrial Accident	48	21	10	2 700	1 150	Explosion, Colombia, 1956
Storm	92	44	50	2 000	52	Landslide, Peru, 1962
Mass Movement Wet	98	27	13	12 000	597	Landslide, Khait (Tadzhikistan), 1949
Miscellaneous accident	58	22	10	3 800	1 096	Fire, Japan, 1923
Storm	507	18	1	300 000	2 443	Storm, Bangladesh, 1970
Transport accident	44	23	10	4 000	5 199	Transport Accident, Philippines, 1987
Volcano	1 058	37	2	30 000	89	Volcanic eruption, Martinique, 1902
Wildfire	20	3	1	1 000	150	Forest fire, USA, 1918

- Frequency: 1970-2014

Type	mean	median	mode	max
Earthquake (seismic activity)	15	16	18	30
Flood	66	54	9	182
Industrial Accident	24	23	5	73
Mass movement dry	2	1	1	5
Mass Movement Wet	12	12	5	31
Miscellaneous accident	23	22	10	49
Storm	52	58	20	96
Transport accident	109	125	12	259
Volcano	2	1	1	6
Wildfire	4	4	1	13

Statistics of frequency by region:

Region	Mean 1970-2014	Mean 1980-2014
Central America	13	15
The Caribbean	8	9
Central Asia	36	42
Eastern Europe	6	7
The Far East	11	13
Sub-Saharan Africa	42	53
The Middle East	15	19
North Africa	11	12
North America	16	18
Oceania	6	7
Southern Africa	7	7
South America	27	32
The Indian sub-continent	42	52
South East Asia	36	42
Southern Europe	13	15
Former republics of Soviet Union/Russia	8	9
Western Europe	10	12

Looking at the statistics with two different lengths of time, we can conclude that there are more and more catastrophes.

It should be noticed that there are some catastrophic events that touch multiple countries. For example: the earthquake-tsunami in 2004 impacted Indonesia (165,000 deaths), India (16,389 deaths), Sri-Lanka (35,399 deaths).

CAT Terrorism:

- GTD (« Global Terrorism Database ») : http://www.start.umd.edu/gtd/, 3 main variables can be found from this database:
  - year of occurrence

- country name
- number of deaths

eventid	Claim_year	Country	Killed
200109110004	2001	United States	1 382
200109110005	2001	United States	1 382
199404130008	1994	Rwanda	1 180
200403210001	2004	Nepal	518
197808190004	1978	Iran	422
198707180001	1987	Mozambique	388
199605230007	1996	Burundi	375
200409010002	2004	Russia	344
198506230001	1985	Canada	329
199802010001	1998	Sri Lanka	320
199607200008	1996	Burundi	304

Figure 1.9: GTD database extration

1.3.3 The role of traditional reinsurance for a life insurer

Reinsurance plays very important roles in insurance activities and risk management:

- Protect the insurer against the occurrence of extreme events or the accumulation of many normal events.
- Reduce the volatility of insurance portfolio.
- Increase the underwriting capacity of insurer. Reinsurance can help insurers underwrite the risk requiring an amount of solvency capital greater than their own.
- Play the role as consulting. Reinsurer has sometimes more experiences and data in the market due to the fact that they work with many insurers at the same time. Reinsurer can therefore provide some services to insurer such as: launching new product, pricing, medical underwriting, etc.

1.4 Reinsurance in Solvency II context

1.4.1 Reinsurance in calculation of solvency capital requirement

1.4.1.1 Reinsurance under Solvency I

Solvency Margin and relief through reinsurance in Solvency I framework are measured very easily based on factors on volumes and limited by arbitrary factors:

$$Solvency\ Margin = 4\%\ of\ statutory\ reserves + 0.3\%\ of\ Sum\ At\ Risk$$

Capital relief through reinsurance as % of the Solvency Margin:

$$Min(reinsured\ reserves/\ total\ reserves, 15\%) + Min(Reinsured\ SAR/Sum\ At\ Risk, 50\%)$$

1.4.1.2 Reinsurance under Solvency II

Under Solvency II, reinsurance has double effects which generally increase the SII Solvency Ratio: $\frac{BOF}{SCR}$

- Reduction of Risk Margin leads to the increase of Basic Own Funds
- Reduction of SCR

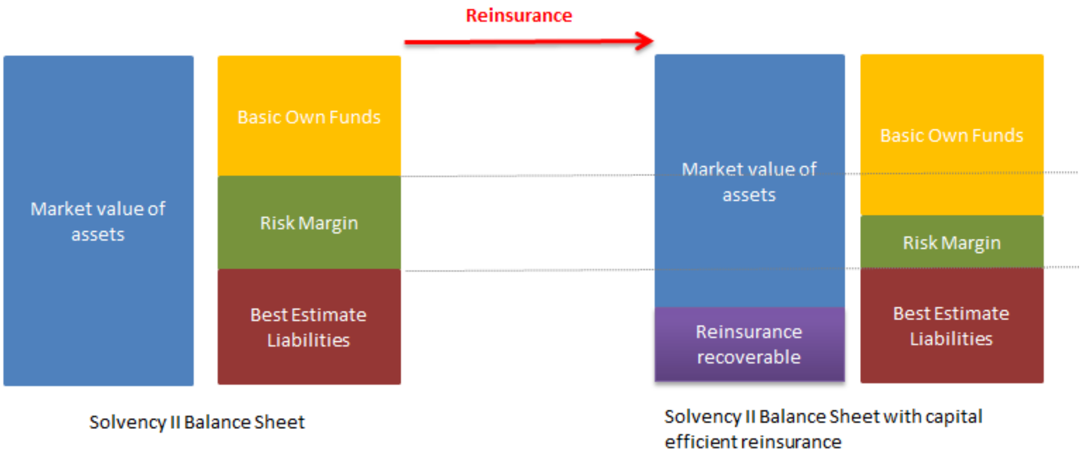


Figure 1.10: Impact of reinsurance under Solvency II

Chapter 2

Non-proportional reinsurance pricing models

In this thesis, we aim to propose the reinsured risk-costing models for non-proportional reinsurance. "Risk-costing" means that we do not set any margin above three traditional components of the price: pure premium, cost of capital and expense. We apply the proposed models in an example of reinsurance program containing:

- XL per life treaty for per life level.
- XL per event (XL CAT) for the part retained net of the XL per life treaty.

The taken example uses an insurance portfolio based in France. The currency is EUR. The structures of the reinsurance treaties are:

- XL per life: EUR 5M XL 1M. The number of reinstatement: 15.
- XL CAT: EUR 100M XL 10M. The number of reinstatement: 1. The minimum number of victims to trigger the treaty: 5.

The information for each kind of treaty is given as following:

XL per life: The treaty is priced for the 2017 coverage.
List of data available and assumptions:

- 2010-2016 historical claims data above the retention of XL per life treaty including the development of claims

Status	Occurrence	2010	2011	2012	2013	2014	2015	2016
Closed	2010	2 100 000	2 200 000	2 200 000	2 200 000	2 200 000	2 200 000	2 200 000
Closed	2010	1 500 000	1 700 000	1 700 000	1 700 000	1 700 000	1 700 000	1 700 000
Closed	2010	1 300 000	1 300 000	1 400 000	1 400 000	1 400 000	1 400 000	1 400 000
Closed	2010	1 250 000	1 290 000	1 350 000	1 350 000	1 350 000	1 350 000	1 350 000
Closed	2010	1 800 000	1 900 000	1 950 000	1 950 000	1 950 000	1 950 000	1 950 000
Closed	2010		1 200 000	1 250 000	1 250 000	1 250 000	1 250 000	1 250 000
Closed	2010		1 300 000	1 300 000	1 300 000	1 300 000	1 300 000	1 300 000
Closed	2010			1 050 000	1 050 000	1 050 000	1 050 000	1 050 000
Closed	2011		1 050 000	1 050 000	1 050 000	1 050 000	1 050 000	1 050 000
Closed	2011		1 150 000	1 150 000	1 150 000	1 150 000	1 150 000	1 150 000
Closed	2011		1 100 000	1 100 000	1 100 000	1 100 000	1 100 000	1 100 000
Closed	2011		2 300 000	2 300 000	2 300 000	2 300 000	2 300 000	2 300 000
Closed	2011		1 200 000	1 400 000	1 400 000	1 400 000	1 400 000	1 400 000
Closed	2011			1 100 000	1 100 000	1 100 000	1 100 000	1 100 000
Closed	2011			1 500 000	1 500 000	1 500 000	1 500 000	1 500 000
Closed	2011				1 600 000	1 900 000	1 900 000	1 900 000
Closed	2011				1 010 100	1 010 100	1 010 100	1 010 100
Open	2012			1 050 000	1 200 000	1 200 000	1 200 000	1 200 000
Closed	2012			1 040 000	1 050 000	1 040 000	1 040 000	1 040 000
Closed	2012			1 400 500	1 400 500	1 400 500	1 400 500	1 400 500
Closed	2012			1 502 000	1 760 000	1 760 000	1 760 000	1 760 000
Closed	2012				1 202 000	1 202 000	1 202 000	1 202 000
Closed	2012				1 040 000	1 040 000	1 040 000	1 040 000
Closed	2012				1 050 000	1 050 000	1 050 000	1 050 000
Closed	2012					1 900 000	2 100 000	2 100 000
Open	2012					1 800 000	1 900 000	1 900 000
Open	2012						1 100 000	1 200 000
Closed	2013				2 300 000	2 300 000	2 300 000	2 300 000
Closed	2013				1 203 000	1 203 000	1 203 000	1 203 000
Closed	2013				1 010 000	1 200 000	1 200 000	1 200 000
Closed	2013				2 130 000	2 102 000	2 102 000	2 102 000
Closed	2013					1 400 500	1 500 000	1 500 000
Open	2013						1 405 650	1 405 650
Closed	2014					2 101 000	2 101 000	2 101 000
Closed	2014					2 300 000	2 300 000	2 300 000
Closed	2014					1 010 010	1 500 000	1 500 000
Closed	2014						1 300 210	1 300 210
Closed	2014						1 210 210	1 210 210
Closed	2014						1 030 210	1 030 210
Open	2014							1 100 100
Open	2014							1 000 230
Closed	2015						1 800 400	1 800 400
Closed	2015						1 701 000	1 701 000
Closed	2015						1 340 000	1 340 000
Closed	2015						1 450 000	1 450 000
Open	2015						1 430 000	1 490 000
Open	2015							1 200 000
Open	2015							1 100 000
Open	2016							3 000 000
Open	2016							1 020 000
Open	2016							1 200 000
Open	2016							1 010 000

Figure 2.1: XL per life - List of claims higher than retention

Note that the amounts didn't take into account inflation rate. They are reported amounts.

- 2010-2016 historical total ceded sum at risk
- 2010-2016 historical earned reinsurance premium
- Model point as in example in chapter 1
- Assuming that there is no special acceptance
- Assuming that there was no change in structure over past years

XL per event (CAT treaty): The treaty is priced for the 2017 coverage.
List of data available:

- Total sum at risk and number of heads
- Sum at risk by range
- Model point in example in chapter 1
- Assuming that there is no special acceptance
- Concentration site information with the most concentrated groups in the portfolio.

2.1 XL per life

2.1.1 Some important definitions

IBNyR and IBNeR

- IBNyR: Incurred But Not yet Reported. This term mentions claims happened during the covered period of underlying treaty but not yet reported at the end of this period. For example, at the end of year N, there are 10 reported claims happened during year N. In the year N + 1,there are 2 claims happened during year N are reported. At the end of year N, these 2 claims are viewed as IBNyR.
- IBNeR: Incurred But Note enough Reported. This term signifies the part of claim that is not reported yet. For example at the end of year N, the claim is reported with the amount of 100. The ultimate amount of this claim is 150 so that at the time the amount of 100 is reported, IBNeR for this claim is 50.

Occurrence year: The year when the claim occurs.
Declaration year: The year when the claim is declared.
Underwriting year: The year in which the portfolio is covered by the treaty, i.e. the treaty is underwritten. For Excess of loss reinsurance, when we take all claims in the underwriting year N, it means that all claims having occurrence year N.
Accounting year: When we take all claims by accounting year M, it means that all claims having declaration year M.
Ultimate claim: The final total amount of claim when it's concluded as not having anymore IBNR. We can mention about the ultimate amount of one claim or the ultimate amount of claims per occurrence year.
Notion:

- $X_{k,j,i}$ is the amount of claim k incurred in year i, reported in year j ($j \geq i$). As we take into account historical claims in 2010-2016, k and j takes value from $N-7$ to $N-1$ where N is next underwriting year, i.e. 2017.

- EPI_k is the earned premium income of year k in the insurance portfolio which is used as an indicator for the basis of the volume of portfolio.
- a_k is inflation rate of year k which is used as an indicator for the valuation of amounts in year k.

2.1.2 Pricing model

2.1.2.1 Grand principle

The premium for the XL per life treaty underwritten in year N is the combination of:

- Expected claim amount The expected claim amount in year N could be estimated by different methods:
  - Frequency - severity
  - Burning Cost
  - Incident rates
  - Combined method

  In the following sections, we will go through each method and discuss their advantage and limitation.
- Cost of capital or variance: When reinsurer underwrites a treaty, it takes risks. Therefore, an amount of solvency capital is required. Though the solvency required capital is normally calculated in the portfolio level, an amount allocated per treaty should be taken into the price. The calculation of cost of capital per treaty will be described in chapter 3.
- Expenses: Once the treaty is signed, reinsurer has to spend fees on administration, acquisition, claim management, etc. The expense is often taken as fixed percentage of the commercial premium for all treaties. In this thesis, we consider the expenses rate used in pricing is 10%.

2.1.2.2 Frequency-severity approach

We call the claims triggering XL per life retention or around the retention "atypical claims".
Principle: The frequency of and severity of atypical claims incurred in year N are calibrated and then simulated in 500K scenarios. The expected value of claim amount is taken as the average amount of 500K scenarios.
We have:

$$S = \sum_{i=1}^{N} X_i^{re}$$

Where:

- S is the annual reinsured claim amount
- N is the annual number of claims

- X_i^{re} is reinsured amount of claim i

The price of XL per Life treaty is a composition of the expected value, the cost of risk which represents the volatility of result undertaken by Reinsurer or the cost of require capital and the expense issued by the management of the reinsurance treaty.

$$P = \frac{E(S) + Cost.of.risk}{1 - expense}$$

The calibration of severity is done by 2 fitting methods: MLE (Maximum Likelihood Estimation) and MME (Method of Moment Estimation).

Maximum Likelihood Estimation (MLE)

Suppose there is a sample $x_1 \dots x_n$ of n independent and identically distributed (i.i.d) observations, coming from a distribution with an unknown probability density function $f(.)$. It is assumed that the function f belongs to a certain family of distributions $\{f(.|\theta), \theta \in \Theta\}$, called the parametric model, so that $f = f(.|\theta_0)$. Θ is the definition interval of the function's parameters. The value θ_0 is unknown and is referred to as the "true value" of the parameter. It is desirable to find an estimator which would be as close as possible to the true value θ_0.

To use the method of maximum likelihood, one first specifies the joint density function for all observations. For an iid sample, this joint density function is:

$$f(x_1, x_2, ..., x_n|\theta) = f(x_1|\theta).f(x_2|\theta)...f(x_n|\theta)$$

The observed values $x_1, x_2, ..., x_n$ are fixed realizations of this function, whereas θ will be the function's parameters and allowed to vary freely; this function will be called the likelihood:

$$\mathcal{L}(\theta|x_1, x_2, ..., x_n) = f(x_1, x_2, ..., x_n) = \prod_{i=1}^{n} f(x_i|\theta)$$

The log likelihood function is more often used in practice:

$$\mathcal{LL}(\theta|x) = ln(\mathcal{L}(\theta|x_1, x_2, ..., x_n)) = \sum_{i=1}^{n} ln(f(x_i|\theta))$$

The method of maximum likelihood estimates θ_0 by finding a value of θ that maximizes $\mathcal{LL}(\theta|x)$. This method of estimation defines a maximum-likelihood estimator (MLE) of θ_0:

$$\widehat{\theta}_{mle} \subseteq \{arg \max_{\theta \in \Theta} \mathcal{LL}(\theta|x)\}$$

Method of Moment Estimation (MME)

The method of moments is a method of estimation of population parameters such as the mean, variance, median, etc., by equating sample moments with unobservable population moments and then solving those equations for the quantities to be estimated.

Let $(x_1, x_2, ..., x_n)$ be an iid sample of the random variable X which is assumed to have a density where

is a vector parameter:
Empirical mean of X is given by:

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Empirical variance of X, the unbiased estimator of the variance, is given by:

$$S_X^2 = \frac{1}{n-1}\sum_{i=1}^{n} (X_i - \bar{X})^2$$

Empirical standard deviation of $X = S_X$
Empirical distribution function of $X = \widehat{F_X}$ which is defined by :

$$\forall x \in \mathbb{R}, \hat{F}_X(x) = \frac{1}{n}Card\{i \in [1;n]/X_i < x\}$$

Empirical α-quantile of X is defined by:

$$\forall \alpha \in]0;1], \hat{F_X^{-1}}(\alpha) = Inf\{x \in Supp(X)/\hat{F}_X(x) \geq \alpha\}$$

Criteria to choose the best fitting quality distribution

- **Akaike Information Criterion (AIC)**

  $$AIC = 2k - ln(\mathcal{L})$$

  Where:

  - k denotes the number of parameters of the statistical model
  - $\mathcal{L}$ denotes the maximized value of the likelihood function

- **Kolmogorov Smirnov criterion (KS)**

  The KS statistic quantifies a distance between the empirical distribution function of the sample and the cumulative distribution function of the reference distribution.

  $$D_n = sup_x|F_n(x) - F(x)|$$

- **Cramer-von Mise**

  Let $x_1, x_2, ...x_n$ be the observed values in increasing order. The Cramer-von Mises criterion uses the fact, that if is the underlying random variable X with the continuous distribution function $F(X)$, then $F(X)$ follows the uniform distribution $U(0,1)$. Cramer-von Mise statistic:

  $$T = \frac{1}{12n} + \sum_{i=1}^{n}\Big[\frac{2i-1}{2n} - F(x_i)\Big]$$

- **Anderson-Darling**

  The Anderson-Darling statistic:

$$AD = n\int_{-\infty}^{\infty}\Big[F_n(x) - F(x)\Big]^2[F(x)(1-F(x)]dF(x)$$

(a) Frequency:

Frequency of atypical claims could be modelled by discrete distributions such as: Poisson, Binomial or Negative Binomial distributions. Each of them has different states of relationship between mean and variance, i.e:

- Negative Binomial: Mean $\leq$ Variance
- Poisson: Mean $>$ Variance

The reason why we won't use Binomial distribution is that it will simulate the number of claims limited to a parameter n, which is not realistic.

