EURIA — EURo Institut d'Actuariat

Mémoire présenté devant le jury de l'EURIA en vue de l'obtention du

Diplôme d'Actuaire EURIA

et de l'admission à l'Institut des Actuaires

le 3 septembre 2012

Par : Jeremy LESNE

Titre : Calculation of the required capital for interest rates and equity risks for variable annuity products

Confidentialité : non.

Membre présent du jury de l'Institut des Actuaires

Signature :

Membres présents du jury de l'EURIA

Entreprise
Deloitte
Signature :

Directeur de mémoire en entreprise
Alan PICONE
Signature :

Invité

Signature :

Autorisation de publication et de mise en ligne sur un site de diffusion de documents actuariels

Signature du responsable entreprise

Signature du candidat

Bibliothèque :

Secrétariat :

EURIA
EURo Institut d'Actuariat

6, avenue le Gorgeu
CS 93837
29238 Brest Cedex 3

T +33 (0)2 98 01 66 55
F +33 (0)2 98 01 66 57
euria@univ-brest.fr

Abstract

Keywords:

Solvency II, Variable Annuities, Best Estimate, quantile, Solvency Capital Requirement, partial internal model, nested simulations, asset modeling, interest rate modeling, asset and liability management.

The Solvency II Directive will bring about very significant changes to the solvency assessment process in the insurance industry. This new directive requires the calculation of a capital, the Solvency Capital Requirement (SCR), in order to reinforce companies' solvency and to protect policyholders. Insurers have the possibility to use a standard approach to calculate this capital but this formula seems inappropriate for certain products like Variable Annuities. Variable Annuities are unit-linked insurance products that give the beneficiary an opportunity to benefit from financial investment. They also guarantee a minimum amount on death or survival during the contract period. Sellers of Variable Annuities have to be ready to estimate their capital requirement with an internal model approach, and this is the reason why in this thesis we present a method that calculates the aggregated required capital for interest rates and equity risks with an internal model. The method used for this calculation is called "Nested Simulations". It is often used to better estimate certain aggregated risk-modules or submodules of the standard formula (in our case the aggregated interest rates and equity risk-submodules). Thus in this thesis we will exemplify not only the calculation of the aggregated capital for interest rates and equity risk-submodules but also the calculation of the capitals for the life risk-submodules by using the shocks given by the standard formula.

Résumé

Mots-clés:
Solvency II, Variable Annuities, Best Estimate, quantile, Solvency Capital Requirement, modèle interne partiel, simulations dans les simulations, modélisation de l'actif, modèles de taux, asset and liability management.

L'arrivée de la Directive Solvabilité II entraine des changements conséquents dans le système d'évaluation de la solvabilité des entreprises d'assurances. Cette nouvelle règlementation requiert le calcul d'un capital économique, le Solvency Capital Requirement (SCR), visant à renforcer la solvabilité des assureurs afin de protéger au mieux les assurés. Pour calculer ce capital, les assureurs ont la possibilité d'utiliser une formule standard. Cependant cette méthode s'avère peu adaptée et est peu recommandée pour certains types de produits tels que les Variable Annuities. Les variable annuities sont des produits d'assurance vie qui permettent de bénéficier des mouvements favorables des marchés financiers en investissant sur un fond, tout en offrant des garanties en cas de vie ou de décès. Les assureurs commercialisant ce type de produits ne pourront plus utiliser la formule standard pour déterminer leur besoin en capital, c'est pourquoi dans ce mémoire nous présentons l'estimation du capital requis aggrégé pour les risques de taux d'intérêts et actions. La methode utilisée pour cette estimation est appelée "Simulations dans les simulations". Cette méthode est souvent utilisée pour estimer le capital requis représentant l'aggrégation de plusieurs modules ou sous-modules de risques (dans notre cas, l'aggrégation entre le capital pour le risque de taux d'intérêts et le capital pour le risque actions). Ainsi dans ce mémoire nous présenterons l'estimation du capital aggrégé pour les risques actions et taux d'intérêts à l'aide d'un modéle interne utilisant la méthode des simulations dans les simulations, mais nous présenterons également dans notre application le calcul des capitaux requis pour les sous-modules du risque vie en utilisant les chocs donnés par la formule standard.

Executive summary

Keywords:

Solvency II, Variable Annuities, Best Estimate, quantile, Solvency Capital Requirement, partial internal model, nested simulations, asset modeling, interest rate modeling, asset and liability management.

Since 2002, Solvency I is the solvency regime applied by insurance companies. Although the Solvency I regime was tested during the recent crisis, it is being criticized: Solvency I is said to be too simple and does not take into account all the risks borne by the companies. In January 2014, the current system will be replaced by a new regime: Solvency II. The Solvency II Directive represents major changes in the solvency system of insurance companies, and this is the reason why the effective date of this new directive has been postponed several times. To protect policyholders, the Solvency II Directive requires the calculation of a capital ensuring the solvency of the company over a one-year period subject to a confidence level of 99.5%: the Solvency Capital Requirement (SCR). This calculation is complicated and can be performed using a standard formula or a full or partial internal model. However, the use of the standard formula is not recommended for certain types of products since the structure of the standard formula does not specifically fit the risk profile of the company. This is the case for Variable Annuity products. In November 2010, the CEIOPS published the consultation paper n° 83 that alarms the insurers: sellers of variable annuities must prepare to estimate their solvency capital requirement by using an internal model since the standard formula can no more be used for these products. This is the reason why, in this thesis we will estimate the aggregated required capital for the interest risk and the equity risk with an internal model using the nested simulation method for variable annuity products, and more precisely for the in-case-of death guarantee called Guaranteed Minimum Death Benefit (GMDB), and the in-case-of survival guarantee called Guaranteed Minimum Accumulation Benefit (GMAB). We will also present the calculation of the life risk capital by using the shocks given by the standard formula.

Our study is close to the study of a partial internal model. Indeed, a partial internal model is used to calculate one or more risk modules, or sub-modules of the basic SCR, and the standard formula is used for the calculation of the remaining modules. In our study, we have estimated the required capital for market risk [1](noted SCR_{Mkt}) with an internal model and we have estimated the required capital for life risk module by using the standard formula. Indeed, market risk is a significant risk for GMAB and GMDB products and this is highlighted in the consultation paper n° 83.

The SCR_{Mkt} calculation requires the calculation of a quantile 99.5%. To do it, several methods exist but we have chosen the method of nested simulations. This method is based on an economic view of the balance sheet and requires two levels of simulations: a first simulation under real world probability for the first year, and then it is necessary to evaluate the liability of the company at the end of the first year with a second level of simulations under risk-neutral world.

Since variable annuity products are generally invested in the monetary market, bonds, and equities, we have used an interest rate model, and an equity model.

We have presented two equity models: the Black and Scholes model and the Merton model. The Black and Scholes model is very used in finance and has the advantage of providing a closed-form solution for the price of equities. However, the Black and Scholes model under-estimates the tails of distribution and in the context of a quantile calculation, this could lead to significant under-estimation of the SCR. This is the reason why we have used the Merton model. This model introduces a Poisson jump used to model discontinuities in stock prices due to the brutal arrival of good or bad information. The Merton model also provides fatter tails of distribution than the Black and Scholes model.

We have presented two one-factor interest rate models: the Vasicek model and the Cox-Ingersoll-Ross (CIR) model. These models have the advantage to provide closed-form solutions for zero-coupon prices. Nevertheless, the Vasicek model can produce negative interest rates and this is the reason why we have used the CIR model that only produces positive interest rates model only produces positive interest rates in its continuous form, when simulating using the explicit Euler scheme we observe that the CIR process can produce negative interest rates. To solve this problem we have used the implicit Euler scheme that produces only positive interest rates.

Calibration is an important part of the model. Indeed, although a model is theoretically relevant, a bad calibration will lead to a bad prediction quality of the model. We have presented two methods to calibrate the Merton model: the method of moments and the maximum likelihood method. Results show that the second method leads to a larger part of the variance explained by the Poisson jump (about 60%). By comparing the adjustment of the Black and Scholes model and the Merton model to the empirical dis-

[1]Here we talk about the capital for market risk by abuse of language (noted SCR_{Mkt}), but in fact we should call it : the aggregation of the required capital for interest rate risk (SCR_{rate}) and the required capital for equity risk (SCR_{equity})

tribution, we have observed that the Merton model better fits the empirical distribution and also presents fatter tails of distribution. For the CIR model, we have distinguished the risk-neutral calibration and the historical calibration. Under the historical world, we have used historical data to calibrate with the maximum likelihood method. Under risk-neutral world, we have used prices of interest rate instruments, thus we have minimized the distance between the prices of zero-coupon bonds observed on the market and the prices of zero-coupon bonds given by the CIR model. We have observed that even if the CIR model is very convenient to use thanks to the closed-form solutions it provides, a one-factor model has the disadvantage of explaining the whole interest rates curve with just the short rate, which assumes a correlation between the rates of different maturities. Thus the adjustment to the interest rate curve is a limit of our model, and this problem essentially comes from the choice of a one-factor model.

In order to simulate the evolution of interest rates and stock returns in a consistent manner, we have introduced a correlation between the Brownian motion of the interest rate model and the Brownian motion of the equity model. The estimation of the correlation coefficient was made using historical data of the last ten years, between the Euro Stoxx 50 and the OverNight Index Average (EONIA). We have observed that the estimation depends on the choice of data and it appears that the correlation has decreased over time.

