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Credit risk measure and optimization of a bond portfolio

Santiago Tavolaro

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Contents

Chapter 1

Introduction

This work is composed of two main objectives:

- The first objective is to measure credit risk of a fixed rate corporate bond portfolio based on historical data under a through-the-cycle regime and a stress period. It is crucial that this be accurate, robust, operational and as simple as possible. I will therefore try to avoid strong assumptions in this work.
- The second objective is to find the best portfolio under constraints using our credit risk assessment with $CVaR$ as the risk measure. More precisely, I distinguish in an optimization framework the part of the portfolio that is already invested from the one that has to be newly invested to determine a portfolio strategy.

For insurance companies, corporate bonds are very often the largest asset class of their investments after sovereign bonds. At the same time, it is not easy to measure the credit risk of a bond portfolio. Credit risk refers to the potential failure of a counterparty to make a contractual payment.

The current situation in which corporate bond spreads are significantly low naturally raises questions about the market's ability to properly assess the underlying risk. The unconventional activities of central banks have been going on for a while and are partly influencing bond prices. In addition, credit risk events and their magnitude are difficult to predict. They very often appear as a surprise. These facts encourage the measurement of credit risk based primarily on past long-term information rather than current market information.

Moreover, there is a significant specificity of the bond portfolios of French life insurers. These portfolios and their associated investment strategies are subject to French accounting rules for the realization of gains and losses. A specific rule ("Code of Insurance - Article A333-3") may discourage the sale of fixed rate bonds, zero coupon bonds or inflation-linked bonds prior to maturity in the case of a life insurance business. This accounting rule has been established to prevent life insurance companies from having an incentive to sell bonds in a declining interest rate environment. In this situation, a life insurance company selling its bonds will make profits over the purchased price, and then will have to reinvest its cash in lower coupon bonds than the first ones.

This rule protects clients against potentially overly aggressive investment strategies in which the pursuit of short-term profits could destroy the future profits of customers or affect the ability of a life insurance company to honor their contracts. In this context, the investment decisions of French life insurance companies may differ from those of other asset managers who simultaneously seek coupon gains and bond market price gains. A life insurance company should therefore focus more than other asset managers on credit risk because of their reduced ability to modify their portfolios. This constraint often involves a "buy-and-hold" strategy. This leads us to measure the credit risk of bond portfolios assuming that there is no significant investment decision in the portfolio other than holding the securities to maturity and reinvesting cash flow.

Optimizing a portfolio often addresses a two-dimensional risk-return problem. However, as far as bonds are concerned, duration is a third major dimension that it is more than legitimate to introduce into the optimization problem. The sensitivity of a bond portfolio to market price fluctuations is directly related to its duration. In addition, the interactions between the assets and liabilities of an insurance company highly depend on the adequacy of their duration. A duration constraint should therefore be naturally added to the classic optimization problem. Another important aspect of portfolio optimization is related to investment dynamics. When a new investment has to be chosen, the positions already held must be taken into account. I propose to make this remark clear in the optimization problem.

This work is composed of nine chapters. After this brief introduction, I will provide an overview of the main factors that affect the assessment of credit risk: underlying information, credit risk models and risk measures. Next, the third chapter presents the credit risk modeling framework. The fourth part is related to parameter estimation techniques. The fifth part presents the estimated values for each parameter. The sixth part introduces some basic descriptive elements about the empirical portfolio used to illustrate the results that can be obtained from the credit risk framework. Results related to the measurement of credit risk losses are discussed in chapter Seven. A portfolio optimization technique using $CVaR$ as a risk measure with some variants is then developed in chapter Eight. A conclusion and some perspectives constitute the last chapter of this work.

Chapter 2

Key features around credit risk assessment

This chapter analyzes successively the three main themes of any credit risk assessment: underlying information, credit risk models and risk measures. A last section gives my point of view on these three subjects.

2.1 Underlying data

The underlying data is probably the most important dimension of a credit risk assessment. In this section, I propose to highlight the main elements that differentiate sources of information with respect to credit risk. I categorize the sources of information into four distinct sets: accounting data or business data, market data, historical default event data, and macroeconomic data. These four sources of information are complementary and should therefore naturally, but also in different ways, be useful for assessing credit risk.

Accounting or business data

Accounting data, mainly based on balance sheets, profit and loss statements, income-related information and press releases, provide idiosyncratic information on the financial health of a firm. In assessing the risk of default, the assets and liabilities that the company holds and expects to hold are particularly useful information for assessing the risk of default of a given company. These data are regularly analyzed by the asset managers for all issuers present in their corporate bond portfolio. They also use this information to make new investment decisions, such as buying or not a new bond issued by a given company.

Accounting data, although very useful at the individual level for measuring, financially, a company's specific risk are rather limited in estimating the credit risk of a portfolio made up of a large number of issuers. The lack of information when considering interactions or dependencies between them do not make these data particularly useful for estimating the credit risk of a global set of bonds. Accounting data, however, are very informative in the expert judgment process.

We consider that accounting data, although relevant for asset managers, are not relevant to our work because they are too specific to each issuer to analyze the credit risk of a portfolio.

Market data

Market data relating to credit risk mainly include spread rates and CDS prices. Market data on credit risk provide interesting insights into how market participants measure, on average, the credit risk of an issuer. Market data have the advantage of being at a high frequency and responding to news and any change on a near instantaneous time scale.

The spread, which corresponds to the interest rate between the yield of a bond and that of a bond that can not default, is the rate above the risk-free rate that the operators are willing to receive for being exposed to an issuer. Spreads can then directly and instantly assess credit risk. A CDS is a derivative instrument giving the holder the right to sell a bond at face value in the event of a default by a particular issuer. The price of a CDS is in a sense the quotation of the protection premium for a given period in case of default. CDSs are directly related to failure events.

On the other hand, spreads and CDSs depend on several factors that can easily deviate from the mere prospect of a credit risk assessment. Market data do not necessarily reflect the risk of an issuer's failure. In fact, they reflect market participants' perceptions and not necessarily the fundamental risk of default. The view of market participants is influenced by many factors. For example, liquidity or the global capital available to invest can cause prices to deviate from fundamentals, temporarily affect market prices and create specific situations such as bubbles.

More generally, the parameters other than those related to the credit risk determining the spreads are, among others, the behavior of the financial markets, the investment decisions (the rolling of the portfolios, the hedging strategies), the macroeconomic situation and their forecasts (GDP, CPI, unemployment...), other opportunities in the markets materialized by the price of other assets, trust between financial institutions, policies and activities of central banks (the target changes, unconventional policies, central bank policy rates, public announcements) or regulatory changes. In the end, the spread and the CDS of an issuer are certainly not only related to its credit risk. This fact is undeniable nowadays and, since the financial crisis of 2008, it has even been amplified when one considers the excess of liquidity injected by the various central banks as well as their political actions. Spreads, which are currently very low, are a phenomenon that can not be entirely linked to the reduction of corporate credit risk.

More specifically, concerning CDSs, prices are driven by over the counter banks' market makers bid-offer prices. CDSs market participants willing to exposed themselves to credit risk as if it was a common asset can therefore raise prices to new levels without providing a strong link with fundamental credit risk. Moreover, even if we assume that CDS prices correctly reflect credit risk, the implicit models used to reverse the credit risk parameters imply different results depending on their respective assumptions. For example, by reverse-engineering CDS prices to obtain the probability of default of an issuer, assumptions must be made on the loss given default. It will be set, for instance, to 60% in a reverse-engineering model, whereas it is actually a random variable with a state space $[0, 1]$. Changing this type of assumption regarding certain risk parameters can have significant effects on others.

Market data is obviously important when it comes to making an investment decision at the best price on the financial market. However, when measuring the risk of failure of an institution, the importance of these data decreases because their ability to reflect the reality of failures is not obvious.

Default data

Information related to defaults is directly derived from past events. this information has the specificity, compared to other sources of information, to be historical. In fact, market data encapsulate present and forecast considerations.

Default information is very informative and accurate regarding default events. Rating agencies provide this information on the basis of a methodology grouping issuers into homogeneous classes with regards to their credit quality. In this approach, information with regards to default is then related to each rating and two firms having the same rating will automatically have the same expected probability of default [1]. The information provided includes the number of defaults, per year and per rating, the loss given default per year and per seniority class and rating transition matrices are as well provided. Rating transition matrices allow to know the likelihood of an issuer to increase or to decrease its credit quality over a given period.

Historical default values are provided by rating agencies such as Standard & Poor's, Moody's or Fitch. This information, provided on an annual frequency, is detailed by rating. The rating agencies give a rating to each company requesting the service. It is important to note that many companies simply do not want to be rated.

The first limitation, or strength, of default information is that they contain no predictive elements. Second, the homogeneity property inside each rating class is not always obvious.

Macroeconomic information

Macroeconomic information provides an insight on the health of the economy and its forecasts. This information is often common and shared by the entire market. This data is of great interest when considering predictive credit risk models. For example, using econometric methods and the present macroeconomic information, we may be able to predict rates of default for each rating for the next year. More importantly, we may be able to predict the next recession that will surely go hand

1. The clustering of issuers is a key process because it must ensure homogeneity within each rating class with respect to the probability of default.

in hand with an increase in default rates.

However, it is well known that financial phenomena are not easily predictable and a lot less when dealing with extreme events such as crises. Crises are surprises, at least for the date, for everyone except for a few (who are not the same with each crisis). Moreover, econometric models are very often difficult to calibrate and have a low level of prediction and many statistical assumptions are not fully verified.

2.2 Credit risk Models

Most models have strong assumptions about the distributions underlying default events. In addition, model calibration and the statistical quality of historical data are not particularly analyzed. Many assumptions are encapsulated, such as default distribution, common probabilities, or correlations of default events. When extreme events are followed, these remarks will also amplify operational impacts in a measure of credit risk.

To work, a model will always have to reduce the complexity of the real risk and take into account the constraint of the available information. In this context, expert judgment is irreplaceable and retains a prominent place in decision-making. But, at the same time, this is not enough when the risk involves many distinct and interactive dimensions that become difficult to manage without adequate tools.

The impacts of models in credit risk assessment can be significant, as shown by Frey and Mac Neil in 2002 [14]. It is essential to consider the assumptions, limitations and strengths of a credit risk model when used from a risk management perspective. They can indeed have a significant impact on investment decisions. In addition, it is important to keep in mind that the term "credit risk" has a broad definition and incorporates several different valuations.

Credit risk modeling has grown considerably since the 90s. Most of the time, literature reviews of credit risk models break down models according to how defaults are modeled. Default event modeling is one of the most important parameters in credit risk assessment, but not the only one. Frey and McNeil [15] (2003) or Joe [18] (1997) propose to divide models into two groups: structural models and reduced form models.

Jarrow and Protter [17] (2004) present structural and reduced-form models with an abstract approach. They argue that structural models assume that the model contains the same information as the company, namely a complete knowledge of the assets and liabilities of the company. On the other hand, reduced-form models have the same information as the market, an incomplete knowledge of the company's situation.

Structural models

Structural models are related to the Merton [23] fundamental model (1974) where firms are in default if the value of their assets is less than the value of their liabilities. This model is based on the valuation methodology developed for the options by Black and Scholes (1973) [4]. In practice, this model requires detailed and specific information from the company which implies great difficulties in the implementation process. As a result, the structural model approach simplifies the modeling of failure events by using random variables passing below a threshold.

Then, dependency between default events of issuers must be integrated when analyzing a bond portfolio. In this context, the dependence is estimated through the assets held by each issuer. The correlation of these assets is operationally difficult to estimate. The structural models use rather the correlation between equities.

Reduced form models

Pfister *et al.* [28] conducted a recent (2015) and very complete literature review regarding reduced form models. The reduced form approach models defaults through an intensity function. They include Duffie and Singleton [11] (1999; a conditional independent default model), Frey and McNeil [15] (2003; a copula model) and Frey and Backhaus [13] (2004; an interacting intensity model).

Industry models

The mains industry credit risk models are:

- CreditRisk+ developed by Credit Suisse financial products' [8] (1997);
- Credit Portfolio View developed by McKinsey & Co. (1998);
- Credit Monitor developed by KMV corporation [19] (2001);
- CreditMetrics originally developed by J.P. Morgan [31] (1997).

These models have been analyzed a tremendous amount of times. To have them all in a simple, synthetic and complete form, please report to the analysis provided by Brassard [5] (2002) or to Crouchy *et al.* [9] (2000). I give here a very short presentation of these models.

CreditRisk+ analyzes only the defaults and not ratings migration. Instead of using historical default rates, CreditRisk+ models the total number of defaults in a portfolio through a Poisson process. Then, a recovery rate is applied to the securities in default.

Credit Portfolio View is a model that uses macroeconomic information to estimate the probability of default. This approach integrates business cycles into the credit risk assessment. However, in finance, past events are not always informative for predictions. Even if they were, their heterogeneity over time is important. In addition, each crisis has always had different impacts on credit events.

CreditMonitor can be broken down into three steps: estimating the market value and volatility of firms' assets, calculating the distance to default, transforming the distance to default into a default rate. This approach is based on strong assumptions and relies on market data to estimate its own parameters.

CreditMetrics is often considered as the most operational model for measuring credit risk. It takes default values into account and evaluates migration events. The necessary data are: firms' rating, recovery rates, a transition rating matrix, correlations between securities and all the usual information on bonds (price, maturity, spread...). This methodology is decomposed into three steps: assessing the credit risk of individual securities, estimating the correlations between the securities and finally the value at risk of the portfolio. The assessment of individual credit risk is based on the historical average of default rates and transition matrices applied to each security taking into account its rating, market price and spreads from the market. The individual losses are then materialized by the price difference between the new estimated price and the current price. In case of default, the model also uses a recovery rate modeled by a Beta distribution and taking into account the seniority of each security. The second step is to estimate the correlations between each security with respect to transitions and default events. The credit metric reduces the number of combinations to be estimated by using a sector-country correlation matrix applied to each firm based on its level of participation in each category. Sector and country correlations are estimated from market indices. The third step is to simulate the multivariate random variables associated with all securities in the portfolio analyzed with the estimated correlation matrix.

2.3 Risk measures

The main purpose of risk management is to assess the performance against the risk taken. A loss distribution L is defined by the difference in value V_t of a portfolio between $t+1$ and t: $L = -(V_{t+1} - V_t)$. In t, L is a random variable as V_{t+1} is random. L is defined by its cumulative distribution function $F(l) = P(L \leqslant l)$. A risk measure is naturally linked to a distribution function of losses, the most exhaustive information, and is only a tool to reduce, more or less effectively, the information coming from this function.

The most common risk measures are the Value at Risk (VaR) and the Conditional Value at Risk ($CVaR$), also called Expected Shortfall:

- VaR_α is the value of the loss distribution function at quantile $\alpha \in]0,1[$

$$VaR_\alpha(L) = Inf\{l : P(L \leqslant l) \geqslant \alpha\}$$

- $CVaR_\alpha$ is the expected value of the loss distribution above the quantile $\alpha \in]0,1[$

$$CVaR_\alpha(L) = E[L|L \geqslant VaR_\alpha(L)] = \frac{1}{1-\alpha}\int_{VaR_\alpha(L)}^{\infty} l dP(l)$$

A stressed VaR and a stressed $CVaR$ can also be calculated. The only difference is related to the L loss function that is here estimated on a stressed sample.

In parallel of these measures, the concept of coherent risk measures has been developed in [1] (1997) and [2] (1999). A coherent risk measure m is a function that satisfies, for G, the set of loss distributions :

- monotonicity: for $X, Y \in G$ and $X \leqslant Y$, $m(X) \leqslant m(Y)$;
- positive homogeneity: $\forall \lambda \geqslant 0$ and $X \in G$, $m(\lambda X) = \lambda m(X)$;
- translation invariance: for $X \in G$, $a \in \mathbb{R}$, $m(X + a) = m(X) + a$;
- subadditivity: for $X, Y \in G$, $m(X + Y) \leqslant m(X) + m(Y)$.

The notions of subadditivity and positive homogeneity imply convexity [2]: for $X, Y \in G$ and $\lambda \in [0, 1]$, $m(\lambda X + (1 - \lambda)Y) \leqslant \lambda m(X) + (1 - \lambda)m(Y)$.

VaR is a standard reference for risk measurement. However, VaR is not a coherent measure because it does not have the property of subadditivity neither convexity [1] (1997). Subadditivity encourages diversification in finance. For example, without subadditivity, the VaR of a portfolio may be greater than the sum of $VaRs$ of the individual assets of that portfolio. VaR is a coherent risk measure if the loss distribution function is normally distributed. In addition, minimizing the VaR of a portfolio may have several local minimums because it is not a convex function.

$CVaR$ is, instead, always a coherent measure regardless of the loss distribution function and has therefore the convexity property ([26] and [29]).

Others risks measures are as well often used:

- the expected loss, $E[L]$, is the average loss that we should experience; it can be expressed as a $CVaR$ for a continuous loss function:
$$E[L] = \int_{-\infty}^{\infty} l dP(l) = \int_{VaR_{0^+}}^{\infty} l dP(l) = CVaR_{0^+}(L)$$
- the unexpected loss at the α level, $UL_\alpha[L] = VaR_\alpha(L) - E[L]$, is the loss difference between the VaR_α and the expected loss.

2.4 Remarks

This section presents my view on these three precedent key features with regards to our objectives before moving on to the model. The first objective is to measure the current credit risk of a portfolio. I do not consider in this work interest rate and spread risks changes. These risks should however be taken into account if we plan to optimize the Solvency 2 ratio. The second objective is to find an optimal portfolio under risk-return considerations.

2. Equality in this expression is convexity and non equality is strict convexity.

Data

Since our goal is to measure the credit risk of a corporate bond portfolio, I chose to use historical default data as much as possible as we want to measure the risk of a default in a portfolio in which all securities are held until maturity. The goal of an insurance company asset manager is, first of all, to take advantage of the coupons rather than the price movements of the securities. The risk associated with this objective is mainly related to defaults events where the coupons would no longer be paid and the principal partially lost.

Market, corporate and macroeconomic data are more than useful in the expert judgment process and in any decision-making approach. However, market data relating to credit risk, spreads and CDS prices, do not always reflect the credit risk of issuers. Accounting data is too granular to assess the risk of a portfolio, but rather the most useful for tracking an issuer specifically. Macroeconomic information is reliable for estimating the health of the economy, but crises have always been a surprise for most investors.

Credit risk models

Regarding credit risk models, they are first numerous. All of them contain several strong assumptions. Depending on the purpose of the work, some are more suitable than others. Reduced form models are more related to market conditions and then well adapted to pricing, for instance, derivatives products. Structural models seem more suited to the purpose of this work.

The industry model that seems the most suited to our task is probably the one implemented in the CreditMetrics approach. This makes it possible to take into account historical information rather than forecasts or macroeconomic information. Considering that a credit crisis is very often a surprise for any investor, a historical approach is very well adapted to integrate these events into the risk assessment.

However, when estimating risk under high quantiles, a long-term average approach such as the one used in the CreditMetrics becomes a limitation in the evaluation process. The time heterogeneity of default events, migration events, and default losses can no longer be captured. This average technique is in fact based on the assumption according to which the annual observations of each of these three parameters are realizations of independent and identically distributed random variables. This assertion is unlikely to be verified. It is obvious that credit events are strongly linked to business cycles [25].

Another limitation of the CreditMetrics approach is the way correlations between securities are modeled. CreditMetrics uses asset price correlation, taking into account sectors and countries, to incorporate in the valuation the relationship between securities with respect to credit events. Some critics appear around this correlation approach. The correlations between market prices are naturally very different from correlations related to credit events. For example, market prices may fluctuate in the same way after a specific announcement and snowball effects are very common in the market. In these situations, the correlations will increase or decrease between assets with no guarantee that we could have the same thing between credit events.

I therefore opt to use past default rates and transition rating matrices with a sampling technique to reproduce the results encountered in each past years instead of estimating averages. Correlations between securities with respect to default events will therefore be as well integrated in my approach through the heterogeneity of past events.

Risk measures

VaR is not suitable for measuring risk. The assumption of a normal loss distribution guaranteeing its coherence is simply not respected. $CVaR$ is much more useful with its coherence property. Moreover, its convexity has also the very strong advantage to be very useful in an optimization problem.

Chapter 3

The credit risk model

This chapter introduces the credit risk model by separating default loss modeling from migration loss. The link of these two risks is taken into account in this approach. Separating both phenomena allows to model them effectively, with their own parameters, variables and framework. A probably more generic and very common point of view regards defaults as another rating (just below C) and then only focus on rating transitions. In this case, the presentations specify in a second step the differences in terms of losses for defaults and migrations.

The associated losses are very distinct between default events and migration events. Accounting rules are very different for these two risks. In addition, in the event of a default, the liquidation process is very long, while a change in rating goes hand in hand with a change in the market value of the asset. An asset manager then naturally considers default and migration differently, at least not with the same risk aversion or utility function. This loss decomposition is simply expressed as:

$$L_{Credit} = L_{Default} + L_{Migration}$$

The next two sections describe how default and migration risks are measured over a one-year horizon. I do not consider in this work a multi-period evaluation (please report to [28] for an example). The presentation starts from the simplest case to extend the approach towards the considered functionalities.

3.1 Default risk

Our default loss model is closely related to the generic loss expression of a portfolio:

$$L_{Default} = \sum_{i=1}^{I} \mathbb{1}_{(D_i=1)}.LGD_i.EAD_i$$

where:

- I is the number of issuers;
- D_i is a univariate Bernoulli random variable with binary outcomes chosen from $\{0, 1\}$ and probability p_i to be in the state 1 (issuer i defaults in state 1); its probability density function is $f_{D_i}(d_i) = p_i^{d_i}(1 - p_i)^{1-d_i}$;

- LGD_i is a random variable within the state space $[0,1]$ expressing the percentage of loss given default on securities issued by issuer i; LGD_i is defined by its cumulative distribution function $F_{LGD_i}(x) = \int_0^x f_{LGD_i}(y)d(y)$;

- EAD_i is a deterministic value expressing the exposure amount on issuer i at default.

Very often, a corporate bond portfolio is made up of several securities issued by the same issuer. The losses given default can then be set at the security level. As we will see below, losses given default from securities issued by the same issuer are not independent from each other. Exposures at defaults can easily be expressed by security rather than by issuer. The expression then becomes:

$$L_{Default} = \sum_{i=1}^{I} \mathbb{1}_{(D_i=1)}. \sum_{s_i=1}^{S_i} LGD_{s_i}.EAD_{s_i}$$

where:

- S_i is the number of securities issued by i $\forall i \in [\![1, I]\!]$;

- LGD_{s_i} is the loss given default of security s_i issued by i;

- EAD_{s_i} is the exposure at default that the portfolio holds on the security s_i.

For a given issuer i, each security $s_i \in [\![1, S_i]\!]$ may hold different seniority levels. In case of default of issuer i, its securities should have the same loss given default for a same seniority level. The securities of issuer i can therefore be regrouped by seniority level. The expression then becomes:

$$L_{Default} = \sum_{i=1}^{I} \mathbb{1}_{(D_i=1)}. \sum_{Sen=1}^{N_{Sen}} LGD_{i,Sen}.EAD_{i,Sen}$$

where:

- $LGD_{i,Sen}$ is the loss given default of issuer i for the securities having the same seniority level Sen $\forall Sen \in [\![1, N_{Sen}]\!]$;

- $EAD_{i,Sen}$ is the sum of the exposures at default associated to issuer i for all securities s_i having the same seniority Sen.

Issuers may exhibit a dependency between each other with regards to default events [1]. Issuers' default events should be eventually modeled as correlated binary random variables. As previously stated, we have a default event for issuer i if $D_i = 1$ or not if $D_i = 0$. Instead of considering I independent univariate Bernoulli random variables, we now consider only one I-multivariate Bernoulli random vector $D = (D_1, ..., D_i, ..., D_I)$ of possible correlated random variables taking values in the

1. The main common factors found in the literature regarding the dependence of credit events are the rating classes, the business cycles, the domiciles and the business sectors of the issuers.

Cartesian product space $\Omega = \{0,1\}^I$. The size of Ω increases sharply with the number of issuers (for instance, with 100 issuers, $\#\Omega = 2^{100} > 1,26 \times 10^{30}$). The probability density function for $I = 2$ can be written as:

$$P(D_1 = d_1, D_2 = d_2) = P(1,1)^{d_1 d_2} P(1,0)^{d_1(1-d_2)} P(0,1)^{(1-d_1)d_2} P(0,0)^{(1-d_1)(1-d_2)}$$

with $P(1,1) + P(1,0) + P(0,1) + P(0,0) = 1$

Any marginal distribution of D follows a multivariate random Bernoulli distribution [10]. For instance the marginal distribution of D_i has density:

$$P(D_i = d_i) = \sum_{d_1=0}^{1} ... \sum_{d_{i-1}=0}^{1} \sum_{d_{i+1}=0}^{1} ... \sum_{d_I=0}^{1} P(d_1, ..., d_{i-1}, d_i, d_{i+1}, ..., d_I) \tag{3.1}$$

The loss default expression is then

$$L_{Default} = \sum_{i=1}^{I} \mathbb{1}_{(D_i=1)} . \sum_{Sen=1}^{N_{Sen}} LGD_{i,Sen} . EAD_{i,Sen}$$

with all D_i $\forall i \in [\![1, I]\!]$ having for density the expression 3.1.

A last step is related to loss given default. As presented later on, the loss given default should be unique for each seniority level of a given issuer. Securities issued by a same issuer are now aggregated by seniority level to measure risk. But another important aspect must be taken into account. In fact, the higher the priority of a creditor is in a defaulting issuer's liquidation process, the lower should its loss given default be (loss given default increases as the seniority level decreases). For an issuer i and $\forall$ $Sen \in [\![1, N_{Sen}]\!]$ ordered by decreasing seniority ($Sen = 1$ is the more senior level and $Sen = N_{Sen}$ is the less senior level), then if i defaults, I introduce new truncated random variables denoted $LGD^{TR}_{i,Sen} \forall Sen \in [\![1, N_{Sen}]\!]$ with state space $[0, 1]$ and defined by the following cumulative distribution function:

$$F_{LGD^{TR}_{i,Sen}}(x) = \begin{cases} \int_0^x f_{LGD^{TR}_{i,Sen}}(y)dy & \forall Sen \in [\![2, N_{Sen}]\!] \\ \int_0^x f_{LGD_{i,Sen}}(y)dy & \text{if } Sen = 1 \end{cases} \tag{3.2}$$

with

$$\begin{aligned} f_{LGD^{TR}_{i,Sen}}(x) &= f_{LGD_{i,Sen}}(x | LGD^{TR}_{i,Sen} > x_{i,Sen-1}) \\ &= \frac{\mathbb{1}_{(x > x_{i,Sen-1})} f_{LGD_{i,Sen}}(x)}{1 - F_{LGD_{i,Sen}}(x_{i,Sen-1})} \end{aligned}$$

and $x_{i,Sen-1}$ being a realization of the random variable $LGD^{TR}_{i,Sen-1}$.