Poisson distribution

N is a Poisson random variable if it takes non-negative integer value: 0,1,2,... and its probability function is as following:

$$P(N = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Mean and variance:

$$E(N) = Var(N) = \lambda$$

The parameter λ can be easily estimated by the empirical mean of the given sample.

Negative binomial distribution

We consider a sequence of independent trials with each following a Bernoulli distribution with probability of success p in each trial. We are observing the sequence until there are r times of failure. The number of success we have seen, K, will have Negative Binomial distribution. It takes non-negative integer value and its probability function is as following:

$$f(k, r, p) = P(K = k) = C^k_{k+r-1}.p^k.(1-p)^r$$

Mean and variance:

$$E(K) = \frac{rp}{1-p} = \mu$$

$$Var(K) = \frac{rp}{(1-p)^2} = \sigma^2$$

By consequence, the parameters r and p are estimated easily by Method of Moment, i.e, with empirical mean and variance of the given sample:

$$p = \frac{\sigma^2 - \mu}{\sigma^2}$$

$$r = \frac{\mu^2}{\sigma^2 - \mu}$$

Consider that $r \to \infty$, denote mean of K as λ then

$$\lambda = \frac{rp}{1-p} \Rightarrow p = \frac{\lambda}{r+\lambda}$$

Hence, the probability mass function will be:

$$f(k;r,p) = \frac{\Gamma(k+r)}{k!.\Gamma(r)}.p^k.(1-p)^r = \frac{\lambda^k}{k!}.\frac{\Gamma(r+k)}{\Gamma(r).(r+\lambda)^k.\frac{1}{(1+\frac{\lambda}{r})^r}}$$

If we consider $r \to \infty$ then $\frac{\Gamma(r+k)}{\Gamma(r).(r+\lambda)^k} \to 1$ and $\frac{1}{(1+\frac{\lambda}{r})^r} \to \frac{1}{e^\lambda}$ and finally,

$$\lim_{r\to\infty} f(k,r,p) = \frac{\lambda^k e^{-\lambda}}{k!}$$

which is the probability mass function of Poisson distribution.
In other words:

$$Poisson(\lambda) = \lim_{r\to\infty} NB(r, \frac{\lambda}{\lambda + r})$$

The Negative Binomial distribution allows to take into account the over-dispersion problem (variance is higher than mean) which can not be shown in Poisson distribution (where variance is equal to mean).

Binomial distribution

When variance is observed to be smaller than mean, one can use the Binomial distribution. K is the number of successes in a sequence of n independent trials where each has the probability of success p. We note K follows a binomial distribution with parameters $n \in N$ and $p \in [0,1]$, i.e, $K \sim B(n,p)$. The probability mass function of K is as following:

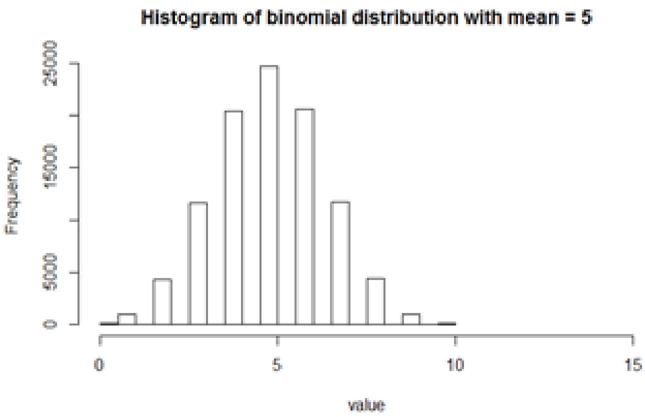
$$P(K=k) = C_n^k \; p^k (1-p)^{n-k}$$

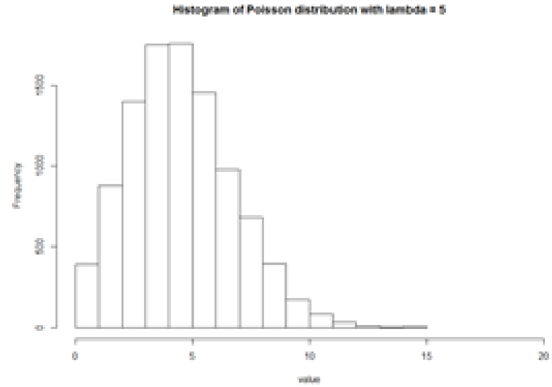
Mean and variance:

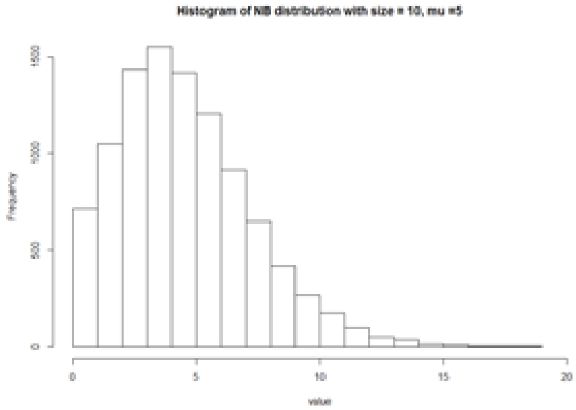
$$E(K) = n.p$$

$$Var(K) = n.p.(1-p)$$

Similar to the 2 frequency distributions above, the parameters of Binomial distribution can be easily estimated by the empirical mean and variance of the given sample.
The following histograms show the difference in variance of those distributions. The negative binomial distribution gives larger dispersion than the Poisson distribution does and the binomial distribution gives the lowest dispersion.







(b) Severity:

As claim amounts in XL per life reinsurance are above the retention, they are considered as "atypical" claims, i.e. high amounts of claims. The normal distribution with generally "short tail" is not appropriated for this kind of claim amounts. The distributions with "intermediate" or "long" tails are more relevant.

We will study in this stage two different fitting approaches:

- Traditional approach where we use usual distribution to fit the severity
- Second approach where we test the "truncated" distributions to see whether they are more adapted than usual distribution.

b.1. Traditional approach: usual distributions

Following usual distributions are tested:

- Log normal
- Gamma
- Weibull
- Pareto
- GPD (Generalized Pareto Distribution)

Log normal distribution

X follows log-normal distribution with two parameter μ and σ that are respectively mean and standard deviation of X if and only if:

$$X = e^{\mu+\sigma.Z}$$

where Z is standard normal random variable.

Apparently, X has positive values.

Probability density function:

$$f(x, \mu, \sigma) = \frac{1}{x\sigma\sqrt{2\pi}}\ exp[-\frac{(lnx-\mu)^2}{2\sigma^2}]$$

The mean and variance of a random variable following a Log-normal distribution: Mean and variance:

$$E(X) = e^{\mu+\frac{\sigma^2}{2}}$$

$$Var(X) = e^{2\mu+\sigma^2}(e^{\sigma^2}-1)$$

MLE

$$\mathcal{LL}(\mu, \sigma|x) = ln(\mathcal{L}(\mu, \sigma|x_1, x_2, ..., x_n)) = \sum_{i=1}^{n} ln(f(x_i|\mu, \sigma)) = -\sum_i (lnx_i\sigma\sqrt{2\pi}) - \sum_i \frac{(lnx_i-\mu)^2}{2\sigma^2}$$

The MLE parameters optimizing this function are:

$$\hat{\mu} = \frac{\sum_i^n lnx_i}{n}$$

$$\hat{\sigma} = \sum_i^n \frac{(lnx_i-\hat{\mu})^2}{n}$$

MME

The parameters μ and σ could be obtained by following equations:

$$\mu = ln\Big(\frac{E[X]^2}{\sqrt{Var[X]+E[X]^2}}\Big)$$

$$\sigma^2 = ln(1+\frac{Var[X]}{E[X]^2})$$

By using the empirical mean and variance, we can calculate the corresponding estimators for μ and σ.

Weibull distribution

The cumulative distribution function of a Weibull variable X:

$$F(x, \beta, \alpha) = 1 - e^{-(\frac{x}{\alpha})^\beta}$$

for $x \geq 0$ and $F(x, \beta, \alpha) = 0$ for $x < 0$

The probability density function:

$$f(x, \alpha, \beta) = \begin{cases} \frac{\beta}{\alpha}\big(\frac{x}{\alpha}\big)^{\beta-1} e^{-(\frac{x}{\alpha})^\beta} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Where $\beta > 0$ is the shape parameter and $\alpha > 0$ is the scale parameter of the distribution.
The mean and variance of a random variable following a Weibull distribution:

$$E(X) = \alpha\ \Gamma(1 + \frac{1}{\beta})$$

$$Var(X) = \alpha^2 [\Gamma(1 + \frac{2}{\beta}) - (\Gamma(1 + \frac{1}{\beta}))^2]$$

<u>MLE</u>

$$\mathcal{LL}(\alpha, \beta|x) = ln(\mathcal{L}(\alpha, \beta|x_1, x_2, ..., x_n)) = \sum_{i=1}^{n} ln(f(x_i|\alpha, \beta)) =$$

The MLE method for Weibull requires some additional steps. We firstly recall the location-scale property of $ln(X)$:
We note $Y = ln(X)$ then the cumulative probability function of Y is:

$$\begin{aligned} P(Y < y) = P(ln(X) \leq y) = P(X \leq exp(y)) = 1 - exp\Big[-\Big(\frac{exp(y)}{\alpha}\Big)^{\beta}\Big] \\ = 1 - exp\Big[-exp\Big\{\Big(y - ln(\alpha)\Big).\beta\Big\}\Big] = 1 - exp\Big[-exp\Big(\frac{y - ln(\alpha)}{\frac{1}{\beta}}\Big)\Big] \\ = 1 - exp\big[-exp\Big(\frac{y-u}{b}\Big)\big] \qquad\qquad (b = \frac{1}{\beta}, u = ln(\alpha)) \end{aligned}$$

The corresponding sample of Y is: $y_1, y_2, ..., y_n \quad y_i = ln(x_i)$
We have $F(y) = 1 - exp\Big[-exp(\frac{(y-u)}{b})\Big] = G(\frac{y-u}{b})$ with $G(z) = 1 - exp(-exp(z))$ with $g(z) = G^{'}(z) = exp(z - exp(z))$. Thus,

$$f(y) = F^{'}(y) = \frac{d}{dy}F(y) = \frac{1}{b}g(\frac{y-u}{b})$$

with

$$ln(f(y)) = -ln(b) + \frac{y-u}{b} - exp(\frac{(y-u)}{b})$$

As partial derivatives of $ln(f(y))$ with respect to u and b we get

$$\frac{\delta}{\delta u}ln(f(y)) = -\frac{1}{b} + \frac{1}{b}exp(\frac{y-u}{b})$$

$$\frac{\delta}{\delta b}ln(f(y)) = -\frac{1}{b} - \frac{1}{b}\frac{y-u}{b} + \frac{1}{b}\frac{y-u}{b}exp(\frac{y-u}{b})$$

and thus as likelihood equation:

$$0 = -\frac{n}{b} + \frac{1}{b}\sum_{i=1}^{n} exp(\frac{y_i - u}{b})$$

or

$$exp(u) = [\frac{1}{n}\sum_{i=1}^{n} exp(\frac{y_i}{b})]^b$$

and

$$0 = -\frac{n}{b} - \frac{1}{b}\sum_{i=1}^{n}(\frac{y_i - u}{b}) + \frac{1}{b}\sum_{i=1}^{n}\frac{y_i - u}{b}exp(\frac{y_i - u}{b})$$

[5] shows that β is the root of following equation:

$$0 = \sum_{i=1}^{n} y_i w_i(\beta) - \frac{1}{\beta} - \bar{y} \qquad where \quad w_i(\beta) = \frac{exp(y_i\beta)}{\sum_j^n exp(y_j\beta)} \quad with \quad \sum_i^n w_i(\beta) = 1$$

Then the estimation of β could be solved by numerical method.

MME

The two first moments are:

$$m_1 = \hat(\mu) = \frac{1}{\alpha}^{\frac{1}{\beta}} \Gamma\Big(1 + \frac{1}{\beta}\Big)$$

$$m_2 = \hat{\mu}^2 + \hat{\sigma}^2 = \Big(\frac{1}{\alpha}\Big)^{\frac{1}{\beta}} \Big[\Gamma(1 + \frac{2}{\beta}) - (\Gamma(1 + \frac{1}{\beta}))^2\Big]$$

Then:

$$\frac{\hat{\sigma}^2}{\hat{\mu}^2} = \frac{\Gamma(1+\frac{2}{\beta}) - \Gamma^2(1+\frac{1}{\beta})}{\Gamma(1+\frac{1}{\beta})} = \frac{\Gamma(1+\frac{2}{\beta})}{\Gamma(1+\frac{1}{\beta})} - \Gamma(1 + \frac{1}{\beta})$$

Which depends only on parameter β. Therefore, the estimator of the shape parameter is obtained using the function:

$$h(c) = \Big(1 + \frac{S_x^2}{\bar{x}^2}\Big)\Big(\Gamma(1 + \frac{1}{\beta})\Big) - \Gamma(1 + \frac{2}{\beta})$$

The MME estimator of β is the solution of $h(\beta) = 0$. After the estimation of β, the scale parameter is estimated by:

$$\hat{\alpha} = \Big(\frac{\bar{x}}{\Gamma(1+\frac{1}{\hat{\beta}})}\Big)^{\hat{\beta}}$$

Pareto distribution

The cumulative probability function of the Pareto random variable is as following:

$$F_X(x) = \begin{cases} 1 - (\frac{x_m}{x})^{\alpha} & x \geq x_m \\ 0 & x < x_m \end{cases}$$

The probability density function:

$$f_X(x) = \begin{cases} \frac{\alpha x_m^{\alpha}}{x^{\alpha+1}} & x \geq x_m \\ 0 & x < x_m \end{cases}$$

The mean and variance of a random variable following a Pareto distribution:

$$E(X) = \begin{cases} \infty & \alpha \leq 1 \\ \frac{\alpha x_m}{\alpha - 1} & \alpha > 1 \end{cases}$$

$$Var(X) = \begin{cases} \infty & \alpha \in (1, 2] \\ (\frac{x_m}{\alpha-1})^2 . \frac{\alpha}{\alpha-2} & \alpha > 2 \end{cases}$$

MLE

$$ln f_X(x) = ln\alpha + \alpha.ln x_m - (\alpha + 1) ln x$$

Hence,

$$\sum_{i=1}^{n} f(x_i|\alpha) = n.ln\alpha + n.\alpha.ln x_m - (\alpha + 1)\sum_{i=1}^{n} x_i$$

The MLE estimation of parameter α is the solution of the equation:

$$\frac{\delta}{\delta\alpha}\sum_{i=1}^{n} f(x_i|\alpha) = \frac{n}{\alpha} + nlnx_m - \sum_{i=1}^{n} lnx_i = 0$$

Hence,

$$\hat{\alpha} = \frac{n}{\sum_i lnx_i - n.lnx_m}$$

MME

The MME estimation of α is deduced by the first moment equation.

$$\hat{\alpha}_{MME} = \frac{\bar{x}}{\bar{x} - x_m}$$

Generalized Pareto Distribution

The cumulative distribution function of random available X following the Generalized Pareto Distribution:

$$F_{(\xi,\mu,\delta)}(x) = \begin{cases} 1 - \left(1 + \frac{\xi(x-\mu)}{\delta}\right)^{-\frac{1}{\xi}} & for\ \xi \neq 0 \\ 1 - exp(-\frac{x-\mu}{\delta}) & for\ \xi = 0 \end{cases}$$

The probability density function:

$$f_{(\xi,\mu,\delta)}(x) = \frac{1}{\delta}\Big[1 + \frac{\xi(x-\mu)}{\delta}\Big]^{(-\frac{1}{\xi}-1)}$$

for $x \geq \mu$ when $\xi \geq 0$ and $\mu \leq x \leq \mu - \frac{\delta}{\xi}$ when $\xi < 0$

The mean and variance of a random variable following a Generalized Pareto Distribution:

$$E(X) = \mu + \frac{\delta}{1-\xi},\ \xi < 1$$

$$Var(X) = \frac{\delta^2}{(1-\xi)^2(1-2\xi)},\ \xi < \frac{1}{2}$$

If $\xi \geq 1/2$ variance doesn't exist, if $\xi \geq 1$ even mean doesn't exist.

With shape $\xi > 0$ and location $\mu = \sigma/\xi$, the GPD is equivalent to the Pareto distribution with scale $x_m = \sigma/\xi$ and shape $\alpha = 1/\xi$.