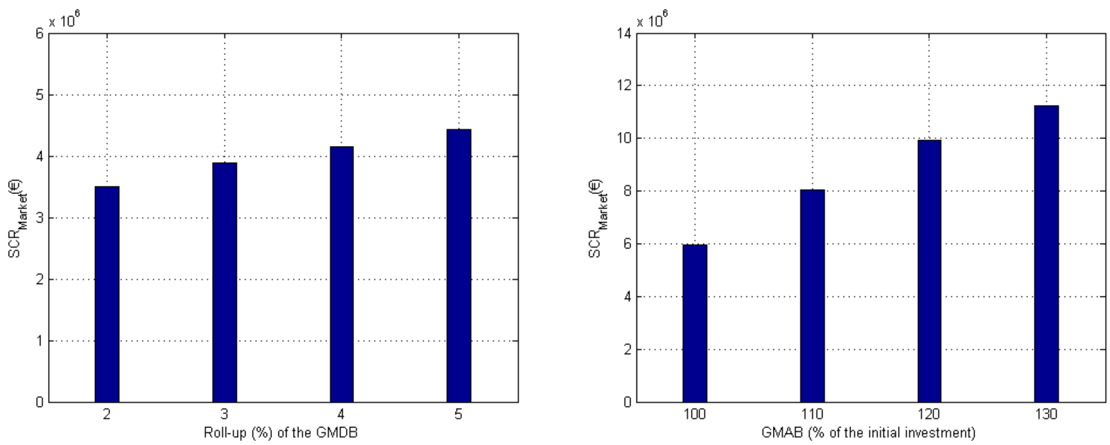
On the liability side, the mortality has been taken deterministic (mortality table TH0002) and the lapse has not been considered. Interactions between the asset and the liability have been modeled with an asset and liability model (ALM). We note, that in order to pay the guarantee of the policyholder, the insurer must retrieve from the fund only the surrender value of the part of the policyholder, and the guarantee is paid with the own capital of the insurer. Every year, the fund is rebalanced as the initial investment risk profile.

The nested simulations concept is very time and storage consuming, thus a particular attention was granted to the optimization of the computer code. We used MATLAB which is well adapted to matrix calculation. It therefore has been preferred matrix calculation, and we have also tried to minimize loops. Using three dimensional matrices has permitted us to store the intermediate values.

In order to illustrate the theoretical part of this thesis, we have made a study of the sensitivity of the SCR_{Mkt} which also permitted us to criticize the results produced by our implementation. First we have checked whether our model satisfies the test of the martingale, and we found a relative error of 0.30%. In order to determine the number of inner scenarios for our example, we have tested the convergence of the Best Estimate at the end of the first year, and we have built a confidence interval for it. We have taken 6000 inner scenarios. Then we have determined the number of outer simulations. To do so, we have calculated the value of the quantile with 5000 outer scenarios (and 6000 inner scenarios), and we have calculated the relative error compared to this value obtained when decreasing the number of outer scenarios. With a relative error of 0.16%, we have taken 4000 outer scenarios. Given the number of simulations, we have begun the sensitivity study of the SCR_{Mkt}. We have considered that the investment is allocated as 50% equities, 20% monetary, and 30% bonds and the maturity of the contract is 10 years.

We have first tested the sensitivity of the SCR_{Mkt} to the in-case-of survival guarantee(GMAB). To do so, we have compared the SCR_{Mkt} obtained depending on the amount of the guarantee expressed in percentage of the initial investment value. Results show that the SCR_{Mkt} increases of about 90% when the guaranteed amount goes from 100% to 130% of the initial amount invested. Then we have tested the sensitivity of the SCR_{Mkt} to the in-case-of death guarantee(GMDB). When the rolled-up guaranteed amount (in case of death) goes from 2% to 5% then the SCR_{Mkt} increases of about 26%. The following graphs illustrate the results:

Figure 1: Impact of the guarantees on the capital requirement for market risk

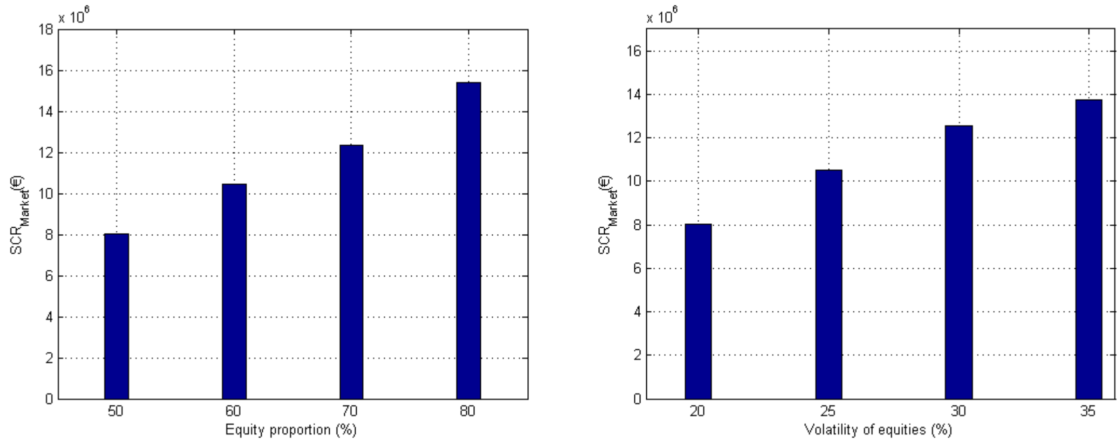


Impact of the GMDB guarantee

Impact of the GMAB guarantee

We have then tested the sensitivity of the SCR_{Mkt} to the allocation of the fund, by varying the part of equities and bonds (the monetary part remains the same). We observe that when the part of equities goes from 50% to 80%, the SCR_{Mkt} increases by about 91%. This brutal variation is explained by an increase in the volatility of the fund when the equity part is increased. In order to illustrate the previous result, we have tested the sensitivity of SCR_{Mkt} to the volatility of the equity part. The SCR_{Mkt} increases by about 70% when the volatility of equities goes from 20 to 35%.

Figure 2: Impact of the investment in equity on capital requirement for market risk



Impact of the proportion of equity

Impact of the volatility of equity

In order to show that our tool is sensitive to the characteristics of policyholders, we have tested the sensitivity of the SCR_{Mkt} to the age of the policyholders. The test of sensitivity was made in the case that all the policyholders own the same contract that includes a GMAB and a GMDB. We have observed that SCR_{Mkt} increases when the age of policyholders increases. This shows that depending on the characteristics of the guarantees, and when the contract includes a GMAB and a GMDB, insurers are exposed to the risk of longevity or to the risk of mortality. In our case, the insurer is more exposed to the risk of mortality as the GMAB guarantee is smaller than the GMDB guarantee which is also paid later.

After performing this sensitivity study, we presented the calculation of the SCR on a simplified case using our model. The capital requirement for market risk, SCR_{Mkt}, was estimated at 8 046 202 €. The required capitals for the life risk module have been calculated using the shocks given by the standard formula. As we have considered that policyholders all own the same kind of contract that includes a GMDB and a GMAB, the capital for the longevity risk sub-module was estimated to be equal to zero. This result is consistent with the study of sensitivity to the age of the policyholder. By applying the mortality shock given by the standard formula, the capital for the mortality risk sub-module was estimated at 162 623€. Similarly, by applying the catastrophe shock we found the catastrophe risk sub-module, estimated at 1 055 921€. After aggregating the life risk sub-modules, we find a capital requirement for the life risk module of 1 107 939€. The EIOPA recommends for partial internal model, to use the correlation matrix of the standard formula to aggregate the risk modules. This is what we did for the life risk module and the market risk module. With the assumptions made on our hypothetical example, the adjustment and the operational risk were equal to zero. Thus we find a SCR estimated at 8 392 034€. We note that in our case where contracts include a GMDB and a GMAB, the market risk represents a major part of the SCR which also justifies the use of a partial internal model to better capture the market risk for these products. After estimating the SCR, we have calculated the risk margin with the simplification using the duration in the technical specifications. This calculation done, we have been able to build the balance sheet within the meaning of the Solvability II Directive, and calculate its solvency ratio.

In this thesis we have presented the calculation of the required capital for interest rates and equity risks by using the method of nested simulations and the calculation of the required capital for the life risk module using the standard formula for GMDB and GMAB products. The results give us a first view of the evolution of the required capital depending on the characteristics of the products. The results show that insurers can reduce their exposure to market risk by decreasing the proportion of the fund invested in equities or by investing in less volatile equities. The importance of market risk for GMAB and GMDB products was highlighted in the application, and this justifies the use of a partial internal model to better assess this risk. We have presented a partial internal model but one possible perspective of this project is to extend our model to a full internal model. However, in order to implement a full internal model, we have presented a method that could reduce the run time and the storage used by the nested simulations: the Least Squares Monte Carlo approach that limits the number of inner scenarios to a single one. Thus as shown in this thesis, means to comply with the Solvency II regulatory framework are heavy and the effective date of the directive is still being discussed. Solvency II is a long run project and it marks the arrival of a new culture of risk.

Synthèse

Mots-clés:

Solvency II, Variable Annuities, Best Estimate, quantile, Solvency Capital Requirement, modèle interne partiel, simulations dans les simulations, modélisation de l'actif, modèles de taux, asset and liability management.

Depuis 2002, Solvabilité I est le régime appliqué par les entreprises d'assurances. Bien que ce système ait pu être testé durant la récente crise, il est aujourd'hui remis en question. Probablement à cause de sa simplicité, Solvabilité I ne permet pas de prendre en compte tous les risques encourus. Ainsi en janvier 2014, le système actuel sera remplacé par un nouveau régime : Solvabilité II. La Directive Solvabilité II représente un changement considérable dans le système de solvabilité des compagnies d'assurances, c'est d'ailleurs pour cette raison que sa date d'entrée en vigueur a été reportée à plusieurs reprises. Afin de protéger les assurés, la Directive Solvabilité II requiert le calcul d'un capital requis destiné à assurer la solvabilité des compagnies à horizon 1 an avec une probabilité de 99.5%: le Solvency Capital Requirement (SCR). Ce calcul est complexe et peut être effectué à l'aide d'une formule standard ou encore à l'aide d'une approche par modèle interne. Cependant la formule standard n'est pas recommandée pour certains types de produits à cause de sa structure rigide et de son manque de souplesse au profil de risque. C'est le cas pour les produits de type Variable Annuities. En novembre 2010, le CEIOPS publie le consultation paper n° 83 qui alarme les assureurs : les vendeurs de variable annuities doivent se préparer à estimer leur capital requis à l'aide d'un modèle interne car l'utilisation de la formule standard ne sera plus autorisée pour ces produits. C'est pourquoi dans ce mémoire nous allons estimer le capital requis aggrégé pour les risques de taux d'intérêts et actions à l'aide d'un modèle interne utilisant la méthode des simulations dans les simulations pour des produits de type variable annuity, et plus particulièrement pour une garantie en cas de décès appelée Guaranteed Minimum Death Benefit (GMDB), et une garantie de capital en cas de vie appelée Guaranteed Minimum Accumulation Benefit (GMAB). Nous présenterons également le calcul des capitaux requis pour les risques du module vie en utilisant les chocs donnés par la formule standard.