The default loss expression is then :

$$L_{Default} = \sum_{i=1}^{I} \mathbb{1}_{(D_i=1)} . \sum_{Sen=1}^{N_{Sen}} LGD^{TR}_{i,Sen} . EAD_{i,Sen}$$

This model, measuring default risk, captures simultaneously :

- at issuer level, the correlation between default events ;
- at security level, the loss given default according to the seniority with a truncation approach.

3.2 Migration risk

The migration of ratings is the other important element in the credit risk assessment of a bond portfolio. Migration risk is related to credit quality changes that may occur in the coming year. During this period, issuers may be in a weaker, stronger or in an identical position with respect to their ability to meet their financial commitments.

The migration of ratings is applied to issuers that do not default after applying the default risk assessment. It is important to well distinguish both phenomena in our modeling. Default risk involves a resolution process with levels of LGD depending on the seniority of the securities. This is quite different from rating migrations (excluding default) where no resolution process is activated. They would however imply market gains or losses as the prices of the securities could vary according to the new rating class of each issuer. A downgraded issuer will cause losses through market prices corrections for its securities.

Rating migrations are very often modeled using a Markov chain where possible states are ratings : AAA, AA, A, BBB, BB, B, C and Default. As explained, we limit here the possible states to AAA, AA, A, BBB, BB, B and C in order to have a modeling approach consistent with the previous section 3.1.

The underlying phenomenon involving market price movements comes from market participants who expect their income to correspond to the new, higher (lower) probability of default of a downgraded (upgraded) issuer. Since the coupons associated with each security issued by a downgraded issuer are fixed at maturity [2], the only way to offset an increase in the probability of default of a "fixed income" instrument is to correct the market price of the security to a level reflecting the new expectations of investors in terms of risk-return.

These price movements can be modeled according to the concept of "yield to maturity" which we will call "yield". The yield of a security is the internal rate of return of the financial transaction of buying the security now and holding it until maturity. In other words:

$$-\frac{P_{s_i,0}}{(1+Yield_{s_i,0})^0} + \sum_{m=M_{s_i}-\lfloor M_{s_i}\rfloor}^{M_{s_i}} \frac{Coupon_{s_i}}{(1+Yield_{s_i,0})^m} + \frac{Principal_{s_i}}{(1+Yield_{s_i,0})^{M_{s_i}}} = 0$$

2. This is true for fixed rate bonds but not for floating rate bond or inflation linked bonds.

or

$$P_{s_i,0} = \sum_{m=M_{s_i}-\lfloor M_{s_i} \rfloor}^{M_{s_i}} \frac{Coupon_{s_i}}{(1+Yield_{s_i,0})^m} + \frac{Principal_{s_i}}{(1+Yield_{s_i,0})^{M_{s_i}}}$$

where

- $P_{s_i,0}$ is the market price of security s_i in $t=0$;
- $Yield_{s_i,0}$ is the yield of security s_i in $t=0$;
- $Coupon_{s_i}$ is the value of the annual coupon of security s_i (we suppose that coupons are annual);
- $Principal_{s_i}$ is the value of the principal of security s_i;
- M_{s_i} is the residual maturity, in years, of security s_i.

In the case of a rating migration, the yield changes. We assume that the new yield integrates the spread difference between the new rating class and the old one. This can be expressed as:

$$Yield_{s_i,1} = Yield_{s_i,0} + \Delta Spread_{r0(i),r_1(i),M_{s_i},Sen_{s_i}}$$

where

- $Yield_{s_i,1}$ is the new yield due and only due to an eventual rating migration of security s_i; for instance, if there is no rating change for i then $Yield_{s_i,1} = Yield_{s_i,0}$;
- $\Delta Spread_{r0(i),r_1(i),M_{s_i},Sen_{s_i}} = Spread_{r1(i),M_{s_i},Sen_{s_i}} - Spread_{r0(i),M_{s_i},Sen_{s_i}}$ is the difference of spreads between the current rating and the rating after one year of issuer i for the residual maturity M_{s_i} and the seniority level Sen_{s_i} of the security s_i [3].

The loss migration can therefore be expressed as follows:

$$L_{Migration} = -\sum_{i=1}^{I}\sum_{s_i=1}^{S_i}(P_{s_i,1} - P_{s_i,0})$$

where $P_{s_i,0}$ is the price of security s_i in $t=0$ and

$$P_{s_i,1} = \sum_{m=M_{s_i}-\lfloor M_{s_i} \rfloor}^{M_{s_i}} \frac{Coupon_{s_i}}{(1+Yield_{s_i,1})^m} + \frac{Principal_{s_i}}{(1+Yield_{s_i,1})^{M_{s_i}}}$$

$\Delta Spread_{r0(i),r_1(i),M_{s_i},Sen_{s_i}}$ is estimated through a Markov chain approach where rating changes have a probability associated to the rating level held in $t=0$. The only random variable included in $P_{s_i,1}$ is $r_1(i)$, the new rating of i in $t=1$. $r_1(i)$ is

3. Spreads take into account two different seniority levels. The associated yields are presented in figure 5.7 page 38.

the outcome of the random migration process that we model according to a Markov chain approach. The transition matrix, defined as a $n \times n$ matrix where n is the number of ratings, is such that:

$$TM = \begin{pmatrix} tm_{r_1,r_1} & tm_{r_1,r_2} & ... & tm_{r_1,r_n} \\ tm_{r_2,r_1} & tm_{r_2,r_2} & ... & tm_{r_2,r_n} \\ ... & ... & ... & ... \\ tm_{r_n,r_1} & tm_{r_n,r_2} & ... & tm_{r_n,r_n} \end{pmatrix}$$

where $tm_{r_k,r_l} \geqslant 0 \ \forall \{r_k, r_l\} \in [\![1, n]\!]^2$ and $\sum_{l=1}^{n} tm_{k,l} = 1 \ \forall k \in [\![1, n]\!]$.

tm_{r_k,r_l} is the probability of an issuer rated r_k to be rated r_l one year later. For an issuer i and its $r_0(i)$ rating, by simulating TM, we obtain its new rating associated to TM and then $\Delta Spread_{r_0(i),r_1(i),M_{s_i},Sen_{s_i}}$ for each security s_i of issuer i.

I suppose that the rating change occurs almost instantaneously in $t = 0 + \epsilon$ for ϵ very small. I therefore use the spreads known in $t = 0$ for those in $t = 0 + \epsilon$.

Migrations modeled through a Markov chain have two strong assumptions:

- the Markov behavior, meaning that the future cannot be better estimated with past information when we already uses the present

$$P(X_{t+1}/X_t = x_t, X_{t-1} = x_{t-1}, \dots, X_0 = x_0) = P(X_{t+1}/X_t = x_t)$$

- the time homogeneity, where transition probabilities are independent of time

$$P(X_{t+1}/X_t = x) = P(X_t/X_{t-1} = x) \ \forall t$$

It is not obvious that the Markov behavior is verified. However, alternative models are very complex to implement. The time homogeneity of the Markov chain is also debatable. It is natural to expect that economic cycles, recessions or any change in the macroeconomic situation will more or less change the migration of ratings through time [25]. This last point will be treated in the next chapter.

Chapter 4

Methods for estimating parameters

This chapter develops methodologies for estimating the different parameters when necessary. We will successively discuss the loss given default, the probability of default including default correlation, rating migration and exposure at default.

4.1 Loss given default

The Loss given default (LGD) is less analyzed in credit risk models compared to probability of default. At the same time, many articles are exclusively related to modeling the LGD as a random variable (see for example [16] for a comprehensive literature review on this topic).

When measuring risk under low probabilities (extreme events) the LGD should be modeled as a random variable with state space $[0, 1]$ rather than a deterministic value. The Value at Risk at 99% can considerably be underestimated by a model with a fixed LGD instead of using a probability distribution covering all the possible state space.

The number of parameters interacting with the LGD are numerous and quite strongly specific to each security: the issuer, the security seniority, the economic environment, the underlying jurisdiction of the recovery process...

Several years may pass before the end of the recovery. Failing issuers can therefore see the value of their assets, which will likely be sold, evolve over time. Creditors also carry an opportunity cost as they can not reinvest now the value they will recover only at the end of the recovery process.

Forecast LGD is quite difficult to handle as we should, for instance, consider: the time period of the recovery process; returns rate of missed investment opportunities during this period; the price of the assets held by the defaulted issuer until the end of the recovery process.

There exist two main historical LGD values that we can potentially integrate in the default loss measurement:

- workout LGD, based on the ultimate recovery, at the end of the entire workout process (often several years);
- market LGD, calculated from market prices 30 days after default.

I opt to use the market LGD in the modeling. This implies a short-term estimate of the loss that could be useful in the event that the asset manager wishes to liquidate his position before embarking on the recovery process. Specific institutions purchase defaulted bonds to speculate on the recovery process.

The Beta distribution is a very appropriate random variable for integrating a stochastic LGD into credit risk models. The Beta distribution can be calibrated using the mean and the standard deviation of the past market $LGDs$. Its $[0, 1]$ state space allows to express the loss as a percentage of the exposure at default. The assumption of a Beta distribution for the LGD is usual in the academic literature as well as in the industrial default risk models. This distribution leads to a family of continuous probability distributions.

As mentioned in the previous chapter, the risks associated with securities issued by the same issuer are closely linked. The seniority dimension implies a priority among creditors in the recovery process. At the same time, creditors of a given security should legally have the same recovery rate. In this situation, we should consider two main assumptions when modeling $LGDs$ related to the same defaulted issuer:

- the LGD of a low seniority security is higher than the LGD of securities with higher seniority ;
- creditors with the same seniority level have the same LGD on issuer i.

After considering our previous remarks, the LGD modeling approach estimates an LGD random variable by seniority level. Seniority is obviously one of the most important parameter discriminating $LGDs$ [1].

For a given security with seniority Sen issued by issuer i, we estimate the distribution $LGD_{i,Sen}$. Based on historical data by seniority level provided by the historical data report from Moody's [24], I calibrate, for each level of seniority, a Beta distribution based on the mean and the variance.

A Beta distribution is entirely determined by two non-negative parameters denoted (a, b). The probability density function of $X \sim Beta(a, b)$ is :

1. The $LGDs$ are assumed to be independent between the issuers in this work. This hypothesis is not insignificant. For example, when two liquidation processes are launched at the same time for two different issuers, the respective assets sold will be subject to the same market conditions, possibly at the same discount price or under the same conditions of sale. This will automatically correlate the amounts available for recovery.

$$f(x,a,b) = \begin{cases} \frac{\Gamma(a+b)}{\Gamma(a)+\Gamma(b)} x^{a-1}(1-x)^{b-1} = \frac{x^{a-1}(1-x)^{b-1}}{\int_0^1 u^{a-1}(1-u)^{b-1}\mathrm{d}u} & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

where Γ is the Gamma function.

The expectation of X is $E[X] = \frac{a}{a+b}$.

The variance of X is $V[X] = E[X^2] - E[X]^2 = \frac{ab}{(a+b)^2(a+b+1)}$.

To understand the shape of the Beta distribution, the interpretable parameters are $a+b$ and $\frac{a}{a+b}$ instead of expectation and variance. The second parameter is useful for giving a measure of the asymmetry of the distribution as a function of its position with respect to 0.5 and the higher $a+b$ is, the more the distribution is concentrated.

This distribution is very flexible and allows many forms of curves:

- if $a = 1$ and $b = 1$, we obtain a uniform distribution;
- if $a = b$, we have a symmetric distribution in the point 0.5; if they are moreover both higher than 1 the curve has a "bell" form; if they are smaller than 1 the curve has a "U" form;
- if $a > b$, the curve is asymmetric to the right side (negative skewness).

The $LGDs$ related to an issuer i are then

$$\begin{pmatrix} LGD_1 \\ \vdots \\ LGD_{N_{Sen}} \end{pmatrix} = \begin{pmatrix} B(a_1, b_1) \\ \vdots \\ B(a_{N_{Sen}}, b_{N_{Sen}}) \end{pmatrix}$$

There are two main general parametric techniques to estimate (a,b):

- the method of moments;
- the maximum likelihood.

The method of moments is certainly the easiest way to determine (a,b). Based on a set of n pasts historical loss given default values denoted $(x_1, \ldots, x_n)$ assumed to be independent and identically distributed, we obtain :

$\hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} x_i$, an estimator of $E[X]$

and

$\hat{\sigma}^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \hat{\mu})^2$, an estimator of $V[X]$

so

$$\begin{cases} \hat{\mu} = \frac{a}{a+b} \\ \hat{\sigma}^2 = \frac{ab}{(a+b)^2(a+b+1)} \end{cases}$$

from which we easily have an estimation of (a, b):

$$\begin{cases} \hat{a} = \hat{\mu}(\frac{\hat{\mu}(1-\hat{\mu})}{\hat{\sigma}^2} - 1) \\ \hat{b} = (1-\hat{\mu})(\frac{\hat{\mu}(1-\hat{\mu})}{\hat{\sigma}^2} - 1) \end{cases}$$

The maximum likelihood estimation method is also very often used. Consider the family of probability distribution function f_θ. From the observations $(x_1, ..., x_n)$, we compute the multivariate probability distribution function $f_\theta(x_1, ..., x_n|\theta)$.

The likelihood function is $L_\theta = f_\theta(x_1, ..., x_n|\theta)$ and the likelihood method consists in estimating θ by maximizing the likelihood function $L_\theta : \hat{\theta} = argmax_\theta L_\theta$.

By considering that $(x_1, ..., x_n)$ are independent and identically distributed, the likelihood function can be simplified by the product of n univariate probability densities: $L_\theta = \prod_{i=1}^n f_\theta(x_i|\theta)$.

Then the log-likelihood transforms the product into a sum: $\log L_\theta = \sum_{i=1}^n \log f_\theta(x_i|\theta)$. The maximum of $\log L_\theta$ can be found by various optimization techniques.

The maximum likelihood technique estimating the parameters of a Beta distribution is as follows. From a set of independent and identically Beta distributed observations $(x_1, ..., x_n)$, we compute the likelihood function : $L_{(a,b)} = \prod_{i=1}^n B_{a,b}(x_i|a, b)$.

Using the expression of the density of a Beta distribution, we directly obtain the following log-likelihood:

$$\log L_{(a,b)} = \sum_{i=1}^n \log\left(\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\right) + (a-1)\log(x_i) + (b-1)\log(1-x_i)$$

The maximum of $\log L_{(a,b)}$ is reached for $(\hat{a}, \hat{b})$ defined by :

$$(\hat{a}, \hat{b}) = argmax_{(a,b)} \log L_{(a,b)}$$

Beside these two parametric approaches, there is the non parametric method based on the empirical cumulative distribution function defined as follows [2]:

$$F_n(x) = \frac{1}{n}\sum_{i=1}^n \mathbb{1}_{(x_i \leqslant x)}$$

2. To go further on the non-parametric approach, Chen (1999) [7] proposes a Beta kernel estimator.

Returning to our model and considering an issuer with N_{Sen} successive levels of seniority ($sen = 1$ being more senior than $Sen = 2$) with the random variables $LGD_{i,1}$, $LGD_{i,2}$, ..., $LGD_{i,N_{Sen}}$, we can incorporate them in the truncated distribution expression defined in the previous chapter in 3.2 page 17.

4.2 Probability of default

Probability of default (PD) is related to issuers rather than securities. PD is unique for each issuer. It is, however, quite difficult to estimate every individual PDs when information regarding default events is often very scarce or nonexistent at the issuer level.

A common method for estimating the PD of issuers is to assign each issuer a rating class. Then, we can associate to each rating class an estimate of its PD assuming that all the issuers belonging to the same class are identically distributed. Rating agencies operate in practice this way. They associate to each issuer a rating [3] that is then used by market participants to assess the credit quality of their portfolio, to make new investment choices or to determine their regulatory capital requirements. Assuming default events are identically distributed within the same rating class, the complexity of the estimate of the probability of default moves from I values (number of issuers) to R values (rating classes) with $R \ll I$. We denote by D_r the default probability distribution of an issuer rated r. We assume that D_r follows a Bernoulli distribution with parameter p_r, the probability of default.

Figure 5.4 page 35 shows the number of corporate defaults registered for 10 000 issuers per rating and per year. These results are provided by the Standard & Poor's annual corporate default study (please report to [34] for the 2018 issue).

Another important dimension regarding PDs is their correlation. Correlations are expected between issuers with respect to default events. For example, the macroeconomic environment, the likelihood of a financial crisis, political instability, regulatory changes, changes in central bank monetary policies or pro-cyclical effects can affect, in different ways, multiple issuers simultaneously [4].

However, whereas the model described in the previous chapter makes use of a multivariate Bernoulli distribution to model default events, the simulation of correlated binary data is not trivial and research in this domain remains active. Several solutions exist under more or less restrictive assumptions. Papers in this topic are numerous [5] since Bahadur (1961) [3]. Many methods involve adjusting a continuous multivariate random variable to verify the Bernoulli marginals and their correlation matrix. For instance, Emrich and Piedmonte method (1991) [12] uses a multivari-

3. Being rated by a rating agency is not compulsory. This is a paid service offered by rating agencies to issuers of securities.

4. We exclude causality considerations in this work.

5. For some examples, please refer to [21], [27] or [30].

ate normal distribution with specific means and correlation matrix. This method involves solving nonlinear equations via numerical integration and does not always guarantee a solution.

Moreover, in 2006, Chaganty and Joe [6] show that there are positive-definite correlation matrices of order $d \geqslant 3$ that are not compatible with the marginal probabilities of any d-multivariate Bernoulli. This is a major drawback for models using correlations other than those related to default events (for instance correlations between equities) as there is no guarantee that they will be compatible with default probabilities.

In parallel, the estimation of correlations is based on two strong assumptions. for instance, the method of moments, presented below, assumes that the data sample of default rates per rating and year is composed of independent and identically distributed data over time.

There is a clear time dependence of default events coming from economic cycles in which, after a recession characterized by higher default levels, the beginning of a new business cycle will be followed by fewer failures as the weaker issuers are gone (defaulted) and stronger remain present with an increasingly favorable economy.

The assumption of identically distributed data is also easily questionable for the same reasons. In times of economic recession, the increase in defaults compared to a normal situation is significant. Figure 5.4 page 35 indicates the number of defaults per rating and per year $y \in [\![1981, 2017]\!]$ for 10 000 issuers. We clearly see that before years in crisis, the number of defaults is low compared to an average rate and strongly increases the year of the crisis. Then, the number of defaults slowly decreases over several years before reaching a new low level.

Using average default rates may underestimate the number of defaults we may encounter. We mitigate the effects of years experiencing extreme numbers of defaults with an average-based approach. The following figure 4.1 gives an overview of how an average-based approach can underestimate default events. I assume, per rating, that historical defaults are independent observations generated from the same Binomial random variable (identically distributed) with the following parameters: average default rate between 1981 and 2017 and the number of rated issuers by Standard & Poor's in 2017. I can then estimate the confidence interval, for instance at quantile 0.95, and count the number of years that are not in the confidence interval. At quantile 0.95, if the aforesaid assumptions are verified, we should have 1.85 year over 37 years that are not in the confidence interval. Figure 4.1 shows that ratings from A to C have clearly more than 1.85 year out of the confidence interval. For instance, There are 24 years outside the confidence interval for BBB issuers. Assumptions of independent and identically distributed data are not verified for most ratings.

Figure 4.1 – Confidence intervals of default rates assuming annual observations per rating are independent and identically distributed

	AAA	AA	A	BBB	BB	B	C
Average default rate (1981 - 2017)	0,00%	0,01%	0,06%	0,21%	0,90%	4,34%	24,07%
Number of issuers rated in 2017 (S&P)	14	336	1355	1800	1317	1834	244
Default rate - Confidence interval - 2,5%	0,00%	0,00%	0,00%	0,06%	0,46%	3,44%	18,85%
Default rate - Confidence interval - 97,5%	0,00%	0,30%	0,22%	0,44%	1,44%	5,29%	29,51%
Number of years out of the confidence interval (over 37 years)	0	1	3	24	22	29	26

Given the assumptions underlying the estimation process of p_r $\forall r \in [\![1, R]\!]$ and pair-wise correlations, the strong limitations regarding the simulation of correlated binary data and the risk of underestimating long-term risk, I opt for a direct simulation of observed past data. This technique consists in simulating randomly the years, each having the same probability to be selected. Then, I use the default rates of each rating of that given year to obtain one set of simulations for the I issuers. We repeat this operation for each simulation: choosing randomly a year and then using the default rate of the selected year for the I issuers. By doing this way, we naturally capture the dependency between ratings with regards to default events that was observed each year in the past. This approach replaces the need to estimate correlations of default events.

I present however, below, how to estimate correlations of default events. Two main ways are possible. First, assuming that all issuer are independent, we can estimate the impact of exogenous variables on each rating. Then, the probability of a number of defaults for a given portfolio conditioned by exogenous variables can be estimated using a mixture model as presented by Frey and McNeil [15] (2003). Second, we can directly estimate the correlations between pairs of ratings without considering the sources of dependency as presented in the seminal and very intuitive paper of Lucas [20] (1995). Since, our objective is to capture through-the-cycle (long-term estimation) and stress period credit risk instead of forecasting credit events depending on current macroeconomic information, I present the second approach. It is important to keep in mind that this correlation, although it exists, is small as many empirical studies have already shown. This point encourages diversification because its gain is very strong.

The correlation expression between two Bernoulli random variables X and Y is:

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma(X)\sigma(Y)}$$

$$\rho_{X,Y} = \frac{E(XY) - E(X)E(Y)}{\sqrt{(E(X^2) - E^2(X))(E(Y^2) - E^2(Y))}}$$

This expression can be written with events probabilities if we consider, for two Bernoulli random variables denoted X and Y, that:

$$E(X) = \sum_{x=0}^{1} xP(X = x) = P(X = 1)$$

and

$$E(XY) = \sum_{x=0}^{1}\sum_{y=0}^{1} xyP(X = x, Y = y) = P(X = 1, Y = 1)$$

Then, ρ can be expressed as:

$$\rho_{X,Y} = \frac{P(X = 1, Y = 1) - P(X = 1)P(Y = 1)}{\sqrt{P(X = 1)(1 - P(X = 1))P(Y = 1)(1 - P(Y = 1))}}$$

Using R rating classes following Bernoulli random variables, the correlation between two issuers X and Y with respectively ratings r and $r' \in [\![1, R]\!]$ depends on the following parameters:

- $P_r(X = 1)\ \forall r \in [\![1, R]\!]$;
- $P_{r,r'}(X = 1, Y = 1)\ \forall (r, r') \in [\![1, R]\!]^2$.

The correlation between X and Y is therefore expressed by:

$$\rho_{r,r'} = \frac{P_{r,r'}(X = 1, Y = 1) - P_r(X = 1)P_{r'}(Y = 1)}{\sqrt{P_r(X = 1)(1 - P_r(X = 1))P_{r'}(Y = 1)(1 - P_{r'}(Y = 1))}}$$

From this simple $\rho_{r,r'}$ expression, we clearly see the parameters needed to estimate a portfolio default loss distribution function taking into account possible correlations between issuers:

- $P_r\ \forall r \in [\![1, R]\!]$, individuals probabilities of default per rating;
- $P_{r,r'}\ \forall \{r, r'\} \in [\![1, R]\!]^2$, joints probabilities of defaults for all pairs of ratings.

As for the estimation of the LGD, I now present the parametric method of moments [6] to estimate P_r and the $P_{r,r'}$. The estimation of the joint probabilities $P_{r,r'}$ is less intuitive than the estimate of the marginals P_r. Lucas [20] (1995) explains the meaning of $P_{r,r'}$ and uses the method of moments to identify it.

The method of moments provides a simple estimate of PDs per rating. It is fairly easy to estimate a long-term average default probability per rating given that:

$$P_r = 1P_r + 0(1 - P_r) = E(D_r)$$

The method of moments provides the following estimator of $E(D_r)$:

$$\mu_r = \frac{1}{Y}\sum_{y=1}^{Y} P_{r,y}$$

6. For a Maximum likelihood approach based on mixture models, we may refer to Frey and McNeil [15] (2003).

where $P_{r,y}$ is the historical default rate of issuers rated r in year y. We transform this default rate expression for rating r and year y into numbers of defaults and numbers of issuers for rating r and year y to get:

$$\mu_r = \frac{1}{Y} \sum_{y=1}^{Y} \frac{\#d_{r,y}}{\#I_{r,y}}$$

where $\#d_{r,y}$ is the number of defaulted issuers rated r in year y and $\#I_{r,y}$ is the number of issuers (not in default) at the start of the period rated r in year y [7]. This expression can be written as:

$$\mu_r = \frac{1}{Y} \sum_{y=1}^{Y} \frac{\binom{\#d_{r,y}}{1}}{\binom{\#I_{r,y}}{1}}$$

with $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

This last expression clearly shows the simplicity behind this estimation technique. for r and y given, we are choosing, at the numerator, one issuer between the defaulted issuers, and, at the denominator, one issuer between all the issuers at the start of the year (not in default). It is the probability of default of one and only one issuer with rating r.

The simultaneous probability of default, in a year, of two issuers with same rating r denoted $P_{r,r}$ is similarly estimated as follows:

$$\hat{P}_{r,r} = \frac{1}{Y} \sum_{y=1}^{Y} \frac{\binom{\#d_{r,y}}{2}}{\binom{\#I_{r,y}}{2}}$$

For two distinct ratings r and r', the probability is estimated by:

$$\hat{P}_{r,r'} = \frac{1}{Y} \sum_{y=1}^{Y} \frac{\binom{\#d_{r,y}}{1}\binom{\#d_{r',y}}{1}}{\binom{\#I_{r,y}}{1}\binom{\#I_{r',y}}{1}}$$

As presented in the credit risk model (chapter 3), the probabilities of default are modeled through a multivariate Bernoulli distribution to be able to integrate eventual correlation effects with regards to default events between issuers. With this estimation approach, all the parameters defining the ratings' multivariate distribution, of dimension R, are now available. We then can move to the issuer level with a I-dimensions multivariate Bernoulli distribution such that:

- $P_i = P_{r(i)} \ \forall i \in [\![1, I]\!]$,
- $P_{i,i'} = P_{r(i),r(i')} \ \forall (i, i') \in [\![1, I]\!]^2$,

where $r(i)$ is the rating of issuer i.