MLE

$$\frac{\mathcal{LL}(u,\sigma,\xi|x)}{n} = -ln(\sigma) - (1 + \frac{1}{\xi}).\sum_{i=1}^{n} ln\Big(1 + \xi\frac{x_i - \mu}{\sigma}\Big)$$

We take $\rho = \frac{\xi}{\sigma}$, then estimator of ξ and ρ are solution of:

$$(\hat{\xi},\hat{\rho}) = argmax\{-ln(\frac{\xi}{\rho}) - (1 + 1/\xi).\sum_{i=1}^{n} ln(1 + \rho(x_i - \mu))\}$$

MME

The estimators of ξ and σ are:

$$\hat{xi} = \frac{1}{2}\Big[1 - \frac{(\bar{x}-\mu)^2}{S_x^2}\Big]$$

$$\hat{\sigma} = \frac{1}{2}(\bar{x} - \mu)\Big[1 + \frac{(\bar{x} - \mu)^2}{S_x^2}\Big]$$

b.2. Truncated distributions

Why truncated distribution?

In statistical theory, we knew about usual distributions for extreme value studies, for example: Weibull, log-normal, log-gamma, etc. However, those distributions have ranges of variable's values in entire set R or R+ while as discussed previously, we're interested in atypical claims, i.e, whose amounts are higher than certain thresholds. That's why we will analyse the use of the "truncated distributions" - whose range are bounded at some levels.

Definition:

If g(x) and G(x) is the density and cumulative distribution functions of a random variable X, then the density distribution function of the truncated variable Y = X/a<X<b is as following:

$$f_{Y,a,b}(x) = \frac{g(x)}{G(b) - G(a)} * I_{a \leq x \leq b}$$

Note from this formula that f(x) has exactly the same parameters as g(x).

In our case, we will use left-truncated distribution since our data accepts values in the interval [R, $+\infty$) where R is the threshold of atypical claims.

We call g(x) as original distribution, and f(x) as truncated distribution.

The following example shows the difference between the original distribution and its truncated version:

Example 1: two-side truncated logistic distribution and logistics distribution.

Following is the histogram of X with logistics distribution and Y with truncated logistics distribution. Its parameters are:

- Location = 0
- Scale = 2

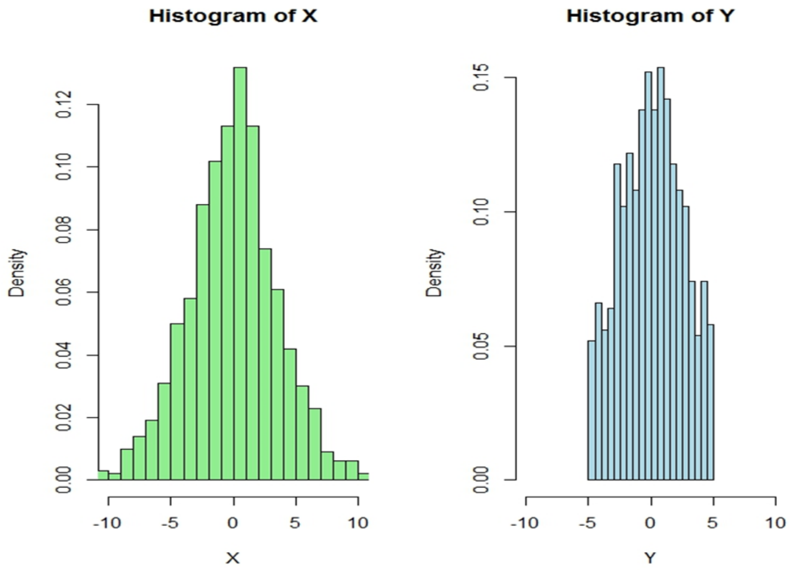


Figure 2.2: Example of truncated distribution - Logistic distribution

The density of Y concentrates in the interval (-5, 5) that makes higher density values of Y in this region than density values of X.

Cumulative probability function and inverse probability function are defined as following:

$$F_Y(x) = \frac{G(max(min(x,b),a)) - G(a)}{G(b) - G(a)}$$

$$F_Y^{-1}(p) = G^{-1}(G(a) + p.(G(b) - G(a)))$$

The mean and variance are calculated as following:

$$E(X) = \int_a^b x f_X(x) dx$$

,

$$Var(X) = \int_a^b \{x - E(X)\}^2 f_X(x) dx$$

In this document, one of our objective is to compare the efficiency of left-truncated distributions and their usual version in fitting severity of atypical claims. We will try to test the following truncated distributions:

- Left-truncated Log-normal
- Left-truncated Weibull

Left-truncated Log-normal

We consider the variable X following left-truncated log-normal distribution then $X \in [u, +\infty]$.

Mean and variance:

$$E(X) = u + exp(\mu + \frac{\sigma^2}{2})$$

$$Var(X) = exp(2.\mu + \sigma^2).(exp(\sigma^2) - 1)$$

<u>MLE</u>

The MLE estimators for μ and σ are:

$$\hat{\mu} = \sum_{i=1}^{n} ln(x_i - u)$$

$$\hat{\sigma} = \frac{n}{n-1} \sum_{i=1}^{n} \Big[ln(x_i - u) - \hat{\mu} \Big]^2$$

<u>MME</u>

The estimators MME for μ and σ are:

$$\hat{\sigma}^2 = ln\Big[1 + (\frac{S_x^2}{\bar{x} - u})^2\Big]$$

$$\hat{\mu} = ln(\bar{x} - u) - \frac{\hat{\sigma}^2}{2}$$

Left-truncated Weibull

The cumulative distribution function of left-truncated Weibull:

$$F(x) = \Big[1 - exp\Big(- (\frac{x-u}{\alpha})^\beta \Big)\Big]$$

Where u is the atypical thresold, β the shape parameter, α the scale parameter.

The mean and variance are given by:

$$E(X) = u + \alpha.\Gamma\Big(1 + \frac{1}{\beta}\Big)$$

$$Var(X) = \alpha^2 [\Gamma(1 + \frac{2}{\beta}) - (\Gamma(1 + \frac{1}{\beta}))^2]$$

Similar to the usual version Weibull distribution, we obtain following estimators of α and β

<u>MLE</u>

The shape parameter c is the solution of $h(\beta) = 0$ where:

$$h(\beta) = \frac{\sum_{i=1}^{n} \Big((x_i - u)^\beta . ln(x_i - u) \Big)}{\sum_{i=1}^{n} (x_i - u)^\beta} - \frac{1}{\beta} - \sum_{i=1}^{n} (x_i - u)^\beta$$

Then the estimation of the scale parameter is:

$$\hat{\alpha} = \Big[\sum_{i=1}^{n} (x_i - u)^{\hat{\beta}} \Big]^{\frac{1}{\hat{\beta}}}$$

Advantage and limitation

<u>Advantage</u>

- This method helps to have very good views on frequency and severity of claim amounts. It helps to predict better the tail of the distribution of claim amounts.

- When there are few claim data higher than the retention, external or internal reinsurer may ask for claim amounts lower than the retention and use this method to simulate claim amount higher than the retention.

Limitation

- It's not applicable when there is no possibility to obtain sufficient claim data.
- The model is purely based on claim data therefore doesn't consider the risk profile of the portfolio (age, sum at risk distribution, etc.)

2.1.2.3 Burning Cost method approach

Definition: "Burning Cost ratio"

Burning Cost ratio for underwriting year N is ratio between the ultimate claim amount occurred during year N and the exposure basis of year N. The earned premium income or the total sum at risk of year N could play the role as "exposure basis" which represents the size of portfolio. The larger portfolio is likely to have bigger ultimate claim amount.

$$BC_i = S_i/C_i$$

Where: BC_i, S_i and C_i are the burning cost ratio, the ultimate claim amount and the exposure basis for underwriting year i.

The "exposure basis" isn't obliged to be the earned premium income or total sum at risk. One can replace it by another indicator which allows having a reference of the development of insurance/reinsurance portfolio, for example: the total ceded Sum at Risk.

If we use "The total Sum at risk" as "exposure basis" for the calculation of burning cost ratio of a reinsurance portfolio:

$$BC_i = \frac{S_i}{SAR_i}$$

Where:

- S_i is the total ultimate reinsured claim amount occurred during year i
- SAR_i is the total SAR in the underwriting year i

Reinsurance pricing XL per life by Burning Cost ratio Principle: The burning cost ratio for the pricing underwriting year is estimated by the (weighted) average of last n years burning cost ratio.

$$\widehat{BC_N} = \frac{1}{n}.(BC_{N-1} + BC_{N-2} + ... + BC_{N-n})$$

Or, weighted average by premium basis:

$$\widehat{BC_N} = \frac{BC_{N-1}.C_{N-1} + BC_{N-2}.C_{N-2} + ... + BC_{N-n}.C_{N-n}}{C_{N-1} + C_{N-2} + ... + C_{N-n}}$$

Finally, the pure premium is calculated by:

$$E(S) = \widehat{BC_N} * SAR_N$$

Calculation of IBNR

In the numerator of the Burning Cost ratio, we have to calculate the "ultimate" amount of claims occurred in the year. The current information gave the view at 31/12/N-1. Wae have to estimate the level of IBNR of reinsured claims at this time. There is a number of traditional approaches for the estimation of IBNR, for example: Chain ladder, Bonheuter Ferguson, Boostrap. However, from the practical point of view, we will consider in this thesis the Chain ladder approach which is deterministic and easy to use.

As mentioned, the estimation of IBNR consists two parts: IBNyR (Incurred But Not yet Reported) and IBNeR (Incurred But Not enough Reported).

IBNyR

The chain ladder method applies to the number of claims $M_{i,j}$ triangle.

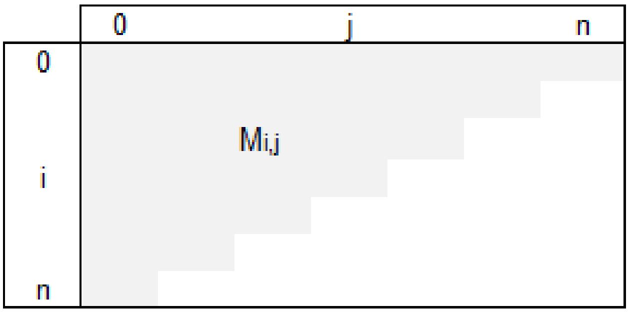


Figure 2.3: Number of claims triangle

The development factor of the number of claims from year j to year j +1:

$$\hat{f}_j = \frac{\sum_{i=1}^{n-j} Mi, j+1}{\sum_{i=1}^{n-j} M_{i,j}} \quad , j = 1, \ldots, n-1$$

Where $M_{i,j}$ is the number of claims occurred in year i and reported before or in year j.

From the development factor we can estimate the ultimate number of claims:

$$\widehat{M}_{i,n} = M_{i,n-i+1} \times \prod_{j=n-i+1}^{n-1} \hat{f}_j$$

IBNeR

The IBNeR effect takes into account the development in terms of amount of occurred claim.

$$\hat{g}_j = \frac{\sum_{i=1}^{n-j} Si, j+1}{\sum_{i=1}^{n-j} S_{i,j}} \quad , j = 1, \ldots, n-1$$

Where $S_{i,j}$ is the total amount of claims occurred in year i and reported before or in year j. It means, we consider also the development of late declared claims. The "double" triangle is used for the calculation of Chain-ladder factors:

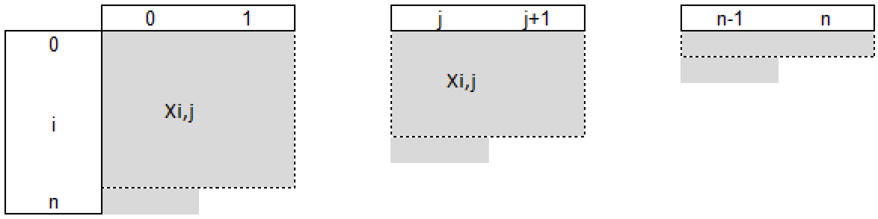


Figure 2.4: Amount of claim triangle

Ultimate claim amounts

As in deterministic approach, we don't aim to add "new claim" in the list, we will apply both IBNyR effect and IBNeR effect in the claim amount. We have the ultimate single claim amount:

$$X_{i,n} = X_{i,n-i+1} \times \prod_{j=n-i+1}^{n-1} \hat{f}_j \times \prod_{j=n-i+1}^{n-1} \hat{g}_j$$

The ultimate reinsured claim amount of the year of occurrence i:

$$S_i = \sum^{N_i} X_{i,n}$$

Where N_i is the number of claims occurred in year i.

Remark: "AS-IF" calculation for Burning Cost approach:

The ultimate claim amounts and exposure basis should be recalculated in order to have equivalent values in the current year (N) (taking into account inflation impact). This "AS-IF" calculation is described in the section 2.1.3.3.

Finally, we have the result for pure premium:

$$E(S) = \widehat{BC_N} * SAR_N$$

However, this deterministic methode doesn't give us a view on volatility of reinsured annual claim amount. The Cost of capital part in the pricing formula could be calculated by using a hybrid approach: using life shock described in Appendix A.

Advantage and limitation

Advantage

- It's deterministic approach which is easy to implement and it helps to have a global view on the claim amounts.
- In case that there are few claims, the Burning Cost method can still give a price.

Limitation

- The Burning Cost approach bases only on the historical claims, it doesn't consider the detail on the exposure of portfolio (age, volatility of SAR, number of insured, etc.)

- The approach is not suitable when there are no historical reinsured claims. When there are few claims, the method still works but the result isn't very reliable due to the fact that in general, reinsurance claims are very volatile, the method doesn't help to predict potential reinsured claims.

- The Burning Cost approach is a deterministic approach which relies on the average of annual results therefore it doesn't take into account the volatility of reinsured claim amount.

2.1.2.4 Incident rate approach

Sometimes, the XL per life treaty isn't working and we don't get enough claim data to perform frequency and severity model. The Burning Cost method can give in some cases a price but it has the limitation to do not take into account the portfolio's risk profile and volatility of claims.
In the data communicated by insurer, we can obtain a model point of the insurance portfolio: the average age and the number of insured per range of SAR, i.e:

Min SAR	Max SAR	Average SAR	Nb of insured lives	Average Age	Total SAR	Total SAR Retained by insurer	Total SAR Ceded to XL per Life
-	500 000	250 000	70 000	35	17 500 000 000	17 500 000 000	-
500 000	1 000 000	750 000	12 000	45	9 000 000 000	9 000 000 000	-
1 000 000	1 500 000	1 250 000	3 000	47	3 750 000 000	3 000 000 000	750 000 000
1 500 000	2 000 000	1 750 000	1 000	50	1 750 000 000	1 000 000 000	750 000 000
2 000 000	2 500 000	2 250 000	600	55	1 350 000 000	600 000 000	750 000 000
2 500 000	3 000 000	2 750 000	300	57	825 000 000	300 000 000	525 000 000
3 000 000	3 500 000	3 250 000	120	60	390 000 000	120 000 000	270 000 000
3 500 000	4 000 000	3 750 000	100	59	375 000 000	100 000 000	275 000 000
4 000 000	4 500 000	4 250 000	58	58	246 500 000	58 000 000	188 500 000
4 500 000	5 000 000	4 750 000	7	60	33 250 000	7 000 000	26 250 000
5 000 000	5 500 000	5 250 000	6	61	31 500 000	6 000 000	25 500 000
5 500 000	6 000 000	5 750 000	2	60	11 500 000	2 000 000	9 500 000
6 000 000	++++	-	-	-	-	-	-
Total			87 193	42	35 262 750 000	31 693 000 000	3 569 750 000

Figure 2.5: Model point for XL per Life pricing

Methodology:
Assuming that we can obtain the mortality Best Estimate of the mortality insurance portfolio, i.e. q_{x_i} table, for each range of sum at risk i, the random variable amount of claims follows a binomial law. We have following parameters applied in the method:

- The total number of experiments, i.e. the number of insureds N_i

- The probability of death, q_{x_i}

Given $\overline{SAR_{re_i}}$, the average sum at risk of the range, the mean and variance of the reinsured claim amount belonging to the range i are calculated by:

- $E(X_i) = q_{x_i}.N_i.\overline{SAR_{re_i}}$

- $Var(X_i) = q_{x_i}.(1 - q_{x_i}).N_i.\overline{SAR_{re_i}}^2$

It is then possible to sum all those variables together to get the amount of claims ceded to reinsurer.

- $E(S) = \sum_i q_{x_i} N_i \overline{SAR_{re_i}}$
- $Var(S) = V^t M V$ with V the vector of standard deviation of X and M the matrix of covariance reflecting the dependency between bands (correlation assumed).

The pricing formula could be the traditional combination of mean and variance:

$$P = \frac{E(S) + \beta\sigma(S)}{1 - expense}$$

Advantage and limitation

Advantage

- The incident rate approach takes into account the portfolio's information such as age, average SAR per range, etc.
- Compared to Burning Cost approach, mortality incident rate approach gives an idea about the volatility of the claim amount.
- The method works even when we have very few claim data
- Simplicity of the calculation: Once mortality Best Estimate rate is available, the expected value and variance of the claim amount are calculated quite easily.

Limitation

- There is a basic risk in the calculation. The Best Estimate mortality rate represents the insurance claims while we are in the issue of estimating reinsured claims. The mortality rate may be lower because of a better underwriting process for high sum insured cases. It doesn't fully take into account historical reinsured claims.

2.1.2.5 Combined approach

The combined approach between Burning Cost which uses the historical reinsurance claim and the incident rate approach which uses the portfolio information (age, average SAR per range, etc.) could help to use the maximum amount of available information.

Principle: We suppose that the historical reinsured claim information represents correctly the average claim amount but it doesn't sufficiently illustrate the variance, for example, due to limited claim data. The idea is to keep this average view and use the variance view given by the qx Best Estimate as described in incident rate method.