Notre étude se rapproche de celle d'un modèle interne partiel. En effet, un modèle interne partiel permet de modéliser le capital requis d'un ou plusieurs modules ou sous modules de risques, puis les autres modules de risques sont calculés à l'aide de la formule standard. Dans notre étude, nous avons estimé le capital requis pour le risque de marché[2] (noté SCR_{Mkt}) à l'aide d'un modèle interne puis nous avons calculé le capital requis pour le module des risques vie à l'aide de la formule standard. En effet le risque de marché associé aux produits de type GMAB et GMDB est un risque très significatif ce qui est d'ailleurs souligné dans le consultation paper n° 83.

Le calcul du capital requis pour le risque de marché requiert le calcul d'un quantile à 99.5%. Pour ce faire, différentes méthodes existent mais nous avons décidé de retenir la méthode naturelle des simulations dans les simulations. Cette méthode est basée sur une vision économique du bilan, et requiert deux niveaux de simulations: des simulations monde réel la première année, puis il est nécessaire de valoriser le passif de la compagnie à un an avec un deuxième niveau de simulations effectuées en risque neutre.

Les produits variable annuities étant généralement alloués dans les marchés monétaires, obligataires et actions, nous avons utilisé un modèle de taux ainsi qu'un modèle action.

Nous avons présenté deux modèles actions: le modèle de Black and Scholes et le modèle de Merton. Le modèle de Black and Scholes est très utilisé en finance et a l'avantage d'être facilement implémentable grâce à la formule fermée qu'il fournit. En revanche, le modèle de Black and Scholes a tendance à sous-estimer les queues de distributions ce qui dans le cadre du calcul d'un quantile pourrait conduire à sous-estimer considérablement le SCR. C'est pourquoi nous avons utilisé le modèle de Merton, qui introduit un saut de poisson afin de modéliser les discontinuités de cours boursiers dues à l'arrivée d'information bonne ou mauvaise sur le titre. Le modèle de Merton présente des queues de distribution plus épaisses que le modèle de Black and Scholes, et reste simple à mettre en pratique puisqu'il fournit une formule fermée pour les cours des actions.
Nous avons présenté deux modèles de taux à un facteur: le modèle de Vasicek et le modèle de Cox-Ingersoll-Ross(CIR). Ces modèles présentent l'avantage de fournir des formules fermées pour le prix des obligations zero-coupon. Néanmoins, le modèle de Vasicek produit des taux négatifs, c'est pourquoi nous avons retenu le modèle de Cox-Ingersoll-Ross qui palie à cette incohérence. Toutefois, bien que le modèle CIR ne produise que des taux positifs sous certaines contraintes en temps continu, la discrétisation de ce processus selon le schéma explicite d'Euler n'assure plus la positivité des taux produits. Ainsi nous avons utilisé le schéma d'Euler implicite qui permet de ne produire que des taux positifs.

[2]nous parlerons de risque de marché par abus de langage, noté SCR_{Mkt}, mais en réalité cette notation fait référence à l'aggrégation du capital pour le risque de taux SCR_{rate} et du capital pour le risque actions SCR_{equity}.

La calibration est une partie importante du modèle interne, car bien qu'un modèle soit pertinent sur le plan théorique, une mauvaise calibration conduira à une mauvaise qualité structurelle et prédictive du modèle. Nous avons présenté deux méthodes de calibration du modèle de Merton: la méthode des moments et la méthode du maximum de vraisemblance. Les résultats montrent que la seconde méthode accorde une part de variance plus importante à la composante poissonnienne, et environ 60% de la variance est expliquée par la composante à saut. En comparant les ajustements du modèle de Merton et du modèle de Black and Scholes, nous avons pu observer que le modèle de Merton est mieux ajusté à la distribution et présente des queues de distribution plus épaisses. Pour le modèle de taux d'intérêt Cox-Ingersoll-Ross, nous avons distingué la calibration risque neutre de la calibration historique. En risque historique, la calibration utilise un historique de données: nous avons calibré notre modèle avec la méthode du maximum de vraisemblance. La calibration risque neutre utilise les prix des instruments de taux observés sur le marché, ainsi nous avons utilisé la méthode des moindres carrés afin de minimiser l'écart entre les prix observés des obligations zero-coupon et les prix théoriques des obligations zero-coupon donnés par le modèle CIR. Nous avons observé que bien que très pratique du fait de la formule fermée qu'il fournit, ce modèle de taux à un facteur présente toutefois l'inconvénient d'expliquer la courbe des taux avec uniquement le taux court, ce qui suppose une corrélation entre les taux des différentes maturités. L'ajustement du modèle à la courbe des taux est une limite de notre étude, mais ceci provient essentiellement du choix d'un modèle à un facteur.

Afin de simuler de façon cohérente l'évolution des taux d'intérêts et des rendements actions, nous avons introduit une corrélation entre les browniens du modèle de taux et du modèle action. L'estimation de ce coefficient de corrélation a été effectuée sur un historique de données de taille 10 ans, l'indice Euro Stoxx 50 et l'Euro OverNight Index Average. En faisant varier la taille de l'historique choisi, nous avons pu remarquer que la corrélation diminue au cours du temps.

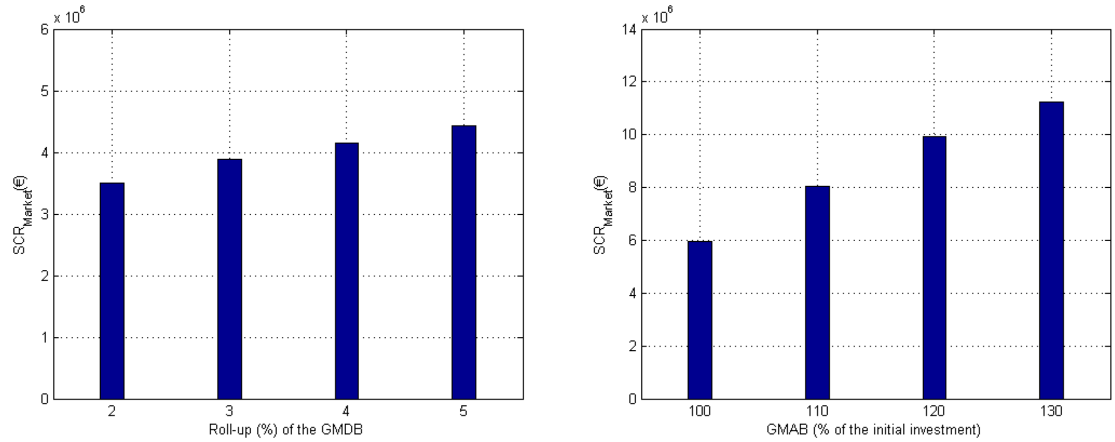
Au passif, la mortalité a été prise déterministe (table TH0002) et le rachat n'a pas été considéré. Les intéractions Actif-Passif ont pu être modélisées à l'aide d'un modèle d'actif-passif, souvent appelé asset and liability model. Il est important de noter qu'afin de payer les garanties contractées par les assurés, l'assureur doit retirer la valeur de la garantie de ses fonds propres, et seule la valeur de rachat de la part du fond des assurés quittant le fond doit être retirée du fond. Chaque année, le fond d'investissement est rebalancé selon le profil de risque initial.

La méthode des simulations dans les simulations est très consommatrice de temps et de stockage, c'est pourquoi une attention particulière a été portée à l'optimisation des procédures. Nous avons utilisé le logiciel MATLAB qui est particulièrement bien adapté au calcul matriciel. Nous avons privilégié le calcul matriciel et nous avons cherché à minimiser les recours aux boucles. L'utilisation de matrices tridimensionnelles a permis de stocker les valeurs intermédiaires.

Afin d'illustrer la partie théorique de ce mémoire, nous avons effectué une étude de sensibilité du capital requis pour le risque de marché qui de plus nous a permis de critiquer la cohérence des résultats fournis par notre implémentation. Au préalable nous avons vérifié que notre modélisation d'actif satisfait au test de la martingale, et nous avons obtenu une erreur relative d'environ 0.30%. Afin de déterminer le nombre de simulations secondaires pour notre exemple, nous avons testé la convergence de l'estimateur du Best Estimate à la fin de la première année et nous avons construit un intervalle de confiance afin de connaitre la précision de nos estimations. Nous avons retenu 6000 simulations secondaires. Ensuite nous avons déterminé le nombre de simulations primaires: pour ce faire nous avons calculé l'erreur relative entre le résultat obtenu en effectuant 5000 simulations primaires, et les résultats obtenus lorsque l'on diminue le nombre de simulations primaires (pour 6000 simulations secondaires). Avec une erreur relative de 0.16%, nous avons retenu 4000 simulations primaires. Le nombre de simulations étant déterminé, nous avons pu commencer à tester la sensibilité du capital requis pour le risque de marché. Nous avons considéré un profil d'investissement correspondant à 50% d'actions, 20% de monétaire et 30% d'obligataire, pour un contrat de maturité 10 ans.

Nous avons d'abord testé la sensibilité à la garantie en cas de vie contractée. Pour ce faire, nous comparons le SCR_{Mkt} obtenu suivant le montant de la garantie exprimé sous la forme d'un pourcentage de la valeur initale de l'investissement. Les résultats montrent que lorsque la garantie en capital au terme du contrat en cas de vie de l'assuré passe de 100% du montant initialement investi à 130%, alors le capital requis pour le risque de marché augmente d'environ 90%. Nous avons ensuite testé la sensibilité du SCR_{Mkt} à la garantie en cas de décès. Lorsque le taux de capitalisation garanti par an en cas de décès (mécanisme roll-up) passe de 2% à 5% alors le capital pour le risque de marché augmente d'environ 26%.