7. It is important to keep in mind that issuers may encounter a rating change over the considered year y period; point treated in the next section related to migration risk

We obtained an estimation process determining the Bernoulli marginals and the pairwise correlations of issuers regarding default events.

4.3 Rating migration

Rating migration is modeled as a Markov chain without the default and the NR (Not Rated) states. As already explained in the previous chapter, since default events are already modeled in the default risk assessment, we apply the migration risk assessment to issuers that have not defaulted in the default risk assessment.

Most of the time an average historical transition matrix is used. As seen in the previous section regarding probability of defaults, I use the same random technique to choose years and their associated transition matrix for one simulation of the entire set of issuers. For one simulation of the I issuers, the year chosen randomly is naturally the same for default and migration to guarantee their coherence. The annual past transition matrices provided by Standard & Poor's have been resized excluding the NR and default states. Please refer to the annex for a complete list of annual transition matrices from 1981 to 2017.

As for default rates, with this approach, years in which transitions were more important or less important than an average-based approach will be considered. This is useful when measuring extreme events rather than expectations.

4.4 Market yields

Market corporate yields are used in the migration loss assessment. They are calculated by rating (AAA, AA, A, BBB, BB, B and C), seniority (senior and subordinated) and by maturity (with an annual granularity). For each combination of these three features, an average yield is calculated by weighting the amount issued for each security on the market. These values are based on the information of 2 722 euro denominated corporate securities. These data are presented in more details in figure 8.1 page 68.

4.5 Exposure at default

The exposure at default, EAD is the maximum amount that could, in the eventuality of a credit event, never be recovered. The EAD does not require an important level of modeling compared to other parameters. The EAD is deterministic and, at a given moment t, an asset manager knows its exposure to a given security.

Chapter 5

Parameter estimation results

This part presents the estimates of the different risk parameters introduced in the previous chapter. The estimation of each parameter is carried out under two regimes: a through-the-cycle and a stress period. The through-the-cycle estimate is linked to long-term available historical information while the stress period estimate is related to the worst year for credit risk of corporate bonds over the last 37 years, 2008 [1].

I use the default event history from 1981 to 2017 recorded by Standard & Poor's and Moody's presented on an annual basis. They provide annual default rates per rating, annual recovery rates per seniority and annual transition rates.

An exhaustive set of market securities is as well used to determine the yields per rating, seniority and maturity.

5.1 Loss given default

Losses given default (LGD) are estimated by seniority. Figure 5.1 page 32 shows the historical market LGD [2] averages by year and seniority computed from the recovery rates (denoted rr) recorded by Moody's [24] between 1982 and 2017 ($LGD = 1 - rr$). From this table we can calculate the mean and the standard deviation for each seniority level (senior secured, senior unsecured and subordinated), then obtain an estimate of the two coefficients (a, b) of a Beta distribution by seniority level. The same operation is carried out over the most unfavorable years in order to determine the same parameters for the stress period. I estimate the stress period parameters over a minimum set of worst years to be able to obtain a standard deviation even if the number of observations is very small.

The set of historical $LGDs$ are annual averages. A more advanced and accurate technique would consist to use individual recorded $LGDs$ instead of annual averages. We could then estimate an LGD distribution for each year and each seniority instead of one by seniority level. In this case, the simulation technique of years used for default rates and rating migration could then be also applied to LGD figures. This

1. 2008 is the worst year for AA and A default rates as illustrated by the figure 5.4 page 35.
2. Please refer to section 4.1 for a definition.

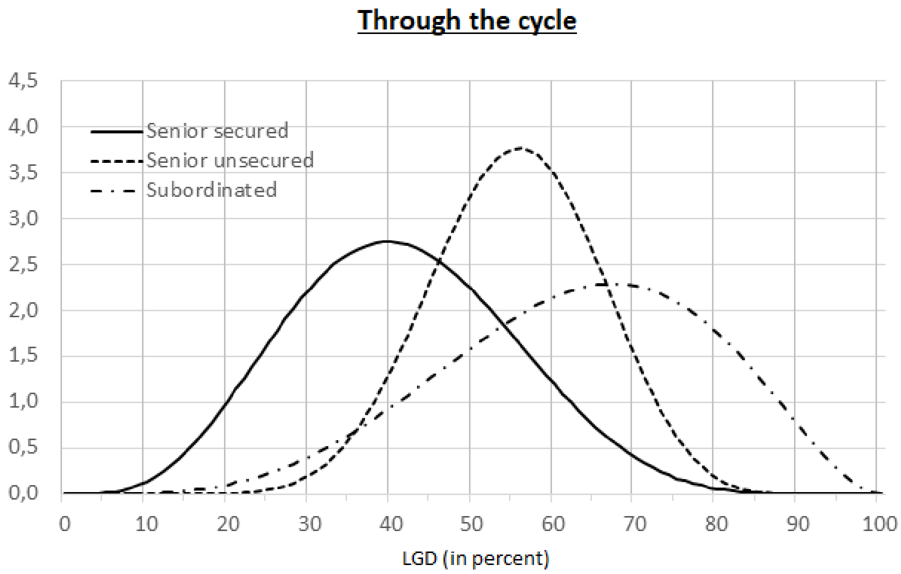
would increase the coherence of the approach as market $LGDs$ and default rates are correlated through time.

Figure 5.1 – Loss Given Default historical rates by year and seniority (source: Moody's [24]; market $LGDs$ 30 days after default)

Year	Through The Cycle			Stress Period		
	Senior Secured	Senior Unsecured	Subordinated	Senior Secured	Senior Unsecured	Subordinated
1982	28%	64%	70%			
1983	60%	47%	59%			
1984		51%	56%			
1985	16%	40%	61%			
1986	41%	47%	57%			
1987	29%	37%	53%			
1988	45%	55%	66%			
1989	53%	56%	74%			
1990	66%	63%	81%			
1991	52%	63%	76%			
1992	38%	51%	62%			
1993		63%	56%			
1994	31%	46%	62%			
1995	38%	52%	58%			
1996	52%	37%	77%			
1997	25%	44%	64%			
1998	53%	58%	82%			
1999	57%	62%	64%			
2000	61%	76%	68%	61%	76%	68%
2001	62%	79%	84%	62%	79%	84%
2002	52%	70%	75%	52%	70%	75%
2003	37%	58%	88%	37%	58%	88%
2004	27%	48%	6%			
2005	28%	45%	49%			
2006	25%	45%	44%			
2007	19%	47%				
2008	42%	66%	76%	42%	66%	76%
2009	63%	63%	55%	63%	63%	55%
2010	41%	55%	66%			
2011	15%	11%	17%			
2012	42%	57%	63%			
2013	32%	55%	74%			
2014	26%	54%	61%			
2015	45%	62%	42%			
2016	51%	69%	76%			
2017	38%	46%	26%			
Mean	41%	54%	61%	53%	69%	74%
Std	14%	13%	18%	11%	8%	12%
Beta coefficient						
a	4,9	12,4	4,9	11,3	12,1	9,0
b	7,0	10,1	2,9	9,8	6,3	3,4

The two following figures 5.2 and 5.3 present probability distribution functions and cumulative distribution functions for each regime and seniority level. Differences of distribution law between seniority levels are important. For instance, under the through-the-cycle regime, the senior secured securities have a median of 40% while the subordinated class is at 65%.

Figure 5.2 – Loss Given Default - Probability Distribution Function by seniority level



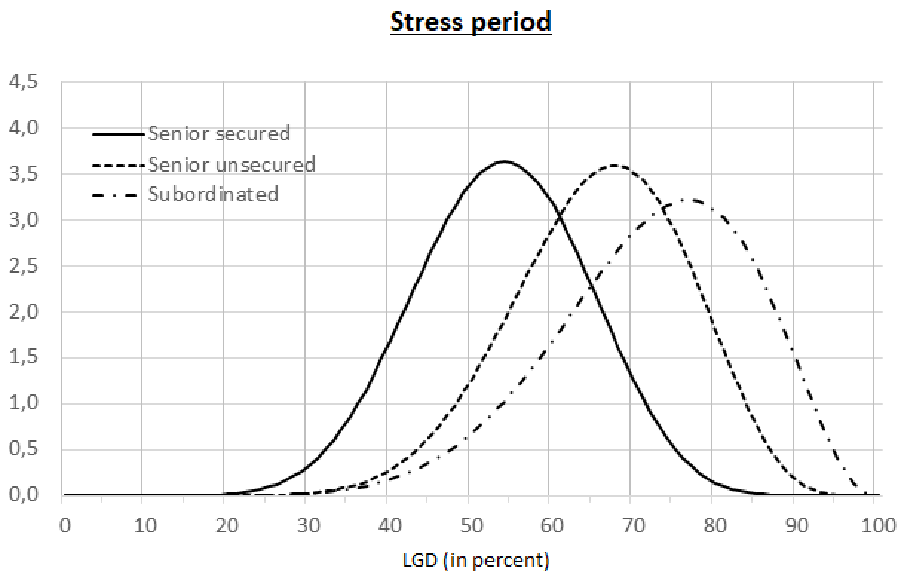
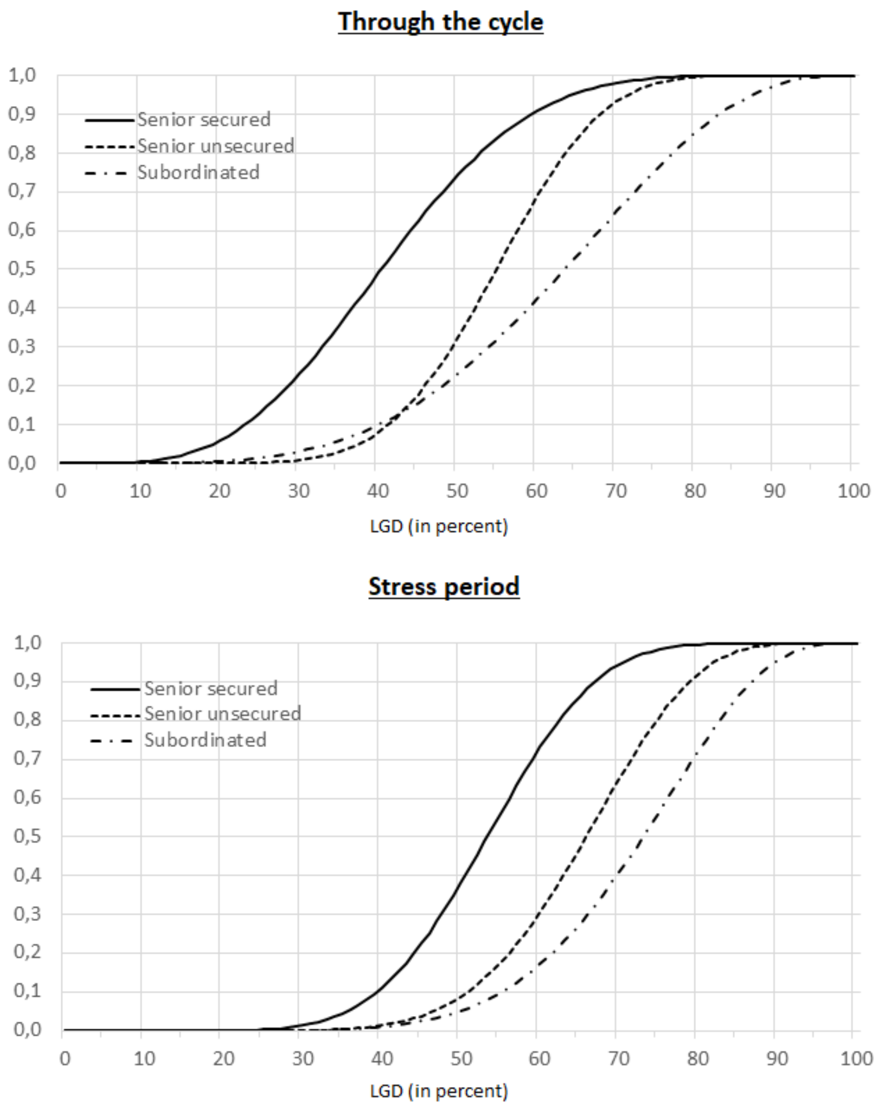


Figure 5.3 – Loss Given Default - Cumulative Distribution Function by seniority level



5.2 Probability of default

Figure 5.5 page 36 presents the estimates using the method of moments on the Standard & Poor's information of default rates presented in figure 5.4 page 35. We successively obtain the average default rate per rating, the joint probability distribution of two simultaneous defaults and the correlation matrix between ratings. These estimates are here presented for illustration purposes. As explained in the precedent chapter, they are not used to obtain loss distributions.

Figure 5.4 – Historical corporate default rates by year and rating (number of defaults for 10 000 corporate issuers by year and rating; source: Standard & Poor's [34])

Year	AAA	AA	A	BBB	BB	B	C
1981	0	0	0	0	0	227	0
1982	0	0	21	34	422	313	2143
1983	0	0	0	32	116	458	667
1984	0	0	0	66	114	341	2500
1985	0	0	0	0	148	647	1538
1986	0	0	18	33	131	836	2308
1987	0	0	0	0	38	308	1228
1988	0	0	0	0	105	363	2037
1989	0	0	18	60	72	338	3333
1990	0	0	0	58	357	856	3125
1991	0	0	0	55	169	1384	3387
1992	0	0	0	0	0	699	3019
1993	0	0	0	0	70	262	1333
1994	0	0	14	0	28	308	1667
1995	0	0	0	17	99	458	2800
1996	0	0	0	0	45	291	800
1997	0	0	0	25	19	351	1200
1998	0	0	0	41	82	463	4286
1999	0	17	18	20	95	729	3333
2000	0	0	27	37	116	770	3596
2001	0	0	27	34	296	1153	4545
2002	0	0	0	101	289	821	4444
2003	0	0	0	23	58	407	3273
2004	0	0	8	0	44	145	1618
2005	0	0	0	7	31	174	909
2006	0	0	0	0	30	82	1333
2007	0	0	0	0	20	25	1524
2008	0	38	39	49	81	408	2727
2009	0	0	22	55	75	1092	4946
2010	0	0	0	0	58	86	2262
2011	0	0	0	7	0	167	1630
2012	0	0	0	0	30	157	2752
2013	0	0	0	0	10	164	2450
2014	0	0	0	0	0	78	1742
2015	0	0	0	0	16	240	2651
2016	0	0	0	6	47	370	3317
2017	0	0	0	0	8	98	2623

The mean of default rates already exhibit interesting things. For instance:

- a AAA rating has never defaulted in 37 years (some of them experienced a downgrading in the year);
- a AA rating has, on average, 1 chance out of 10 000 to default in the year;
- a A rating has 6 times more chances to default than a AA rating;
- a BBB rating has 3.5 times more chances to default than a A rating;

- a BB rating has 4.3 times more chances to default than a BBB rating;
- a B rating has 4.8 times more chances to default than a BB rating;
- a C rating has 5.5 times more chances to default than a B rating.

The two-dimensional joint probabilities of default presented in figure 5.5 page 36 are obviously very small and mechanically smaller than individual marginals default probabilities. For instance, two issuers rated respectively B and BB have on average 0.06% chance to default simultaneously in a year whereas an issuer rated B has 4.34% chance to default in a year and an issuer rated BB 0.90%.

Correlations (figure 5.5 page 36) are as well very small but not null. Any correlation between ratings AAA, AA, A, BBB and BB is below 1.10%.

Figure 5.5 – Estimations by the method of moments - Probability of default (mean and variance), joint probability of default and correlation between ratings

	AAA	AA	A	BBB	BB	B	C
Mean	0,00%	0,01%	0,06%	0,21%	0,90%	4,34%	24,07%
Variance	0,00%	0,01%	0,06%	0,20%	0,89%	4,15%	18,27%

Joint probability of default

	AAA	AA	A	BBB	BB	B	C
AAA	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
AA	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
A	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,02%
BBB	0,00%	0,00%	0,00%	0,00%	0,00%	0,01%	0,07%
BB	0,00%	0,00%	0,00%	0,00%	0,02%	0,06%	0,26%
B	0,00%	0,00%	0,00%	0,01%	0,06%	0,29%	1,28%
C	0,00%	0,00%	0,02%	0,07%	0,26%	1,28%	7,10%

Correlation

	AAA	AA	A	BBB	BB	B	C
AAA	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	0,00%
AA	0,00%	0,29%	0,14%	0,05%	-0,01%	0,04%	0,14%
A	0,00%	0,14%	0,18%	0,09%	0,14%	0,26%	0,44%
BBB	0,00%	0,05%	0,09%	0,30%	0,36%	0,55%	0,91%
BB	0,00%	-0,01%	0,14%	0,36%	1,10%	0,92%	1,07%
B	0,00%	0,04%	0,26%	0,55%	0,92%	2,52%	2,67%
C	0,00%	0,14%	0,44%	0,91%	1,07%	2,67%	7,17%

5.3 Rating migration

As explained in the previous chapter, instead of using an average transition matrix for the through-the-cycle approach, I use all the annual transition matrices provided by standard & Poor's, presented in the annex, to simulate first, the year and then, the associated rating migration matrix as done for default events.

Otherwise, Standard & Poor's provides a long-term (1981 to 2017) average of transition rates (figure 5.6 page 37). The probability to move into the default state, denoted D, is 0 as we only consider this Markov chain to issuers that have not defaulted in the default loss model. Given the long-term average Markov chain, an issuer rated BB, that has not defaulted in the year, has for instance 7,72% chance to be rated B in the year and 9,02% chance under the stress period.

Figure 5.6 – One-year transition rates excluding default state (source: Standard & Poor's [34])

Average transition rates (1981-2017)

	AAA	AA	A	BBB	BB	B	C	D
AAA	89,91%	9,33%	0,55%	0,05%	0,08%	0,03%	0,05%	0
AA	0,54%	90,45%	8,33%	0,53%	0,05%	0,07%	0,02%	0
A	0,03%	1,86%	92,03%	5,59%	0,34%	0,14%	0,02%	0
BBB	0,01%	0,11%	3,75%	91,41%	4,05%	0,54%	0,13%	0
BB	0,01%	0,03%	0,13%	5,54%	85,88%	7,72%	0,68%	0
B	0,00%	0,04%	0,11%	0,23%	6,12%	88,22%	5,30%	0
C	0,00%	0,00%	0,22%	0,33%	1,09%	22,32%	76,03%	0

Stress period (2008)

	AAA	AA	A	BBB	BB	B	C	D
AAA	87,10%	6,45%	3,23%	0,00%	0,00%	1,08%	2,15%	0
AA	0,00%	81,19%	18,02%	0,60%	0,00%	0,00%	0,20%	0
A	0,00%	1,68%	92,64%	5,20%	0,48%	0,00%	0,00%	0
BBB	0,00%	0,00%	2,75%	92,90%	3,84%	0,29%	0,21%	0
BB	0,00%	0,10%	0,00%	5,40%	84,35%	9,02%	1,14%	0
B	0,00%	0,00%	0,00%	0,16%	4,33%	86,01%	9,50%	0
C	0,00%	0,00%	0,00%	0,00%	0,00%	21,15%	78,85%	0

5.4 Yields

Market yields are used to calculate the new yield of bonds for issuers having experienced a rating migration. When computing the through-the-cycle estimate, we use the 2017 yields. 2008 yields are used for the stress period. We also distinguish yields by seniority level (senior and subordinated).

The spread difference to apply in case of migration can directly be performed on the yield figures.

Figure 5.7 – Yields by rating and maturity (in years) recorded in 2017 and 2008 for senior and subordinated corporate bonds

2017 Maturity (in years)	Senior corporate bonds						
	AAA	AA	A	BBB	BB	B	C
1	-0,30%	-0,14%	-0,07%	0,02%	0,71%	1,39%	1,65%
5	0,18%	0,36%	0,49%	0,67%	2,37%	5,76%	6,67%
10	0,53%	1,03%	1,21%	1,48%	3,20%	5,03%	7,60%
15	1,35%	1,54%	1,75%	1,95%	4,20%	4,56%	6,89%
20	1,43%	1,61%	1,94%	2,79%	6,03%	6,54%	9,88%

2017 Maturity (in years)	Subordinated corporate bonds						
	AAA	AA	A	BBB	BB	B	C
1	-0,09%	0,07%	0,14%	0,35%	1,29%	2,37%	3,60%
5	0,66%	0,83%	0,96%	1,45%	2,52%	4,32%	6,74%
10	1,25%	1,75%	1,93%	2,55%	4,17%	5,54%	8,64%
15	2,07%	2,26%	2,47%	3,49%	5,17%	6,87%	10,71%
20	1,69%	1,87%	2,19%	4,34%	7,00%	9,29%	14,49%

2008 Maturity (in years)	Senior corporate bonds						
	AAA	AA	A	BBB	BB	B	C
1	4,67%	5,31%	7,32%	8,40%	20,93%	33,45%	45,98%
5	5,49%	6,31%	8,28%	9,33%	13,89%	18,44%	23,00%
10	5,61%	6,33%	6,38%	8,35%	13,00%	17,65%	22,30%
15	5,24%	5,96%	6,01%	7,98%	12,63%	17,28%	21,93%
20	4,70%	5,42%	5,47%	7,44%	12,09%	16,74%	21,39%

2008 Maturity (in years)	Subordinated corporate bonds						
	AAA	AA	A	BBB	BB	B	C
1	4,88%	5,52%	7,53%	8,73%	21,51%	34,43%	47,92%
5	5,96%	6,78%	8,75%	10,11%	14,04%	17,00%	23,07%
10	6,33%	7,05%	7,10%	9,42%	13,97%	18,16%	23,34%
15	5,96%	6,68%	6,73%	9,52%	13,60%	19,59%	25,75%
20	4,95%	5,67%	5,72%	8,98%	13,06%	19,50%	26,00%

Chapter 6

A short portfolio description

This chapter provides a brief description of a real insurance company's portfolio used in the next two chapters for numerical applications. It is composed of 127 securities issued by 109 different corporate issuers. I present some elements of this corporate bond portfolio:

- the distribution of the issuers by rating and seniority;
- the distribution of the securities by coupon and yield;
- the distribution of the securities by residual maturity and duration;
- the distribution of the weightings of the invested capital in each security.

A last section presents the main outcomes of this portfolio.

6.1 Rating and seniority

Figure 6.1 presents the rating distribution of the 109 issuers and the same weighted distribution by the amounts of invested capital. The first table includes not rated (NR) issuers, counting for 13% of the issuers. Not rated issuers have many different reasons for not being rated and their level of credit risk can be quite heterogeneous. For risk measure considerations, NR issuers are considered as BB issuers in our approach[1]. The second table of figure 6.1 shows the weights with NR issuers considered as BB rated issuers.

The portfolio is made up for more than 80% of issuers rated at least BBB. Only 1% of the issuers have a rating of less than or equal to B.

1. The solvency capital requirement (SCR) for spread risk under Solvency 2 takes into account specific shocks for unrated securities or issuers. For NR bonds, the shocks to determine the solvency capital requirement are between BBB and BB ratings and closer to BBB than to BB.

Figure 6.1 – Number of issuers by rating

With Not Rated (NR) issuers	AAA	AA	A	BBB	BB	B	C	NR	Total
Number of Counterparties (in %)	4%	11%	32%	35%	5%	1%	0%	13%	100%
Weighted invested capital (in %)	5%	12%	35%	31%	4%	1%	0%	11%	100%

After affecting NR issuers to BB rating	AAA	AA	A	BBB	BB	B	C	NR	Total
Number of Counterparties (in %)	4%	11%	32%	35%	17%	1%	0%	0%	100%
Weighted invested capital (in %)	5%	12%	35%	31%	15%	1%	0%	0%	100%

With regards to the seniority of the securities, the 127 securities are broken down into:

- 3 senior secured;
- 99 senior unsecured;
- 25 subordinated.

6.2 Coupons and yields

Figure 6.2 presents the coupons distribution weighted by the capital. 54% of the invested capital has an annual return between 2% and 4%. The average annual coupon, weighted by the invested capital in each security, is of 2.96%.

Figure 6.2 – Distribution function of annual coupons (expressed in percentage of principal)

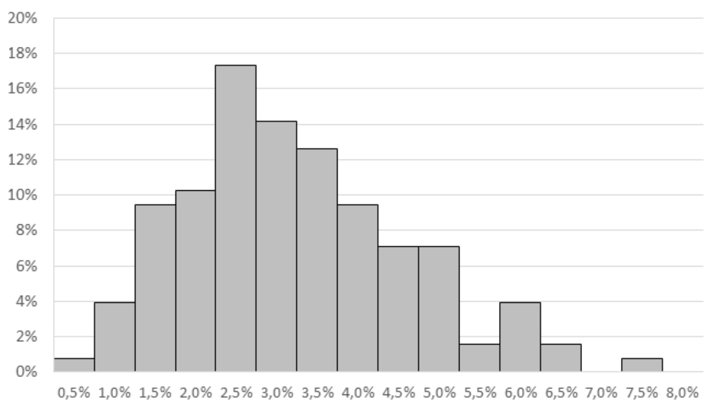


Figure 6.3 shows the portfolio yield distribution. The average yield of the portfolio

is of 1.05%. The yield of a security s satisfies the following equation:

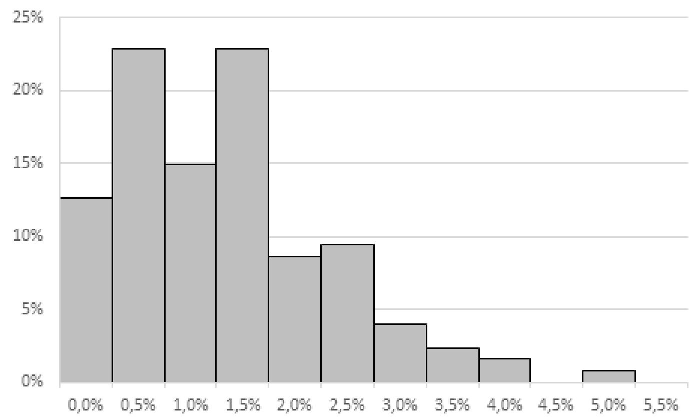
$$Price_s = \sum_{m=M_s-\lfloor M_s \rfloor}^{M_s} \frac{Coupon_s}{(1+Yield_s)^m} + \frac{Principal_s}{(1+Yield_s)^{M_s}}$$

The yield of a portfolio composed of S securities is:

$$Yield = \sum_{s=1}^{S} w_s \times Yield_s$$

where w_s is the market value proportion of s into the portfolio.