Methodology:

From the historical claim amount, we determine the Best Estimate mortality of reinsured portfolio.

$$q_x^{RE} = \frac{S^{real}}{S^{Insurance}_{Best.Estimate}} \times {q_x}^{Insurance}_{Best.Estimate}$$

Where:

- q_x^{RE} is the Best Estimate mortality rate for reinsured claim
- S^{real} is the historical amount of reinsured claim
- $S^{Insurance}_{BestEstimate}$ is the hypothetical reinsured claim amount calculated by using qx Best Estimate of insurance portfolio (as in the section "incident rate approach")
- ${q_x}^{Insurance}_{BestEstimate}$ is the qx Best Estimate rate of the insurance portfolio

Once the q_x^{RE} is calculated, the price is then calculated in the same way as in the incident rate approach.

Advantage and limitation

Advantage

- It takes into account both historical reinsured claim information and portfolio information.

Limitation

- It doesn't keep the view in variance shown by historical reinsured claims information as the frequency-severity does. However, when the variance of reinsured claims view is judged as not creditable (for example, due to limited claim data, the treaty is not very working, etc.), we can consider that this limitation is not significant.

2.1.3 Application of proposed pricing models

2.1.3.1 Data

We will consider the XL per Life reinsurance treaty as described at the beginning of the chapter. The information given in a Renewal period to reinsurers are as following:

- Contractual elements: Draft of reinsurance treaty with condition terms such as: Retention, Limit, the number of reinstatement, etc.
  - Retention: 1M
  - Limit: 5M
  - Number of reinstatement: 15
- Historical claim amounts above the retention level containing:
  - Amount of claim at 1st declaration date

– Date of occurrence. Claim historic from 2010 to 2016. In practice, the year 2016 isn't complete as the Renewal period takes place normally before year end. It requires in the modelling some treatment for example: proportionally adding the claim amount in order to obtain full year claims. However, in this case study, for the sake of simplification, we assume the claim data is updated to full year 2016.

– The claim amount and status (open/closed) at the end of each year

- Portfolio profile: The model point presented in chapter 1 is used:

Min SAR	Max SAR	Average SAR	Nb of insured lives	Average Age	Total SAR	Total SAR Retained by insurer	Total SAR Ceded to XL per Life
-	500 000	250 000	70 000	35	17 500 000 000	17 500 000 000	-
500 000	1 000 000	750 000	12 000	45	9 000 000 000	9 000 000 000	-
1 000 000	1 500 000	1 250 000	3 000	47	3 750 000 000	3 000 000 000	750 000 000
1 500 000	2 000 000	1 750 000	1 000	50	1 750 000 000	1 000 000 000	750 000 000
2 000 000	2 500 000	2 250 000	600	55	1 350 000 000	600 000 000	750 000 000
2 500 000	3 000 000	2 750 000	300	57	825 000 000	300 000 000	525 000 000
3 000 000	3 500 000	3 250 000	120	60	390 000 000	120 000 000	270 000 000
3 500 000	4 000 000	3 750 000	100	59	375 000 000	100 000 000	275 000 000
4 000 000	4 500 000	4 250 000	58	58	246 500 000	58 000 000	188 500 000
4 500 000	5 000 000	4 750 000	7	60	33 250 000	7 000 000	26 250 000
5 000 000	5 500 000	5 250 000	6	61	31 500 000	6 000 000	25 500 000
5 500 000	6 000 000	5 750 000	2	60	11 500 000	2 000 000	9 500 000
6 000 000	++++	-	-	-	-	-	-
Total			87 193	42	35 262 750 000	31 693 000 000	3 569 750 000

- The earned premium income for the period 2010-2016 and the expected premium income for 2017:

Year	2010	2011	2012	2013	2014	2015	2016	2017
Total EPI	47 965 800	49 000 500	50 750 230	53 023 000	54 650 230	55 623 200	56 231 000	57 650 000

- The total SAR for the period 2010-2016 and the expected SAR for 2017.

Year	2010	2011	2012	2013	2014	2015	2016	2017
Total SAR	32 556 213 000	32 906 500 000	33 650 000 000	34 156 230 000	34 609 800 000	34 800 950 000	35 262 750 000	35 500 000 000

2.1.3.2 Claim data descriptive statistics

Frequency development 2010-2016:

The following triangle contains frequency development information of claims above retention 1M with occurrence year and development year viewed at 31/12/2016. Note that the number shown in the diagonal is the number of claims known at 31/12/2016.

Occurrence/Development year	N	N+1	N+2	N+3	N+4	N+5	N+6
2010	5	7	8	8	8	8	8
2011	5	7	9	9	9	9	
2012	4	7	9	10	10		
2013	4	5	6	6			
2014	3	6	8				
2015	5	7					
2016	3						

Severity viewed at present year:

The following tables shows some descriptive statistics about the gross amount of claims (before reinsurance) viewed at 31/12/2016:

Year	Total gross amount of claims	Min	Max	Mean	Median
2010	12 200 000	1 050 000	2 200 000	1 525 000	1 375 000
2011	12 510 100	1 010 100	2 300 000	1 390 011	1 150 000
2012	13 892 500	1 040 000	2 100 000	1 389 250	1 201 000
2013	9 710 650	1 200 000	2 300 000	1 618 442	1 452 825
2014	11 541 960	1 000 230	2 300 000	1 442 745	1 255 210
2015	10 081 400	1 100 000	1 800 400	1 440 200	1 450 000
2016	4 030 000	1 010 000	2 000 000	1 343 333	1 020 000

We should notice that in the claim data, there are "closed" claims which are considered to no longer develop and "open" claims which would have potentially IBNeR amount in the future. The number of "closed" and "open" viewed at 31/12/2016 are resumed in the following table:

Year	Number of "closed" claims	Number of "open" claims
2010	8	0
2011	9	0
2012	7	3
2013	5	1
2014	6	2
2015	4	3
2016	1	2

We can see that all claims occurred in 2010 and 2011 are all considered as "closed". This situation is very common in death claims: in general, all claims are reported and closed after 5 years or less.

2.1.3.3 Pricing

a) Frequency - severity approach

Frequency fitting:

Step 1: Determining the ultimate number of claims:

By using Chain-ladder we can estimate the ultimate number of claims by occurrence year. This step helps us take into account the IBNyR effect.

Occurrence/Development year	N	N+1	N+2	N+3	N+4	N+5	N+6
2010	5	7	8	8	8	8	8
2011	5	7	9	9	9	9	9
2012	4	7	9	10	10	10	10
2013	4	5	6	6	6	6	6
2014	3	6	8	8	8	8	8
2015	5	7	9	9	9	9	9
2016	3	5	6	6	6	6	6

The IBNyR development factor from N+i to N+i+1:

N->N+1	N+1->N+2	N+2->N+3	N+3->N+4	N+4->N+5	N+5->N+6
1,5000	1,2500	1,0313	1,0000	1,0000	1,0000

The IBNyR development factor from each occurence year to it's ultimate number:

2016	2015	2014	2013	2012	2011
1,934	1,289	1,031	1,000	1,000	1,000

Step 2: Fitting the ultimate number of claims

Year	Ultimate number of claims
2 010	8
2 011	9
2 012	10
2 013	6
2 014	8
2 015	9
2 016	6

The ultimated number of claims is fitted with Poisson or Negative Binomial distribution. As the number of data points is small, the MME method could be used.

As $E(N) = 8 > Var(N) = 2.3$, the chosen distribution should be Poisson with parameters: $\lambda = 8$

Severity fitting:

Step 1: Estimation of IBNeR effect

The IBNeR development factor from N+i to N+i+1:

N->N+1	N+1->N+2	N+2->N+3	N+3->N+4	N+4->N+5	N+5->N+6	N+6->N+7
1,04	1,01	1,01	1,00	1,00	1,00	1,00

The IBNeR ultimate development factor of claims for each occurrence year to it's ultimate amount:

2016	2015	2014	2013	2012	2011	2010
1,07	1,02	1,02	1,00	1,00	1,00	1,00

From the IBNeR development factors given by method Chain-ladder, we can see that claims occurred between 2010 and 2014 will no longer develop.

Step 2: Taking into account inflation

Given the inflation rate, we could calculate also the inflation impact based on a 100-scale index:

Year	100-scale inflation
2 010	93,6
2 011	95,3
2 012	97,7
2 013	99,0
2 014	99,7
2 015	99,7
2 016	99,9
2 017	100,0

Step 3: Taking into account evolution of portfolio exposure. Final AS-IF claim amount
Given the total SAR per information, we can calculate the impact of the evolution of portfolio exposure based on a 100-scale index:

Year	SAR	100-scale SAR
2 010	32 556 213 000	92
2 011	32 906 500 000	93
2 012	33 650 000 000	95
2 013	34 156 230 000	96
2 014	34 609 800 000	97
2 015	34 800 950 000	98
2 016	35 262 750 000	99
2 017	35 500 000 000	100

The final AS-IF claim amount is calculated by applying the inflation index and portfolio exposure index. The coefficients are calculated per occurrence year as following:

Year	100-scale inflation	100-scale SAR	AS-IF coefficient
2 010	93,6	92	1,16
2 011	95,3	93	1,13
2 012	97,7	95	1,08
2 013	99,0	96	1,05
2 014	99,7	97	1,03
2 015	99,7	98	1,02
2 016	99,9	99	1,01
2 017	100,0	100	1,00

These coefficients are to be applied to the claim amount in order to have the AS-IF amounts. It means that a claim amount X occurred in 2010 is equivalent to a claim amount of $1.16 \times X$ in 2017. The following table shows the list of 10 largest claims and their AS-IF amount.

#	Year	Claim amount viewed at 31/12/2016	AS-IF claim amount
1	2011	2 300 000	2 603 499
2	2013	2 300 000	2 415 414
3	2014	2 300 000	2 367 424
4	2010	2 200 000	2 561 651
5	2013	2 102 000	2 207 478
6	2014	2 101 000	2 162 591
7	2012	2 100 000	2 268 552
8	2016	2 000 000	2 015 671
9	2010	1 950 000	2 270 554
10	2011	1 900 000	2 150 717

The AS-IF claim amounts are taken as the sample of the calibration of severity. As mentioned above, 6 distributions are tested:

- Log-normal
- Weibull

- Left-truncated Log-normal
- Left-truncated Weibull
- Pareto
- Generalized Pareto Distribution

We have following result for goodness of fit criteria:

Distribution	AIC	Kolmogorov Smirnov Statistic	Cramer von Mises Statistic
Truncated Log-normal MLE	1 446	0,103	6 700
Truncated Log-normal MME	1 560	0,180	8 392
Truncated Weibull MLE	**1 440**	**0,093**	**2 324**
Truncated Weibull MME	1 501	0,109	2 718
Log-normal MLE	1 457	0,154	7 801
Log-normal MME	1 457	0,144	8 407
Weibull MLE	1 468	0,165	9 861
Weibull MME	1 471	0,166	9 863
Pareto MLE	1 521	0,122	7 871
Pareto MME	1 587	0,159	9 783
GPD MLE	1 476	0,112	2 591
GPD MME	1 431	0,101	2 460

Figure 2.6: Goodness of fit criteria

These criteria shows that the "best fit" distribution should be truncated Weibull MLE. The parameters of the calibrated Weibull MLE are:

Parameter	Value
threshold	1000000
shape	1,1215
scale	504 744
mean	1 484 020
sd	432 352
q(99.5%)	3 232 379

The additional checks containing quantile comparison and cumulative distribution function plot are performed in order to validate the choice.

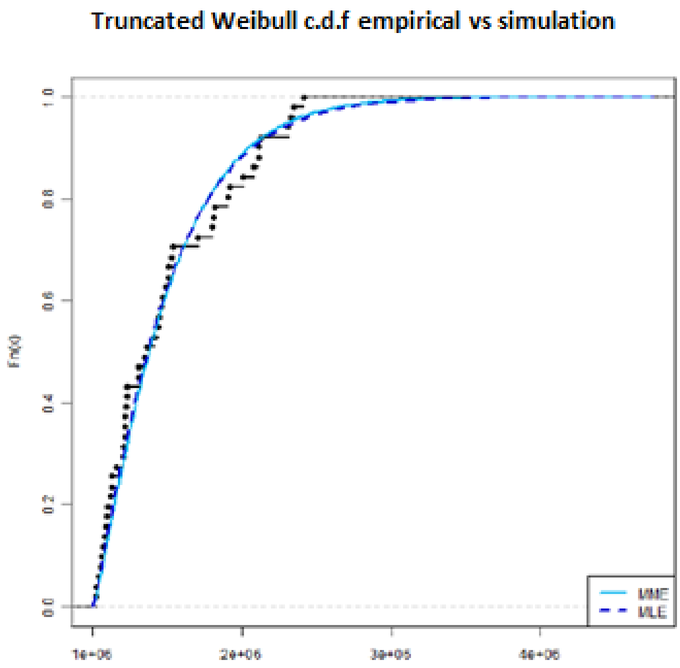


Figure 2.7: Truncated Weibull cumulative distribution function

Distribution	q50	q75	q90	q95	q99	q99.5
Empirical	1 370 883	1 841 894	2 162 591	2 367 424	2 603 499	2 603 499
Truncated Log-normal MLE	1 299 298	1 640 854	2 271 624	2 916 284	5 135 691	6 480 890
Truncated Log-normal MME	1 368 455	1 606 660	1 950 293	2 243 095	3 057 381	3 474 097
Truncated Weibull MLE	**1 364 031**	**1 675 398**	**2 061 818**	**2 342 634**	**2 970 025**	**3 232 379**
Truncated Weibull MME	1 375 050	1 675 980	2 040 441	2 301 217	2 875 262	3 112 585
Log-normal MLE	1 430 326	1 716 094	2 022 725	2 231 386	2 681 354	2 963 777
Log-normal MME	1 433 502	1 707 529	1 999 092	2 197 084	2 620 292	2 982 183
Weibull MLE	1 489 965	1 788 756	2 045 160	2 190 785	2 454 220	2 746 298
Weibull MME	1 496 572	1 770 538	2 002 993	2 135 590	2 370 104	2 752 686
Pareto MLE	1 289 508	1 662 831	2 327 130	3 000 854	5 415 537	6 983 377
Pareto MLE	1 253 758	1 571 910	2 119 626	2 657 499	4 492 817	5 632 907
GPD MLE	1 373 770	1 701 996	2 075 326	2 418 065	2 773 680	3 731 323
GPD MME	1 401 927	1 890 089	2 218 976	2 341 876	2 871 987	3 827 112

Looking at the quantile comparison in the table above, the truncated Weibull distribution gives more prudence in the tail of the distribution and very good fit in the beginning part of the distribution. In addition, given the fact that the existed claims are only at the first ranges of Sum At Risk, we validate the truncated Weibull distribution.

It is interesting to compare the goodness of fit of truncated Weibull with its original (non truncated) version of the distribution:

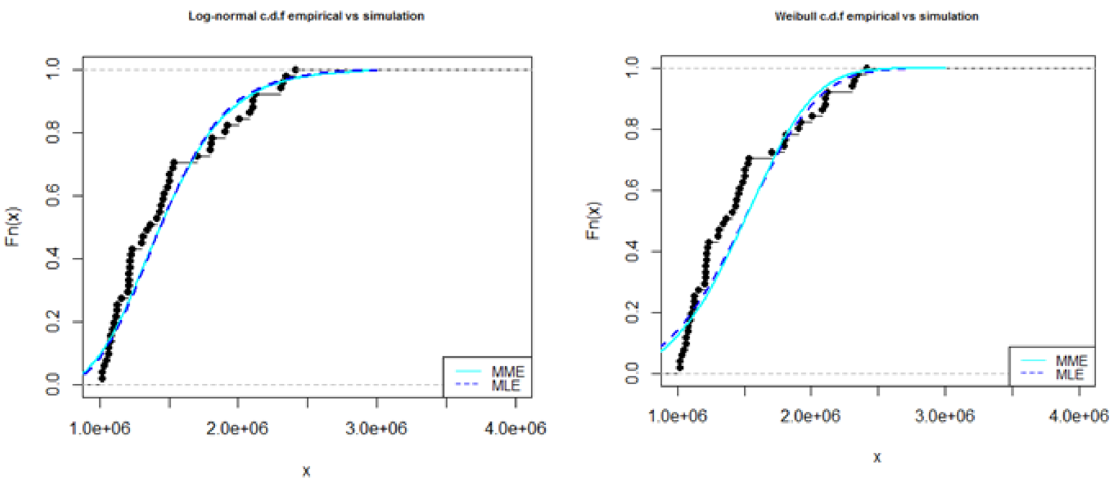


Figure 2.8: Usual Weibull and lognormal cumulative distribution function

From the graph above, we can see that the original version gives a good fit at the tail of the distribution but it has bad quality in the beginning of the distribution.

Another disadvantage of using the original version is that it will give a value lower than the threshold in the simulation phase, which doesn't seem logical when we want to simulate claims higher than the retention.