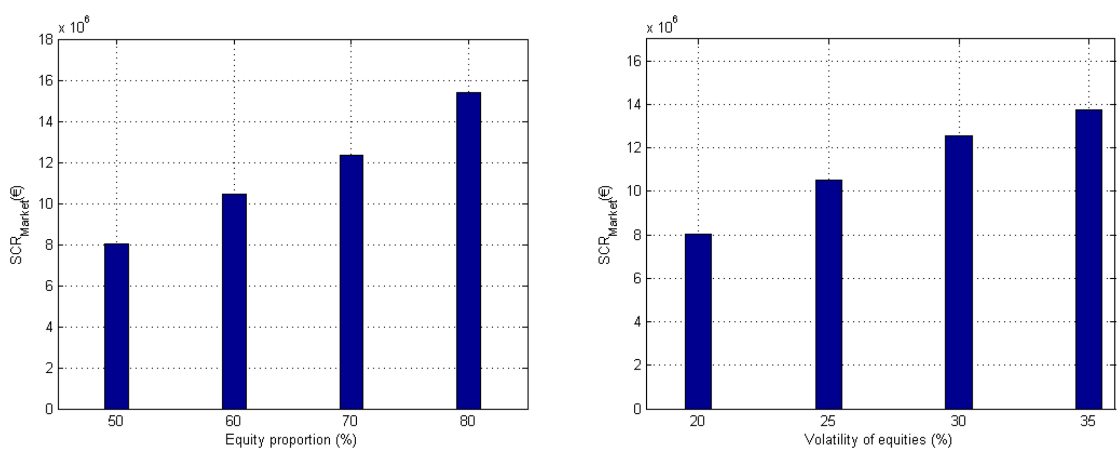
Figure 3: Impact des garanties sur le capital pour le risque de marché



Impact de la garantie GMDB Impact de la garantie GMAB

Nous avons ensuite testé la sensibilité du capital à l'allocation de l'investissement, notamment en faisant varier la part investie en actions et la part investie en obligations (la part investie en monétaire reste fixe). Nous observons que lorsque la part investie en actions passe de 50% à 80%, le SCR_{Mkt} augmente d'environ 91%. Cette forte variation est due à l'augmentation de volatilité du fond lorsque l'on augmente la part action. Afin d'illustrer les résultats précédents, nous avons testé la sensibilité du capital à la volatilité de la part action du fond. Lorsque la volatilité de la part action passe de 20% à 35%, le SCR_{Mkt} subit une augmentation d'environ 70%. Les schémas ci-dessous illustrent les résultats obtenus:

Figure 4: Impact de l'investissement en actions sur le capital pour le risque de marché



Impact de la proportion action Impact de la volatilité Action

Afin de montrer que notre outil est sensible aux caractéristiques des assurés nous avons testé la sensibilité du capital à l'âge de l'assuré. Nous avons effectué cette sensibilité dans le cas où tous les assurés ont souscrit à une garantie GMAB et à une garantie GMDB. Nous avons remarqué que le besoin en capital augmente quand l'âge de l'assuré augmente, ceci montre que l'assureur est suivant les caractéristiques des garanties, plus exposé au risque de mortalité ou au risque de longévité. Dans notre cas, l'assureur est plus exposé au risque de mortalité car la garantie GMAB est plus petite que la garantie GMDB et de plus elle est payée plus tard.

Après cette étude de sensibilité, nous avons présenté le calcul du SCR à l'aide de notre modèle interne partiel. Le SCR_{Mkt} a été estimé à 8 046 202€. Les capitaux pour les risques du module vie ont été calculés à l'aide des chocs fournis par la formule standard. Etant donné que nous avons considéré que les assurés disposent tous du même type de contrat qui comprend une garantie en cas de décès et une garantie en cas de vie, le risque de longévité a été estimé nul ce qui est cohérent avec les résultats obtenus lors du test de sensibilité à l'âge des assurés que nous avons réalisé précédemment. En appliquant le choc de mortalité de la formule standard, le capital requis pour le sous-module mortalité a été estimé à 162 923€. De la même façon, en appliquant le choc catastrophe de la formule standard nous avons trouvé un besoin en capital pour le risque catastrophe de 1 055 921€. Après agrégation des risques du module vie, le capital requis pour le module vie est estimé à 1 107 939€. Afin d'agréger les différents modules, l'EIOPA recommande l'utilisation de la matrice de corrélation fournie par la formule standard ce que nous avons appliqué pour le module des risques vie et le module risque de marché. Avec les hypothèses de notre cas fictif, l'ajustement et le module de risque opérationnel sont nuls ce qui estime le SCR à 8 392 034€. Nous remarquons que dans notre exemple de contrats comportant des garanties GMAB et des garanties GMDB, le risque de marché représente la majeure partie du risque encouru par l'assureur ce qui justifie l'utilisation d'un modèle interne partiel pour mieux appréhender ce risque pour ces types de produits. Une fois le SCR calculé nous avons pu déterminer la marge de risque telle qu'elle est définie dans les spécifications techniques en utilisant la simplification proposée qui utilise la duration. Ce calcul effectué, nous avons pu reconstituer le bilan économique au sens de Solvabilité II et calculer le ratio de solvabilité qui lui est associé.

Ainsi dans ce mémoire nous avons présenté le calcul du capital pour les risques actions et taux à l'aide de la méthode des simulations dans les simulations, et le calcul des capitaux pour le risque vie à l'aide de la formule standard, pour les garanties GMAB et GMDB. Les résultats obtenus nous ont donné une première vision de l'évolution du capital requis en fonction des caractéristiques des produits. Nous avons montré que les assureurs peuvent réduire leur exposition au risque de marché en diminuant la part investie en actions ou en investissant sur des actions moins volatiles. L'importance du risque de marché pour les produits de type GMAB et GMDB a été mise en évidence lors de l'application, ce qui justifie l'emploi d'un modéle interne partiel pour mieux appréhender ce risque. Nous avons présenté un modèle interne partiel simplifié mais une des perspectives de ce projet est d'étendre notre modèle à un modèle interne total. Néanmoins, pour implémenter un modèle interne total, et pour qu'il puisse être utilisé rapidement pour des prises de décisions par exemple, nous avons présenté une méthode qui permettrait de réduire le temps de calcul et l'espace utilisé par la méthode des simulations dans les simulations: le Least Squares Monte Carlo qui réduit le nombre de scénarios primaires à un seul scénario. Ainsi ce mémoire a montré la lourdeur des moyens à mettre en place pour satisfaire à la réglementation Solvabilité II dont la date d'entrée en vigueur est toujours discutée. Solvabilité II est un projet de long terme qui marque l'arrivée d'une nouvelle culture de risque qui devra être adoptée par les assureurs.

Acknowledgment

This thesis would not have been possible without the guidance and the help of several individuals, and I would like to thank them in some words:

I would like to express my gratitude to Alan PICONE and Thierry FLAMAND for offering me this interesting internship. I would like to thank the Actuarial team for their support and their interest in this thesis all over this period.

This project would not have been possible without Olga GORNUSHKINA who was clearly involved in this project and who guided me in my research.

I also want to thank Nicolas BARETH for his advice on the implementation of the project but also Li YAN who advised me on the theoritical part of this project.

My sincere thanks also go to Frédéric PLANCHET and Matthieu CHAUVIGNY for responding to my emails in record time.

I also thank my schoolmate and colleague, Johan ROGEZ, who was a real support during this period.

School-year end is the right moment to thank all the teachers and all the members of the EURIA for the training I received at the school.

Grateful acknowledgement is made to Sophie BOURDET and Fabrice HAMON, who gave me good advice at crucial moments.

I thank Jorge VIGIL for his support during this internship and for his valuable observations after critically reading this document.

I thanks my family for supporting me during my training at the EURIA.

Last but not least, special thanks go to my fiancee who has been a great support during all three years.

Contents

Introduction

The regulation of the insurance sector will experience substantial changes in the near future. Although the current system known as the Solvency I system has shown efficiency in the recent crisis, it does not take into account all risks borne by insurers. Thus, a new directive will replace it in January 2014: the Solvency II Directive. This new regulation is a real change in the solvency assessment process, and to prepare insurers for meeting the requirements, the EIOPA and the European Commission submit Quantitative Impact Study completed with consultation papers. The EIOPA produces standards but it is revealed that for some products those standards are inappropriate. This is the case for certain life insurance products such as Variable Annuities. Variable Annuities are unit-linked products with additional death or survival guarantees that also give beneficiaries the opportunity to benefit from up-movements of financial markets. However, risks associated to these products are complicated. To meet its obligations, the Solvency II Directive requires insurers to calculate a required capital, the Solvency Capital Requirement. With the aim of protecting policyholders this capital must be fairly estimated and the standard formula proposed to simplify the calculation seems inappropriate to variable annuities. In november 2010, the EIOPA published the consultation paper n° 83 that has alarmed the insurers: variable annuity sellers should prepare themselves to assess their solvency capital requirement with an internal model otherwise they could not be allowed to sell their products. Thus, the objective of this thesis is to estimate the required capital for interest rates and equity risks with an internal model using the method of nested simulations for a company selling variable annuities featuring the accumulation (GMAB) and the death (GMDB) guarantees. We will also illustrate the calculation of the required capital for the life risk-module using the standard formula.

This thesis consists of several parts. First, we explain the change in the regulation for required capital by comparing the current framework and the Solvency II framework. Then, we introduce variable annuity products and their characteristics. In the third part, we present our calculation structure that includes calculation with an internal model, as well as the method of nested simulations on which our model is based. The next chapter presents in detail the economic scenario generator and the whole process of assets modeling. Then we explain the way we modeled the liabilities' cash-flows and calculated the Best Estimate. In order to illustrate the theoretical part presented so far, we make a sensitivity study of the capital requirement for market risk, and we finally present a calculation of the solvency capital requirement with our partial internal model.

Before concluding, we present the perspective of the project through the Least Squares Monte Carlo concept which is an alternative method of the full nested simulations.

Chapter 1

From Solvency I to Solvency II

Introduction

This first chapter aims to present the evolution of the regulatory context in the insurance sector. After briefly describing the current solvency regime, we present the solvency II Directive detailing its actors, its pillar structure and the changes in the balance sheet in particular with the new definition of capital requirements.