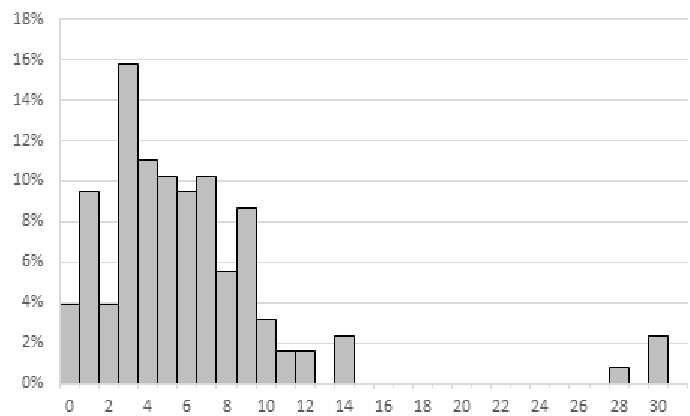
Figure 6.3 – distribution function of yields



6.3 Residual maturity and duration

Figure 6.4 shows the residual maturity distribution function of the portfolio. The average residual maturity of the portfolio is of 6.44 years. 57% of the bonds have a residual maturity between 3 and 7 years.

Figure 6.4 – Distribution function of residual maturities (expressed in years)



The duration of a security s is expressed as:

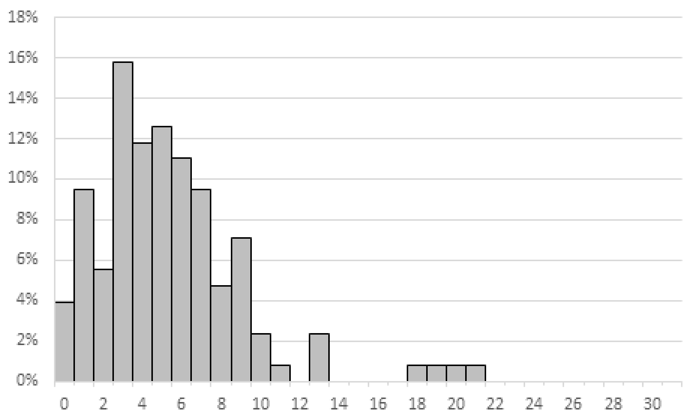
$$Duration_s = \frac{\sum_{m=M_s-\lfloor M_s \rfloor}^{M_s} m \times \frac{Coupon_s}{(1+Yield_s)^m} + M_s \times \frac{Principal_s}{(1+Yield_s)^{M_s}}}{\sum_{m=M_s-\lfloor M_s \rfloor}^{M_s} \frac{Coupon_s}{(1+Yield_s)^m} + \frac{Principal_s}{(1+Yield_s)^{M_s}}}$$

The duration of a portfolio composed of S securities is given by:

$$Duration = \sum_{s=1}^{S} w_s \times Duration_s$$

Figure 6.5 presents the distribution of the duration. The duration of the portfolio is 5.64 years.

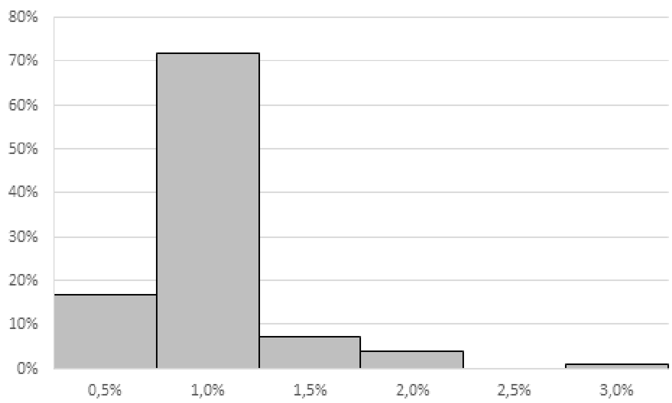
Figure 6.5 – Distribution function of durations (expressed in years)



6.4 Granularity

Portfolio weights range from 0.5% to 3% of the total capital of the portfolio. More than 70% of the investments are made with only 1% of the total invested capital. The diversification of this portfolio is very high.

Figure 6.6 – Securities weight distribution function



6.5 Key features

To summarize, the portfolio used in this study:

- consists of 127 securities issued by 109 different issuers;
- is composed of more than 80% of issuers rated above or equal to BBB with 13% of not rated issuers (considered as rated BB in the analysis);
- is composed of 20% of subordinated securities;
- has a weighted capital average coupon of 2.96%;
- has a weighted capital average yield of 1.05%;
- has a weighted capital average residual maturity of 6.44 years;
- has a weighted capital average duration of 5.64 years;
- is very granular with over 70% of trades with an amount invested representing 1% of the portfolio's total capital.

Chapter 7

Credit risk results

After introducing in the previous chapter, some descriptive elements of the portfolio used for the empirical analysis, I present the main results related to the model and applied to this portfolio. The results are broken down into five sections: number of defaults, default loss, number of migrations, migration loss and the total loss related to credit risk. Each section measures risk considering two regimes: a through-the-cycle regime and a stress period regime. The through-the-cycle regime considers long-term levels for the different parameters while the stress period is only based on 2008 credit risk events.

A first section presents the simulation technique process. The results were obtained with 100 000 simulations of the credit risk model applied to the portfolio presented in the previous chapter.

7.1 Simulation technique

The simulation of default risk is conducted with the following approach for the through-the-cycle regime. For one simulation of the portfolio:

1. I choose randomly one year between 1981 and 2017 with equal probability; this year is denoted y;
2. for each issuer, the parameter of its univariate marginal Bernoulli distribution related to the default event is equal to the default rate of its rating in year y;
3. for each issuer, I simulate its Bernoulli distribution; we obtain the number of defaults of the portfolio;
4. for each issuer having defaulted in the precedent step, I simulate for each of its securities, the through-the-cycle truncated Beta distributions associated to their LGD considering their seniority; I then multiply them by the EAD of each of its securities to get the default loss of the portfolio;
5. for each issuer not having defaulted, I simulate the migration rates of year y (without default state) presented in the annex; we obtain the number of rating migrations of the portfolio;

6. for each issuer having a change of its rating, I calculate the associated market price change of each of its securities; we obtain the migration loss of the portfolio;
7. I add the default loss and the migration loss of the portfolio to get the total loss of the portfolio.

By repeating this algorithm many times, we obtain the empirical distributions presented in the next five sections.

For the stress period, all the steps are the same. Only the values of the parameters change.

7.2 Number of defaults

This section presents the default risk results taking into account the number of defaults. The next section concerns losses due to these defaults.

Figure 7.1 shows the cumulative distribution function of the number of defaults that we may encounter in a year over the 109 issuers for the through-the-cycle and the stress period regimes. The probability of having at least one default in the year is of 25% under the through-the-cycle regime and 43% under the stress period regime. The probability of having at least two defaults is 5% under the through-the-cycle regime while it is twice as high under the stress period. Highest numbers of defaults have then very similar probabilities for the both regimes.

Figure 7.1 – Number of defaults over 109 issuers - Cumulative distribution function

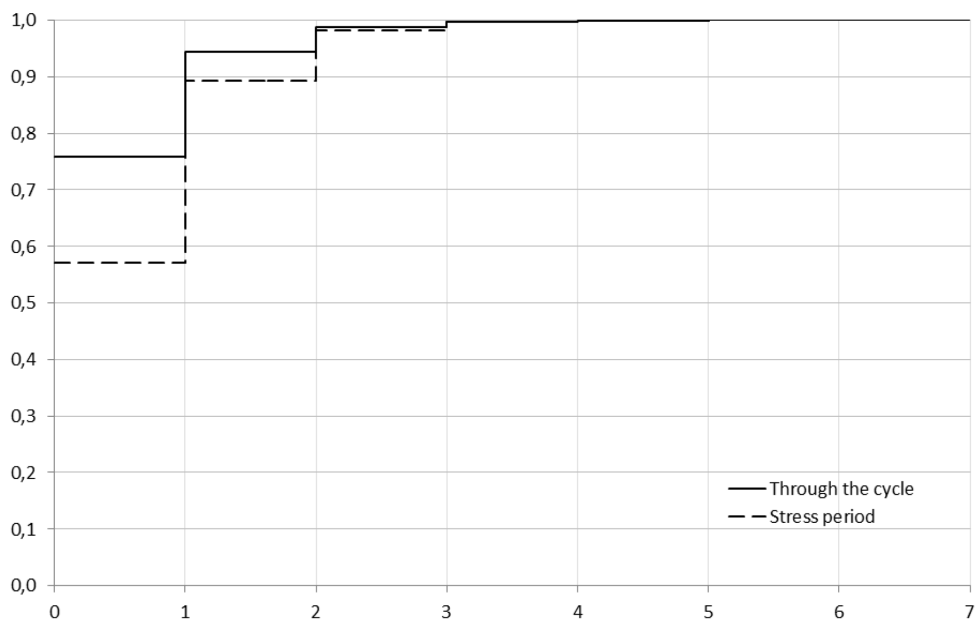


Figure 7.2, introducing the VaR and the $CVaR$ for different quantiles, shows that the number of defaults that we can expect under extreme circumstances (quantiles above or equal to 0.95) is the same whether we are in the through-the-cycle regime or in the stress period regime. Expressed as a percentage of the number of issuers, the VaR and the $CVaR$ at 0.99 are almost the same for both regimes (2.75% for $VaRs$ and 3% for $CVaRs$). For the quantile at 0.999, the estimate of $CVaR$ under the through-the-cycle regime is even greater than that of the stress period while their VaR are identical.

Figure 7.2 – VaR and $CVaR$ - Number of defaults over 109 issuers expressed in percent

	Through the cycle		Stress period	
Quantile	**VaR**	**CVaR**	**VaR**	**CVaR**
0,500	0,00%	0,58%	0,00%	1,03%
0,750	0,00%	1,15%	0,92%	1,39%
0,950	1,83%	2,11%	1,83%	2,23%
0,990	2,75%	2,98%	2,75%	3,02%
0,995	2,75%	3,21%	2,75%	3,29%
0,999	3,67%	4,01%	3,67%	3,96%

Figure 7.3 presents the expected and unexpected losses [1]. The expected loss is of 0.29 default for 100 issuers under the through-the-cycle regime and 0.51 default for 100 issuers under the stress period. The unexpected number of defaults at quantile 0.99 for 100 issuers are at 2.46 defaults under the through-the-cycle regime and 2.24 under the stress period.

In other words, under the through-the-cycle regime, when we normally expect to experience 0.29 default over 100 issuers in a year, in the same time, the unexpected loss, at the 0.99 quantile, is more than 8 times higher (2.46%). In the same time, under the stress period regime, the unexpected loss (2,24%) is lower compared to the through-the-cycle regime and is less than 5 times higher than the expected loss. These results illustrate the common fact that the surprise with regards to default events is higher under a normal regime than under an already stressed regime.

1. The unexpected loss at quantile α is the difference between the VaR_α and the expected loss (please refer to section 2.3 related to risk measures for more details).

Figure 7.3 – Expected and unexpected loss - Number of defaults over 109 issuers expressed in percent

	Through the cycle	Stress period
Expected Loss	0,29%	0,51%
Quantile	**Unexpected loss**	**Unexpected loss**
0,950	1,55%	1,32%
0,990	2,46%	2,24%
0,995	2,46%	2,24%
0,999	3,38%	3,16%

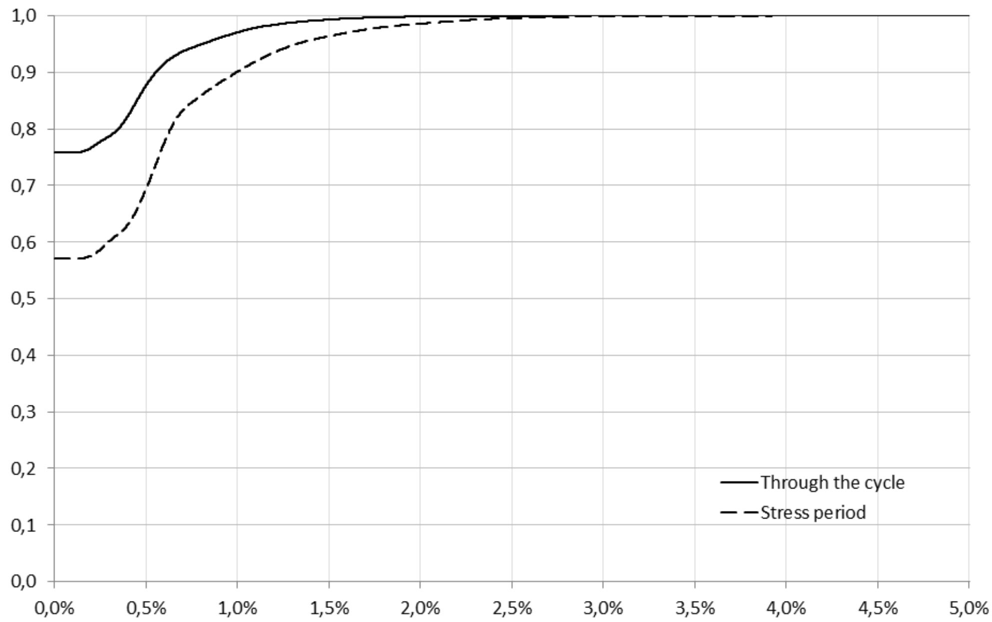
To summarize, under the through-the-cycle regime, we expect 0.29% default in a year with an unexpected loss, at quantile 0.99, 8 times higher. The $CVaR_{0.99}$ is of 2.98% under the through-the-cycle regime and almost the same for the stress period.

7.3 Default loss

This section is dedicated to the measurement of losses due to defaults. To obtain the losses incurred, we use the defaults simulated and presented in the previous section, the loss given default and the exposure at default. The exposure at default is the principal of the security and the loss given default is stochastically modeled as previously presented.

Figure 7.4 shows the default loss cumulative distribution functions for both regimes. The losses obviously start to be strictly positive when we encounter at least one default. The default loss of the stress period regime is significantly higher than the through-the-cycle regime. Since the number of defaults is the same under both regimes for high quantiles (see figure 7.2), the difference in losses is due to the issuers into defaults and to the disparity between losses given default of both regimes (see figures 5.1 and 5.2 pages 32 and 33).

Figure 7.4 – Default loss (in percentage of the exposure amount of the portfolio) - Cumulative distribution function



The VaR is now clearly higher for the stress period (2.16% at the 0.99 quantile) compared to the through-the-cycle regime (1.37% at the 0.99 quantile).

Figure 7.5 – VaR and $CVaR$ - Default loss (in percentage of the exposure amount of the portfolio)

	Through the cycle		Stress period	
Quantile	**VaR**	**CVaR**	**VaR**	**CVaR**
0,500	0,00%	0,29%	0,00%	0,67%
0,750	0,00%	0,59%	0,57%	1,03%
0,950	0,81%	1,16%	1,32%	1,83%
0,990	1,37%	1,71%	2,16%	2,57%
0,995	1,60%	1,93%	2,45%	2,85%
0,999	2,15%	2,46%	3,06%	3,49%

The expected loss under the through-the-cycle regime represents 0.15% of the exposure. This is twice more important under the stress period (0.33%).

Figure 7.6 – Expected and unexpected loss - Default loss (in percentage of the exposure amount of the portfolio)

	Through the cycle	Stress period
Expected Loss	0,15%	0,33%
Quantile	**Unexpected loss**	**Unexpected loss**
0,950	0,66%	0,99%
0,990	1,22%	1,83%
0,995	1,45%	2,12%
0,999	2,01%	2,72%

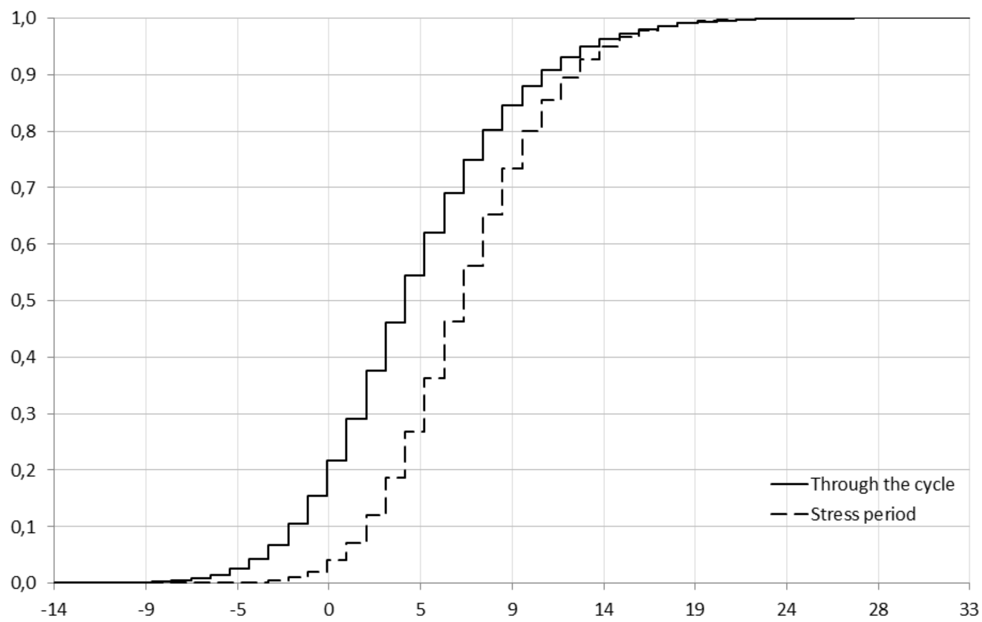
Finally, the VaR at quantile 0.99 for losses due to default risk are respectively estimated at 1.37% of the invested capital under the through-the-cycle regime and at 2.16% under the stress period regime. The expected loss is 0.15% under the through-the-cycle and of 0.33% under the stress period.

7.4 Number of migrations

In this section we explore the migrations of ratings of the issuers. More Specifically, I focus the analysis on the net number of rating changes. Here, an issuer being either downgraded or upgraded by several ratings, for instance by 3 ratings, will be considered as 3 rating changes for the whole portfolio. A net positive number of changes corresponds to a net downgrade of the portfolio.

Figure 7.7 shows the cumulative distribution function of the net number of rating changes. This function shows the low probability to improve the ratings of the issuers of this portfolio. Under the through-the-cycle regime, the probability to improve ratings is below 15% and below 3% for the stress period. The mode is at 4 under the through-the-cycle regime meaning that the highest probability of net rating changes in the year is 4 out of 109 issuers and 6 for the stress period regime.

Figure 7.7 – Net number of rating changes (over 109 issuers) - Cumulative distribution function (if the number of downgrades are higher to the number of upgrades, the net number of changes is positive)



For high quantiles, the VaR under both regimes are very similar. The VaR at the 0.99 quantile of the net number of rating changes is 16.51% for both regimes (figure 7.8).

Figure 7.8 – VaR and $CVaR$ - Net number of rating changes (over 109 issuers)

	Through the cycle		Stress period	
Quantile	**VaR**	**CVaR**	**VaR**	**CVaR**
0,500	3,67%	7,72%	6,42%	9,57%
0,750	7,34%	10,32%	9,17%	11,61%
0,950	12,84%	15,16%	12,84%	15,44%
0,990	16,51%	19,09%	16,51%	18,60%
0,995	18,35%	20,54%	18,35%	19,89%
0,999	22,02%	23,59%	21,10%	22,71%

The expected loss in terms of net rating changes is estimated at 4.07% of the number of issuers under the through-the-cycle regime and at 6.57% under the stress period regime.

Figure 7.9 – Expected and unexpected loss - Net number of rating changes (over 109 issuers)

	Through the cycle	Stress period
Expected Loss	4,07%	6,57%
Quantile	Unexpected loss	Unexpected loss
0,950	3,27%	6,27%
0,990	8,78%	9,94%
0,995	12,45%	11,77%
0,999	14,28%	14,53%

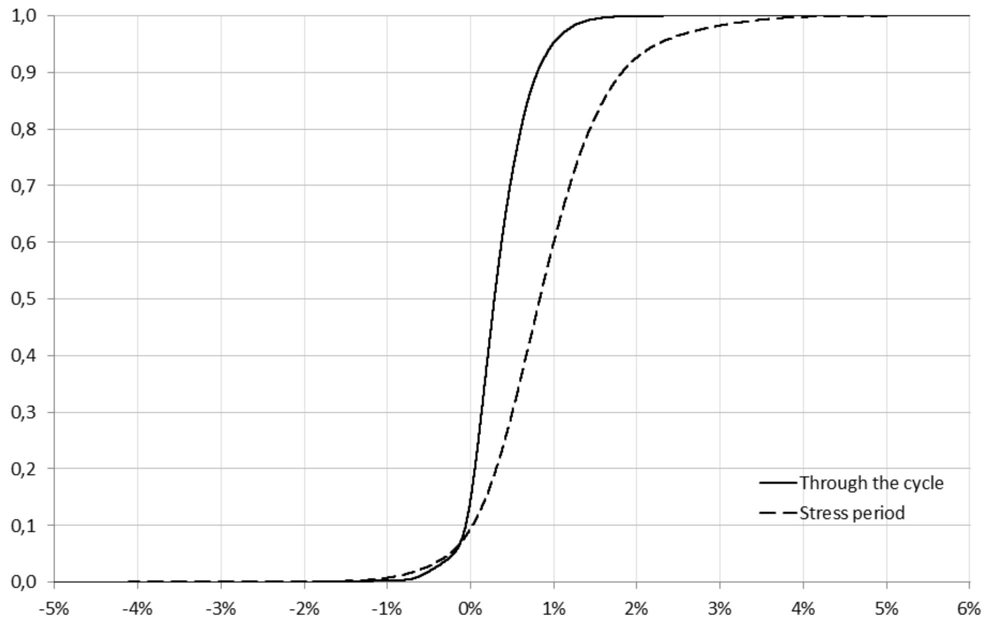
The number of rating changes is obviously much larger than the number of default events. The unexpected net number of rating changes is higher under the stress period regime for 0.95 and 0.99 quantiles, whereas for the higher quantiles we find the opposite. This phenomenon is explained by the fact that some years simulated for the through-the-cycle regime have encountered a number of downgraded ratings more important than the year of the stress period, 2008.

7.5 Migration loss

Losses due to rating migrations are materialized by market price changes. I explore migration considering market losses in the same way as in the default loss section. Prices adjustments are based on yields by rating, maturity and seniority (see section 3.2 page 18 related to the migration risk model; you may refer to section 5.4 page 37 for yield levels).

The cumulative distribution functions for migration losses are shown in figure 7.10. While the losses due to migration imply market value losses, I express them in percent of the sum of the principal of each security. This will allow to add default losses and migration losses expressed in percentage of the principals. The through-the-cycle regime uses 2017 end of year (low) spreads while the stress period is based on 2008 end of year (high) spreads. The shapes of the loss curves are significantly different between the two regimes.

Figure 7.10 – Migration loss (expressed in percentage of the exposure amount of the portfolio) - Cumulative distribution function



The 0.99 quantile VaR of the migration loss is 1.37% under the through-the-cycle regime and 3.34% for the stress period (see figure 7.11).

Figure 7.11 – VaR and $CVaR$ - Migration loss (expressed in percentage of the exposure amount of the portfolio)

	Through the cycle		Stress period	
Quantile	**VaR**	**CVaR**	**VaR**	**CVaR**
0,500	0,29%	0,61%	0,83%	1,47%
0,750	0,54%	0,81%	1,30%	1,89%
0,950	0,98%	1,22%	2,24%	2,89%
0,990	1,37%	1,58%	3,34%	3,77%
0,995	1,53%	1,73%	3,66%	4,06%
0,999	1,85%	2,07%	4,31%	4,65%

The expected migration loss is of 0.32% of the invested capital for the through-the-cycle regime and 0.89% for the stress period.

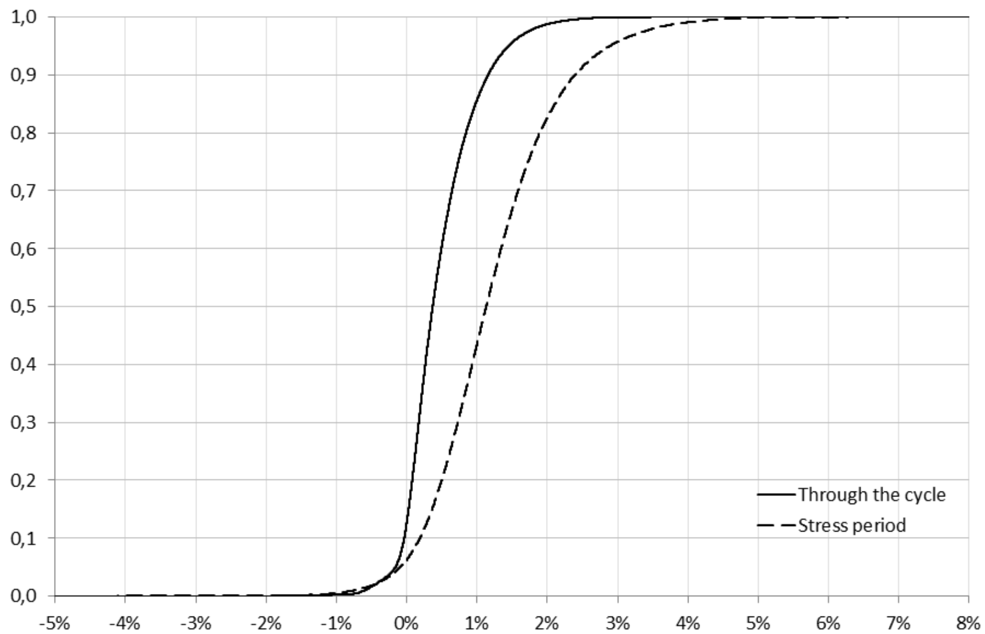
Figure 7.12 – Expected and unexpected loss - Migration loss (expressed in percentage of the exposure amount of the portfolio)

	Through the cycle	Stress period
Expected Loss	0,32%	0,89%
Quantile	Unexpected loss	Unexpected loss
0,950	0,66%	1,35%
0,990	1,04%	2,45%
0,995	1,20%	2,77%
0,999	1,52%	3,42%

7.6 Total credit risk loss

The total loss of credit risk is the sum of the default loss and migration loss (figure 7.13).

Figure 7.13 – Total loss (expressed in percentage of the exposure amount of the portfolio) - Cumulative distribution function



At the 0.99 quantile, the VaR is 2.09% under the through-the-cycle regime and 3.96% under the stress period regime (figure 7.14).