Distribution	Log-normal MLE	Weibull MLE
% Number of claim < threshold	8%	14%

Pricing result

From the simulation we obtain following result for mean, variance of annual reinsured claim amount and price of the treaty based on these two elements:

E(S)	σ(S)	Price	Currency
4 221 049	1 886 949	4 899 715	EUR

The result above corresponds to the following assumption: $\beta = 10\%$ and $expense = 10\%$ in the formula $S = \frac{E(S)+\beta\times\sigma(S)}{1-expense}$

b) Burning Cost approach

As described, the Burning Cost is a deterministic approach which apply both IBNeR and IBNyR effects in the development of claim amounts. The ultimate claim amounts are then recalculated in "AS-IF" amounts taking into account the inflation impact. The pattern of ultimate claim amount, exposure basis per occurrence year and the Burning Cost based on the simple average are shown in the table below:

Claims	2010	2011	2012	2013	2014	2015	2016
Total claim amount viewed at 2016	4 200 000	3 510 100	3 892 500	3 710 650	3 541 960	3 081 400	1 030 000
Ultimate claim amount with IBNyR	4 200 000	3 510 100	3 892 500	3 710 650	3 902 646	5 995 555	4 792 383
Ultimate claim amount with IBNyR+IBNeR	4 200 000	3 510 100	3 892 500	3 714 301	3 936 284	6 108 449	5 062 956
Ultimate claim amount with IBNyR+IBNeR+Indexation	5 027 545	4 126 343	4 225 470	3 815 601	3 978 106	6 146 489	5 071 825
Total SAR	32 556 213 000	32 906 500 000	33 650 000 000	34 156 230 000	34 609 800 000	34 800 950 000	35 262 750 000
Total SAR AS-IF	34 764 550 629	34 527 462 053	34 456 510 772	34 512 407 644	34 731 063 090	34 901 941 661	35 301 539 025
Burning Cost	0,0145%	0,0120%	0,0123%	0,0111%	0,0115%	0,0176%	0,01%
Average burning cost	**0,0133191%**						

The pure premium is the product of average burning cost and the expected exposure basis of 2017. Pure premium given by Burning Cost approach: EUR 4,728,270. The reinsurer can add their cost of capital and expense into the price. The cost of capital is calculated as described in appendix A (at about EUR $445,952 \times 110\% = 490,547$ assuming a transmission factor of 110%). Finally, we have reinsurance premium at about EUR $5,798,686$

c) Incident approach

Expected value and the variance of annual reinsured claim amount calculated by qx Best Estimate insurance can be shown in the following table:

Min SAR	Max SAR	Total SAR Ceded to XL per Life	qx BE insurance	E(Xi)	Var(Xi)
-	500 000	-	0,08926%	-	-
500 000	1 000 000	-	0,23098%	-	-
1 000 000	1 500 000	750 000 000	0,27256%	2 044 202	509 657 677 787
1 500 000	2 000 000	750 000 000	0,33318%	2 498 868	1 867 906 332 878
2 000 000	2 500 000	750 000 000	0,46660%	3 499 482	4 353 941 705 728
2 500 000	3 000 000	525 000 000	0,52961%	2 780 444	4 840 007 930 109
3 000 000	3 500 000	270 000 000	0,64557%	1 743 040	3 896 521 125 754
3 500 000	4 000 000	275 000 000	0,60236%	1 656 488	4 527 901 755 878
4 000 000	4 500 000	188 500 000	0,56376%	1 062 690	3 434 273 038 634
4 500 000	5 000 000	26 250 000	0,64557%	169 462	631 380 737 969
5 000 000	5 500 000	25 500 000	0,69496%	177 214	747 924 384 906
5 500 000	6 000 000	9 500 000	0,64557%	61 329	289 432 947 818
6 000 000	++++	-	-	-	
Total		**3 569 750 000**	**Total**	**15 693 219**	**25 098 947 637 463**

Figure 2.9: Incident approach

Suppose that there is no correlation between different ranges of SAR, then $Var(S) = \sum_i Var(X_i)$. In final, we have:

- E(S) = 15.7m
- $\sigma(S) = 5m$

If we apply the same pricing formula as in the frequency-severity approach ($\beta = expense = 10\%$), then we have P = 17.9m.
This price has only indicative sense since it has no link with the historical reinsured claim amount.

d) Combined approach

The combined approach between Burning Cost and incident rate uses both results of two approaches. From the expected annual claim amount estimated in Burning Cost and incident rate approaches, we estimate the discount rate in qx Best Estimate:

$$q_x^{RE} = \frac{4.7}{15.7} \times q_x^{BE} = 30.1\% \times q_x^{BE}$$

By using the same calculation as in the incident rate approach and the same pricing formula with $\beta = expense = 10\%$, we obtain:

- $E(S) = 4.7\ m$
- $\sigma(S) = 2.7\ m$
- $Final\ premium = 5.56\ m$

e) Conclusion

In summary, we have following results given by 4 approaches:

Method	Pure premium	Possible to integrate volatility	Standard deviation	Final premium(*)
Frequency-severity	4 221 049	Yes	1 886 949	4 899 715
Burning Cost	4 728 270	No	Not available	5 798 686
Incident rate	15 700 000	Yes	5 000 000	17 900 000
Combined approach	4 728 270	Yes	2 700 000	5 560 000

Figure 2.10: Summary - results of XL per life reinsured risk-costing models

(*): All prices are calculated based on the same pricing formula using pure premium, standard deviation and expense except for Burning Cost approach as this deterministic approach doesn't calculate the variance.

We have following conclusion:

- The frequency and severity approach purely based on the historical reinsured claim information. Its outputs are therefore expected value and variance based on historical amounts.
- The Burning Cost approach also purely based on the historical reinsured claim information. However, it's deterministic approach which applies both IBNeR and IBNyR effect on the claim amount viewed at 31/12/N-1. The method doesn't calculate the variance of annual reinsured claim. The price is calculated based on pure premium and cost of capital estimated from a hybrid method (life shocks for SCR and PC approach for pure premium).
- The incident rate approach in this case has only indicative purpose because it takes into account only the portfolio information, the best estimate mortality view is based on insurance portfolio (for internal reinsurer) and reinsurer's experience in the market (for external reinsurer).
- Combined method between Burning Cost and Incident Rate could help to return to the expected claim amount level as in Burning Cost calculation and in addition, to have a view on variance when there are few historical claims.

From the different views on the price, an internal reinsurer could be able to define their strategy of cession and an insurer can challenge the price of external reinsurers.

2.2 XL per event

2.2.1 Risks covered by XL per event (CAT treaty)

The CAT treaty covers following risks:

- Earthquake
- Tsunami
- Flood
- Storm
- Volcanic activities
- Forest fires
- Industrial hazard
- Terrorism

And the following risks are not covered by CAT treaty:

- Famine
- Epidemic, pandemic
- Extreme temperature
- Drought

The reason for that they are not covered is these types of event normally happen during a long period and it's difficult to conclude exactly the cause of the death for a big number of victims.

2.2.2 Treaty conditions

The treaty CAT covers accidents caused by the types of events mentioned above but there are in addition other conditions for example:

- The CAT event occurs in the covered period of the treaty, i.e. from 1 January to 31 December;
- There are at least M victims caused by the event and the amount of claim exceeds the retention of the treaty. In general, M is often taken between 3 and 5 deaths.
- The CAT treaty covered victims during first three days of the event. After that, it's often considered as a second event.

2.2.3 Data analysis

We recall the available data for the pricing of CAT treaty:

Portfolio data:

	Group			Individual		
Risk class	*SAR*	*Nb of insureds*	*Average SAR*	*SAR*	*Nb of insureds*	*Average SAR*
Mortality	20 502 000 000	51 000	402 000	11 191 000 000	36 193	309 203

And,

Min SAR	Max SAR	Average SAR	Nb of insured lives	Average Age	Total SAR	Total SAR Retained by insurer	Total SAR Ceded to XL per Life
-	500 000	250 000	70 000	35	17 500 000 000	17 500 000 000	-
500 000	1 000 000	750 000	12 000	45	9 000 000 000	9 000 000 000	-
1 000 000	1 500 000	1 250 000	3 000	47	3 750 000 000	3 000 000 000	750 000 000
1 500 000	2 000 000	1 750 000	1 000	50	1 750 000 000	1 000 000 000	750 000 000
2 000 000	2 500 000	2 250 000	600	55	1 350 000 000	600 000 000	750 000 000
2 500 000	3 000 000	2 750 000	300	57	825 000 000	300 000 000	525 000 000
3 000 000	3 500 000	3 250 000	120	60	390 000 000	120 000 000	270 000 000
3 500 000	4 000 000	3 750 000	100	59	375 000 000	100 000 000	275 000 000
4 000 000	4 500 000	4 250 000	58	58	246 500 000	58 000 000	188 500 000
4 500 000	5 000 000	4 750 000	7	60	33 250 000	7 000 000	26 250 000
5 000 000	5 500 000	5 250 000	6	61	31 500 000	6 000 000	25 500 000
5 500 000	6 000 000	5 750 000	2	60	11 500 000	2 000 000	9 500 000
6 000 000	++++	-	-	-	-	-	-
Total			87 193	42	35 262 750 000	31 693 000 000	3 569 750 000

Contractual elements:

Treaty name	Retention	Limit	Number of reinstatement	AAL	Min.victim
CAT-A-France	10 000 000	100 000 000	1	200 000 000	5

The treaty covers both natural catastrophes (CAT NAT), terrorism events, and also man-made non terrorism events (traffic accidents, industrial accidents, etc.)

Public catastrophes database - EMDAT

EMDAT database contains historical natural and man-made non terrorism catastrophic events.

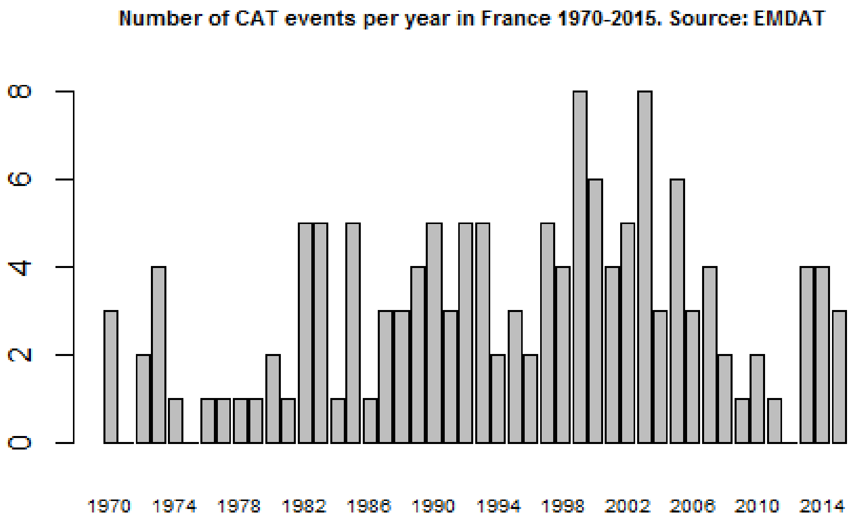
The treaty covers an insurance company based in France. As at the renewal period, we don't have the full year 2016 database, we suppose to use the historical catastrophic events during the period 1970-2015 in order to calibrate the risk. We don't consider the events happened before 1970 to make sure that the data is enough reliable and that we have the similarity of condition (infrastructure, prevention, etc.) as today.

Frequency:

Frequency 1970-2015

Min	Max	Median	Mean	1st quantile	3rd quantile
0	8	3	3,087	1	4,75

In average, there are 3 events per year are recorded in EMDAT.

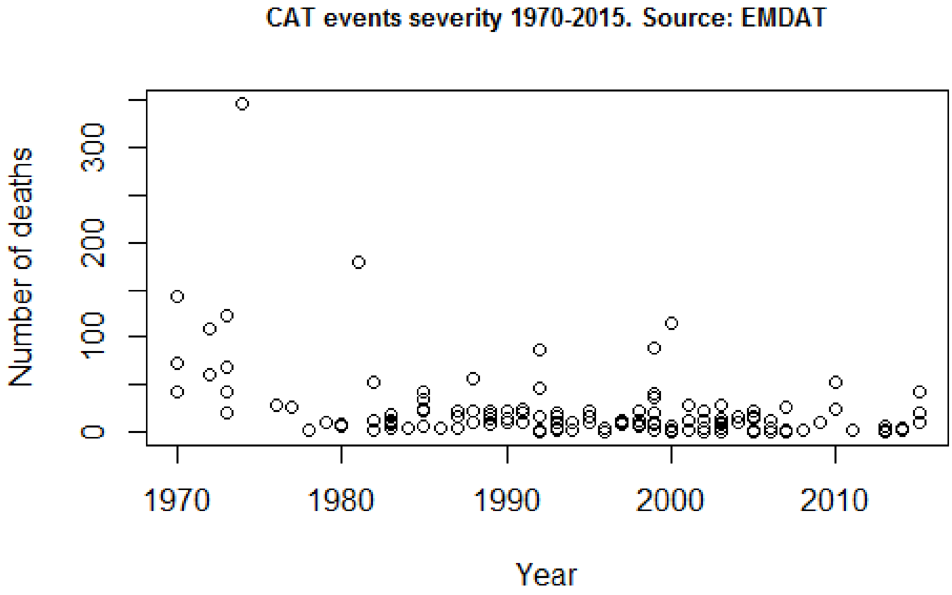


Severity:

Severity 1970-2015

Min	Max	Median	Mean	1st quantile	3rd quantile
1	346	11	21,98	4	22,75

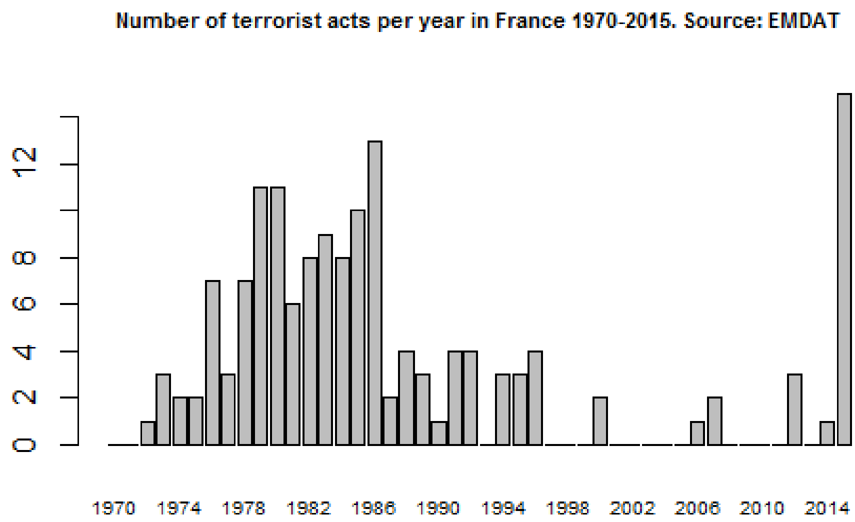
It's interesting to note that the median and mean are far lower than the max. In order to have a better view on the number of victims, we plot them per occurrence year:



There are 6 events with more than 100 deaths in the tail of the sample. Most of the remaining events concentrate in the range $[1, 50]$. The median of the number of deaths is 11.

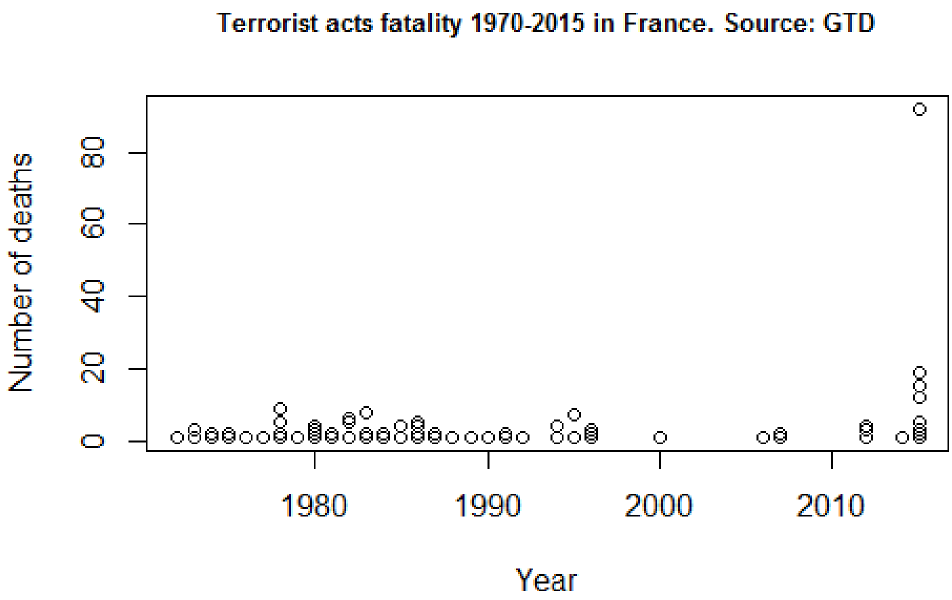
Public terrorist acts database - GTD

Frequency:



The graphic above shows that there were a lot of terrorist acts in the period 1975-1995 then it was very few events until recent period. However, in 2015, it was suddenly recorded the largest number of terrorist acts in France (15 attacks).

Severity:



The graph above about fatality of terrorist acts in France shows that almost all of events have less than 20 deaths. There is only one irregular point: 92 people were killed in Bataclan attack.

Population:

The population of France is taken from http://data.worldbank.org/ with 66.8 millions of inhabitants in 2016.

2.2.4 Pricing model

2.2.4.1 Deterministic model

When we say "deterministic", it means the method doesn't require Monte-Carlo simulation.
One of the first pricing models was developed by Strickler (1960): The reinsurer obligation is modeled by 2 factors:

- The frequency distribution of the events which cause more than M deaths
- The distribution of the reinsured claim amount

Distribution of claim amount:

Denote C the claim amount of one insured life in a CAT event. $S\bar{A}R$ the average sum at risk per head. According to Strickler's assumption, C follows the following distribution: $C \sim \overline{SAR} \times Exp(1)$ where $Exp(1)$ is the exponential distribution with mean 1. Then, in an event of n deaths, the total claim amount of this event, denoting by Z, follows the following distribution: $Z \sim \overline{SAR} \times \Gamma(n, 1)$ where

$Gamma(n,1)$ is Gamma distribution. The underlying assumption here is the claim amounts per life in one CAT event are independent, then the sum of independent exponential distribution is a Gamma distribution.