1.1 Solvency I

Since 2002, Solvency I is the solvency regime applied by the European Union's insurers. The purpose is to maintain efficient and safe insurance market for the benefit and protection of policyholders. Following this reform, insurers have to conduct a review of their solvency requirements governed by three prudential principles:

1. Companies must maintain at all times sufficient technical provisions to meet the commitments towards the clients. As a prudential rule, technical provisions amount should be higher than the expected amount of claims.

2. Investment must be done having regard to safety and returns. The assets must be diversified and adequately spread.

3. A sufficient amount of own capital is essential to protect against future losses: this surplus is called the solvency margin. In life insurance, the solvency margin is calculated as a fixed ratio of technical provisions and of the sum at risk. For example, in life insurance 0.3% of the sum at risk and 4% of mathematical provisions or 1% for unit-linked business. The Authorities require a minimum level of margin, under which they can take emergency measures necessary for the

well-being of the company. Those measures can result in a refinancing operation, in a capital increase, or in a reinsuring plan that decreases the required capital.

1.2 The limits

Through the above description, it is notable that a certain importance has been granted to the simplicity of the approach. Allowing for simplicity is a great value especially for the compliance costs of both implementing and maintaining compliance following its implementation. The fixed percentage method used for the calculation of solvency margin is easy to understand, and brings transparency. Although we haven't observed ruins during the last crisis in the insurance sector, simple reasoning and case based reasoning lead to founded criticism of the current system.

Solvency I exclusively deals with the quantitative aspect and does not provide incentive for the qualitative aspect, abandoning risk management.

The quantitative approach uses formulas based on provisions of the accounting year. This practice is not consistent with the forward-looking idea. Moreover, the prudential rule translated in the formulas for the regulatory margin results in an overestimation which penalizes not only the shareholders but also the clients of the insurance company especially in the context of competition. Also, the risk margin calculation does not take into consideration the market risk: if we assume two insurance companies with the same liabilities but with different assets composed of 100% equities for the first one and 100% bonds for the second one, then the solvency margin amount will be the same under Solvency I. No distinction is done between the different risks, only the subscription risk impacts the margin. The simplicity of Solvency I leads to unaffordable and unacceptable mistakes, and a simple formula cannot match every insurance risk profile. This current system is inconsistent with the market.

National regulation systems are quite different across Europe and technical provisions calculations can differ widely from one country to another, so it is an uneven playing field across Europe and this creates competitive arbitrage. In a European context, characterized by competition but also by a unique market, regulation must be unified. Today's challenge is not only about risk quantification, but also about risk management. Regarding all these lacunas, the European Commission has tried to work out a new system: "Solvency II".

1.3 Solvency II

Nominally fully operational in 2014, Solvency II is a mandatory regulation that will affect all insurers and reinsurers in the European Union. Solvency II's primary objective is to strengthen policyholders' protection as well as the competitiveness of the insurance sector and the wider Europe economy by aligning capital requirements more closely with the risk profile of European companies. Indeed, contrary to Solvency I, Solvency II is not only about technical calculation of capital reserves, but also about each company's approach to risk management. This new directive exemplifies the current trend in solvency regulation towards comprehensive risk-and-economic based regulatory regimes. With regards to compliance, Solvency II represents the biggest ever exercise in establishing a single set of rules governing insurer creditworthiness and risk management.

1.3.1 The actors

In order to build this new revolution in the whole insurance industry, many different-skilled actors have worked on the project:

- The European Commission coordinates and drives the project by transmitting directives, recommendations and tests.

- European Insurance and Occupational Pensions Authority (EIOPA), composed of high-level representatives from the insurance and occupational pensions supervisory authorities of the European Union's Member States, advises the Commission in particular with technical aspects, and with the consistency implementation of Solvency II.

- The European Insurance Federation (formerly the CEA), has the objective to adapt the regulation with the development of the European insurances.

- National authorities must ensure the good implementation of the directive in their respective country. It also provides quantitative impact studies to the insurance companies. In Luxembourg, the national authority is called "Commissariat aux Assurances".

- Professionals from the insurance sector are called for advice by the EIOPA and the European Commission.

1.3.2 The pillar structure

Built on a Lamfalussy process and widely inspired by the bank regulation Basel II, Solvency II is structured around three guiding principles which cut across market, credit, liquidity, operational and insurance risks.

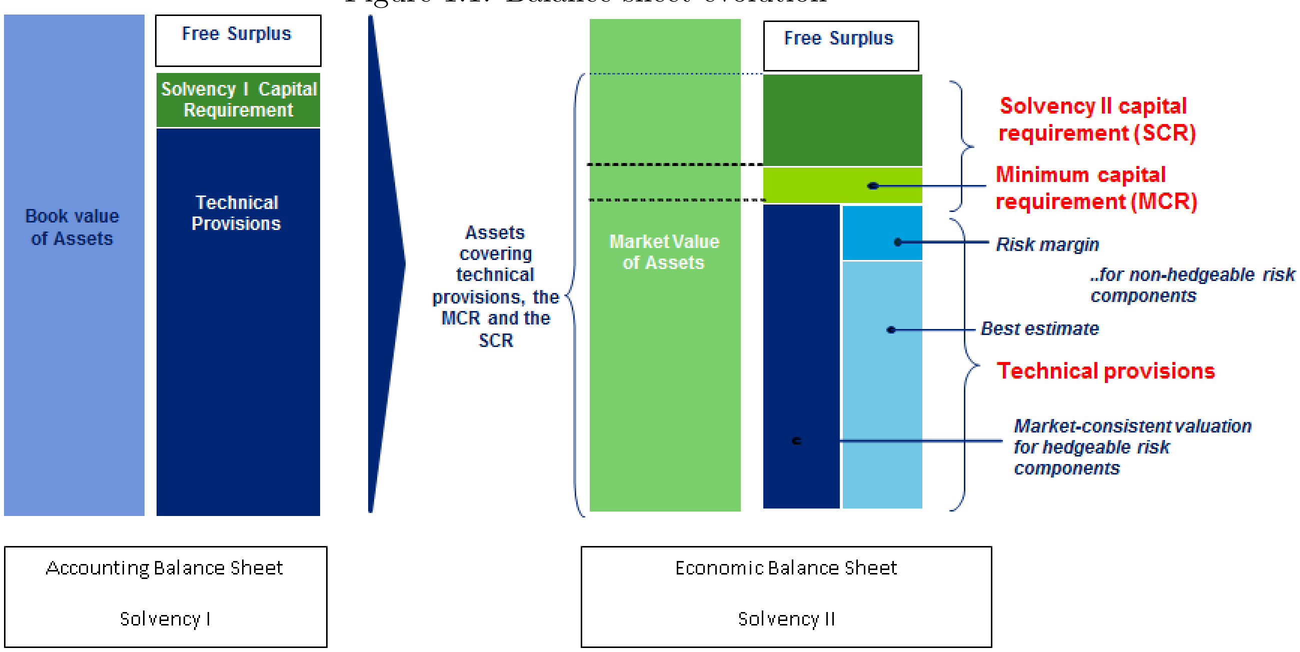
Pillar I :

Pillar I is the quantitative component of the new regulation. It deals with the capital requirements of insurers wishing to provide coverage in the European Community market.

MCR and SCR

Solvency II contains two levels of capital requirements: the Solvency Capital Requirement (**SCR**) and the Minimum Capital Requirement (**MCR**). The MCR is the minimum under which the company is technically insolvent, and the risk considered is unacceptable for policyholders. Authorities will automatically intervene and take measures to solve the situation. The SCR is the new solvency target level for firms: Article 101 of the directive requires that "*the SCR shall correspond to the Value-at-Risk of the basic own funds of an insurance or reinsurance undertaking subject to a confidence level of 99.5% over a one-year period*". Pillar I presents some innovations in the balance sheets. Both assets and liabilities are to be fair-valued (market value of assets and liabilities). A risk margin (market value margin) is to be added to the fair value of the liabilities, also called the Best Estimate, to give the technical provisions.

Figure 1.1: Balance sheet evolution[1]



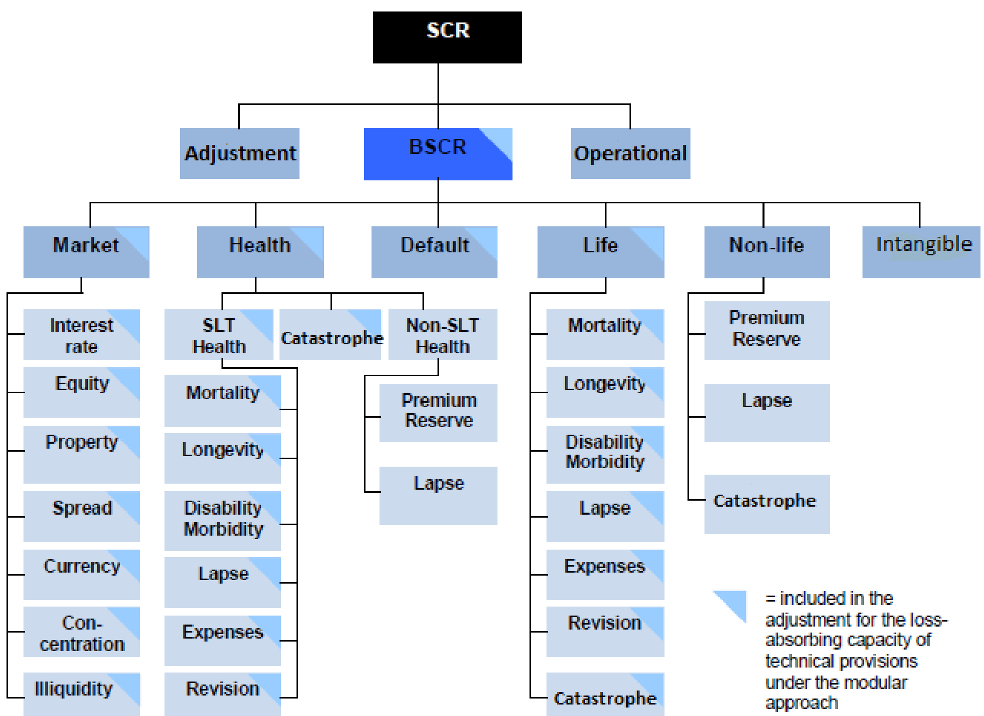
To comply with Solvency II, the SCR must be calculated annually but in theory it should be calculated as soon as the risk profile of the company has changed significantly. The SCR must be reported to the supervisor and published. Insurance companies can use the standard formula or they can demonstrate to the regulator their ability to quantify their risk exposure, and meet the requirement with a full or partial internal model. Considering the large resource commitment from both staffing and financial perspectives, some companies such as small to mid-size players are generally expected to use the standard formula while large global players are typically opting for the internal

[1]Sources : Internal training within Deloitte

model approach that best fits the risk profile.