Figure 7.14 – VaR and $CVaR$ - Total loss (expressed in percentage of the exposure amount of the portfolio)

	Through the cycle		Stress period	
Quantile	**VaR**	**CVaR**	**VaR**	**CVaR**
0,500	0,37%	0,86%	1,13%	1,92%
0,750	0,73%	1,18%	1,73%	2,44%
0,950	1,44%	1,84%	2,89%	3,54%
0,990	2,09%	2,44%	3,96%	4,49%
0,995	2,35%	2,67%	4,34%	4,85%
0,999	2,87%	3,17%	5,18%	5,62%

The expected loss is 0.47% of the total invested capital under the through-the-cycle regime and 2.5 times higher under the stress period (figure 7.15). The unexpected loss at quantile 0.99 is 1.62% under the through-the-cycle regime and less than twice higher under the stress period. When we are already in a stress period, the surprise is, in a sense, proportionally smaller with respect to the expected loss than under the through-the-cycle regime.

Figure 7.15 – Expected and unexpected loss - Total loss (expressed in percentage of the exposure amount of the portfolio)

	Through the cycle	Stress period
Expected Loss	0,47%	1,22%
Quantile	**Unexpected loss**	**Unexpected loss**
0,950	0,97%	1,67%
0,990	1,62%	2,73%
0,995	1,88%	3,12%
0,999	2,40%	3,96%

7.7 Main outcomes

This specific portfolio presents quite interesting risk results. Under the through-the-cycle regime:

- The number of defaults expected in a year is of 0.29% of the number of issuers (1 issuer out of 344) while the net number of rating changes is of 4.07% (1 rating change out of 24 issuers) rating changes.

- When considering adverse low probability events, $VaRs$ as well as $CVaRs$ at quantiles 0.950 and 0.999 are significantly different. For instance, the total loss 0.999 quantile VaR is (2.87% of the total exposure) twice higher than the total loss 0.95 quantile VaR (1.44%).
- The expected default loss is 0.15% of the total exposure whereas the expected migration loss is 0.32%.
- The $VaRs$ at quantile 0.99 are the same for the default loss and migration loss (1.37% of the total exposure).
- The VaR at quantile 0.999 for the default loss is higher (2.15%) than the one for migration loss (1.85%).

The model allows to decompose in several elements the risk analysis of a portfolio:

- The number of defaults and the net number of migrations distribution functions give a general idea of how many credit events we can encounter in a year.
- The default loss and the migration loss functions estimate how much we can lose as a percentage of the exposure. The aggregation of the two losses corresponds to the total credit loss function.
- The synthetic risk measures (VaR and $CVaR$) provide information about the risk that can be encountered under different levels of probability.
- Expected and unexpected losses are useful indicators to measure how much we should expect to lose and how much we should be surprised to lose in a year.
- Considering two distinct regimes gives strong indications of risk from a long-term economic perspective and on the risk under a very adverse event.

Chapter 8

Portfolio optimization

This chapter makes use of a risk-return optimization framework to find an optimal portfolio of corporate bonds under constraints.

Most of the time, portfolio optimization is related to the seminal Markowitz approach [22] based on variance to measure risk. This method uses basic concepts of probability theory and leads to convenient expressions to find an optimal portfolio, for instance, for stocks under some assumptions. Stocks returns are random variables. When using their expected return, their variance and covariance, we can formulate a portfolio optimization problem that minimizes variance for a return objective.

When looking at credit risk, variance is not an appropriate risk measure. Instead of stocks, fixed rate bonds have a fixed and known return, the coupon, with no concept of variance. They are rightly called fixed-income securities. Fixed-income securities promise the holder a fixed and defined income over a period of time. However, they carry the risk that issuers will not honor their commitments. Risk appears through default and migration rating events captured by their associated loss distribution as presented and calculated in the previous chapters.

VaR and $CVaR$ are synthetic measures of risk. They reduce the information of a loss distribution function to a singleton. Since $CVaR$ is a coherent measure holding the convexity property while VaR does not (please refer to Chapter 2 for more details), I opt for an optimization framework based on return and $CVaR$.

This chapter is composed of five sections. The first section presents, the general optimization framework using $CVaR$ as the risk measure. A second section introduce some optimization variants with among others a duration constraint. A third section defines an optimized dynamic portfolio strategy. The fourth section is related to the corporate bonds market used in the last section for an empirical example of the dynamic approach.

8.1 Optimization framework

The optimization problem consists in finding the optimal portfolio composition, through the invested capital weights to associate with each security, minimizing $CVaR_\alpha$ for a fixed return objective denoted R. Having S possible securities, the weights allocation is defined by the set $w = \{w_1, ..., w_s, ...w_S\}$ such that:

$$\sum_{s=1}^{S} w_s = 1 \text{ and } w_s \geqslant 0, \forall s \in [\![1, S]\!]$$

meaning that all the available capital is invested and that we do not short sell assets [1].

Each corporate bond security has its own fixed-income. We denote returns [2] associated with the set of securities as $Ret = \{R_1, ..., R_s, ..., R_S\}$.

The exposures at default ($EADs$) appearing in Chapter 3 (Credit risk modeling) are now, in some sense, replaced by w. To integrate our decision vector w in the loss function, I denote $f(w, L)$ a loss univariate random variable. $f(w, L)$ is the loss distribution associated simultaneously to weights w and to the S-dimensional loss random variable L with cumulative distribution probability

$$G(l) = G(l_1, ..., l_s, ..., l_S) = P(L_1 \leqslant l_1, ..., L_s \leqslant l_s, ..., L_S \leqslant l_S)$$

where L_s is the marginal random variable related to the loss distribution of one unit of security s. The cumulative distribution function of the loss function for a fixed w decision vector is given by:

$$F(w, x) = \int_{f(w,l) \leqslant x} dG(l)$$

The optimization problem can then be expressed as:

$$\begin{array}{ll} min_w & CVaR_\alpha(f(w, L)) \\ \text{subject to} & \sum_{s=1}^{S} w_s R_s = R \\ & \sum_{s=1}^{S} w_s = 1 \\ & w_s \geqslant 0, \forall s \in [\![1, S]\!] \end{array}$$

Before going further, we should have a look at the VaR and the $CVaR$ expressions for a continuous f loss function and a fixed w.

1. These two conditions can be easily released.
2. In the case of non fixed-income assets, such as stocks, returns are random variables that should be defined as an S-multivariate random variable.

At quantile α, VaR is

$$VaR_{\alpha}(f(w,L)) = min_{l \in \mathbb{R}} f(w,l) \geqslant \alpha$$

and $CVaR$ is

$$CVaR_{\alpha}(f(w,L)) = E[f(w,L)|f(w,L) \geqslant VaR_{\alpha}(f(w,L))]$$

$$= \frac{1}{1-\alpha} \int_{f(w,l) \geqslant VaR_{\alpha}(f(w,L))} f(w,l) dG(l)$$

$$= VaR_{\alpha}(f(w,L)) + \frac{1}{1-\alpha} \int_{l \in \mathbb{R}^S} (f(w,l) - VaR_{\alpha}(f(w,L)))^{+} dG(l)$$

with $x^{+} = max\{0, x\}$

The resolution of the previous optimization problem is not obvious because $CVaR$ does not appear as a simple function of w and depends on VaR. An optimization framework of $CVaR$ has been presented in the seminal paper of Rockafellar and Uryasev [32] in 2000. The objective function of the previous optimization problem can indeed be expressed as a convex function and then be solved by classic non-stochastic methods considering the two following theorems:

- Rockafellar and Uryasev first theorem (proof in [32]):

  $$H_{\alpha}(f(w,L),\gamma) = \gamma + \frac{1}{(1-\alpha)} \int_{l \in \mathbb{R}^S} (f(w,l) - \gamma)^{+} dG(l)$$

  is convex as a function of γ and $CVaR_{\alpha}(f(w,L)) = min_{\gamma \in \mathbb{R}} H_{\alpha}(f(w,L),\gamma)$

- Rockafellar and Uryasev second theorem (proof in [32]):

  $$min_{w} CVaR_{\alpha}(f(w,L)) = min_{w,\gamma \in \mathbb{R}} H_{\alpha}(f(w,L),\gamma)$$

  and $H_{\alpha}(f(w,L),\gamma)$ is convex as a function of w and γ.

The optimization problem can then be expressed as:

$$\begin{aligned} min_{w,\gamma \in \mathbb{R}} \quad & H_{\alpha}(f(w,L),\gamma) \\ \text{s.t.} \quad & \textstyle\sum_{s=1}^{S} w_s R_s = R \\ & \textstyle\sum_{s=1}^{S} w_s = 1 \\ & w_s \geqslant 0, \forall s \in [\![1, S]\!] \end{aligned}$$

However, the credit risk loss measure defined in this work and estimated empirically by simulations is not a continuous function. Its empirical estimate is similar to a S-dimensional discrete random variable function composed of the N simulated scenarios as space of possible states. The $CVaR$ expression for a discrete loss function is:

$$CVaR_\alpha(f(w,L)) = VaR_\alpha(f(w,L)) + \frac{1}{(1-\alpha)N}\sum_{n=1}^{N}(f(w,l_n) - VaR_\alpha(f(w,L)))^+$$

with $f(w,l_n) = \sum_{s=1}^{S} l_{n,s}w_s$ and $l_{n,s}$ being the loss under scenario n for one unit of security s (all the $l_{n,s}$ $\forall s \in [\![1,S]\!]$ are related to each other via the previously presented credit risk model).

We can now write $CVaR$ as :

$$CVaR_\alpha(f(w,L)) = VaR_\alpha(f(w,L)) + \frac{1}{(1-\alpha)N}\sum_{n=1}^{N}\left(\sum_{s=1}^{S} l_{n,s}w_s - VaR_\alpha(f(w,L))\right)^+$$

The optimization problem is then:

$$\begin{aligned} min_w \quad & CVaR_\alpha(f(w,L)) = VaR_\alpha(f(w,L)) + \tfrac{1}{(1-\alpha)N}\textstyle\sum_{n=1}^{N}\left(\sum_{s=1}^{S} l_{n,s}w_s - VaR_\alpha(f(w,L))\right)^+ \\ s.t. \quad & \textstyle\sum_{s=1}^{S} w_s R_s = R \\ & \textstyle\sum_{s=1}^{S} w_s = 1 \\ & w_s \geqslant 0, \forall s \in [\![1,S]\!] \end{aligned}$$

Rockafellar and Uryasev published in 2002 a second important article generalizing their theorems to non-continuous loss functions [33]. The previous optimization problem is then equivalent to the following one:

$$\begin{aligned} min_{w,\gamma} \quad & H_\alpha(f(w,L),\gamma) = \gamma + \tfrac{1}{(1-\alpha)N}\textstyle\sum_{n=1}^{N}\left(\sum_{s=1}^{S} l_{n,s}w_s - \gamma\right)^+ \\ s.t. \quad & \textstyle\sum_{s=1}^{S} w_s R_s = R \\ & \textstyle\sum_{s=1}^{S} w_s = 1 \\ & w_s \geqslant 0, \forall s \in [\![1,S]\!] \\ & \gamma \in \mathbb{R} \end{aligned}$$

To solve this optimization problem as a linear program, we introduce artificial variables $z = \{z_1, ..., z_n, ...z_N\}$ to replace $\sum_{s=1}^{S}(l_{n,s}w_s - \gamma)^+$ with two new sets of constraints:

$$z_n \geqslant \sum_{s=1}^{S} l_{n,s}w_s - \gamma \text{ and } z_n \geqslant 0, \forall n \in [\![1, N]\!]$$

The problem is then equivalent to:

$$\begin{aligned}
min_{w,\gamma,z} \quad & \gamma + \tfrac{1}{(1-\alpha)N} \textstyle\sum_{n=1}^{N} z_n \\
s.t. \quad & z_n \geqslant 0, \forall n \in [\![1, N]\!] \\
& z_n \geqslant \textstyle\sum_{s=1}^{S} l_{n,s}w_s - \gamma, \forall n \in [\![1, N]\!] \\
& \textstyle\sum_{s=1}^{S} w_s R_s = R \\
& \textstyle\sum_{s=1}^{S} w_s = 1 \\
& w_s \geqslant 0, \forall s \in [\![1, S]\!] \\
& \gamma \in \mathbb{R}
\end{aligned}$$

All the constraints being linear, the problem is a linear programming problem that can be solved, for instance, with the simplex method.

8.2 Optimization variants

The optimization framework seen in the previous section can be adapted to other possible characteristics. This framework makes it possible to include other constraints or to modify the objective function to be minimized.

First, the objective function can weight the default loss and migration loss functions as desired (the weights should be in any case positive to keep the convexity property). For example, if we consider that the default losses should be avoided more than migration losses, for example twice more, we can write the function to be minimized as follows:

$$H_\alpha(f(w, L'), \gamma) = \gamma + \frac{1}{(1-\alpha)N} \sum_{n=1}^{N} \left(\sum_{s=1}^{S} l'_{n,s} w_s - \gamma \right)^+$$

with $l'_{n,s} = 2l_{Default,n,s} + l_{Migration,n,s}$

It can be noted that $VaR_\alpha(f(w, L'))$ does not appear in the optimization process. It is therefore not necessary to calculate it even if the loss function has been modified.

The problem of minimization is then:

$$
\begin{aligned}
min_{w,\gamma,z} \quad & \gamma + \frac{1}{(1-\alpha)N}\sum_{n=1}^{N} z_n \\
s.t. \quad & z_n \geqslant 0, \forall n \in [\![1, N]\!] \\
& z_n \geqslant \sum_{s=1}^{S}(2l_{Default,n,s} + l_{Migration,n,s})w_s - \gamma, \forall n \in [\![1, N]\!] \\
& \sum_{s=1}^{S} w_s R_s = R \\
& \sum_{s=1}^{S} w_s = 1 \\
& w_s \geqslant 0, \forall s \in [\![1, S]\!] \\
& \gamma \in \mathbb{R}
\end{aligned}
$$

Second, in terms of constraints, insurers often set a duration target for their bond portfolio. Duration has several implications for the activity of an insurer. Here are some points illustrating how the duration intervenes in its activity:

- bond price movements may have significant impacts on a Solvency 2 prudential balance sheet depending on the duration mismatch between assets and liabilities;
- Solvency 2's standard capital requirements increase with duration. The solvency 2 paradigm for spread risk depends on the rating and duration of the individual securities; the longer the duration and the lower the rating are, the higher the capital required is;
- the portfolio strategy with respect to expected future interest rates and spreads movements will at a certain level define a portfolio duration objective in order to obtain earnings from futures market conditions; an asset manager expecting a rise of interest rates should hold a shorter duration portfolio than an asset manager expecting the opposite;
- the value of a bond portfolio is more sensitive to interest rates changes with a higher duration.

We would naturally like to incorporate a duration constraint into the classic two-dimensional approach of portfolio optimization based on risk and return. The determination of a duration target for a corporate bond portfolio is generally defined by a forward-looking analysis of financial markets and an asset and liability management analysis.

Assuming that the duration target, denoted D, is already defined, we add the following additional linear constraint to the optimization problem:

$$\sum_{s=1}^{S} w_s D_s = D$$

where D_s is the duration of security s.

$$\begin{aligned}
min_{w,\gamma,z} \quad & \gamma + \tfrac{1}{(1-\alpha)N}\textstyle\sum_{n=1}^{N} z_n \\
s.t. \quad & z_n \geqslant 0, \forall n \in [\![1,N]\!] \\
& z_n \geqslant \textstyle\sum_{s=1}^{S} l_{n,s}w_s - \gamma, \forall n \in [\![1,N]\!] \\
& \textstyle\sum_{s=1}^{S} w_s R_s = R \\
& \textstyle\sum_{s=1}^{S} w_s D_s = D \\
& \textstyle\sum_{s=1}^{S} w_s = 1 \\
& w_s \geqslant 0, \forall s \in [\![1,S]\!] \\
& \gamma \in \mathbb{R}
\end{aligned}$$

By calculating the previous optimization problem for different D durations and R return levels, we get a three-dimensional efficient frontier surface. Each result of w is an optimal combination considering simultaneously return, duration and $CVaR_\alpha$.

8.3 Dynamic portfolio management framework

This section proposes to go a little further by considering the case where only a part of the portfolio is reinvested while the other is unchanged. This situation is similar to that of a life insurance company often encouraged to hold its bonds until maturity and to invest only the new available capital. This new capital, free of any investment, is the net sum of the cash flows that will occur in the next period. We set this period to one year. For reasons of simplification, we assume that the new capital to be invested and free from any commitment is made up of the principal of bonds maturing in the year. In other words, the capital invested in bonds with a residual maturity of more than one year represents the unavailable portion of capital to make new investments, while bonds with a residual maturity of less than one year determine the capital available for investment at the end of the year.

The optimization problem of the previous section is therefore slightly modified to take into account the distinction between the fixed part of the portfolio and the capital available for new investments in the objective function. The available capital can be invested in all securities, denoted u $\forall u \in [\![1,U]\!]$, available on the market. The previous expression of $CVaR$:

$$CVaR_\alpha(f(w,L)) = VaR_\alpha(f(w,L)) + \frac{1}{(1-\alpha)N}\sum_{n=1}^{N}\left(\sum_{s=1}^{S} l_{n,s}w_s - VaR_\alpha(f(w,L))\right)^+$$

is then replaced by the following one:

$$CVaR_\alpha(f(w,L)) = VaR_\alpha(f(w,L)) + \frac{1}{(1-\alpha)N}\sum_{n=1}^{N}\left(\sum_{s'=1}^{S'} l_{n,s'}w_{s'} + \sum_{u=1}^{U} l_{n,u}w_u - VaR_\alpha(f(w,L))\right)^+$$

where:

- $w_{s'}$ $\forall s' \in [\![1, S']\!]$ is the set of unchanged weights associated to securities with more than one year residual maturity in relation to the unavailable portion of capital to make new investments;
- w_u $\forall u \in [\![1, U]\!]$ is the set of weights associated to the available part of the capital to invest which need to be optimally determined;
- $l_{n,u}$ $\forall u \in [\![1, U]\!]$ and $\forall n \in [\![1, N]\!]$ represents the discrete distribution loss function of securities available on the market; $l_{n,u}$ $\forall u \in [\![1, U]\!]$ and $\forall n \in [\![1, N]\!]$ are estimated simultaneously with all the others $l_{n,s'}$ $\forall s \in [\![1, S']\!]$ and $\forall n \in [\![1, N]\!]$ to integrate dependency effects between securities in the credit risk model.

The objective function can then be changed into the following one:

$$CVaR_\alpha(f(w,L)) = VaR_\alpha(f(w,L)) + \frac{1}{(1-\alpha)N}\sum_{n=1}^{N}\left(C_{l_n} + \sum_{u=1}^{U} l_{n,u}w_u - VaR_\alpha(f(w,L))\right)^+$$

where $C_{l_n} = \sum_{s'=1}^{S'} l_{n,s'}w_{s'}, \forall n \in [\![1, N]\!]$, are constants given that $w_{s'}$, $\forall s' \in [\![1, S']\!]$, are now fixed.

The optimization objective function being only minimized on weights w_u, $\forall u \in [\![1, U]\!]$, the problem becomes:

$$\begin{aligned}
&min_{w_u \forall u \in [\![1,U]\!]} && VaR_\alpha(f(w,L)) + \tfrac{1}{(1-\alpha)N}\textstyle\sum_{n=1}^{N}\left(C_{l_n} + \sum_{u=1}^{U} l_{n,u}w_u - VaR_\alpha(f(w,L))\right)^+ \\
&s.t. && C_w + \textstyle\sum_{u=1}^{U} w_u = 1 \\
& && C_R + \textstyle\sum_{u=1}^{U} w_u R_u = R \\
& && C_D + \textstyle\sum_{u=1}^{U} w_u D_u = D \\
& && w_u \geqslant 0, \forall u \in [\![1, U]\!]
\end{aligned}$$

with C_w, C_R and C_D constants such that :

$$C_w = \sum_{s'=1}^{S'} w_{s'}$$

$$C_R = \sum_{s'=1}^{S'} w_{s'} R_{s'}$$

$$C_D = \sum_{s'=1}^{S'} w_{s'} D_{s'}$$

that we can transform as previously into:

$$\begin{aligned}
min_{w_u \forall u \in [\![1,U]\!], \gamma, z} \quad & \gamma + \frac{1}{(1-\alpha)N} \sum_{n=1}^{N} z_n \\
s.t. \quad & z_n \geqslant 0, \forall n \in [\![1, N]\!] \\
& z_n \geqslant C_{l_n} + \sum_{u=1}^{U} l_{n,u} w_u - \gamma, \forall n \in [\![1, N]\!] \\
& C_w + \sum_{u=1}^{U} w_u = 1 \\
& C_R + \sum_{u=1}^{U} w_u R_u = R \\
& C_D + \sum_{u=1}^{U} w_u D_u = D \\
& w_u \geqslant 0, \forall u \in [\![1, U]\!] \\
& \gamma \in \mathbb{R}
\end{aligned}$$

This problem can be applied to the already analyzed empirical portfolio to which we add the securities available on the market.

In addition, insurers' internal management rules can be added to this problem. For instance, two linear and therefore convex rules can be added to the optimization problem:

- individual exposure to issuers rated A or above should not exceed 3% of the total invested capital;
- issuers rated BBB or less should be below 1.5% of the total invested capital.

$$
\begin{aligned}
min_{w_u \forall u \in [\![1,U]\!], \gamma, z} \quad & \gamma + \frac{1}{(1-\alpha)N} \sum_{n=1}^{N} z_n \\
s.t. \quad & z_n \geqslant 0, \forall n \in [\![1, N]\!] \\
& z_n \geqslant C_{l_n} + \sum_{u=1}^{U} l_{n,u} w_u - \gamma, \forall n \in [\![1, N]\!] \\
& C_w + \sum_{u=1}^{U} w_u = 1 \\
& C_R + \sum_{u=1}^{U} w_u R_u = R \\
& C_D + \sum_{u=1}^{U} w_u D_u = D \\
& w_u \leqslant 0.030 \text{ for } u \text{ rated AAA, AA, A} \\
& w_u \leqslant 0.015 \text{ for } u \text{ rated BBB, BB, B, C} \\
& w_u \geqslant 0, \forall u \in [\![1, U]\!] \\
& \gamma \in \mathbb{R}
\end{aligned}
$$

8.4 Corporate bonds market data

As stated in the previous section, the capital available for investment is represented by all the corporate bonds publicly available. The number of corporate bonds denominated in euros is obviously very large. We have identified 2 722 potential corporate bonds denominated in euros for a total market value of 2 230 billions euros at 2017 year-end.

The number of securities (2 722) is quite important for an optimization framework solved with a single processor. The linear problem defined previously would be composed of 2 722 dimensions, for the securities available on the market, 127 dimensions for the securities included in the portfolio at the start of the period, 100 000 dimensions (the number of simulations) for the z_n and 1 dimension for γ. In addition, the 2 722 securities have to be previously evaluated, in terms of risk, simultaneously with the 127 securities already included in the portfolio.

To reduce the size of operational calculations and to integrate the fact that insurers are major players on the corporate bond primary market rather than on the secondary market, I use securities available on the market to build simple generic artificial securities considering the 7 rating classes (AAA, AA, A, BBB, BB, B, C), 2 seniority levels (senior and subordinated), all the residual maturities (on an annual granularity). I can then calculate the average yield for of each index, one for each combination of rating, seniority and maturity. They are calculated by weighting yields by the amount in euros issued for each security. We thus obtain 134 artificial securities for which market data are available. I assume that they are issued by 134 different issuers. Then, with a simple arbitrage argument, we can consider that for each combination of the three features (rating class, seniority and residual maturity), the yield is equivalent to the coupon of a bond priced at par that would be issued on the primary market with maturity, the residual maturity of the index and with the same rating and seniority characteristics.

A description of available bonds

Figure 8.1 provides a general overview of the set of securities listed on the market and used for the implementation of indices. It successively presents by combination, the number of securities, the capital market value expressed in percent of the total capital market and the average yield weighted by the capital market value.

We can first observe the kind of corporate bonds (denominated in euros) available on the market. Most of the bonds are concentrated on ratings BBB and A with high seniority and less than 10 years residual maturity. The mode of this distribution is reached for the following characteristics: BBB, senior with a residual maturity of 4 years (125 securities).

Regarding the capital (second table of figure 8.1), its distribution is strongly linked to the number of bonds issued. The mode of this distribution is the same as for the previous distribution (senior, BBB and 4 years of residual maturity representing 4.6% of the total market value).

Finally, the third table of figure 8.1 presents the yields. The average yield weighted by the capital amount issued is of 1.05% at 2017 year-end. This table regarding yields shows how much the market thinks it needs to be rewarded considering rating, seniority and maturity.

Expected loss of available bonds

To give some indications with regards to risk, I apply the credit risk model on this set of indices to get their loss default distribution, loss migration distribution and total loss distribution. The results are based on 100 000 simulations.

Figure 8.2 presents the expected losses in percentage of the exposure. For instance, the expected default loss (first table of figure 8.2) of a senior BBB security is of 0.08% of the invested capital. Default losses do not depend on the maturity of the security (please refer to section 3.1 page 15 related to the default loss model for more details). For a given combination of rating and seniority, the discrepancies for default losses that we may encounter are due to noise around simulations. By increasing the number of simulations, the discrepancies over the maturity dimension decrease.

The expected migration loss for the combination BBB, senior and 4 years maturity is of 0.39% of the exposure (please report to the second chart of figure 8.2). For migration loss, maturity is an important factor of the model. For instance, a 10 years maturity, BBB, senior security has an expected loss for migration twice more important than a 4 years maturity, BBB, senior.

As we know, the higher the rating is, the lower the probability of default is. But, in the same time, higher is the rating, higher is the contribution of migration loss to the total loss. This effect is amplified with the increase of maturity. For instance, the expected default loss of a AA senior 5 years maturity security is of 0.01% and its expected migration loss is of 0.08%, 8 times more important. On the other hand, a B senior 5 years maturity security has an expected default loss (1.79%) and an expected migration loss (1.71%) almost equal. When observing the same values for a 10 years maturity, the expected migration loss (0.14%) is 14 times higher than the expected default loss for a AA senior security and less than twice higher for a B senior (3.37%).

CVaR of available bonds

Figure 8.3 presents the 0.99 quantile $CVaR$, denoted $CVaR_{0.99}$, for default loss, migration loss and total loss in percentage of the exposure. Again, the $CVaR$ of default loss does not depend on maturity as the default loss distribution calculation does not rely on maturity. The differences for a given combination of rating and seniority through maturities are due to the number of simulations. By increasing the number of simulations, for a given combination of rating and seniority, the $CVaR$ default losses values will converge to the same value independently of the maturity. The discrepancies are moreover more important for the $CVaRs$ than for the expected losses as only the worst 1% cases (1 000 simulations) are integrated in the $CVaR_{0.99}$ calculation while the 100 000 simulations are used to compute the expected losses. Lastly, we can observe that higher is the rating, higher are the differences between the simulations for the $CVAR_{0.99}$ as the number of defaults events are rarer. These operational issues can easily be mitigated by increasing the number of simulations with more adapted computer resources.