If $w_n(z)$ is the density distribution function of Z, then:

$$w_n(z) = e^z \frac{z^{n-1}}{(n-1)!}$$

<u>Number of victims:</u>

By analyzing the SBMLIC database (Statistical Bulletin of the Metropolitan Life Insurance Company), Strickler found that the annual number of victims among 1 million people of the general population due to CAT events with more than n deaths could be estimated by:

$$A(n) = 8 \times 100^{1/n} \times n^{1/3}$$

A(1) represents the total number of victims over 1 million people due to CAT events during the year.

<u>The annual number of CAT events:</u>

The number of CAT events which cause exactly n deaths over 1 million people of general population is calculated by:

$$H(n) = \frac{A(n) - A(n-1)}{n}$$

Hence, the probability that an event causes exactly n deaths when it occurs is:

$$h(n) = \frac{H(n)}{\sum_{i=0}^{\infty}(H(i))}$$

Note that when $n \to \infty$, $H(n) \to 0$ and $h(n) \to 0$. Strickler used a conservative assumption that $H(n) \to 0$ when $n > 1500$.

<u>Distribution of the claim amount:</u>

As mentioned, in CAT treaty, the reinsurer will pay only when the event causes from M deaths. The density probability function of the amount of claim:

$$w(n) = \sum_{n=M}^{\infty} h(n) w_n(z)$$

<u>**Limitation of the Strickler model:**</u>

- The function A(n) is predefined and applied only for the US database at the giving time of the extraction of data.
- The assumption that H(n) = 0 when n > 1500 is a strong assumption. In history, it exists some CAT events that cause a much higher number of death. Example: Tsunami in Indonesia in 2014 killed 165,708 people.
- The assumption about the claim amount which follows Gamma distribution is a strong assumption.

Strickler model has strong assumption but given the availability of data at that time, it already demonstrated a good approach and give us the idea to develop new model.

2.2.4.2 Model by simulation

Erland Ekheden (2008) published in his thesis new model which at the same time reuses Stricker's ideas and takes into account some improvements compared to Strickler's model:

- The frequency and severity of CAT events are calibrated from an international source of data
- The distribution of claim amount divided by the average sum at risk is again assumed to be Gamma distributed

Notion:

- K: the number of catastrophic events occurred during the period of contract (we consider the case of 1 year period);
- X_k: the number of victims in the general population due to k^{th} event.
- Y_k: the number of victims in the insured portfolio
- Z_k: the amount paid by the insurer for the event k
- Z_k^{re}: the amount paid by the reinsurer for the event k
- SAR_{ind}: the average sum at risk of individual business

$$SAR_{ind} = \frac{\text{Total sum at risk individual business}}{\text{Total number of heads individual business}}$$

- SAR_{grp}: the average sum at risk of group business

$$SAR_{grp} = \frac{\text{Total sum at risk group business}}{\text{Total number of heads group business}}$$

- $\overline{SAR}$: The average sum at risk of the whole portfolio (both individual and group)
- q: insurance penetration rate

$$q = \frac{\text{the number of insured}}{\text{population of the covered zone}}$$

- $pr.region$: sub-region penetration rate

$$pr.region = \frac{\text{population of the covered zone}}{\text{population of the simulation zone}}$$

- R: retention of the treaty
- L: limit of the treaty

Frequency:

Note Km the number of CAT events which cause at least m deaths in the calibration region. We suppose that K_m is:

- A Poisson variable if $E(K_m) \geq Var(K_m)$
- A Binomial Negative distribution if $E(K_m) < Var(K_m)$. In general, we observe a over-dispersion (variance > mean). The BN distribution gives us this property.

The parameters of frequency are calibrated easily by the method of moment.
In some cases, depending on the countries, the covered zone doesn't contain enough data, we have to calibrate based on a larger region by adding some neighbor countries.
Example: Luxembourg. We would regroup it with Germany, France, Belgium. We note « the region »: Luxembourg, Germany, France, Belgium and « the sub-region » : Luxembourg.
The the *pr.region* parameter is used:

$$pr.region = \frac{\text{population of the covered zone}}{\text{population of the simulation zone}}$$

We note J_m the number of CAT events which cause at least m deaths in the covered region:

$$J_m \sim Binom(K_m, pr.region)$$

Severity:
We suppose that the severity in sub-region has the same characteristic as in the region. We calibrate the parameters of severity distribution in the regional level. We note X the number of deaths in the region due to a CAT event. X can receive extreme values in the tail of distribution. Therefore, we can model X by a GPD (Generalized Pareto Distribution):

$$X \sim GPD(m, \sigma, \xi)$$

We note Y the number of death in the insured portfolio due to the event that caused X death in the region. .
We suppose that given X, Y follows Binomial distribution.

$$Y|X \sim Binom(X, p)$$

In order to take into account the dependency of victims in a CAT event, we introduce here a parameter θ.
Then, p follows Beta distribution given X:

$$p|X \sim Beta(q\theta ln(X), (1-q)\theta ln(X)), \theta \in R$$

where q is the insured penetration rate: $q = \frac{\text{total number of insureds}}{\text{population of the covered zone}}$
Remark:

- if $\theta ln(X) \to \infty$ then $Y|X \sim Binom(X, q)$
- if $\theta ln(X) \to 0$ then $P(Y=0|X) = 1 - P(Y=X|X) = 1-q$

The parameter θ represents dependency. A small θ implicates high dependency and a big θ implicates the independence among the victims.

The reinsured claim amount due to CAT events:

In Erland Ekheden's thesis, the Strickler's assumption is applied. The amount of claim paid by the insurer follows gamma distribution multiplied by the average sum at risk per head.

$$Z = \overline{SAR} \times \Gamma(y, 1)$$

Application of reinsurance treaty:

$$Z^{re} = min(max(Z - R, 0), L)1_{Y>=m}$$

Hence, the annual reinsured claim amount:

$$S = min(\sum_{i}^{J_m} Z_i^{re}, AAL)$$

Pricing formula:

It should take into account following main components: pure premium, cost of required capital, expense, number of reinstatement:

$$P_{final} = \frac{PP + CoC}{(1 - expense) * f_{reinstatement}}$$

Where:

- PP: Pure premium
- CoC: Cost of capital
- $f_{reinstatement}$: factor which represents the fact that insurer has to pay reinstatement premium.

Reinstatement factor:

In order to quantify the impact of reinstatement, we consider following consequent cases:

a) We consider the following reinsurance contract: **L XS F** (without AAL or reinstatement).

S is the sum of N claims of reassured amounts $Y_{i,i=1..N}$, i.e.

$$S = \sum_{i=1}^{N} Y_i$$

where Y is the reassured amount for sinister X:

$$Y = min[max((X - F), 0), L]$$

In this case, we can say that there is no limit for S in a year, when $N \to \infty$ then $S \to \infty$.

The pure premium of reinsurance:

$$P_1 = E(S)$$

b) We consider the following reinsurance contract: **L XS F @ 1 reinst 0 %** (free reinstatement). Note that in this case, the reinstatement is *free* and $AAL = (1 + nb.of.reinst) * L =$ 2L.
We denote S' the total annual charge for the reinsurance company, we get:

$$S' = \text{Min [S, AAL]} = \text{Min [S, 2L]}$$

Then we get the premium P_2 for this 1 free reinstatement treaty:

$$P_2 = E(S')$$

c) We consider the following reinsurance contract: **L XS F @ 1 reinst 100 %** (paying reinstatement). Note that in this case, the reinstatement is *paying* and $AAL = (1 + nb.of.reinst) * L =$ 2L.
We denote S'' the total annual charge for the reinsurance company then

$$S'' = S' - \textit{premium for reinstatement}$$

In fact, the company must pay S' and receive an amount called *premium for reinstatement* denoting P_{reinst} from the re-insured each time the re-insured want to use the reinstatement. P_{reinst} is a random factor which depends on the number of claims and claim amounts.

$$P_{reinst} = \begin{cases} 0 & \text{if } S' = 0 \\ P_3 * \frac{S'}{L} & \text{if } 0 < S' < L \\ P_3 & \text{if } S' \geqslant L \end{cases}$$

We can re-write:

$$P_{reinst} = P_3 * \frac{S'}{L} * \mathbb{I}_{\{0<S'<L\}} + P_3 * \mathbb{I}_{\{S' \geqslant L\}}$$

We denote P_3 the pure premium for this 1 paying reinstatement treaty then we get:

$$\begin{aligned} P_3 &= E(S'') \\ &= E(S') - E(P_{reinst}) \\ &= P_2 - P_3 * [E(\frac{S'}{L} * \mathbb{I}_{\{0<S'<L\}}) + E(\mathbb{I}_{\{S' \geqslant L\}})] \\ &= P_2 - P_3 * E(\frac{Min(S', L)}{L}) \end{aligned}$$

Finally,

$$P_3 = \frac{P_2}{1 + \frac{\overline{Min}(S', L)}{L}}$$

Generally, if $nb.reinst > 1$ we can derive also

$$P_3 = \frac{P_2}{1 + \frac{\overline{Min}(S', n*L)}{L}}$$

where n is number of reinstatement.
Suppose that other charges and expenses are hold from free reinstatement treaty to paying reinstatement treaty so we can apply this formula for the full reinsurance premium.

Cost of capital:

The SCR corresponding to the quantile 99.5% of the distribution is in general high for CAT risk. It creates a significant cost of capital for Reinsurer. By consequence, Reinsurer tends to impose the cost of capital in the price.
The quantile 99.5% normally corresponds to the module SCR Life CAT which doesn't represent the full risk taken by Reinsurer. A transmission factor is usually used in order to extrapolate to the full SCR per treaty. The cost of capital rate is defined based on reinsurer's view.

$$CoC = CoC_{rate} \times SCR_{treaty.level}$$

In this thesis, we assume the cost of capital rate for reinsurer at 8% and the transmission factor from SCR_{CAT} to the total SCR per treaty is 110%.
The final pricing formula applied to our case:

$$P = \frac{PP + 8\% \times 110\% \times SCR_{CAT}}{(1 - expense)(1 + \frac{\overline{Min}(S', nL)}{L})}$$

2.2.4.3 Development of model by simulation

Advantage and limitation of model by simulation Erland Ekheden

Advantage:

- The model by simulation improved two first limitations of Strickler's model: Based on two rich historical databases, it calibrated more exactly the frequency and severity of the catastrophes.
- It partially applied the portfolio information (average sum at risk and number of heads), included dependency in order to identify if the victims are more likely to be in collective or individual business.

Limitation:

- It still relies on the strong assumption about the distribution of reinsured claim amount (Gamma distribution).
- The severity distribution relies directly on Generalized Pareto Distribution without deep quantitative justification.
- It doesn't take into account geographical location information of insureds. For example: concentration sites such as building, tower, etc. The simulated scenarios purely based on historical catastrophic events recored in EMDAT and GTD.

We'll try to overcome these limitations of the model by proposing following approaches:

(1) For each simulation, when the number of deaths in the insured portfolio is simulated (denoted as y), we can use the portfolio composition information (model point) to take randomly y heads in the model point. Hence, the claim amount gross of the reinsurance treaty.

$$Z = \sum_{i}^{y} SAR_i$$

(2) From the calibration of atypical claims in XL per Life pricing section above, we conclude that truncated distributions seem to get better quality of fit when we calibrate the data from the certain threshold. In general, having some extreme claims in the tail of the distribution is one characteristic of CAT claims. Therefore, we will propose here to test different fat tail distribution and compare their quality of test in terms of the Maximum Likelihood value:

- Truncated log-normal
- Truncated Pearson V
- Truncated log-gamma
- Truncated Log-logistic
- Truncated Burr
- GPD

(3) Suppose that we have information about 5 most concentrated sites in terms of Sum at Risk. We suppose that we can estimate the loss in a worst case scenario for each site but we have no idea about it's probability of occurrence. By consequence, we add this 5 scenarios in our 1M scenarios simulated by the model frequency - severity and reassess the simulated distribution of losses.

2.2.4.4 Application

Model's parameters:

- Region (Used for the calibration): France. Given the descriptive statistics, we conclude that we have enough data for France and we don't have to extend to the neighbour countries.
- Sub-region (The covered region): France
- Frequency: Poisson or Negative Binomial
- Severity: Fat tail distributions

Frequency & severity calibration:
The first step consists the choice of threshold for truncated distribution. We noticed that the database are generally more reliable from 10 deaths.
In addition, for GPD distribution, we use the classic Hill plot in order to determine the threshold.

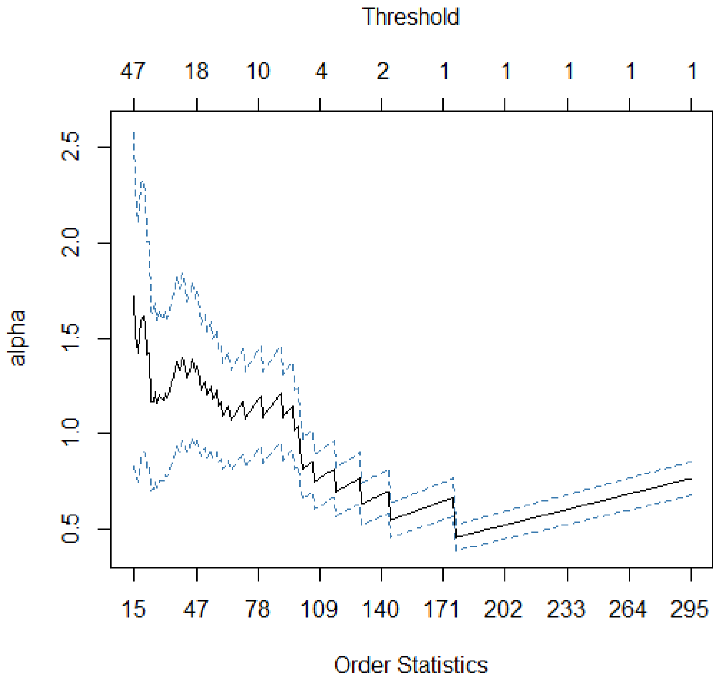


Figure 2.11: Hill plot

From the Hill plot, we can see that the threshold could be chosen at 10 where the alpha index has stabilized.

We compare the quality of fit of GPD versus other distribution:

Distribution	Log-likehood value	AIC
Truncated log-normal	-350,33	700,66
Truncated Pearson V	-350,42	700,84
Truncated Log-gamma	-350,45	700,91
Truncated Log-logistic	-350,46	700,92
Truncated Burr	-350,46	700,92
GPD	-327,32	**660,64**

Figure 2.12: Best fit distribution for the number of victims

From the result table, we always retain GPD distribution which gave the lowest AIC. However, in practice, we may choose another best fit distribution in another case.

In addition to the calculation of AIC, we try to look at the cumulative distribution function and we focus on the tail of the distribution to verify the retained conclusion.

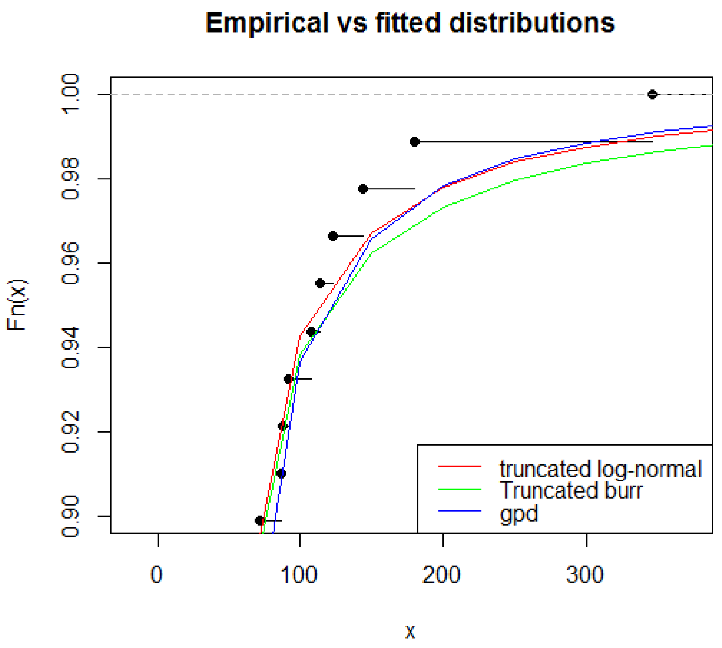


Figure 2.13: C.d.f of GPD vs other distributions

From the cumulative distribution function graph, we conclude that GPD seems to fit the best the empirical distribution. The traced c.d.f line for GPD is quite closed to the c.d.f line for truncated log-normal but slightly more prudent in the tail of the distribution after the quantile 0.98.
Comment: We validate the GPD distribution but it was interesting to test other distributions.

By retaining GPD, we have following estimated parameters for frequency and severity distribution:

Mean frequency	Variance frequency	Lambda Poisson	Size Negative Binomial	Mu Negative Binomial	xi GPD	Beta GPD
1,9348	2,9512	1,9348	3,1051	1,9348	0,7304	9,6225

As $E(Freq) < Var(Freq)$, the chosen distribution for frequency should be Negative Binomial.
Result:
We generate 1 M scenarios with each corresponds to one year period. The result given by the model by simulation in 2.2.4.2 shows that there are 481 scenarios where the treaty is trigger. Hence, the return period is 2079 years. By consequence, the quantile 99.5% is 0 (the return period greater than 200 years).