The standard formula is a simplified approach for capital requirement calculation. It is a common methodology for European insurance companies based on a modular approach. The respective merits of a "top-down" versus a "bottom-down" method were considered by the CEIOPS, but for practical reasons the "bottom-down" or the so-called "modular" approach was chosen. Indeed, a modular approach allows to test a number of different modeling treatments for the same risk-module (composed of several sub-modules). The following picture illustrates the overall structure of the standard formula.

Figure 1.2: Overall structure of the SCR[2]

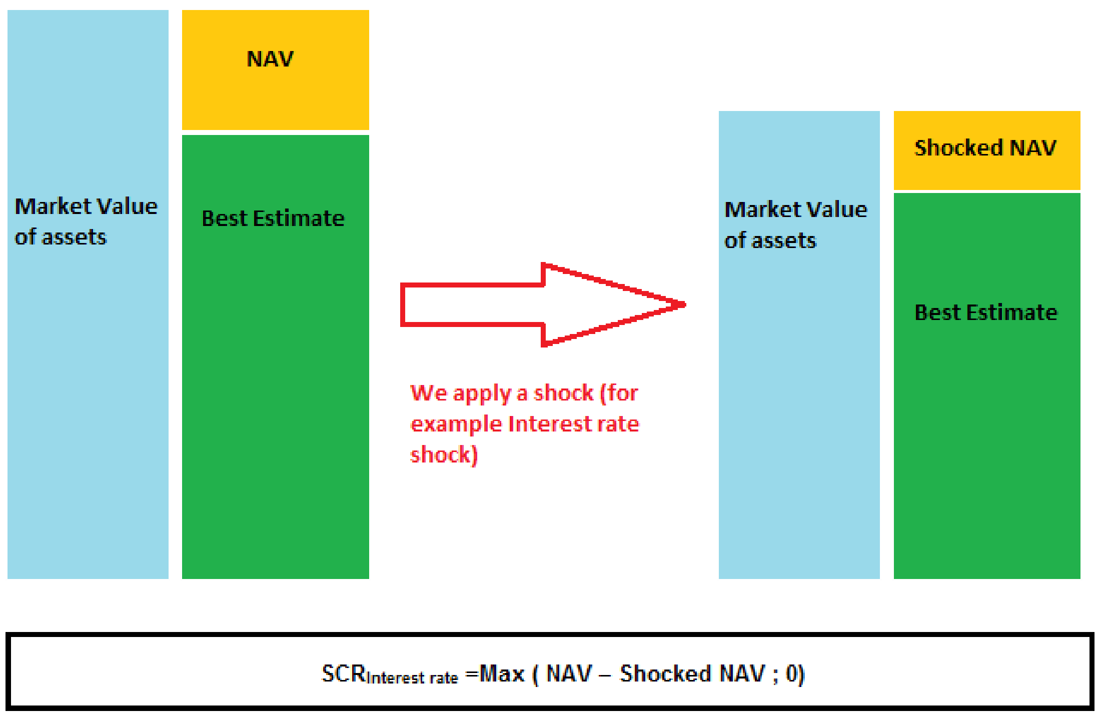


The standard approach is said to be "factor-based" which means that coefficients are assigned to distinguish the different risks. The specifications of the calculation are given in the **Technical Specifications of the QIS5**. All these specifications have been adjusted in cooperation with the professional world. Correlation matrices are given to aggregate between the risk sub-modules and across the risk modules. The Basic Solvency Capital Requirement (BSCR) is the sum of the aggregated modular SCR and the intangible SCR. The mechanism to calculate the sub-modules SCR is based

[2]Sources : Technical Specifications of the QIS 5 , Article SCR.1.1.,[7]

on the calculation of the **Net Asset Value**[3] after applying a shock. This method is generally called the Delta NAV method.

Figure 1.3: Delta NAV method



Even if the standard formula is an option proposed by the authorities to comply with the pillar I requirement, all companies should ensure that the standard approach is a good fit. Indeed, as it is a standard formula, a large proportion of the calibration results relies heavily on "expert opinion", judgment, and subjectivity. And of course, this is also a limit of a standard approach.

[3]NAV = Asset - Liability

Pillar II: System of Governance

Robust governance requirements being a pre-requisite for an efficient solvency system, the pillar II deals with qualitative requirements and the prudential supervision. Insurance companies are requested to comply with a set of requirements:

- The fit and proper rules
- The existence of a proper risk management system
- The Own risk and Solvency Assessment (ORSA)
- The internal control framework
- The role of internal audit, the actuarial function and risk management function

As a key part of pillar II, the ORSA is a strategic analysis process requiring a multi-functional approach that links together the outputs of risk, capital and strategy planning. Risk management information is essential for success, and insurers will need to measure, report and communicate key risk performance metrics. The aim of the ORSA is to facilitate the links between the pillar I capital requirement and the pillar II risk assessment. This will demand a lot of effort and investment, but it's a long run project and those who will embrace the ORSA, will get a return on the investment in the future, especially in terms of stakeholders' expectations.

Pillar III: Supervisory reporting and public disclosure

Even if Pillar I and II are the primary focus, it is important not to lose sight of the pillar III disclosure. This pillar requires to submit to the supervisory authorities the information which is necessary for the purposes of supervisions, but also to publicly disclose on an annual basis, a report on their solvency and financial condition.

Enhancing disclosure requirements will help to increase market transparency, which is important to compare one insurer to another. Supervisors can use this important source of information to conduct the supervisory review process and evaluate risk profiles.

1.3.3 The internal model

The Solvency II Directive gives organizations the opportunity to develop and use internal models for calculating and reporting their solvency capital requirements to the supervisor. There is no exact definition of the internal model and this allows for flexibility in the development. The internal model is the collection of processes, systems and calculations that together quantify and rank the risks faced by the business. An internal model quantify risks and can therefore replace parts (partial internal model), or even all the standard formula. Implementing an internal model requires companies to understand the specific risk of their portfolios. The internal model must be able to provide an appropriate calculation of the SCR, but it must also be an integrated part of the undertaking's risk management process and system of governance. As every model, the internal model must satisfy some tests and requirements:

- Statistical quality standards
- Calibration standards
- Profit & loss attribution
- Validation standards
- Documentation standards

Thus, the internal model is an important but expensive project for companies. However using an internal model will give a better estimation of their solvency capital requirement, and this is a real asset for the management of their risks. Estimating the solvency capital requirement with an internal model generally leads to a smaller value than if the standard formula had been used (there are exceptions). Thus, companies will have more assets available for their activities than if they had used the standard formula. Moreover, the internal model as it leads to a better understanding of the risk of the company, will be a real decision making tool, and will definitely be an asset to develop competitive advantage. Although the implementation of an internal model is an expensive investment for companies, the better risk vision that it provides will be profitable on the long run. The internal model represents the philosophy of the Solvency II Directive: implementing Solvency II requires understanding of several concepts relating to governance, risk management and capital setting.

Chapter 2

Variable Annuities

Introduction

In this chapter, we present variable annuity products. In a first step it is important to define and describe those products. Then, we focus on the GMDB and the GMAB products which are respectively an in case of death guarantee and an in case of survival guarantee. We present the risks associating to these products and we also present a method that could be used for their pricing. Finally, we talk about hedging and reinsurance.

Introduced in the middle of the twentieth century in the United States of America by AXA, Variable Annuities is the US term to describe unit-linked products with secondary guarantees. Variable annuities are basically unit-linked contracts with additional guarantees. Often used for retirement products, the structure is also used for any life product. This product knows a great success these times. Indeed, in the context of perturbed financial market and retirement system changes, people need guaranteed and performing investments. Variable annuities can offer them those guarantees. They are known as Guaranteed Minimum x Benefit (GMxB).

2.1 Unit-Linked Products

Unit-linked product is a type of life insurance where the cash value of a policy varies according to the current value of the net asset value of the underlying investment assets. The premium paid is used to purchase units in investment asset chosen by the policyholder. The investment risk is borne exclusively by the policyholder.

2.2 The GMxB

As the investment risk with unit-linked products is exclusively borne by policyholders, insurers have added guarantees to these products for attractiveness reasons. A guarantee minimizes the loss under the conditions of the contract. If the insured situation happens (death, ...) then the beneficiary receives:

$$max(G_t; FV_t)$$

Where G_t is the value of the guarantee at t and FV_t the value of the fund at t.
In order to calculate the guaranteed amount at time t, G_t, different mechanisms exist:

- **Premium Return:** the guaranteed capital is equal to the maximum between the initial investment and the value of the fund when the guarantee is applicable.

- **The Roll-up guarantee:** the capital guaranteed is the greater between the initial investment value increased at a specified rate of interest and the value of the fund when the guarantee is applicable.

- **The Ratchet or contract anniversary value:** the guaranteed amount is equal to the greater of the contract value at guarantee application time, or premium payments, or the contract value on a specified date. The specified date is generally the contract anniversary dates.

Guarantees can be grouped into two types of guarantees: In-case-of-death and in-case-of-survival guarantees. The most famous guarantees are known as GMxB for Guaranteed Minimum x Benefit. These guarantees form a toolbox that responds to the different needs of the client.

2.2.1 Guaranteed Minimum Income Benefit (GMIB)

This guarantee provides the owner with a base amount of lifetime income after a given period regardless of how the investments have performed. This guarantee only takes effect if the owner decides to annuitize the contract.