From figure 8.3 we see, for instance, that a BBB senior 4 years maturity security has a default loss $CVaR_{0.99}$ of 9.1%, a migration loss $CVaR_{0.99}$ of 17% and a total loss $CVaR_{0.99}$ of 23.2%. The total loss manifests a clear behavior with regards to risk: higher is the rating, higher is the contribution of migration loss to the total loss $CVaR_{0.99}$; lower is the rating, higher is the importance of default loss in the determination of the $CVaR_{0.99}$ total loss. This fact is directly implied by the 1% worst losses counting for the computation of the $CVaR_{0.99}$ total loss. As the rating increases, extreme losses are dominated by migration losses and, conversely, as the rating decreases, extreme losses are dominated by default losses. For instance, $CVaR_{0.99}$ total loss of B and C rated securities is equal to their default loss meaning that over the 100 000 simulations, the 1 000 worsts with respect to the total losses are all default losses and no one is a migration loss. This implies that, in the case of extreme adverse events, the highest-rated securities carry potential (market value) losses that are more sensitive to maturity (a characteristic integrated in the migration loss model) than the lowest-rated ones. In the end, while lower ratings are riskier (with eventual losses related to defaults), their extreme risk depends less on maturity.

Figure 8.1 – Market securities per residual maturity, rating and seniority (2017 year-end; euro denominated): number of securities, market values and yields

Number of securities

Maturity	AAA		AA		A		BBB		BB		B		C		Total
	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	
1			8		43	5	57	7	18	5	1				144
2	2		17		84	5	92	15	24	12	7	1			259
3			28		100	11	115	24	41	16	16	3			354
4	2		32		105	6	125	17	38	11	28		3	1	368
5			30		91	7	124	26	45	11	41	2	4		381
6	1		13		75	6	111	14	43	11	26	1	2		303
7			11		72	4	111	25	54	5	18		1		301
8			12		51	5	68	20	12	5	2		1		176
9			14		53	4	68	17	10	1	2				169
10			10		31	4	39	7	9	1	2				103
11	2		4		17		20	2	3		1				49
12			5		13		17	1	1						37
13			1		5										6
14					3		7								10
15	1				9		8		1						19
16					3		2		1						6
17	1		2		5		1								9
18					4	1	1								6
19			1		6		1								8
20			1		5		3								9
22					1										1
23					1										1
27							1								1
29									1						1
37									1						1
Total	9		189		777	58	971	175	302	78	144	7	11	1	2722

Market value (in%)

Maturity	AAA		AA		A		BBB		BB		B		C		Total
	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	
1			0,3		1,7	0,3	1,9	0,4	0,6	0,2	0,0				5,5
2	0,1		0,9		3,6	0,2	3,4	0,8	0,7	0,4	0,1	0,0			10,2
3			1,4		4,2	0,6	4,1	1,1	1,2	0,5	0,3	0,1			13,4
4	0,1		1,4		4,3	0,4	4,6	0,7	0,9	0,3	0,7		0,0	0,0	13,5
5			1,2		3,9	0,3	4,4	1,2	1,3	0,3	0,7	0,0	0,0		13,3
6	0,0		0,5		3,0	0,3	3,8	0,6	1,2	0,4	0,6	0,0	0,1		10,5
7			0,4		2,7	0,3	3,7	1,0	1,5	0,2	0,4		0,0		10,1
8			0,7		2,0	0,2	2,2	0,9	0,4	0,1	0,0		0,0		6,7
9			0,7		2,3	0,2	2,5	0,8	0,2	0,0	0,1				6,8
10			0,4		1,1	0,1	1,4	0,3	0,2	0,0	0,0				3,7
11	0,1		0,2		0,6		0,7	0,1	0,1		0,0				1,8
12			0,2		0,4		0,6	0,1	0,0						1,4
13			0,0		0,2										0,2
14					0,1		0,2								0,3
15	0,0				0,3		0,4		0,1						0,8
16					0,1		0,1		0,0						0,2
17	0,1		0,1		0,1		0,1								0,4
18					0,1	0,0	0,1								0,3
19			0,0		0,3		0,1								0,4
20			0,0		0,2		0,1								0,3
22					0,0										0,0
23					0,0										0,0
27							0,0								0,0
29									0,0						0,0
37									0,0						0,0
Total	0,5		8,6		31,3	3,0	34,5	7,9	8,5	2,5	3,0	0,1	0,2	0,0	100

Yield (in %)

Maturity	AAA		AA		A		BBB		BB		B		C		Total
	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	
1			-0,14		-0,07	0,14	0,02	0,35	0,71	1,29	1,39				0,14
2	0,01		-0,09		0,00	0,20	0,05	0,67	0,75	1,13	3,31	2,20			0,20
3			0,08		0,13	0,54	0,24	0,85	1,68	1,63	4,99	3,62			0,56
4	0,10		0,20		0,32	0,79	0,46	1,21	2,10	2,51	5,23		6,23	6,19	0,86
5			0,36		0,49	0,96	0,67	1,45	2,37	2,52	5,76	4,32	6,67		1,17
6	0,54		0,52		0,67	1,20	0,88	1,86	2,44	2,47	4,78	3,59	6,13		1,35
7			0,65		0,85	1,63	1,04	1,95	2,89	2,60	4,70		5,71		1,52
8			0,83		0,99	1,60	1,25	2,34	2,78	2,86	4,99		5,34		1,46
9			0,95		1,16	1,88	1,44	2,35	3,15	2,68	3,53				1,51
10			1,03		1,21	1,93	1,48	2,55	3,20	4,17	5,03				1,64
11	1,01		1,11		1,35		1,62	2,49	3,98		4,32				1,61
12			1,22		1,48		1,83	3,37	4,25						1,74
13			1,38		1,59										1,56
14					1,53		2,16								2,02
15	1,35				1,75		1,95		3,24						1,98
16					1,82		2,02		3,58						2,24
17	1,43		1,70		1,74		2,79								1,82
18					1,81	2,06	2,11								1,99
19			1,70		2,06		2,97								2,25
20			1,61		1,94		2,79								2,24
22					2,21										2,21
23					2,66										2,66
27							2,61								2,61
29									3,83						3,83
37									4,37						4,37
Total	0,64		0,44		0,58	0,97	0,78	1,58	2,22	2,06	4,99	3,55	6,16	6,19	1,05

Figure 8.2 – Expected loss of market securities (expressed as a percentage of the exposure amount): default loss, migration loss and total loss

Default loss (Expected Loss in %)

Maturity	AAA		AA		A		BBB		BB		B		C		Total
	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	
1			0,01		0,02	0,04	0,08	0,15	0,36	0,52	1,77				0,12
2	0,00		0,01		0,02	0,03	0,08	0,13	0,37	0,60	1,80	2,73			0,12
3			0,01		0,02	0,04	0,08	0,12	0,37	0,56	1,81	2,66			0,15
4	0,00		0,01		0,02	0,05	0,09	0,13	0,37	0,57	1,81		9,93	15,12	0,23
5			0,01		0,03	0,03	0,08	0,13	0,37	0,57	1,79	2,62	10,02		0,23
6	0,00		0,00		0,02	0,03	0,08	0,12	0,37	0,53	1,80	2,74	10,01		0,26
7			0,01		0,02	0,04	0,09	0,12	0,36	0,57	1,75		9,94		0,19
8			0,01		0,02	0,04	0,09	0,13	0,38	0,58	1,81		9,99		0,13
9			0,01		0,02	0,04	0,08	0,12	0,38	0,57	1,75				0,09
10			0,01		0,02	0,03	0,09	0,14	0,37	0,57	1,77				0,10
11	0,00		0,00		0,02		0,09	0,15	0,38		1,81				0,09
12			0,00		0,02		0,10	0,11	0,36						0,06
13			0,01		0,02										0,02
14					0,03		0,09								0,07
15	0,00				0,02		0,08		0,38						0,08
16					0,02		0,08		0,38						0,11
17	0,00		0,00		0,02		0,07								0,02
18					0,03	0,04	0,09								0,06
19			0,01		0,03		0,09								0,04
20			0,01		0,02		0,08								0,04
22					0,03										0,03
23					0,03										0,03
27							0,08								0,08
29									0,38						0,38
37									0,34						0,34
Total	0,00		0,01		0,02	0,04	0,08	0,13	0,37	0,56	1,79	2,69	9,98	15,12	0,17

Migration loss (Expected Loss in %)

Maturity	AAA		AA		A		BBB		BB		B		C		Total
	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	
1			0,02		0,02	0,03	0,07	-0,01	0,14	0,71	1,49				0,09
2	0,03		0,03		0,04	0,06	0,20	0,06	0,34	1,05	1,62	3,72			0,18
3			0,04		0,06	0,06	0,30	0,13	0,22	1,06	1,08	3,62			0,23
4	0,06		0,05		0,07	0,13	0,39	0,24	0,37	1,41	1,09		4,59	4,69	0,31
5			0,08		0,07	0,13	0,39	-0,08	0,09	1,50	1,71	3,95	4,46		0,31
6	0,21		0,07		0,07	0,22	0,50	0,00	-0,19	1,58	2,02	4,24	5,22		0,39
7			0,06		0,10	0,19	0,66	-0,17	0,14	1,84	2,74		5,65		0,42
8			0,12		0,10	0,32	0,49	-0,18	0,42	2,16	2,40		6,26		0,29
9			0,07		0,11	0,20	0,62	0,34	-0,60	2,52	2,96				0,35
10			0,14		0,15	0,16	0,83	0,54	0,19	2,37	3,37				0,50
11	0,15		0,24		0,20		1,03	0,99	-1,74		4,05				0,52
12			0,26		0,21		1,08	0,10	-1,44						0,57
13			0,07		0,48										0,41
14					0,56		1,06								0,95
15	0,33				0,25		1,15		-1,96						0,55
16					0,27		1,36		-1,48						0,38
17	0,41		0,07		0,30		1,14								0,37
18					0,82	1,31	1,34								1,15
19			0,53		0,74		1,33								0,87
20			0,55		0,86		1,44								1,05
22					0,83										0,83
23					0,87										0,87
27							1,50								1,50
29									-8,53						-8,53
37									-10,2						-10,16
Total	0,16		0,08		0,10	0,15	0,48	0,06	0,02	1,36	1,78	3,77	5,04	4,69	0,32

Total loss (Expected Loss in %)

Maturity	AAA		AA		A		BBB		BB		B		C		Total
	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	
1			0,03		0,04	0,07	0,16	0,14	0,50	1,23	3,26				0,21
2	0,03		0,03		0,06	0,09	0,29	0,19	0,71	1,65	3,43	6,45			0,30
3			0,05		0,08	0,10	0,38	0,25	0,59	1,62	2,90	6,28			0,38
4	0,06		0,06		0,09	0,17	0,48	0,38	0,74	1,97	2,90		14,52	19,81	0,54
5			0,08		0,09	0,16	0,48	0,05	0,47	2,07	3,50	6,58	14,48		0,54
6	0,21		0,07		0,09	0,25	0,58	0,13	0,18	2,11	3,82	6,98	15,23		0,66
7			0,07		0,12	0,23	0,75	-0,05	0,50	2,41	4,49		15,59		0,61
8			0,13		0,13	0,35	0,58	-0,05	0,80	2,74	4,21		16,24		0,42
9			0,08		0,14	0,23	0,70	0,47	-0,22	3,09	4,71				0,44
10			0,15		0,17	0,20	0,91	0,69	0,56	2,94	5,14				0,60
11	0,15		0,24		0,22		1,12	1,13	-1,36		5,86				0,61
12			0,27		0,23		1,18	0,22	-1,07						0,63
13			0,08		0,51										0,43
14					0,59		1,14								1,02
15	0,33				0,28		1,22		-1,58						0,63
16					0,29		1,44		-1,10						0,50
17	0,41		0,07		0,32		1,21								0,39
18					0,85	1,35	1,43								1,21
19			0,54		0,77		1,42								0,91
20			0,55		0,88		1,52								1,10
22					0,85										0,85
23					0,90										0,90
27							1,58								1,58
29									-8,15						-8,15
37									-9,81						-9,81
Total	0,16		0,08		0,12	0,19	0,56	0,19	0,39	1,92	3,57	6,43	15,02	19,81	0,49

Figure 8.3 – $CVaR_{0.99}$ of market securities (expressed as a percentage of the exposure amount): default loss, migration loss and total loss

Default loss (CVaR 0.99 in %)

Maturity	AAA		AA		A		BBB		BB		B		C		Total
	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	
1			0,5		2,3	4,0	8,4	15,0	35,6	51,8	59,9				11,0
2	0,0		0,7		2,2	3,2	8,3	13,2	36,5	60,4	60,0	83,2			10,3
3			0,6		2,1	3,7	8,0	12,3	37,2	55,8	60,0	82,8			11,5
4	0,0		0,6		2,3	4,8	9,1	13,5	36,6	56,6	60,4		71,2	91,5	12,0
5			0,6		2,6	3,4	8,4	12,8	37,3	57,2	60,2	83,1	70,9		13,4
6	0,0		0,4		2,2	3,1	8,2	12,4	36,6	52,5	60,2	83,3	71,2		14,4
7			0,6		2,1	3,5	8,6	12,4	36,1	57,0	59,8		71,2		13,6
8			0,6		2,4	3,8	8,5	13,4	37,6	58,0	59,7		71,4		9,7
9			0,6		2,4	3,6	8,0	12,3	37,6	56,8	60,0				7,8
10			0,7		2,3	3,2	8,6	14,4	36,7	57,1	59,7				8,9
11	0,0		0,3		2,1		8,5	14,7	37,9		60,0				7,2
12			0,3		2,1		9,6	11,2	36,4						6,4
13			0,8		2,1										1,8
14					2,5		8,7								7,4
15	0,0				2,3		7,8		37,9						8,4
16					1,9		7,8		37,9						11,1
17	0,0		0,4		2,2		7,3								2,0
18					2,5	3,5	8,6								5,6
19			0,6		2,6		9,1								4,0
20			0,5		2,2		8,4								4,5
22					2,5										2,5
23					2,7										2,7
27							8,0								8,0
29									37,6						37,6
37									34,4						34,4
Total	0,0		0,6		2,3	3,6	8,4	13,0	36,8	56,3	60,1	83,1	71,2	91,5	11,4

Migration loss (CVaR 0.99 in%)

Maturity	AAA		AA		A		BBB		BB		B		C		Total
	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	
1			0,7		1,2	1,0	3,8	1,8	3,1	6,6	15,7				2,6
2	1,3		1,3		3,1	1,8	8,9	5,6	5,7	9,8	17,9	38,6			5,7
3			2,1		4,5	3,8	12,5	7,9	9,3	14,7	17,3	38,7			8,2
4	2,3		3,1		6,1	5,5	17,0	10,6	12,8	19,8	20,1		29,8	29,2	11,3
5			3,7		6,6	4,9	16,6	7,2	12,1	19,4	27,2	45,8	28,4		11,7
6	4,8		3,6		8,1	6,4	18,1	7,9	11,2	21,1	27,4	46,8	33,1		13,4
7			4,7		9,0	6,4	21,3	8,3	14,7	24,4	35,2		36,3		15,3
8			5,2		9,2	8,5	14,8	7,3	15,2	26,8	32,5		40,5		11,3
9			5,4		11,0	6,2	25,0	20,7	12,7	29,7	47,1				17,1
10			5,9		14,2	15,1	21,9	23,4	25,4	39,8	50,4				18,4
11	6,3		7,4		15,2		24,3	32,0	12,6		50,2				18,3
12			8,8		15,7		26,8	26,7	16,3						20,0
13			9,8		21,8										19,8
14					21,2		30,1								28,1
15	8,5				18,1		29,4		20,0						24,1
16					19,3		31,7		23,1						25,1
17	10,2		7,9		21,5		30,4								16,6
18					33,9	33,3	43,9								38,6
19			19,0		32,1		42,5								33,7
20			19,7		33,4		43,5								36,0
22					39,9										39,9
23					40,6										40,6
27							57,8								57,8
29									44,0						44,0
37									44,0						44,0
Total	4,9		3,8		7,8	5,7	17,0	9,8	11,9	17,3	26,3	41,0	32,3	29,2	12,0

Total loss (CVaR 0.99 in %)

Maturity	AAA		AA		A		BBB		BB		B		C		Total
	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	Sen.	Sub.	
1			1,2		3,5	5,0	11,5	16,4	36,1	53,0	59,9				12,7
2	1,3		2,0		5,3	4,9	15,6	18,0	37,3	60,7	60,0	83,2			14,4
3			2,7		6,6	7,5	18,7	19,0	38,2	57,6	60,0	82,8			17,2
4	2,3		3,7		8,4	10,1	23,2	22,2	38,1	58,5	60,4		71,2	91,5	19,8
5			4,3		9,2	8,2	22,2	19,0	38,5	59,4	60,2	83,1	70,9		21,0
6	4,8		3,9		10,3	9,4	23,3	19,1	37,9	56,2	60,2	83,3	71,2		23,1
7			5,4		11,1	9,8	26,1	19,7	38,2	59,9	59,8		71,2		23,8
8			5,7		11,5	12,0	20,7	19,9	39,1	60,5	59,7		71,4		18,3
9			6,0		13,3	9,5	29,2	29,7	38,9	60,0	60,0				21,9
10			6,6		16,4	18,2	26,4	33,4	40,9	62,0	59,7				23,1
11	6,3		7,7		17,2		28,4	40,6	39,3		60,0				22,2
12			9,1		17,7		31,0	34,3	38,8						23,5
13			10,6		23,6										21,4
14					23,3		33,4								31,1
15	8,5				20,2		32,5		40,5						28,3
16					21,1		34,3		41,3						30,3
17	10,2		8,3		23,5		33,0								17,9
18					35,7	35,7	45,4								40,4
19			19,5		34,0		44,1								35,5
20			20,2		35,0		44,8								37,3
22					41,5										41,5
23					42,2										42,2
27							58,0								58,0
29									55,0						55,0
37									54,6						54,6
Total	4,9		4,4		10,0	9,3	22,6	21,3	38,3	58,2	60,1	83,0	71,1	91,5	20,1

Artificial securities for optimization

The 134 indices presented in the previous three figures are then used to define the list of assets available to make new investments in the primary market. The benefit of using yields lies in the fact that, under a non-arbitrage assumption, we can create artificial primary issued securities with a price at par, with an annual coupon equal to the yield and with a maturity equal to the residual maturity of market securities. This approach leads to 134 artificial securities. However, this number of artificial securities is small considering the importance of diversification in determining an optimal portfolio. In this case, there would be only one artificial security for each combination of rating, seniority and maturity while it might be optimal to invest several times in securities having the same features but issued from different issuers. $x\%$ invested in a portfolio composed of two securities having the same profile makes a portfolio having the same return but less risky than a portfolio with $x\%$ invested in only one of these two securities. To solve this problem, the 134 indices are transformed into 228 securities where some indices are repeated to create other new artificial securities. They are created in proportion to the number of securities in each index as follows:

- the combinations having less than 15 securities are present once as artificial securities;
- the combinations having between 16 and 25 securities are present twice as artificial securities;
- the combinations having between 26 and 60 securities are present three times as artificial securities;
- the combinations having more than 61 securities are present four times as artificial securities.

This leads to 228 artificial securities available for the amount of capital to be invested in the dynamic portfolio framework. We assume that none of them are issued by a same issuer. This approach indirectly assumes, for instance that when having less than 15 securities on the market for a given combination of rating, seniority and maturity, asset managers will be able to consider only one security for their potential investments. Asset managers often reduce the space of potential securities by using their expert judgment. For instance, a concentration into a sector or a country or, be more or less exposed to some kind of securities are choices that will automatically eliminate several securities. In addition, the information specific to each issuer defines the views of asset managers and reduces again the number of securities that can be considered as potential attractive investments. Some issuers will exhibit better expectations than others.

8.5 Dynamic portfolio management results

In this section, we apply the dynamic optimization framework simultaneously to the empirical portfolio aged by one year presented in chapter 6 and to the 228 artificial securities created from the securities available on the market.

The optimization problem seen in the precedent section is:

$$
\begin{array}{ll}
min_{w_u \forall u \in [\![1,U]\!], \gamma, z} & \gamma + \frac{1}{(1-\alpha)N} \sum_{n=1}^{N} z_n \\
s.t. & z_n \geqslant 0, \forall n \in [\![1,N]\!] \\
 & z_n \geqslant C_{l_n} + \sum_{u=1}^{U} l_{u,s} w_u - \gamma, \forall n \in [\![1,N]\!] \\
 & C_w + \sum_{u=1}^{U} w_u = 1 \\
 & C_R + \sum_{u=1}^{U} w_u R_u = R \\
 & C_D + \sum_{u=1}^{U} w_u D_u = D \\
 & w_u \leqslant 0.030, \forall u \text{ rated AAA, AA, A} \\
 & w_u \leqslant 0.015, \forall u \text{ rated BBB, BB, B, C} \\
 & w_u \geqslant 0, \forall u \in [\![1,U]\!] \\
 & \gamma \in \mathbb{R}
\end{array}
$$

The portion of the portfolio that is still invested after one year has the following characteristics:

- 109 securities are still left instead of 127 at the beginning of the period;
- represents $C_w = 90.04\%$ of the total capital available;
- has a return $C_R = 2.74\%$;
- has a duration (one year later) $C_D = 5.38$ years.

The amount of capital available after one year represents 9.96% of the total capital. If we are seeking the minimal $CVaR_{0.99}$ for a portfolio return of 2.80% and a 5.50 years duration, the 9.96% to reinvest should generate a return of 3.31% and have a duration of 6.55 years. The figure 8.4 below presents the levels that the part to invest must achieve in terms of return and duration to reach different objectives. Some of them are not feasible. For instance, it is not possible to build a portfolio with a return of 2.40% (without short selling) given that the still invested part of the portfolio has a return of 2.74%. Indeed, even if we were not investing the 9.96% of capital available, the return would be of $90.04\% \times 2.74 + 9.96\% \times 0 = 2.47\%$.

Figure 8.4 – Decomposition of return (in %) and duration (in years) objectives between the still invested part of the portfolio and the available part for new investments

Return objective (in %)		2,30	2,40	2,50	2,60	2,70	2,80	2,90	3,00	3,10
Part already invested	(90,04%)	2,74	2,74	2,74	2,74	2,74	2,74	2,74	2,74	2,74
Part to invest	(9,96%)	*not feasible*	*not feasible*	0,29	1,30	2,30	3,31	4,31	5,31	6,32

Duration objective (in years)		4,75	5,00	5,25	5,50	5,75	6,00	6,25	6,50	6,75
Part already invested	(90,04%)	5,38	5,38	5,38	5,38	5,38	5,38	5,38	5,38	5,38
Part to invest	(9,96%)	*not feasible*	1,53	4,04	6,55	9,06	11,57	14,08	16,59	19,10

Figure 8.5 represents the minimal $CVaR_{0.99}$ surface solved by the previous optimization problem for many combinations of returns and durations. This surface is convex on both dimensions and shows how credit risk, return and duration interact.

Figure 8.5 – Dynamic portfolio management efficient frontier surface (with management rules) - $CVaR_{0.99}$ and return in percentage of the exposure amount and duration in years

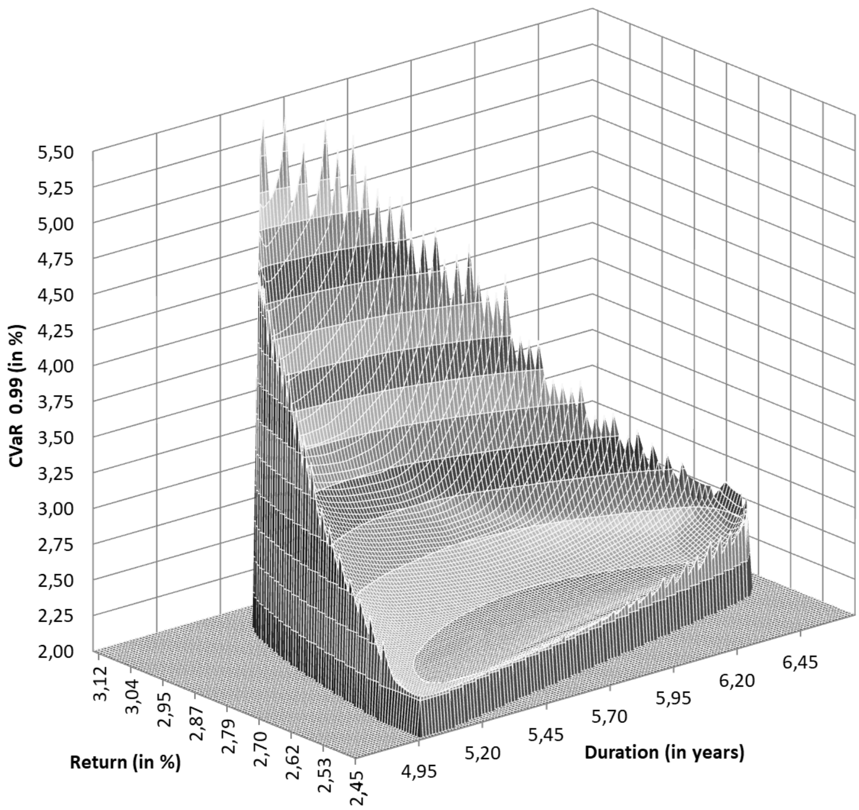


Figure 8.6 presents the minimal $CVaR_{0.99}$ values of the overall portfolio for combinations of return and duration objectives with management rules. A dot indicates a non-feasible combination (For instance, the 2.9% return and 6.5 years duration combination is not feasible).

We deduce from this table some elements. For instance:

- the combination having the lowest $CVaR_{0.99}$ (2.23%) is reached for a portfolio return of 2.6% and a duration of 5.5 years;
- lowering the duration to 5.25 years for the same 2.6% objective return increases by 0.02 point the $CVaR_{0.99}$ (2.25%);
- with an objective return of 2.7%, the fact to move duration from 5.5 years to 5.25 years increases now by 0.1 point the $CVaR_{0.99}$ [3];

Figure 8.6 – $CVaR_{0.99}$ dynamic portfolio efficient frontier (with management rules) - $CVaR_{0.99}$ and return in percentage of the total exposure and duration in years

Duration (in years)	Return (in %)									
	2,4	2,5	2,6	2,7	2,8	2,9	3,0	3,1	3,2	3,3
4,75	.	.	.	.	.	.	.	.	.	.
5,00	.	2,27	2,48	.	.	.	.	.	.	.
5,25	.	2,23	2,25	2,37	2,54	2,77	3,11	.	.	.
5,50	.	.	2,23	2,27	2,38	2,57	3,11	.	.	.
5,75	.	.	2,24	2,25	2,34	2,53	3,78	.	.	.
6,00	.	.	2,26	2,26	2,36	2,72	.	.	.	.
6,25	.	.	2,48	2,30	2,46	.	.	.	.	.
6,50	.	.	.	2,46	.	.	.	.	.	.
6,75	.	.	.	.	.	.	.	.	.	.
7,00	.	.	.	.	.	.	.	.	.	.