E	std	quantile 99.5%	nb.triggered.scen	total nb.scen	Return period
6 978	559 110	-	481	1 000 000	2 079

In this case, the Reinsurer doesn't incur a loss in an 1/200 event. However, it takes always the risk which represents by the variance of the annual claim amount. Assume that Reinsurer places its

SCR_{CAT} as based on the standard error. The SCR_{CAT} could be prudently calculated by the following formula:

$$SCR_{CAT} = max[qt_{99.5\%}^{simulated}, qt_{99.5\%}^{normality}]$$

The $qt_{99.5\%}^{normality}$ is the hypothetical quantile 99.5% assuming the normality: the annual reinsured claim amount follows the centered normal distribution $N(0, \sigma)$. Hence, $qt_{99.5\%}^{normality} = \Phi_{99.5\%}^{-1}(0,1) \times \sigma \approx 2.5758.\sigma$.

The table below shows the pricing result based on 3 cases: (a) the reinsured claim amounts follow Gamma distribution; (b) the reinsured claim amounts are simulated by using model point and randomly choosing the life insureds and (c) adding deterministic scenarios.

For (c): We assume that Reinsurer estimated the 5 biggest deterministic losses are: Building collapse (100m), and others (25m, 21m, 20m and 15m). We have following results:

	E	std	f_reinstatement	STEC_CAT	Price
(a) Gamma assumption	6 978	559 110	1,000070	1 440 156	148 558
(b) Using model point	6 830	490 489	1,000069	1 263 402	131 112
(c)Adding deterministic scenarios	7 011	502 263	1,000069	1 293 729	134 279

Comment:

- The reinstatement factor is very closed to 1. It means, there is very small probability that Insurer has to pay the reinstatement premium
- The price given by the Gamma assumption is at 148K. When we use the information in model point, the price is lower. This is due to the lower standard error of claim amount. It can be explained by the fact that the model point after per life coverage is quite homogeneous in terms of sum at risk, there is less volatility in this case.
- When we add the deterministic scenarios, both expected value and standard error increase. It could be explained by the fact that the 5 defined scenarios are more likely in the tail of the distribution.

Chapter 3

Optimisation of reinsurance program

In this chapter, we assume that due to business specificity (for example: lack of detailed data per life which is as mentioned, likely to happen concerning group business), the non-proportional structure is the most suitable type of reinsurance.

The optimisation of reinsurance consists of the determination of the retention and the limit of the treaties. At the first state, we assume that the structure of XL per event treaty is unchanged then we will try to optimize the determination of retention and limit of XL per life treaty by looking at different economic factors.

At the second step, we study the determination of XL per event treaty's structure.

3.1 Retention and limit of XL per life

Besides the regulatory or risk appetite constraint, insurer can define its optimal reinsurance based on its own objectives, for example: optimizing the $P\&L$ ceded given the need of decreasing $x\%$ volatility, minimizing the volatility given $y\%$ of profit ceded to reinsurer, etc. We will study firstly the relation among different factors and then determine the optimal reinsurance given each kind of objective.

Limit:

The sum of retention and limit should cover almost all insureds' sums at risk in the portfolio. The cases where the sums at risk exceed the sum of retention and limit are considered as "Special Acceptances" which should be validated by reinsurer whether it's accepted to be covered in the treaty or not, then the reinsured amount will exceed the limit of the treaty in this case.

Retention:

Setting retention for insurer depends on and impact on different important fields:

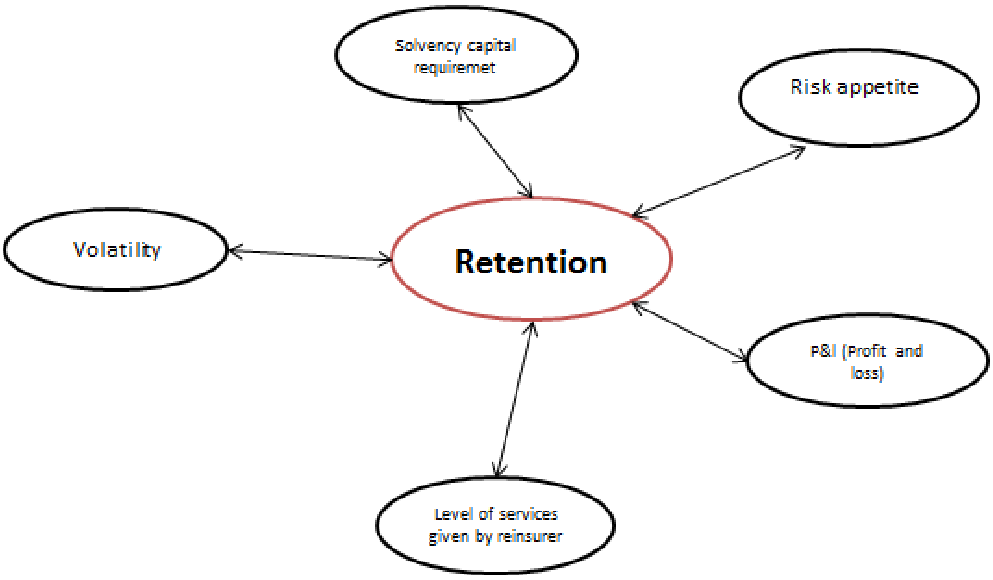


Figure 3.1: Retention per life management

- **Solvency capital requirement**: as discussed in last chapters, the level of retention could impact directly the solvency capital requirement. The higher retention is, the bigger volume is retained by insurer and the higher required solvency capital is.
- **P&L**: The reinsurance means a part of expected profit will be ceded to reinsurer in exchange of other advantages such as volatility reducing, solvency required capital reducing and capacity of underwriting new business increasing
- **Volatility**: Excess of loss reinsurance can reduce the volatility of result since it normally covers atypical risk. One of the reason why insurer calls out reinsurance is that its result is too volatile.
- **Risk appetite**: The volume of business under insurer's retention should respect its risk appetite.
- **Level of services given by reinsurer**: Sometimes, Insurer needs Reinsurer expertise and service in order to launch a new product. However, this case is more likely to happen to proportional reinsurance with quota-share basis which could make Reinsurer as a partner of Insurer during the development phase of new product. When Insurer grows in terms of experience on claim management, underwriting, etc. it can reduce the cession and pass to non-proportional reinsurance. We're studying in this document the second phase: determining the optimal retention when Insurer has already relevant experience on the portfolio.

3.1.1 Different factors impacted by the level of retention

3.1.1.1 Expected profit and loss

The expected profit or loss ceded to reinsurer via the XL per Life reinsurance treaty is given by:

$$P.L_{ceded} = E(S_{re}) - P_{re}$$

Where:

- $P.L_{ceded}$ is the profit or loss ceded via reinsurance treaty.
- S_{re} is the claim amount. $E(S_{re})$ represents the expected reinsured amount of claims.
- P_{re} is the reinsurance premium.

The expected claim amount $E(S_{re})$ is estimated by the average claim amount over last 5 years historical claims.
The challenge for insurer consists of the estimation of reinsurance premium. Insurer can profit the renewal period to obtain prices corresponding to different level of retentions.

3.1.1.2 Solvency capital requirement

The reinsurance has certainly impact on solvency capital requirement. The gain in solvency required capital could be translated in the gain of cost of capital. The cost of capital is defined as the cost of for reinsurer while keeping an amount in the available financial resource without using it, for example, for a financial investment purpose.

$$CoC = Rate_{CoC} \times SCR$$

We will analyse in details in the next section the impact of reinsurance in the solvency capital requirement

3.1.1.3 Volatility

The volatility of the portfolio should be measured net of reinsurance:

$$\sigma = std(P.L_{net.of.reins})$$

3.1.1.4 Risk appetite

Each insurer should have it risk appetite framework. In practice, it's often set as a condition about a maximum annual amount of claims in 1/20 case or the expected volatility.

3.1.1.5 Level of services provided by reinsurer

The increase in retention means less business and less profit for reinsurer. By consequence, the level of services provided by reinsurer would reduce. The decision on retention should be made with consideration about the need of insurer.

In the next section, we will make a focus on the impact of reinsurance on the solvency capital requirement.

3.1.2 Impact of reinsurance in solvency capital requirement

From insurer's perspectives, the traditional reinsurance can mainly impact the SCR_{Life} and $SCR_{default}$. We can find in the following graph the common composition of the total SCR of an insurance company.

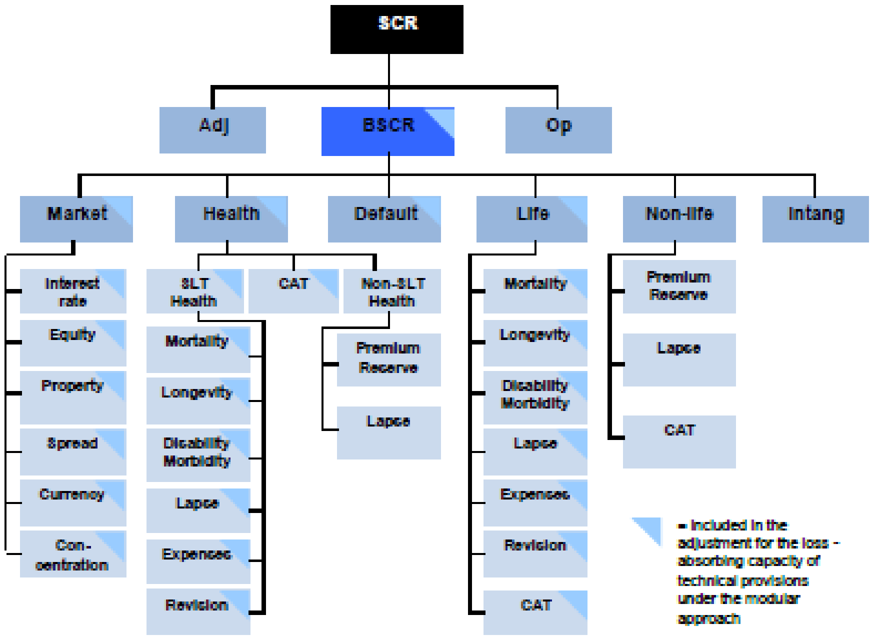


Figure 3.2: Solvency Capital Requirement tree

For the current reinsurance program, we continue to use the model point established in the chapter 2 and study the impact of reinsurance on this portfolio. For the other reinsurance programs, the application is the same.

Min SAR	Max SAR	Average SAR	Nb of insured lives	Average Age	Total SAR	Total SAR Retained by insurer	Total SAR Ceded to XL per Life
-	500 000	250 000	70 000	35	17 500 000 000	17 500 000 000	-
500 000	1 000 000	750 000	12 000	45	9 000 000 000	9 000 000 000	-
1 000 000	1 500 000	1 250 000	3 000	47	3 750 000 000	3 000 000 000	750 000 000
1 500 000	2 000 000	1 750 000	1 000	50	1 750 000 000	1 000 000 000	750 000 000
2 000 000	2 500 000	2 250 000	600	55	1 350 000 000	600 000 000	750 000 000
2 500 000	3 000 000	2 750 000	300	57	825 000 000	300 000 000	525 000 000
3 000 000	3 500 000	3 250 000	120	60	390 000 000	120 000 000	270 000 000
3 500 000	4 000 000	3 750 000	100	59	375 000 000	100 000 000	275 000 000
4 000 000	4 500 000	4 250 000	58	58	246 500 000	58 000 000	188 500 000
4 500 000	5 000 000	4 750 000	7	60	33 250 000	7 000 000	26 250 000
5 000 000	5 500 000	5 250 000	6	61	31 500 000	6 000 000	25 500 000
5 500 000	6 000 000	5 750 000	2	60	11 500 000	2 000 000	9 500 000
6 000 000	++++	-	-	-	-	-	-
Total			87 193	42	35 262 750 000	31 693 000 000	3 569 750 000

The ceded ratio of portfolio is 10% in terms of Sum At Risk and 6% in terms of number of insureds. The average retained Sum At Risk is 363,481.

We will study in the following reinsurance program with 2 levels of protection: per life and per event.

The reinsurance treaties have non.proportional structure: XL per Life and XL per Event. The treaties have following condition:

- XL per Life: 5M XS 1M
- XL per Event: 10M XS 100M.

For the sake of simplification, we suppose that the portfolio covered only the risk of death (mortality risk). The benefit is paid directly one time after the event of death.
The Best Estimate mortality table of the portfolio is assumed to be 80% of the mortality table TH.TF 00.02 in France.
When we try to calculate the average qx weighted by SAR for overall portfolio, we obtain a significant difference between the average qx before and after XL per Life reinsurance. The different between qx of the portfolio net and gross of reinsurance is 13.8%.

Average qx gross of XL per Life	0,198%
Average qx net of XL per Life	0,171%
Δ net/gross	-13,8%

We assume that the Insurer uses the standard formula in their calculation of SCR.
Capital requirement for mortality risk sub.module:
According the simplified formula in standard formula approach, the capital requirement for mortality risk is calculated as:

$$SCR_{mortality} = 0.15 \ . \ SAR \ . \ q \ . \ \sum_{k=1}^{n} \left(\frac{1-q}{1+i_k}\right)^{(k-0.5)}$$

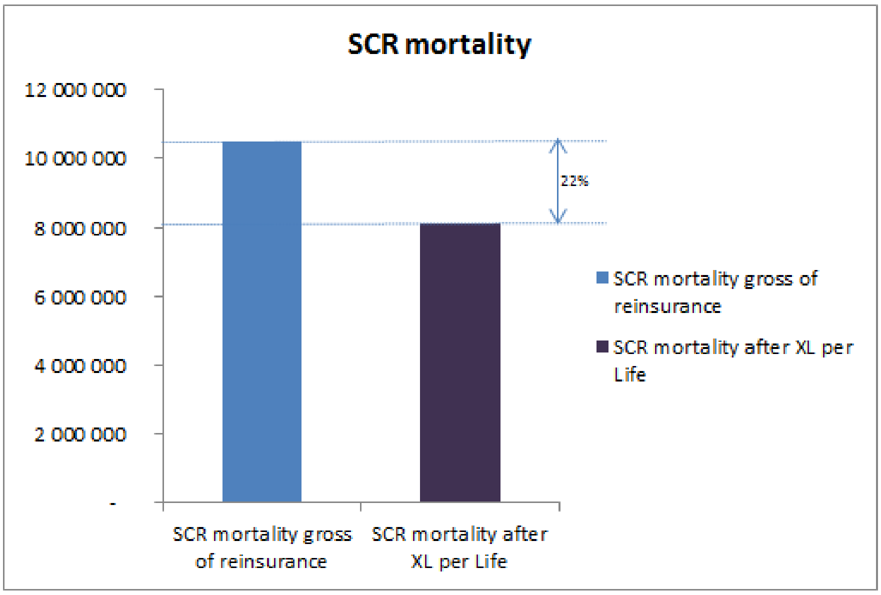
Where:

- q is the expected average mortality risk weighted by the sum assured over the next year. In this case, we consider that q was estimated at the level of
- n denotes the modified duration in years of payments payable on death included in the best estimate projection. In this case, we consider that the payment should be made one time consequently to the event of death. Therefore, $n = 0$ and $\sum_{k=1}^{n} \left(\frac{1-q}{1+i_k}\right)^{(k-0.5)} = 1$
- SAR: total sum at risk of the insurer

Hence, finally:

$$SCR_{mortality} = 0.15 \ . \ SAR \ . \ q$$

In the given example, the XL per life treaty could help insurer to reduce SCR mortality by 22%. It's a result of the fact that insurer ceded 10% SAR to XL per life treaty and the average mortality weighted by sum at risk is lower than the average mortality weighted by the SAR gross of reinsurance by 13.8%

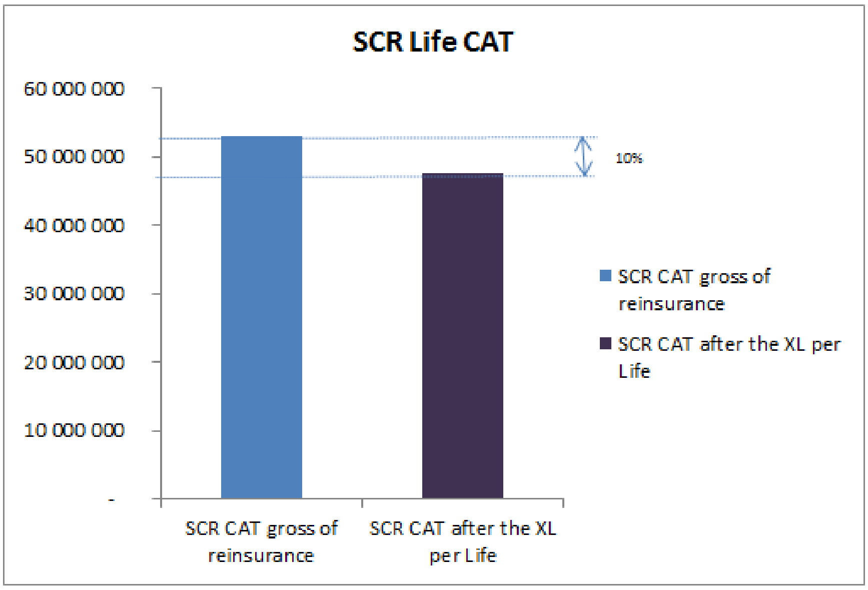


Capital requirement for catastrophe risk sub module:

According the simplified formula in standard formula approach, the capital requirement for life CAT risk sub.module is calculated as:

$$SCR_{Life.CAT} = 0.0015 \ . \ SAR$$

The amount $0.0015 \times SAR$ is also considered as the claim amount in case of an 1/200 CAT event. However, the application of the CAT reinsurance treaty can not reduce the loss amount in case of 1/200 event because the coefficient 0.15% was calibrated in standard formula from a pandemic event while XL CAT treaty doesn't cover pandemic. The only impact comes from XL per Life treaty with 10% of reduction in SCR Life CAT.



Capital requirement for counter party default sub module:

The counter.party default sub.module reflects in this case the loss due to unexpected default of the reinsurer.