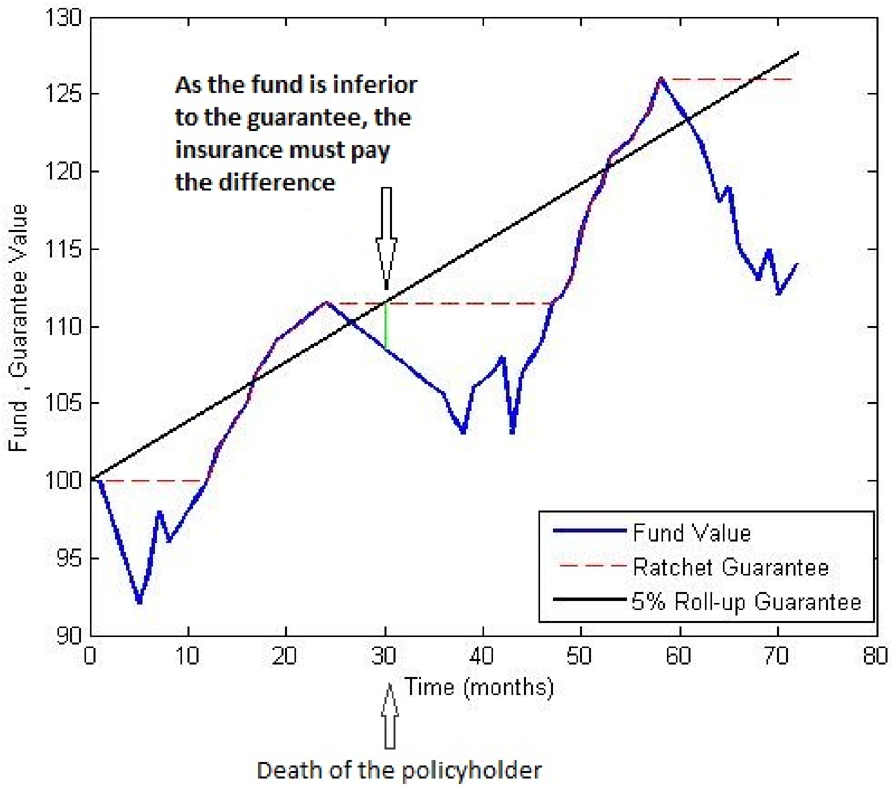
2.2.2 Guaranteed Minimum Withdrawal Benefit (GMWB)

It guarantees that a certain percentage of the amount invested can be withdrawn periodically (more generally annually) until the invested amount is recovered, regardless of market performance. A lifelong version of this product exists (GLWB).

2.2.3 Guaranteed Minimum Death Benefit (GMDB)

This guarantee ensures a payment to a named beneficiary if the contract owner dies in a given period. The GMDB is generally bought by investors who want to invest in the stock market which gives them high returns since they know that their families will be protected against financial loss in the case of an early death. The guaranteed amount is generally a ratchet or a roll-up guarantee. The following picture illustrates a case.

Figure 2.1: Guarantee Minimum Death Benefit



In case of death at time t, the beneficiary receives:

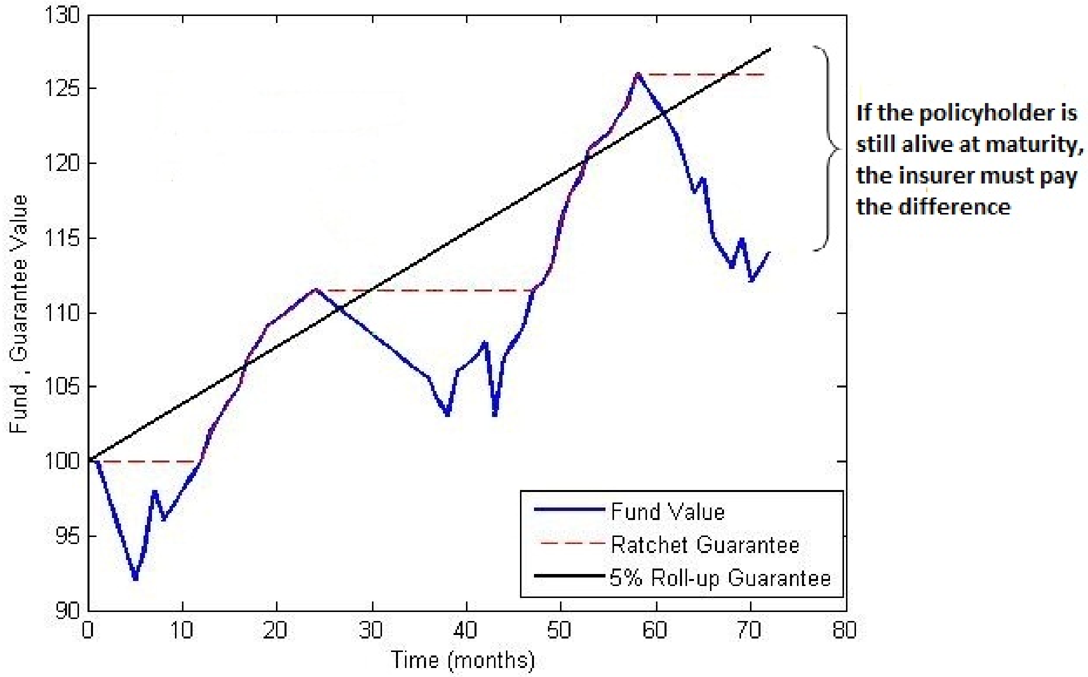
$$\max(FV_t; G_t)$$

where FV_t is the fund value at time t and G_t is the guaranteed amount of the GMDB.

2.2.4 Guaranteed Minimum Accumulation Benefit (GMAB)

This rider is nearly the same as the GMDB but it is the in-case-of-survival guarantee. Indeed, it ensures the owner to receive a minimum percentage of the amount invested after a given number of years (typically the maturity of the contract) regardless of the actual investment performance.

Figure 2.2: Guarantee Minimum Accumulation Benefit



At the applicable date of the guarantee, if the policyholder is still alive, he receives:

$$\max(FV_T; G_T)$$

Where FV_T is the surrender fund value at T (typically the maturity of the contract) and G_T the guaranteed amount of the GMAB.

2.3 Risk mapping

The risk mapping consists in identifying the risks associated to the contract. Variable annuities associate insurance risks and market risks. In the following table, we identify the risks arising from a contract including a GMDB and a GMAB:

<table>
<tr><th>Risk / Contract</th><th colspan="6">Market Risk</th><th colspan="4">Insurance Risk</th><th rowspan="2">Default Risk</th></tr>
<tr><th></th><th>Illiquidity</th><th>Rate</th><th>Currency</th><th>Spread</th><th>Concentration</th><th>Equity</th><th>Mortality</th><th>Longevity</th><th>Lapse</th><th>Expense</th></tr>
<tr><td>GMDB</td><td colspan="6">✓</td><td>✓</td><td></td><td>✓</td><td>✓</td><td>✓</td></tr>
<tr><td>GMAB</td><td colspan="6">✓</td><td></td><td>✓</td><td>✓</td><td>✓</td><td>✓</td></tr>
<tr><td>GMAB + GMDB</td><td colspan="6">✓</td><td>✓</td><td>✓</td><td>✓</td><td>✓</td><td>✓</td></tr>
</table>

Table 2.1: Risk Mapping

The market risk is composed of several risks:

- Illiquidity premium risk arises from the risk of increase of the value of technical provisions due to a decrease in the illiquidity premium. The illiquidity premium is generally represented by an increase in the discount rate applied in the calculation of the present value of future payments under certain (long-term) insurance contracts.

- The interest rate risk is the impact of fluctuations in interest rate on the insurance's profitability. As an example, when interest rates rise, new bonds are sold with higher yields than older bonds, so the price of the older bonds go down. In other words, if the company has to sell bonds before maturity, it may be worth more or less than it paid for it depending on the movement of the interest rates.

- Currency risk arises from changes in the level or volatility of currency exchange rates.

- Spread risk results from the sensitivity of the value of assets, liabilities and financial instruments to changes in the level or in the volatility of credit spreads over the risk-free interest rate term structure.

- Concentration risk is the risk of concentrating the investment in a particular stock, or particular sector. Indeed we cannot predict which sector (or asset) will perform. This is the reason why it is highly recommended to well-diversify the portfolio.

- When the investment contains equities, a risk arises from the possible depreciation of this investment. Exposure to equity risk refers to all assets and liabilities whose value is sensitive to changes in equity prices.

For GMAB and GMDB, the insurance risk is composed of:

- Mortality risk is associated with insurance obligations where an insurance undertaking guarantees to make a single or recurring series of payments in the event of the death of the policyholder during the policy term. In our case, this risk is associated to the GMDB.

- Longevity risk is associated with insurance obligations where an insurance undertaking guarantees to make a single or recurring series of payments in the event of the survival of the policyholder during the policy term. In our case, this risk is associated to the GMAB.

- Lapse risk arises from unanticipated (higher or lower) rate of policy lapses, terminations, changes to paid-up status (cessation of premium payment) and surrenders.

- Expense risk arises from the variation in the expenses incurred in servicing insurance.

The counterparty default risk module should reflect possible losses due to unexpected default, or deterioration in the credit standing, of the counterparties and debtors of undertakings.

2.4 Pricing

The pricing of GMAB and GMDB is similar to the pricing of an option. In this section, we briefly present a possible method of pricing for a contract associating a GMDB and a GMAB.