When excluding management rules (figure 8.7):

- $w_u \leqslant 0.030, \forall u$ rated AAA, AA, A;
- $w_u \leqslant 0.015, \forall u$ rated BBB, BB, B, C;

the number of feasible combinations increases. The new feasible combinations are shown in gray in figure 8.7. Their risk is much higher than that of the other combinations, which means that these management rules only exclude combinations that would have presented levels of risk too important to be considered. Moreover, as one might expect, by removing the management rules, as any other constraint, the minimum values of $CVaR_{0.99}$ are equal or slightly lower than the one including the management rules. The largest decrease in risk is of 0.35 point for the following

3. These increases of risk observed when decreasing duration are due to the fact that the newly invested 9.96% are less diversified and therefore riskier for low duration objectives.

objective: return of 3.0% and duration of 5.75 years.

Figure 8.7 – $CVaR_{0.99}$ dynamic portfolio efficient frontier (without management rules) - $CVaR_{0.99}$ and return in percentage of the total exposure and duration in years

Duration (in years)	Return (in %)									
	2,4	2,5	2,6	2,7	2,8	2,9	3,0	3,1	3,2	3,3
4,75	.	.	.	.	.	.	.	.	.	.
5,00	.	2,27	2,48	4,27	.	.	.	.	.	.
5,25	.	2,23	2,25	2,37	2,54	2,77	3,11	5,55	.	.
5,50	.	.	2,23	2,27	2,38	2,57	3,11	.	.	.
5,75	.	.	2,24	2,25	2,34	2,53	3,43	.	.	.
6,00	.	.	2,26	2,26	2,36	2,68	4,57	.	.	.
6,25	.	.	2,46	2,30	2,46	3,15	.	.	.	.
6,50	.	.	.	2,46	2,82	4,33	.	.	.	.
6,75	.	.	.	4,81	4,43	.	.	.	.	.
7,00	.	.	.	.	.	.	.	.	.	.

The following four figures detail the 228 artificial securities [4] (lines) that should be purchased to get the minimal $CVaR_{0.99}$ for 12 main return and duration objectives (columns) with management rules. For instance, to reach an objective of 2.6% return and 5.5 years duration for the whole portfolio, we have to invest 6.3% of the available capital for investment (representing 9.96% of the total capital) into a AAA, senior security with 2 years duration and a return of 0.01%, 7.0% into a AAA, senior security with 9.6 years duration and a return of 1.01% and so on...

The 3% and the 1.5% management rules constraints are equal to $30.1\%(= 100\% \times 3\%/9.96\%)$ and to $15.1\%(= 100\% \times 1.5\%/9.96\%)$ in this table. For instance, For a 2.5% return and 5 years duration objective, a AA senior security with 1.1 year duration should be invested at 30.1% of the available capital for investment. This means that the 3% limit management rule has been reached for this security.

Many securities are never selected by the optimization framework under those 12 objectives of return and duration. for instance, BBB senior securities have too low returns compared to their risk while BBB subordinated securities present a more interesting risk-return ratio (please refer to the last table of figure 8.1 related to yields and to the last table in figure 8.3 for $CVaRs_{0.99}$ levels of total losses for an overview of individual risks and returns). The opposite occurs for BB and B securities were senior securities are selected while subordinated not.

As the return objective increases, the rating quality of the optimal securities to be

4. The first figure is related to artificial securities rated AAA and AA, the second figure to rated A, the third figure to rated BBB and the forth figure to rated BB, B and C.

held declines rapidly. 0.1% variation in returns implies significant changes in weights as only 9.96% of the capital is newly invested and 90.04% is unchanged. A return objective of 2.6% implies a return of 1.33% for the available capital, while a return objective of 2.7% implies a return of 2.34% for the available capital.

The 5.5 years and 6 years duration objectives have almost the same risk ($CVaR_{0.99}$) for target returns of 2.6%, 2.7% and 2.8%. As a result, an asset manager anticipating a rise of interest rates should choose, for a same level of risk, the 5.5 years duration portfolio rather than the 6 years duration.

Figure 8.8 – Optimal weighting of assets in relation to the capital to be invested for 12 pairs of selected return and duration objectives - part 1/4 (ratings AAA and AA)

Portfolio objectives			Duration (in years)			5,0	5,0	5,5	5,5	5,5	5,5	5,5	6,0	6,0	6,0	6,0	6,5
			Return (in %)			2,5	2,6	2,6	2,7	2,8	2,9	3,0	2,6	2,7	2,8	2,9	2,7
New investments profil (9,96% of the portfolio)			Duration (in years)			1,53	1,53	6,55	6,55	6,55	6,55	6,55	11,56	11,56	11,56	11,56	16,58
			Return (in %)			0,33	1,33	1,33	2,34	3,34	4,34	5,35	1,33	2,34	3,34	4,34	2,34
			Number of assets to buy			16	11	22	27	30	24	15	14	16	18	14	12
CVaR at 0,99 of the portfolio (in %)						2,27	2,48	2,23	2,27	2,38	2,57	3,11	2,26	2,26	2,36	2,72	2,46
Securities available for investment						Assets' weight in % for securities with ratings AAA and AA (total per objective equals 100%)											
Asset number	Rating	Seniority	Return (in %)	Duration (in years)	Maturity (in years)												
1	AAA	SEN	0,01	2,0	2,1			6,3									
2	AAA	SEN	0,10	3,7	4,0								7,0				
3	AAA	SEN	0,54	6,0	6,4												
4	AAA	SEN	1,01	9,6	10,9			7,0					8,0	7,5			
5	AAA	SEN	1,35	12,7	15,3								18,9	11,8			
6	AAA	SEN	1,43	15,0	17,4								2,6				
7	AA	SEN	-0,14	1,1	1,2	30,1	11,6										
8	AA	SEN	-0,09	2,0	2,0												
9	AA	SEN	-0,09	2,0	2,0												
10	AA	SEN	0,08	3,0	3,1												
11	AA	SEN	0,08	3,0	3,1												
12	AA	SEN	0,08	3,0	3,1												
13	AA	SEN	0,20	3,9	4,1												
14	AA	SEN	0,20	3,9	4,1												
15	AA	SEN	0,20	3,9	4,1												
16	AA	SEN	0,36	4,7	4,9												
17	AA	SEN	0,36	4,7	4,9												
18	AA	SEN	0,36	4,7	4,9												
19	AA	SEN	0,52	5,7	6,0												
20	AA	SEN	0,65	6,7	7,0								0,8				
21	AA	SEN	0,83	7,3	8,0												
22	AA	SEN	0,95	8,4	9,0												
23	AA	SEN	1,03	9,2	10,0								16,6				
24	AA	SEN	1,11	10,2	10,8								8,5				
25	AA	SEN	1,22	10,4	11,7												
26	AA	SEN	1,38	12,1	13,3								9,6	6,1			
27	AA	SEN	1,70	14,8	17,2			3,8					20,2	21,1	19,2		14,9
28	AA	SEN	1,61	16,7	19,8								0,1				16,3
29	AA	SEN	1,70	16,8	18,7												16,2

Figure 8.9 – Optimal weighting of assets in relation to the capital to be invested for 12 pairs of selected return and duration objectives - part 2/4 (rating A)

Portfolio objectives			**Duration (in years)**			5,0	5,0	5,5	5,5	5,5	5,5	5,5	6,0	6,0	6,0	6,0	6,5
			Return (in %)			2,5	2,6	2,6	2,7	2,8	2,9	3,0	2,6	2,7	2,8	2,9	2,7
New investments profil (9,96% of the portfolio)			**Duration (in years)**			1,53	1,53	6,55	6,55	6,55	6,55	6,55	11,56	11,56	11,56	11,56	16,58
			Return (in %)			0,33	1,33	1,33	2,34	3,34	4,34	5,35	1,33	2,34	3,34	4,34	2,34
			Number of assets to buy			16	11	22	27	30	24	15	14	16	18	14	12
CVaR at 0,99 of the portfolio (in %)						2,27	2,48	2,23	2,27	2,38	2,57	3,11	2,26	2,26	2,36	2,72	2,46
Securities available for investment						**Assets' weight in % for securities with rating A (total per objective equals 100%)**											
Asset number	**Rating**	**Seniority**	**Return (in %)**	**Duration (in years)**	**Maturity (in years)**												
30	A	SEN	-0,07	1,2	1,2												
31	A	SEN	-0,07	1,2	1,2	27,1											
32	A	SEN	-0,07	1,2	1,2	19,9	18,5	10,6									
33	A	SEN	0,00	2,0	2,0												
34	A	SEN	0,00	2,0	2,0												
35	A	SEN	0,00	2,0	2,0			4,4									
36	A	SEN	0,00	2,0	2,0												
37	A	SEN	0,13	2,9	3,0												
38	A	SEN	0,13	2,9	3,0												
39	A	SEN	0,13	2,9	3,0												
40	A	SEN	0,13	2,9	3,0												
41	A	SEN	0,32	3,9	4,0												
42	A	SEN	0,32	3,9	4,0												
43	A	SEN	0,32	3,9	4,0												
44	A	SEN	0,32	3,9	4,0			0,9									
45	A	SEN	0,49	4,8	5,0			10,5									
46	A	SEN	0,49	4,8	5,0												
47	A	SEN	0,49	4,8	5,0												
48	A	SEN	0,49	4,8	5,0												
49	A	SEN	0,67	5,7	6,0												
50	A	SEN	0,67	5,7	6,0												
51	A	SEN	0,67	5,7	6,0												
52	A	SEN	0,67	5,7	6,0												
53	A	SEN	0,85	6,6	7,0												
54	A	SEN	0,85	6,6	7,0												
55	A	SEN	0,85	6,6	7,0												
56	A	SEN	0,85	6,6	7,0												
57	A	SEN	0,99	7,5	8,1			3,4									
58	A	SEN	0,99	7,5	8,1												
59	A	SEN	0,99	7,5	8,1			11,6									
60	A	SEN	1,16	8,4	9,0			5,2					1,4				
61	A	SEN	1,16	8,4	9,0												
62	A	SEN	1,16	8,4	9,0												
63	A	SEN	1,21	9,3	10,1												
64	A	SEN	1,21	9,3	10,1												
65	A	SEN	1,21	9,3	10,1												
66	A	SEN	1,35	10,2	11,2												
67	A	SEN	1,35	10,2	11,2												
68	A	SEN	1,48	10,6	12,1												
69	A	SEN	1,59	11,6	13,3												
70	A	SEN	1,75	12,5	15,0												
71	A	SEN	1,53	12,6	13,8												
72	A	SEN	1,82	13,2	16,0									2,0			
73	A	SEN	2,21	14,3	21,5												
74	A	SEN	1,74	14,7	17,1								3,3	4,6	2,0		0,2
75	A	SEN	1,81	15,3	18,3												1,8
76	A	SEN	2,66	15,8	22,9										1,6		5,8
77	A	SEN	1,94	15,8	19,9												8,1
78	A	SEN	2,06	16,0	19,2												9,8
79	A	SUB	0,14	1,2	1,2	4,1	9,7										
80	A	SUB	0,20	1,8	1,9	2,7											
81	A	SUB	0,54	3,0	3,1				4,6								
82	A	SUB	0,79	3,7	4,0												
83	A	SUB	0,96	4,6	5,0												
84	A	SUB	1,20	5,3	5,8												
85	A	SUB	1,63	6,5	7,2												
86	A	SUB	1,60	7,1	7,8												
87	A	SUB	1,88	7,9	8,8				2,2								
88	A	SUB	1,93	8,8	10,0												
89	A	SUB	2,06	13,5	17,7												

Figure 8.10 – Optimal weighting of assets in relation to the capital to be invested for 12 pairs of selected return and duration objectives - part 3/4 (rating BBB)

Portfolio objectives			Duration (in years)			5,0	5,0	5,5	5,5	5,5	5,5	5,5	6,0	6,0	6,0	6,0	6,5
			Return (in %)			2,5	2,6	2,6	2,7	2,8	2,9	3,0	2,6	2,7	2,8	2,9	2,7
New investments profil (9,96% of the portfolio)			Duration (in years)			1,53	1,53	6,55	6,55	6,55	6,55	6,55	11,56	11,56	11,56	11,56	16,58
			Return (in %)			0,33	1,33	1,33	2,34	3,34	4,34	5,35	1,33	2,34	3,34	4,34	2,34
			Number of assets to buy			16	11	22	27	30	24	15	14	16	18	14	12
CVaR at 0,99 of the portfolio (in %)						2,27	2,48	2,23	2,27	2,38	2,57	3,11	2,26	2,26	2,36	2,72	2,46
Securities available for investment						Assets' weight in % for securitie with rating BBB (total per objective equals 100%)											
Asset number	Rating	Seniority	Return (in %)	Duration (in years)	Maturity (in years)												
90	BBB	SEN	0,02	1,2	1,2	0,4											
91	BBB	SEN	0,02	1,2	1,2												
92	BBB	SEN	0,02	1,2	1,2												
93	BBB	SEN	0,05	2,0	2,0												
94	BBB	SEN	0,05	2,0	2,0												
95	BBB	SEN	0,05	2,0	2,0												
96	BBB	SEN	0,05	2,0	2,0												
97	BBB	SEN	0,24	2,9	3,0												
98	BBB	SEN	0,24	2,9	3,0												
99	BBB	SEN	0,24	2,9	3,0												
100	BBB	SEN	0,24	2,9	3,0												
101	BBB	SEN	0,46	3,9	4,1												
102	BBB	SEN	0,46	3,9	4,1												
103	BBB	SEN	0,46	3,9	4,1												
104	BBB	SEN	0,46	3,9	4,1												
105	BBB	SEN	0,67	4,7	5,0												
106	BBB	SEN	0,67	4,7	5,0												
107	BBB	SEN	0,67	4,7	5,0												
108	BBB	SEN	0,67	4,7	5,0												
109	BBB	SEN	0,88	5,6	6,0												
110	BBB	SEN	0,88	5,6	6,0												
111	BBB	SEN	0,88	5,6	6,0												
112	BBB	SEN	0,88	5,6	6,0												
113	BBB	SEN	1,04	6,5	7,0												
114	BBB	SEN	1,04	6,5	7,0												
115	BBB	SEN	1,04	6,5	7,0												
116	BBB	SEN	1,04	6,5	7,0												
117	BBB	SEN	1,25	7,4	8,1												
118	BBB	SEN	1,25	7,4	8,1												
119	BBB	SEN	1,25	7,4	8,1												
120	BBB	SEN	1,25	7,4	8,1												
121	BBB	SEN	1,44	8,3	9,1												
122	BBB	SEN	1,44	8,3	9,1												
123	BBB	SEN	1,44	8,3	9,1												
124	BBB	SEN	1,44	8,3	9,1												
125	BBB	SEN	1,48	9,1	10,0												
126	BBB	SEN	1,48	9,1	10,0												
127	BBB	SEN	1,48	9,1	10,0												
128	BBB	SEN	1,62	9,9	10,9												
129	BBB	SEN	1,62	9,9	10,9												
130	BBB	SEN	1,83	10,6	11,9												
131	BBB	SEN	1,83	10,6	11,9												
132	BBB	SEN	1,95	11,3	15,0												
133	BBB	SEN	2,02	12,0	16,0												
134	BBB	SEN	2,16	12,4	14,0												
135	BBB	SEN	2,79	13,9	17,2										0,3		
136	BBB	SEN	2,97	14,2	18,7												
137	BBB	SEN	2,11	14,5	18,2												
138	BBB	SEN	2,79	15,3	19,9										1,0	1,3	1,5
139	BBB	SEN	2,61	20,4	27,2										1,4	7,3	7,1
140	BBB	SUB	0,35	1,2	1,3	5,0	10,3	2,0	4,9								
141	BBB	SUB	0,67	2,2	2,3	2,2			2,5								
142	BBB	SUB	0,67	2,2	2,3												
143	BBB	SUB	0,85	2,9	3,1				0,4								
144	BBB	SUB	0,85	2,9	3,1				6,5								
145	BBB	SUB	1,21	3,7	4,0												
146	BBB	SUB	1,21	3,7	4,0												
147	BBB	SUB	1,45	4,6	5,0				3,4								
148	BBB	SUB	1,45	4,6	5,0				2,9								
149	BBB	SUB	1,45	4,6	5,0			0,4	5,6	1,1							
150	BBB	SUB	1,86	5,4	6,1			6,9	8,3	8,7							
151	BBB	SUB	1,95	6,3	7,0			1,3	5,8					1,9			
152	BBB	SUB	1,95	6,3	7,0			10,4	11,3	10,6				10,5			
153	BBB	SUB	1,95	6,3	7,0				0,4								
154	BBB	SUB	2,34	7,0	8,1				0,2	0,9				0,8	1,7		
155	BBB	SUB	2,34	7,0	8,1			1,5	0,6	0,6							
156	BBB	SUB	2,35	7,8	9,1												
157	BBB	SUB	2,35	7,8	9,1										0,2		
158	BBB	SUB	2,55	8,4	9,9												
159	BBB	SUB	2,49	8,9	10,9												
160	BBB	SUB	3,37	9,9	12,2										5,9		

Figure 8.11 – Optimal weighting of assets in relation to the capital to be invested for 12 pairs of selected return and duration objectives - part 4/4 (ratings BB, B and C)

Portfolio objectives			Duration (in years)			5,0	5,0	5,5	5,5	5,5	5,5	5,5	6,0	6,0	6,0	6,0	6,5
			Return (in %)			2,5	2,6	2,6	2,7	2,8	2,9	3,0	2,6	2,7	2,8	2,9	2,7
New investments profil (9,96% of the portfolio)			Duration (in years)			1,53	1,53	6,55	6,55	6,55	6,55	6,55	11,56	11,56	11,56	11,56	16,58
			Return (in %)			0,33	1,33	1,33	2,34	3,34	4,34	5,35	1,33	2,34	3,34	4,34	2,34
			Number of assets to buy			16	11	22	27	30	24	15	14	16	18	14	12
CVaR at 0,99 of the portfolio (in %)						2,27	2,48	2,23	2,27	2,38	2,57	3,11	2,26	2,26	2,36	2,72	2,46
Securities available for investment						Assets' weight in % for securities with ratings BB, B and C (total per objective equals 100%)											
Asset number	Rating	Seniority	Return (in %)	Duration (in years)	Maturity (in years)												
161	BB	SEN	0,71	1,2	1,2		7,4										
162	BB	SEN	0,71	1,2	1,2		7,9										
163	BB	SEN	0,75	1,9	2,0												
164	BB	SEN	0,75	1,9	2,0												
165	BB	SEN	1,68	2,8	3,1	0,9											
166	BB	SEN	1,68	2,8	3,1					2,0							
167	BB	SEN	1,68	2,8	3,1												
168	BB	SEN	2,10	3,8	4,1												
169	BB	SEN	2,10	3,8	4,1					1,3							
170	BB	SEN	2,10	3,8	4,1												
171	BB	SEN	2,37	4,6	5,0				2,0	6,7	4,7						
172	BB	SEN	2,37	4,6	5,0				2,4	4,3							
173	BB	SEN	2,37	4,6	5,0				0,9	3,2	2,3						
174	BB	SEN	2,44	5,5	6,1					0,6							
175	BB	SEN	2,44	5,5	6,1				0,3	2,3	1,3						
176	BB	SEN	2,44	5,5	6,1						1,4						
177	BB	SEN	2,89	6,2	7,0					2,0	2,1						
178	BB	SEN	2,89	6,2	7,0					1,1	0,4						
179	BB	SEN	2,89	6,2	7,0					0,6	0,2						
180	BB	SEN	2,78	6,9	7,8												
181	BB	SEN	3,15	7,7	9,0				3,9	4,7	6,8			4,2	7,5		
182	BB	SEN	3,20	8,3	9,8												
183	BB	SEN	3,98	9,1	11,1	1,6		4,6	8,4	9,7	12,0			7,2	11,9	11,1	
184	BB	SEN	4,25	9,3	11,7	0,2		1,7	8,5	9,8	11,3	3,9		6,4	13,6	15,1	
185	BB	SEN	3,24	10,0	15,1			2,5	4,2	4,3	2,5			4,8	6,9		
186	BB	SEN	3,58	11,1	15,7			1,1	4,2	2,6	4,2			4,2	9,1	9,1	
187	BB	SEN	3,83	16,6	29,3			1,8	1,4	1,1	2,6	2,9	1,3	3,1	5,4	15,1	5,8
188	BB	SEN	4,37	17,7	37,2	0,4		2,1	2,7	1,8	1,9	14,0	1,9	3,7	7,2	15,1	12,6
189	BB	SUB	1,29	1,3	1,3	0,3	7,7										
190	BB	SUB	1,13	1,9	1,9												
191	BB	SUB	1,63	2,8	3,0												
192	BB	SUB	1,63	2,8	3,0												
193	BB	SUB	2,51	3,7	4,1												
194	BB	SUB	2,52	4,6	5,0												
195	BB	SUB	2,47	5,3	6,0												
196	BB	SUB	2,60	6,3	7,1												
197	BB	SUB	2,86	7,0	8,3												
198	BB	SUB	2,68	7,5	8,7												
199	BB	SUB	4,17	8,2	9,9												
200	B	SEN	1,39	1,2	1,2		4,0										
201	B	SEN	3,31	1,8	1,9		5,4										
202	B	SEN	4,99	2,9	3,2	0,7	6,5			1,9	4,6	2,9					
203	B	SEN	4,99	2,9	3,2	4,1	10,9		1,7	6,0	8,0	4,5			4,0		
204	B	SEN	5,23	3,7	4,2					2,6	5,0	7,9					
205	B	SEN	5,23	3,7	4,2					3,3	6,7	7,4					
206	B	SEN	5,23	3,7	4,2	0,3				1,9	4,7	5,8					
207	B	SEN	5,76	4,3	4,9					3,1	5,2	12,7			1,1	7,7	
208	B	SEN	5,76	4,3	4,9						4,5	13,6				5,7	
209	B	SEN	5,76	4,3	4,9					0,4	4,2	10,9				6,0	
210	B	SEN	4,78	5,2	6,0						0,8						
211	B	SEN	4,78	5,2	6,0												
212	B	SEN	4,78	5,2	6,0					0,8	2,3						
213	B	SEN	4,70	6,0	7,0												
214	B	SEN	4,70	6,0	7,0												
215	B	SEN	4,99	6,5	7,9											0,9	
216	B	SEN	3,53	7,9	9,3												
217	B	SEN	5,03	8,0	10,0							1,4				4,0	
218	B	SEN	4,32	9,1	11,5												
219	B	SUB	2,20	2,3	2,5												
220	B	SUB	3,62	2,8	3,1												
221	B	SUB	4,32	4,4	4,8												
222	B	SUB	3,59	5,1	5,7												
223	C	SEN	6,23	3,5	4,0							2,8					
224	C	SEN	6,67	4,3	5,2							5,7				1,3	
225	C	SEN	6,13	4,9	5,8							3,6				0,4	
226	C	SEN	5,71	6,1	7,4												
227	C	SEN	5,34	6,6	8,0												
228	C	SUB	6,19	3,3	4,0												

Chapter 9

Conclusion and perspectives

The analysis of the key features around credit risk gave us a general overview of the main determinants impacting a credit risk assessment. My choices, motivated by robustness, accuracy and simplicity, lead to choose historical data, to define a model close to the CreditMetrics approach and to consider $CVaR$ as the major risk measure.

The presented model breaks down default from migration events as those risks are different in many respects. I moreover integrated a truncated structure of losses given default into the credit risk model.

Operational and empirical considerations led to consider a simulation of years to estimate defaults and transition rates. The basic assumption of independent and identically distributed sets of data is not verified and the simulation of correlated binary data is still a research field. The simulation of historical data better captures extreme events compared to a long-term average approach.

The credit risk model run on a real corporate bond portfolio of a life insurance company was calculated under two regimes (through-the-cycle and stress period). I showed the risk distribution of different outcomes: number of defaults, default loss, number of migrations, migration loss and total credit risk loss. The results consider several risk measures for different levels of risk: VaR, $CVaR$, expected loss and unexpected loss.

The last chapter related to portfolio optimization presented an application with empirical results derived from the seminal papers of Rockafellar and Uryasev [32] [33]. By introducing a duration constraint, this approach led to a three-dimensional efficient frontier. Considering the rolling mechanism behind a bond portfolio, I integrated in the optimization framework a clear distinction between the part of the capital to newly invest and the part already invested. The weights based on the empirical portfolio and the securities available on the market are also presented.

An analysis of the securities available on the market broken down by rating, seniority and residual maturity gives interesting indications of their expected loss and $CVaR$ for default loss, migration loss and total loss.

Perspectives are numerous.

There is room for refinement of the suggested credit risk framework. We may consider more granular classes than ratings when analyzing default and transition rates. For instance, sector or country dependencies are not considered in this methodology. We indirectly assume that this kind of more granular information is actually part of the expert judgment. The first risk management layer is directly implemented by the financial division with a clear investment policy. The expert judgment of asset managers and the fact that not everything can be incorporated into the models makes this first layer crucial. The second layer consists to assess credit risk in order to measure risk, follow investment decisions and propose new ones in light of the risks incurred.

Extensions can be made to consider the current macroeconomic situation in order to predict next year default and transition rates as well as loss given defaults using well known econometric techniques. This approach would define a third analysis regime: the "forecast regime".

Other risks, such as interest rate risk, can be added to the actual optimization framework by including them in the loss function used in the optimization framework. Other asset classes can also be assessed and added to the optimization framework. Sovereign bonds or stocks, with their own risk function, would allow to find the best allocations by asset class minimizing $CVaR$ for given objectives.

The strategic plan of insurance companies is, since Solvency 2, presented in the RSR and tested in the ORSA. It often incorporates a forecast of market conditions and some stress scenarios. The optimization framework can also use a loss function calculated based on these hypothetical market views to measure, for example, how the optimal strategy based on past and present events (through-the-cycle) deviates from that of the optimal strategy based on forward market assumptions.

Bibliography

[1] ARTZNER P., DELBAEN F., EBER J.M. and HEATH D. (1997) Thinking coherently, Risk, 10, 68-71.