There are two kinds of exposures, noted type 1 and type 2 exposures:

- The class of type 1 exposures covers the exposures which may not be diversified and where the counter-party is likely to be rated. It contains reinsurance arrangements.
- The class of type 2 exposures covers the exposures which are usually diversified and where the counter-party is likely to be unrated.

The reinsurance treaty consists the type 1 exposures.

$$SCR_{def,1} = \begin{cases} 3\ .\ \sqrt{V},\ if\ \sqrt{V} \leq 7\%. \sum_i LGD_i \\ 5\ .\ \sqrt{V},\ if\ 7\%. \sum_i LGD_i < \sqrt{V} \leq 20\%. \sum_i LGD_i \\ \sum_i LGD_i,\ if\ 20\%. \sum_i LGD_i \leq \sqrt{V} \end{cases}$$

Where the sum is taken over all independent counter-parties and:

- LGD_i = Loss given default for type 1 exposure of counter-party i
- V = Variance of the loss distribution of the type 1 exposures
- $\sqrt{V}$ = The standard deviation of the loss distribution of the type 1 exposures

When there is only one counterparty, the variance is measured by Bernoulli distribution with parameter PD.

$$V = PD \times (1 - PD) \times LGD^2$$

Where:

- PD is the probability of default
- LGD is the loss-given-default of the counterparty

The value of PD is given by credit quality steps:

Credit quality step	0	1	2	3	4	5	6
Probability of default PD_i	0.002 %	0.01 %	0.05%	0.24%	1.20%	4.2 %	4.2 %

The equivalent ratings are evaluated by ESMA as following:

Credit Quality Step	Fitch's assessments	Moody's assessments	S&Passessments
1	AAA to AA-	Aaa to Aa3	AAA to AA-
2	A+ to A-	A1 to A3	A+ to A-
3	BBB+ to BBB-	Baa1 to Baa3	BBB+ to BBB-
4	BB+ to BB-	Ba1 to Ba3	BB+ to BB-
5	B+ to B-	B1 to B3	B+ to B-
6	CCC+ and below	Caa1 and below	CCC+ and below

The Solvency Capital Requirement technical specification indicated other cases where credit steps are not available and also when the reinsurance treaty is under pooling risk concept. For the sake of simplification, we suppose that reinsurer has a credit quality step 2 and therefore the probability of default is 0.05%. We assume that the insurer has only one reinsurer which covers both XL per Life and XL CAT treaty.
When there is only one single name exposure, the variance is calculated as following:

$$V = PD.(1 - PD).LGD^2$$

For a reinsurance arrangement i , the loss.given.default should be calculated as follows:

$$LGD_i = max(0, 50\%(Recoverables_i + 50\%RM_{re,i}) + F.Collateral_i)$$

where:

- $Recoverables_i$ is Best Estimate recoverables from reinsurance contract i
- $RM_{re,i}$ is risk mitigating effect on underwriting risk of the reinsurance arrangement

- $Collateral_i$ is Risk-adjusted value of collateral
- F is the factor to take into account the economic effect of the collateral

For the sake of simplification, we suppose that Collateral factor is null.
The risk mitigating effect $RM_{re,i}$ is an approximation of the difference between the (hypothetical) capital requirement for underwriting risk under the condition that the reinsurance arrangement is not taken into account in its calculation and the capital requirement for underwriting risk (without any amendments) (ref. [4])

$$RM_{re,i} = SCR_{UW.gross.of.reins.} - SCR_{UW.net.of.reins.}$$

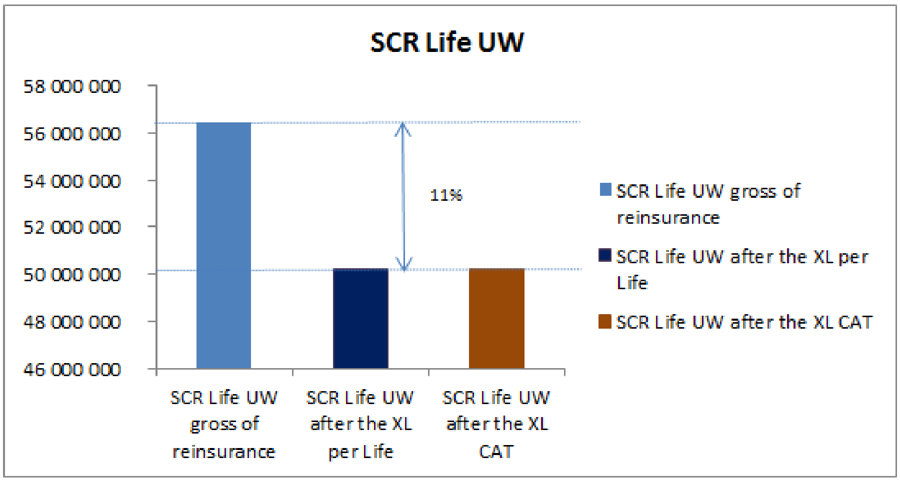
The calculation of $RM_{re,i}$ requires the complete calculation of SCR Life Underwriting on the complete portfolio. For the sake of simplification, we assume that the given example represents the whole portfolio, there is no other risk to be consider in the module SCR Underwriting.
The matrix of correlation could be applied to calculate the SCR Life Underwriting:

	mortality	Longevity	disability	lapse	expenses	revision	CAT
mortality	1						
longevity	-0.25	1					
disability	0.25	0	1				
lapse	0	0.25	0	1			
expenses	0.25	0.25	0.5	0.5	1		
revision	0	0.25	0	0	0.5	1	
CAT	0.25	0	0.25	0.25	0.25	0	1

We assume that SCR lapse, longevity, expense and revision are null, the only sub-modules to be considered are mortality and CAT. In this case:

$$SCR_{Life.UW} = \sqrt{SCR^2_{mortality} + 2.\rho.SCR_{mortality}.SCR_{Life.CAT} + SCR^2_{Life.CAT}}$$

where ρ is correlation between sub-module mortality and sub-module CAT: 0.25



The recoverable best estimate is calculated based on the qx Best Estimate insurance shown in the section 2.1.3.3 (Incident rate pricing approach) which is used in reinsurance pricing basis.
The calculation of SCR_{def} is shown in the following table:

SCR Life UW gross of reinsurance	56 429 845
SCR Life UW after the XL per Life	50 187 689
SCR Life UW after the XL CAT	50 187 689
RM	6 242 156
LGD	9 407 149
Recoverable best estimate	15 693 219
PD	0,05%
V	44 225 099 119
SCR_def	630 893

The final step consists of the application of the correlation matrice in order to have the total SCR:

Corr	Market	Default	Life	Health	Non-Life
Market	100%	25%	25%	25%	25%
Default		100%	25%	25%	50%
Life			100%	25%	0%
Health				100%	0%
Non-Life					100%

For the sake of simplification, we assume that the total SCR consists only Life risk and counterparty risk. Finally,

SCR Life UW after reinsurance	50 187 689
SCR default	630 893
Correlation	25%
Total SCR	**50 349 118**

3.1.3 KPIs of cession

In point of view of Insurer, there are two cases that require the optimisation of reinsurance: optimisation without constraint and optimisation under constraint.
In the first case, we assume that Insurer searches for reinsurance agreement with no specific constraint. The only purpose of reinsurance is to optimize the insurer's economic key performance indicators (KPIs); for example: gain in capital requirement versus ceded profit, gain in volatility reduction versus reinsurance versus ceded profit. We consider following indicators:

3.1.3.1 Cession without constraint

Reinsurance added value (RAV)

Since reinsurance treaty has often negative impact on $P\&L$ and positive impact on solvency capital requirement, we can say that reinsurance is a trade-off between the loss in PL and the gain in solvency capital requirement.
The reinsurance added value is therefore calculated based on: Reinsurance added value (RAV) = Gain in cost of capital – Ceded profit. The first expression of this indicator:

$$RAV_1 = \Big[CoC_{gross.of.reins} - CoC_{net.of.reins}\Big] - \Big[P_{re} - E(S_{re})\Big]$$

As $CoC_{gross.of.reins}$ doesn't depend on reinsurance, we will aim to maximize the second expression of the indicator:

$$RAV_2 = -CoC_{net.of.reins} - P_{re} + E(S_{re})$$

Next, as $S_{re} = S_{gross.of.reins} - S_{net.of.reins}$ and $S_{gross.of.reins}$ doesn't depend on the reinsurance, then, finally, we aim to minimize the following term:

$$RAV = CoC_{net.of.reins} + P_{re} + E(S_{net.of.reins})$$

Insurer could both model the first and third parts of the RAV based on its Best Estimate assumption and its SCR calculation model. The P_{re} is; however, depends on Reinsurer's quotation during the renewal period.

Coefficient of variation (CV)

The coefficient of variation shows the extent of variability in relation to the mean of $P\&L$. Firstly, we aim to minimize the coefficient of variation CV_1.

$$CV_1 = \frac{\sigma}{P.L_{net.of.reins}} = \frac{\sigma}{P.L_{gross.of.reins} - P_{re} + E(S_{re})}$$

Then, as $P.L_{gross.of.reins}$ doesn't depend on reinsurance. $S_{re} = S_{gross.of.reins} - S_{net.of.reins}$ and $S_{gross.of.reins}$ doesn't depend on the reinsurance. Finally, we will aim to maximize following term:

$$CV = \frac{\sigma}{P_{re} + E(S_{net.of.reins})}$$

Insurer could both model the σ and $S_{net.of.reins}$ based on its own Best Estimate mortality. However, as mentioned, P^{re} depends on Reinsurer feedback during the renewal period.

3.1.3.2 Cession with constraints

Insurer could look for reinsurance with various purposes. We will study following examples:

- (1): The reinsurer looks for a reduction of 10% of its cost of capital.
- (2): The reinsurer looks for ceding 20% of its volatility

We limit our scope of study in the optimisation with constraint.
(1) Ceding 10% of the Cost of Capital:
Similarly to the calculation in the sub-section 3.1.2, we calculate the cost of capital for the list of retentions from 0 to the maximum SAR (6M) with the steps of 500K and by taking 8% as the cost of capital rate, we obtain following result in terms of cost of capital:

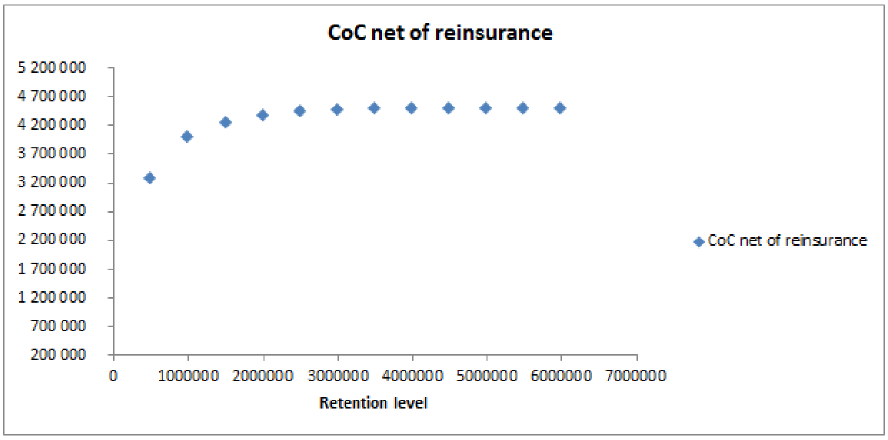


Figure 3.3: Cost of capital net of reinsurance

Retention	0	500 000	1 000 000	1 500 000	6 000 000
CoC net of reinsurance	-	3 276 972	4 015 015	4 266 601	4 514 388
Reduction in CoC vs gross of reinsurance	-100,0%	-27,4%	-11,1%	-5,5%	0,0%

Figure 3.4: Cost of capital reduction after the reinsurance

The result shows that the most suitable retention is 1M.
In general, the insurance should place the offer of reinsurance program with 1M retention in the market and look for the lowest premium.

(2) Ceding 20% of the volatility:

We model the volatility of claim amount net of reinsurance by using Best Estimate rate:

$$Var(S) = \sum_i q_{x_i}.(1 - q_{x_i}).N_i.\overline{SAR_i}^2$$

Where S: annual amount of claims net of reinsurance. SAR_i is the average Sum At Risk amount net of reinsurance of range i.

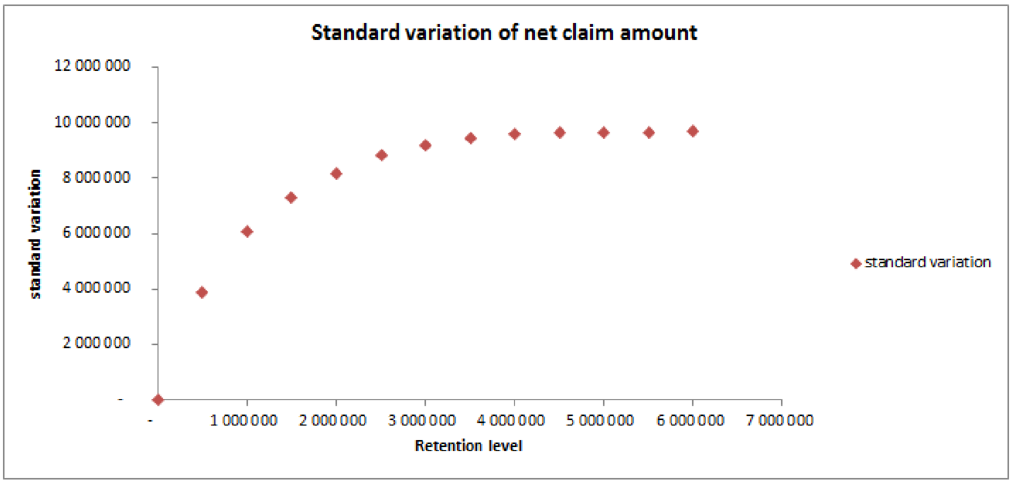


Figure 3.5: Standard deviation of net claim amount

Retention	-	500 000	1 000 000	1 500 000	2 000 000	6 000 000
σ	-	3 906 357	6 098 923	7 328 123	8 200 299	9 686 047
Volatility reduction	-100%	-60%	-37%	-24%	-15%	0%

Figure 3.6: Volatility reduction after reinsurance

The result suggests that the suitable level of retention according to insurers need in this case is between 1.5M and 2M.

3.2 Retention and limit of XL CAT

As described in the previous part, the XL CAT treaty can not help to reduce the SCR. In fact, in general, the main purpose of XL CAT reinsurance is to protect the portfolio against the catastrophic events with extreme losses.

The limit of CAT treaty should be normally set to cover the Probable Maximum Loss in one single event. The PML requests a specific study based on the real exposure of insured portfolio on some very concentrated site: building, working factory, etc. and also eventually geological studies, etc. This is not the purpose of this report and therefore we will just mention about it here without further details.

The retention of CAT treaty should correspond to the maximum loss that the insurance company can sustain in one single event. Each insurance company should define in their risk appetite for one single CAT event. The risk appetite framework of insurance company is also not the scope of our study.

Conclusion

The proposed risk-costing or pricing models for reinsurance treaties could help the internal reinsurer to challenge reinsurer's prices and define the best practice in terms of cession in a big insurance group. Due to the nature of data in reinsurance, it doesn't exist an unique model which fits all cases.

For XL per life, the reinsurer could apply a frequency-severity calibration model purely based on claim data when there is sufficient claim data. Even when the treaty is not working, this model can be always used if reinsurer can obtain the claim data below the retention. This thesis proposes a better quality of fit given by truncated distributions compared to usual distributions. When having less claim data, the reinsurer could use alternative methods such as Burning Cost or incident rate or a combined method. The combined method helps to use both view in Best Estimate of insurance claims and historical reinsured claim data.

For XL per event (CAT treaty), Generalized Pareto Distribution is often used for the calibration of CAT risks. However, it's interesting to test the quality of fits with other truncated "fat tail" distributions. The past models: Strickler's model and Erland Ekheden's CAT model in life were based on some strong assumption using public historical claim data in EMDAT. This report integrated also the terrorism claim data in GTD database. In additional, it proposes to use the portfolio Sum At Risk distribution data in the simulation of claim amounts and also to add the geographical concentration scenarios in the list of scenarios.

The optimization of reinsurance plays also an important role in the reinsurance management. Depending on the need of insurer in terms of volatility or required capital, an optimal reinsurance program can be defined to help insurer to obtain its objectives.

This thesis presented only a small part of life reinsurance activities. In particular, for internal reinsurance activities, the third important role such as capturing diversification is not studied in this thesis. Also, the modelling of proportional reinsurance is not mentioned. In general, with the implementation of solvency II with the requirement of better managing the risk and with the new age of data, reinsurance will have no doubt big advantage in not only risk management but also insurance business.

Appendix A

Example: calculation of Cost of capital in Burning Cost pricing

We suppose that Reinsurer follows standard formula. Reinsurer firstly calculates its SCR Life UW containing mainly SCR CAT and SCR mortality based on the sum at risk and the standard formula shock. The total SCR attached to the reinsurance treaty is calculated by the SCR Life UW times a transmission factor.

The calculation is performed in the following table:

XL per Life	
SAR to be covered for XL per Life	3 569 750 000
Average qx	0,1325%
Mortality shock standard formula	*15%*
SCR mortality XL per Life	709 241
CAT shock standard formula	*0,15%*
SCR Life CAT	5 354 625
Correlation mortality and life CAT	*25%*
SCR Life UW	5 574 396
Transmission factor	110%
SCR total	6 131 835
CoC rate	8%
CoC	490 547

In practice, the qx is calibrated depending on each reinsurer. The qx taken in this example is based on the local entity view of the risk (based on qx Best Estimate).

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