[2] ARTZNER P., DELBAEN F., EBER J.M. and HEATH D. (1999) Coherent measures of risk. Mathematical Finance, 9, 203-228.

[3] BAHADUR R. (1961) A representation of the joint distribution responses to n dichotomus items. In: Solomon, H (Ed.), Studies in item analysis and prediction, Standford mathematical studies in the social sciences, Vol. VI. Standford university press, Standford, california,USA.

[4] BLACK F. and SCHOLES M. (1973) The pricing of options and corporate liabilities, Journal of political economy, 81, 637-654.

[5] BRASSARD D. (2002) Etude comparative des mesures de risque et des modèles d'évaluation du risque de crédit. Mémoire de l'école des hautes études commerciales (université de Montréal).

[6] CHAGANTY N.R. and JOE H. (2006) Range of correlation for dependent Bernoulli random variables. Biometrika, 93, 1, 197-206.

[7] CHEN S. X. (1999) Beta kernel estimators for the density functions. Computational Statistics & Data Analysis.

[8] CREDIT SUISSE FINANCIAL PRODUCTS (1997) CreditRisk+ a Credit Risk Management Framework. Technical document.

[9] CROUCHY M., GLAI D. and MARK R. (2000) A comparative analysis of current credit risk models. Journal of banking and finance, 24, 59-117.

[10] DAI B., DING, S. and WAHBA G. (2013) Multivariate Bernoulli distribution. Bernoulli, 19(4), 1465-1483.

[11] DUFFIE D. and SINGLETON K. (1999) Modelling term structure models of defaultable bonds. Review of financial studies, 12, 687-720.

[12] EMRICH L.J. and PIEDMONTE M.R. (1991) A method for generating high-dimensional multivariate binary variates. The american statistician, Volume 45, Number 4, 302-304.

[13] FREY R. and BACKHAUS J. (2004) Portfolio credit risk models with interacting default intensities: a markovian approach. Preprint, University of Leipzig.

[14] FREY R. and McNEIL A. (2002) VaR and Expected Shortfall in Portfolios of dependent Credit Risks: Conceptual and Practical insights. Journal of banking and finance, 1317-1344.

[15] FREY R. and McNEIL A. (2003) Dependent defaults in models of portofolio credit risk. Journal of risk, Volume 6, Number 1, 59-92.

[16] HURLIN C., LEYMARIE J. and PATIN A. (2018) Loss functions for LGD model comparison. halshs-01516147v3

[17] JARROW R. A. and PROTTER P. (2004) Structural versus reduced form models: a new information based perspective. Journal of investment management, volume 2, number 2, 1-10.

[18] JOE H. (1997) Multivariate models and dependence concepts. Chapman & Hall, London.

[19] KEALHOFER S. and BOHN J. (2001) Portfolio management of default risk. KMV working paper.

[20] LUCAS D. J. (1995) Default correlation and credit analysis. The journal of fixed income, 76-87.

[21] LUNN A. D. and DAVIES. S. J. (1998) A note on generating correlated binary variables. Biometrika, Volume 85, Issue 2, 487-490.

[22] MARKOWITZ H.M. (1952) Portfolio selection. Journal of Finance, 7, 77-91.

[23] MERTON R. (1974) On the pricing of corporate debt: the risk structure of interest rate. Journal of Finance, 28, 449-470.

[24] MOODY'S (2018) Annual Default Study: corporate default and recovery rates, 1920-2017 Moody's Global Credit Policy.

[25] NICKELL P., PERRAUDIN W. and VAROTTO S. (2000) Stability of rating transitions. Journal of Banking & Finance, 24, 203-227.

[26] OGRYCZAK W. and RUSZCZYNSKI A. (2002) Dual stochastic dominance and quantile risk measures. International transactions in operational research 9, 661-680.

[27] PARK C. G., PARK T. and SHIN D. W. PFISTER T. (1996) A simple method for generating correlated binary variates. The american statistician, Volume 50, Number 4, 306-310.

[28] PFISTER T., UTZ S., WIMMER M. (2015) Capital allocation in credit portfolios in a multiperiod setting: a literature review and practical guidelines. Review of Managerial Science, 9, 1-32.

[29] PFLUG G.C. (2000) Some remarks on the value-at-risk and the conditional value-at-risk. In: Uryasev, S.(Ed.), Probabilistic Constrained Optimization: Methodology and Applications. Kluwer Academic Publishers, Dordrecht.

[30] OMAN S. D. and ZUCKER D. M. (2001) Modeling and generating correlated binary variables. Biometrika, 88, 1, 287-290.

[31] RISKMETRIC-GROUP (1997) CreditMetrics - Technical document.

[32] ROCKAFELLAR R. T. , URYASEV S. (2000) Optimization of conditional value-at-risk. Journal of risk, Volume 3, Number 3, 21-41.

[33] ROCKAFELLAR R. T. , URYASEV S. (2002) Conditional value-at-risk for general loss distributions. Journal of banking and finance, 26, 1443-1471.

[34] STANDARD & POOR'S (2018) 2017 annual global corporate default study and rating transitions.

Annex

.

Historical one-year transition matrices excluding default state

1981	AAA	AA	A	BBB	BB	B	C
AAA	0,89	0,11	0,00	0,00	0,00	0,00	0,00
AA	0,02	0,90	0,08	0,00	0,00	0,00	0,00
A	0,00	0,04	0,89	0,07	0,00	0,00	0,00
BBB	0,00	0,00	0,05	0,90	0,05	0,00	0,00
BB	0,00	0,00	0,01	0,06	0,62	0,31	0,00
B	0,00	0,00	0,01	0,00	0,05	0,91	0,03
C	0,00	0,00	0,00	0,00	0,00	0,09	0,91

1982	AAA	AA	A	BBB	BB	B	C
AAA	0,93	0,05	0,02	0,00	0,00	0,00	0,00
AA	0,00	0,91	0,07	0,00	0,01	0,00	0,00
A	0,00	0,04	0,86	0,09	0,01	0,00	0,00
BBB	0,00	0,00	0,02	0,88	0,09	0,00	0,00
BB	0,00	0,01	0,00	0,04	0,86	0,09	0,00
B	0,00	0,00	0,01	0,01	0,03	0,91	0,05
C	0,00	0,00	0,00	0,00	0,00	0,11	0,89

1983	AAA	AA	A	BBB	BB	B	C
AAA	0,81	0,18	0,01	0,00	0,00	0,00	0,00
AA	0,00	0,93	0,05	0,01	0,00	0,00	0,00
A	0,01	0,04	0,90	0,04	0,01	0,00	0,00
BBB	0,00	0,01	0,07	0,86	0,06	0,01	0,00
BB	0,00	0,01	0,01	0,04	0,81	0,13	0,00
B	0,00	0,00	0,01	0,01	0,04	0,94	0,01
C	0,00	0,00	0,00	0,00	0,00	0,21	0,79

1984	AAA	AA	A	BBB	BB	B	C
AAA	0,72	0,27	0,01	0,00	0,00	0,00	0,00
AA	0,02	0,93	0,04	0,01	0,00	0,00	0,00
A	0,00	0,03	0,93	0,03	0,01	0,00	0,00
BBB	0,00	0,00	0,12	0,79	0,06	0,02	0,00
BB	0,00	0,00	0,01	0,08	0,85	0,05	0,00
B	0,00	0,00	0,00	0,01	0,06	0,93	0,00
C	0,00	0,00	0,00	0,00	0,00	0,00	1,00

1985	AAA	AA	A	BBB	BB	B	C
AAA	0,92	0,07	0,00	0,00	0,01	0,00	0,00
AA	0,00	0,88	0,08	0,02	0,00	0,01	0,00
A	0,00	0,02	0,90	0,07	0,01	0,00	0,00
BBB	0,00	0,01	0,08	0,80	0,07	0,04	0,00
BB	0,00	0,00	0,01	0,06	0,81	0,11	0,02
B	0,00	0,00	0,02	0,00	0,03	0,95	0,01
C	0,00	0,00	0,00	0,00	0,00	0,36	0,64

1986	AAA	AA	A	BBB	BB	B	C
AAA	0,93	0,07	0,00	0,00	0,00	0,00	0,00
AA	0,01	0,92	0,05	0,02	0,00	0,01	0,00
A	0,00	0,05	0,82	0,09	0,02	0,02	0,00
BBB	0,00	0,00	0,08	0,80	0,09	0,03	0,00
BB	0,00	0,00	0,00	0,07	0,84	0,07	0,01
B	0,00	0,00	0,00	0,00	0,05	0,82	0,13
C	0,00	0,00	0,00	0,00	0,00	0,00	1,00

1987	AAA	AA	A	BBB	BB	B	C
AAA	0,96	0,03	0,00	0,01	0,00	0,00	0,00
AA	0,02	0,93	0,05	0,00	0,00	0,00	0,00
A	0,00	0,01	0,90	0,06	0,00	0,01	0,00
BBB	0,00	0,01	0,06	0,83	0,07	0,04	0,00
BB	0,00	0,00	0,00	0,08	0,82	0,09	0,00
B	0,00	0,00	0,01	0,00	0,06	0,90	0,03
C	0,00	0,00	0,00	0,02	0,02	0,10	0,85

1988	AAA	AA	A	BBB	BB	B	C
AAA	0,91	0,06	0,02	0,00	0,01	0,00	0,00
AA	0,01	0,84	0,11	0,03	0,01	0,00	0,00
A	0,00	0,02	0,92	0,05	0,01	0,01	0,00
BBB	0,00	0,00	0,10	0,81	0,06	0,02	0,01
BB	0,00	0,00	0,01	0,08	0,80	0,09	0,02
B	0,00	0,00	0,00	0,00	0,06	0,90	0,04
C	0,00	0,00	0,00	0,06	0,06	0,14	0,75

1989	AAA	AA	A	BBB	BB	B	C
AAA	0,94	0,06	0,00	0,00	0,00	0,00	0,00
AA	0,01	0,93	0,07	0,00	0,00	0,00	0,00
A	0,00	0,02	0,89	0,07	0,02	0,00	0,00
BBB	0,00	0,00	0,07	0,85	0,07	0,01	0,01
BB	0,00	0,00	0,01	0,15	0,78	0,06	0,01
B	0,00	0,00	0,00	0,00	0,09	0,86	0,05
C	0,00	0,00	0,04	0,00	0,04	0,00	0,92

1990	AAA	AA	A	BBB	BB	B	C
AAA	0,95	0,05	0,00	0,00	0,00	0,00	0,00
AA	0,01	0,89	0,10	0,00	0,00	0,00	0,00
A	0,00	0,02	0,89	0,08	0,01	0,00	0,00
BBB	0,00	0,00	0,05	0,88	0,05	0,01	0,00
BB	0,00	0,00	0,00	0,08	0,77	0,12	0,04
B	0,00	0,01	0,00	0,01	0,04	0,88	0,07
C	0,00	0,00	0,00	0,00	0,03	0,07	0,90

Historical one-year transition matrices excluding default state

1991	AAA	AA	A	BBB	BB	B	C
AAA	0,87	0,12	0,01	0,00	0,00	0,00	0,00
AA	0,00	0,92	0,08	0,00	0,00	0,00	0,00
A	0,00	0,01	0,93	0,06	0,01	0,00	0,00
BBB	0,00	0,01	0,04	0,89	0,05	0,01	0,00
BB	0,00	0,00	0,00	0,06	0,83	0,08	0,02
B	0,00	0,00	0,00	0,00	0,07	0,88	0,04
C	0,00	0,00	0,00	0,03	0,06	0,11	0,80

1992	AAA	AA	A	BBB	BB	B	C
AAA	0,90	0,10	0,00	0,00	0,00	0,00	0,00
AA	0,01	0,91	0,06	0,01	0,00	0,00	0,00
A	0,00	0,01	0,94	0,04	0,00	0,00	0,00
BBB	0,00	0,00	0,05	0,90	0,04	0,01	0,00
BB	0,00	0,00	0,00	0,13	0,80	0,04	0,03
B	0,00	0,00	0,01	0,02	0,13	0,80	0,05
C	0,00	0,00	0,00	0,00	0,06	0,19	0,75

1993	AAA	AA	A	BBB	BB	B	C
AAA	0,95	0,03	0,02	0,00	0,00	0,00	0,00
AA	0,00	0,94	0,06	0,00	0,00	0,00	0,00
A	0,00	0,01	0,95	0,04	0,00	0,00	0,00
BBB	0,00	0,00	0,05	0,89	0,06	0,00	0,00
BB	0,00	0,00	0,00	0,09	0,81	0,09	0,00
B	0,00	0,00	0,01	0,01	0,16	0,80	0,02
C	0,00	0,00	0,00	0,00	0,04	0,46	0,50

1994	AAA	AA	A	BBB	BB	B	C
AAA	0,91	0,09	0,01	0,00	0,00	0,00	0,00
AA	0,00	0,90	0,09	0,00	0,00	0,00	0,00
A	0,00	0,01	0,94	0,04	0,00	0,00	0,00
BBB	0,00	0,00	0,03	0,94	0,02	0,00	0,00
BB	0,00	0,00	0,00	0,08	0,89	0,03	0,00
B	0,00	0,00	0,00	0,00	0,06	0,91	0,03
C	0,00	0,00	0,00	0,00	0,00	0,15	0,85

1995	AAA	AA	A	BBB	BB	B	C
AAA	0,92	0,08	0,00	0,00	0,00	0,00	0,00
AA	0,01	0,90	0,09	0,00	0,00	0,00	0,00
A	0,00	0,02	0,94	0,04	0,00	0,00	0,00
BBB	0,00	0,00	0,04	0,92	0,04	0,00	0,00
BB	0,00	0,00	0,01	0,07	0,88	0,05	0,00
B	0,00	0,00	0,00	0,01	0,09	0,88	0,02
C	0,00	0,00	0,00	0,06	0,00	0,12	0,81

1996	AAA	AA	A	BBB	BB	B	C
AAA	0,93	0,07	0,00	0,00	0,00	0,00	0,00
AA	0,01	0,94	0,05	0,00	0,00	0,00	0,00
A	0,00	0,03	0,95	0,02	0,00	0,00	0,00
BBB	0,00	0,00	0,06	0,92	0,02	0,00	0,00
BB	0,00	0,00	0,00	0,09	0,85	0,05	0,00
B	0,00	0,00	0,00	0,01	0,10	0,87	0,02
C	0,00	0,00	0,00	0,00	0,11	0,17	0,72

1997	AAA	AA	A	BBB	BB	B	C
AAA	0,95	0,05	0,00	0,00	0,00	0,00	0,00
AA	0,01	0,94	0,04	0,01	0,00	0,00	0,00
A	0,00	0,02	0,93	0,04	0,00	0,00	0,00
BBB	0,00	0,00	0,04	0,93	0,03	0,01	0,00
BB	0,00	0,00	0,00	0,09	0,86	0,05	0,00
B	0,00	0,00	0,01	0,00	0,08	0,88	0,03
C	0,00	0,00	0,00	0,00	0,00	0,24	0,76

1998	AAA	AA	A	BBB	BB	B	C
AAA	0,93	0,07	0,00	0,01	0,00	0,00	0,00
AA	0,00	0,94	0,05	0,00	0,00	0,00	0,00
A	0,00	0,02	0,92	0,06	0,00	0,00	0,00
BBB	0,00	0,00	0,03	0,91	0,06	0,01	0,00
BB	0,00	0,00	0,00	0,06	0,84	0,07	0,03
B	0,00	0,00	0,00	0,01	0,07	0,87	0,06
C	0,00	0,00	0,07	0,00	0,00	0,33	0,60

1999	AAA	AA	A	BBB	BB	B	C
AAA	0,94	0,06	0,00	0,00	0,00	0,00	0,00
AA	0,00	0,92	0,07	0,01	0,00	0,00	0,00
A	0,00	0,02	0,91	0,06	0,00	0,00	0,00
BBB	0,00	0,00	0,03	0,92	0,04	0,00	0,00
BB	0,00	0,00	0,00	0,03	0,88	0,08	0,01
B	0,00	0,00	0,00	0,00	0,03	0,91	0,05
C	0,00	0,00	0,00	0,00	0,00	0,05	0,95

2000	AAA	AA	A	BBB	BB	B	C
AAA	0,95	0,04	0,02	0,00	0,00	0,00	0,00
AA	0,01	0,88	0,11	0,00	0,00	0,00	0,00
A	0,00	0,03	0,89	0,08	0,00	0,00	0,00
BBB	0,00	0,00	0,03	0,92	0,04	0,00	0,00
BB	0,00	0,00	0,00	0,04	0,88	0,07	0,01
B	0,00	0,00	0,00	0,00	0,04	0,90	0,05
C	0,00	0,00	0,00	0,00	0,02	0,10	0,88

Historical one-year transition matrices excluding default state

2001	AAA	AA	A	BBB	BB	B	C
AAA	0,94	0,06	0,00	0,00	0,00	0,00	0,00
AA	0,00	0,88	0,12	0,00	0,00	0,00	0,00
A	0,00	0,02	0,90	0,07	0,00	0,00	0,00
BBB	0,00	0,00	0,03	0,90	0,05	0,01	0,01
BB	0,00	0,00	0,00	0,03	0,83	0,11	0,02
B	0,00	0,00	0,00	0,00	0,04	0,86	0,11
C	0,00	0,00	0,00	0,00	0,00	0,15	0,85

2006	AAA	AA	A	BBB	BB	B	C
AAA	0,98	0,02	0,01	0,00	0,00	0,00	0,00
AA	0,01	0,98	0,01	0,00	0,00	0,00	0,00
A	0,00	0,03	0,93	0,03	0,00	0,00	0,00
BBB	0,00	0,00	0,06	0,90	0,03	0,01	0,00
BB	0,00	0,00	0,00	0,09	0,81	0,09	0,01
B	0,00	0,00	0,00	0,00	0,11	0,81	0,08
C	0,00	0,00	0,00	0,00	0,00	0,26	0,74

2002	AAA	AA	A	BBB	BB	B	C
AAA	0,87	0,12	0,00	0,01	0,00	0,00	0,00
AA	0,00	0,79	0,18	0,02	0,00	0,01	0,00
A	0,00	0,01	0,87	0,11	0,01	0,00	0,00
BBB	0,00	0,00	0,02	0,88	0,07	0,02	0,01
BB	0,00	0,00	0,00	0,04	0,86	0,08	0,02
B	0,00	0,00	0,00	0,00	0,06	0,83	0,11
C	0,00	0,00	0,01	0,00	0,03	0,18	0,78

2007	AAA	AA	A	BBB	BB	B	C
AAA	0,98	0,02	0,00	0,00	0,00	0,00	0,00
AA	0,01	0,96	0,03	0,00	0,00	0,00	0,00
A	0,00	0,03	0,93	0,03	0,00	0,00	0,00
BBB	0,00	0,00	0,04	0,92	0,03	0,01	0,00
BB	0,00	0,00	0,00	0,08	0,85	0,07	0,00
B	0,00	0,00	0,00	0,00	0,09	0,88	0,03
C	0,00	0,00	0,00	0,00	0,00	0,31	0,69

2003	AAA	AA	A	BBB	BB	B	C
AAA	0,89	0,09	0,02	0,00	0,00	0,00	0,00
AA	0,00	0,88	0,11	0,00	0,00	0,00	0,00
A	0,00	0,01	0,92	0,07	0,00	0,00	0,00
BBB	0,00	0,00	0,02	0,93	0,05	0,00	0,00
BB	0,00	0,00	0,00	0,03	0,85	0,11	0,01
B	0,00	0,00	0,00	0,00	0,08	0,87	0,05
C	0,00	0,00	0,00	0,00	0,01	0,20	0,79

2008	AAA	AA	A	BBB	BB	B	C
AAA	0,87	0,06	0,03	0,00	0,00	0,01	0,02
AA	0,00	0,81	0,18	0,01	0,00	0,00	0,00
A	0,00	0,02	0,93	0,05	0,00	0,00	0,00
BBB	0,00	0,00	0,03	0,93	0,04	0,00	0,00
BB	0,00	0,00	0,00	0,05	0,84	0,09	0,01
B	0,00	0,00	0,00	0,00	0,04	0,86	0,09
C	0,00	0,00	0,00	0,00	0,00	0,21	0,79

2004	AAA	AA	A	BBB	BB	B	C
AAA	0,94	0,06	0,00	0,00	0,00	0,00	0,00
AA	0,00	0,96	0,04	0,00	0,00	0,00	0,00
A	0,00	0,01	0,96	0,03	0,00	0,00	0,00
BBB	0,00	0,00	0,02	0,96	0,02	0,00	0,00
BB	0,00	0,00	0,00	0,04	0,89	0,06	0,00
B	0,00	0,00	0,00	0,00	0,08	0,89	0,03
C	0,00	0,00	0,01	0,00	0,01	0,22	0,76

2009	AAA	AA	A	BBB	BB	B	C
AAA	0,91	0,09	0,00	0,00	0,00	0,00	0,00
AA	0,00	0,82	0,17	0,01	0,00	0,00	0,00
A	0,00	0,00	0,91	0,08	0,00	0,00	0,00
BBB	0,00	0,00	0,02	0,90	0,06	0,01	0,00
BB	0,00	0,00	0,00	0,04	0,83	0,13	0,01
B	0,00	0,00	0,00	0,00	0,03	0,86	0,10
C	0,00	0,00	0,00	0,00	0,00	0,19	0,81

2005	AAA	AA	A	BBB	BB	B	C
AAA	0,90	0,09	0,01	0,00	0,00	0,00	0,00
AA	0,00	0,94	0,05	0,01	0,00	0,00	0,00
A	0,00	0,02	0,94	0,05	0,00	0,00	0,00
BBB	0,00	0,00	0,06	0,90	0,03	0,00	0,00
BB	0,00	0,00	0,00	0,06	0,86	0,08	0,00
B	0,00	0,00	0,00	0,01	0,10	0,84	0,05
C	0,00	0,00	0,00	0,01	0,01	0,34	0,63

2010	AAA	AA	A	BBB	BB	B	C
AAA	0,74	0,25	0,00	0,01	0,00	0,00	0,00
AA	0,01	0,88	0,11	0,00	0,00	0,00	0,00
A	0,00	0,01	0,95	0,04	0,00	0,00	0,00
BBB	0,00	0,00	0,03	0,95	0,02	0,00	0,00
BB	0,00	0,00	0,00	0,06	0,90	0,04	0,00
B	0,00	0,00	0,00	0,00	0,07	0,91	0,02
C	0,00	0,00	0,00	0,00	0,01	0,46	0,53

Historical one-year transition matrices excluding default state

2011	AAA	AA	A	BBB	BB	B	C
AAA	0,51	0,49	0,00	0,00	0,00	0,00	0,00
AA	0,00	0,86	0,13	0,01	0,00	0,00	0,00
A	0,00	0,02	0,91	0,07	0,01	0,00	0,00
BBB	0,00	0,00	0,03	0,94	0,03	0,00	0,00
BB	0,00	0,00	0,00	0,06	0,88	0,05	0,01
B	0,00	0,00	0,00	0,00	0,08	0,88	0,04
C	0,00	0,00	0,00	0,00	0,00	0,33	0,67

2016	AAA	AA	A	BBB	BB	B	C
AAA	0,87	0,13	0,00	0,00	0,00	0,00	0,00
AA	0,00	0,93	0,07	0,00	0,00	0,00	0,00
A	0,00	0,01	0,95	0,04	0,00	0,00	0,00
BBB	0,00	0,00	0,03	0,94	0,03	0,00	0,00
BB	0,00	0,00	0,00	0,03	0,90	0,07	0,00
B	0,00	0,00	0,00	0,00	0,05	0,89	0,07
C	0,00	0,00	0,00	0,00	0,02	0,26	0,73

2012	AAA	AA	A	BBB	BB	B	C
AAA	0,88	0,13	0,00	0,00	0,00	0,00	0,00
AA	0,00	0,87	0,12	0,02	0,00	0,00	0,00
A	0,00	0,01	0,92	0,07	0,00	0,00	0,00
BBB	0,00	0,00	0,02	0,94	0,04	0,00	0,00
BB	0,00	0,00	0,00	0,05	0,87	0,08	0,00
B	0,00	0,00	0,00	0,00	0,04	0,91	0,05
C	0,00	0,00	0,00	0,00	0,00	0,25	0,75

2017	AAA	AA	A	BBB	BB	B	C
AAA	0,64	0,36	0,00	0,00	0,00	0,00	0,00
AA	0,00	0,96	0,04	0,00	0,00	0,00	0,00
A	0,00	0,00	0,97	0,02	0,00	0,00	0,00
BBB	0,00	0,00	0,02	0,95	0,03	0,00	0,00
BB	0,00	0,00	0,00	0,04	0,91	0,05	0,00
B	0,00	0,00	0,00	0,00	0,04	0,91	0,05
C	0,00	0,00	0,00	0,01	0,00	0,30	0,69

2013	AAA	AA	A	BBB	BB	B	C
AAA	0,90	0,10	0,00	0,00	0,00	0,00	0,00
AA	0,00	0,97	0,03	0,00	0,00	0,00	0,00
A	0,00	0,01	0,95	0,04	0,00	0,00	0,00
BBB	0,00	0,00	0,04	0,93	0,02	0,00	0,00
BB	0,00	0,00	0,00	0,06	0,89	0,05	0,00
B	0,00	0,00	0,00	0,00	0,06	0,88	0,05
C	0,00	0,00	0,00	0,00	0,00	0,18	0,82

2014	AAA	AA	A	BBB	BB	B	C
AAA	0,79	0,21	0,00	0,00	0,00	0,00	0,00
AA	0,00	0,98	0,02	0,00	0,00	0,00	0,00
A	0,00	0,01	0,96	0,03	0,00	0,00	0,00
BBB	0,00	0,00	0,03	0,95	0,02	0,00	0,00
BB	0,00	0,00	0,00	0,04	0,91	0,04	0,00
B	0,00	0,00	0,00	0,00	0,05	0,92	0,03
C	0,00	0,00	0,00	0,00	0,00	0,12	0,88

2015	AAA	AA	A	BBB	BB	B	C
AAA	1,00	0,00	0,00	0,00	0,00	0,00	0,00
AA	0,00	0,95	0,04	0,00	0,00	0,00	0,00
A	0,00	0,01	0,93	0,06	0,00	0,00	0,00
BBB	0,00	0,00	0,03	0,91	0,05	0,00	0,00
BB	0,00	0,00	0,00	0,04	0,88	0,08	0,00
B	0,00	0,00	0,00	0,00	0,04	0,90	0,05
C	0,00	0,00	0,00	0,00	0,00	0,11	0,89