This method is based on the pricing principle that the price of the guarantee is equal to the expected present value of future costs of the contract.
To simplify the explanation, we consider a contract in which the death guarantee and the accumulation guarantee are fixed (respectively K_1 and K_2). We also consider that the insured person dies at the end of a year. The beneficiary is ensured to receive the maximum between the guarantee and the surrender value. Thus, the insurer guarantees the following payment at t:

$$Max(G_t;FV_t) = FV_t + \underbrace{Max(0;G_t - FV_t)}_{\text{financial risk borne by the insurance}}$$

where FV_t is the surrender fund value at t and G_t is the guaranteed amount at t.
The insurer will pay K_1 if the fund has under-performed, and if the owner dies (for the death guarantee). Similarly, the insurer will pay K_2 if the fund has under-performed at maturity, and if the policyholder is still alive at the maturity of the contract. So in order to quantify the risk, we have to multiply the sum at risk by the probability that the insurer pays which is for each year the probability of death of the policyholder, and for the maturity of the contract, the probability that he is still alive.
The idea is to observe that $Max(0;G_t - FV_t)$ is a put option of strike G_t maturing at t. If we take a contract maturing in N years, a pricing method can be:

$$P = \left[\sum_{i=1}^{N} \frac{Max\left(0;K_1 - FV_i\right)}{(1+r)^i} \times \ _{1/i-1}q_x\right] +_N p_x \times \ Max\left(0;K_2 - FV_N\right)$$

where

- FV_i is the surrender fund value at the end of the year i.
- r is the risk-free interest rate
- $_{1/i-1}q_x$ the probability that a person of age x dies between i-1 and i.
- $_np_x$ the probability that a person of age x is still alive after n years.

We note that this method is rather practical in the case that the fund follows a dynamic like the Black and Scholes dynamics which provides a closed-form solution for the price of put options. However, the fund is generally composed of more than one type of assets and that does not always allow for a closed-form solution. The insurer must then use Monte Carlo estimation.

2.5 Hedging strategy

Hedging strategy can be used to reduce the risk of the insurance. For variable annuities, in order to reduce the risk, the insurer can hedge the financial risks.

Financial Hedging strategy
Sellers of variable annuities can use hedging strategies for the financial risk. Those strategies consist in using the method of replicating portfolio. The seller of variable annuities usually takes positions on the underlying asset or more precisely on options.

In a perfect case, the aim of the replicating method is:

$$\textbf{Mathematical Provisions at t} + \textbf{hedging portfolio at t} = \textbf{guarantee at t}$$

and the aim is that at t+1 the equality is still verified:

$$\textbf{Mathematical Provisions(t+1)} + \textbf{hedging portfolio(t+1)} = \textbf{guarantee(t+1)}$$

In order to realize this kind of hedging strategy, the seller needs to match the sensitivity of the portfolio and the sensitivity of the undertakings. For this, financial Greek letters are used. The most famous example is the delta neutral strategy. The letter delta represents the sensitivity of the option (here a put option) due to the variation of the price of the asset. The delta gives the quantity of the underlying asset to buy to cover a position in call options. Thus the delta neutral strategy means that the overall value of the portfolio will not change for small changes in the price of its underlying asset. Thus, the seller of call options will have a delta neutral position by buying delta equities for each option sold. In the case of GMAB and GMDB contracts, we have shown that the insurers is selling put options. So in order to cover, the insurer can have a delta neutral position in put options. Then similarly, this short position in put options (the position of seller) can then be covered by a short position in the underlying asset.
However, as the delta varies with the value of the underlying asset, the delta neutral strategy has to be done continuously, and this implies transactions fees. So as to reduce the number of transactions, a threshold is usually established in order to determine when the portfolio has to be modified (and to avoid modifying it too often). Other hedging strategies can be used going through the greek letters but we will not present them.

Reinsurance

Reinsurance is the insurance of insurance companies. Reinsurance contracts are generally used to hedge against larger claims. Reinsuring contracts can decrease the risk partially or fully.
In the case of GMDB and GMAB contracts, the insurer can transfer its risk fully or partially to a reinsurer . If the risk is transferred fully then the insurer is just the manager of the contracts and the risk is borne by the reinsurer. However it is important to note that the insurance is exposed to the risk of default of the reinsurer, and if the reinsurer goes bankrupt then the insurance have to bear the guarantee sold. In the context of the recent crisis, this risk is not to be under-estimated.

Chapter 3

Capital calculation with an internal model

Introduction

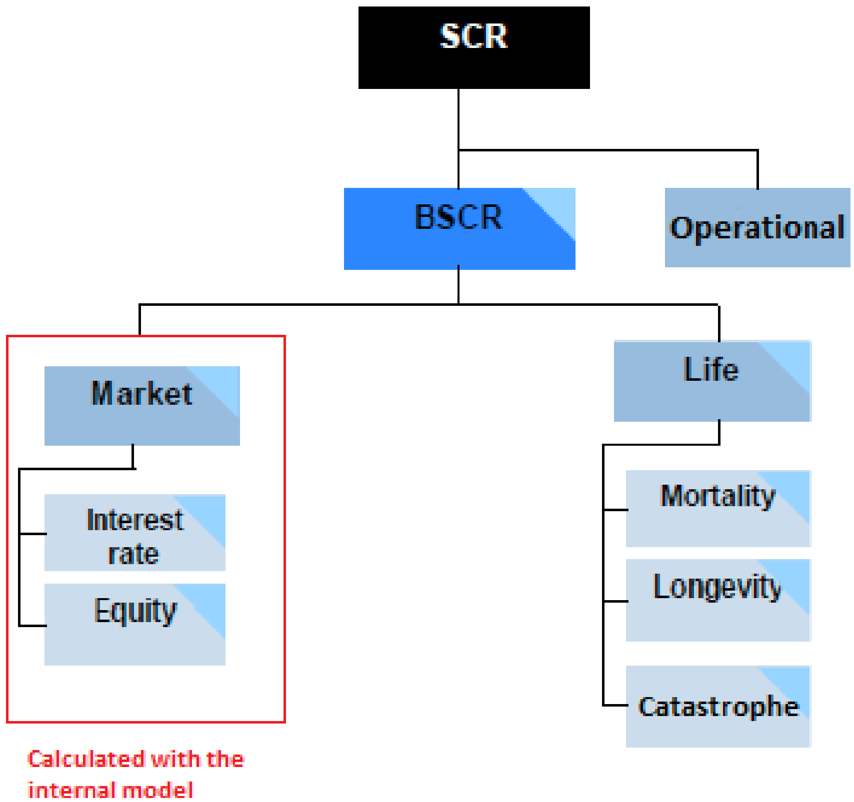
This chapter aims to present the method used to assess the required capital in the application of this thesis. In this aim, the first section presents the regulatory context in terms of capital requirement calculation for variable annuity products (this thesis is realized in Luxembourg and even if the Solvency II Directive is not enforced yet, the regulator already requires the use of an internal model for variable annuity products). Given this regulatory context and given the complexity of a quantile estimation, the method used to estimate the required capital for interest rates and equity risks will be performed through an internal model approach with the nested simulation concept. The description of this method is shared into two parts: the first one tries to explain and justify qualitatively this concept and the second one tries to explain the theoretical concept of this method.

3.1 Internal model for variable annuities

Following the arrival of variable annuity products in Europe, the EIOPA has decided to regulate the sale of these products in the context of Solvency II. Indeed, variable annuities are profitable but also risky products. The risks associated to these products are multiple and complicated. The EIOPA published in November 2010 the Consultation Paper n° 83 which describes particular aspects of variable annuity business. The market risk was highlighted and the standard formula was said to be inappropriate. Thus, the insurance authorities could require a validated internal model (or a partial internal model) to allow the sale of variable annuities. This is an alarming news for small and mid-size companies. When the Solvency II Directive is applied, it will be forbidden to sell variable annuities for companies who do not have a validated full or partial internal model.

A partial internal model is used for the calculation of one or more risk modules, or sub-modules of the basic solvency capital requirement. In our case, we will model the equity and the interest rate sub-modules with the method of nested simulations and we will approximate the market risk by these two sub-modules to present the general mechanism of a partial internal model. The calculation of the life risk module will be performed using the shocks of given by the standard formula. The choice of modeling the market risk with an internal approach is motivated by the fact that it is the most significant risk for GMDB and GMAB products. We will show in our application that the structure of our partial internal model can be illustrated as follows:

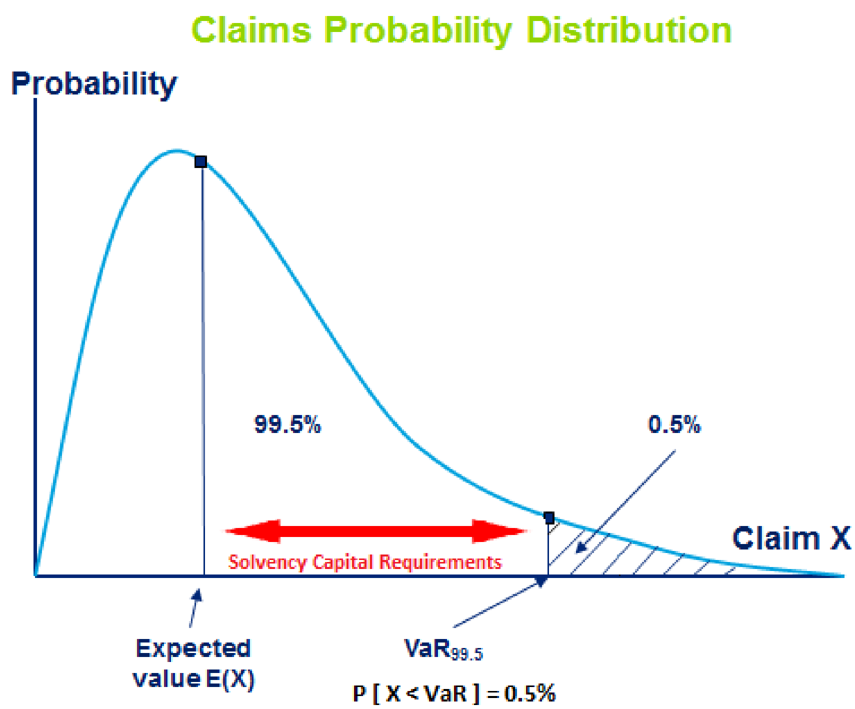
Figure 3.1: Simplified structure of the SCR



Thus the capital requirement for the market risk[1] module will be calculated using the following definition: the capital requirement is the value-at-risk of the basic own funds of an insurance or reinsurance undertakings subject to a confidence level of 99.5% over a one year period. The following picture illustrates this definition:

[1]Here we talk about the capital for market risk by abuse of language (noted SCR_{Mkt}), but in fact we should call it : the aggregation of the required capital for interest rate risk (SCR_{rate}) and the required capital for equity risk (SCR_{equity})

Figure 3.2: Solvency Capital Requirement



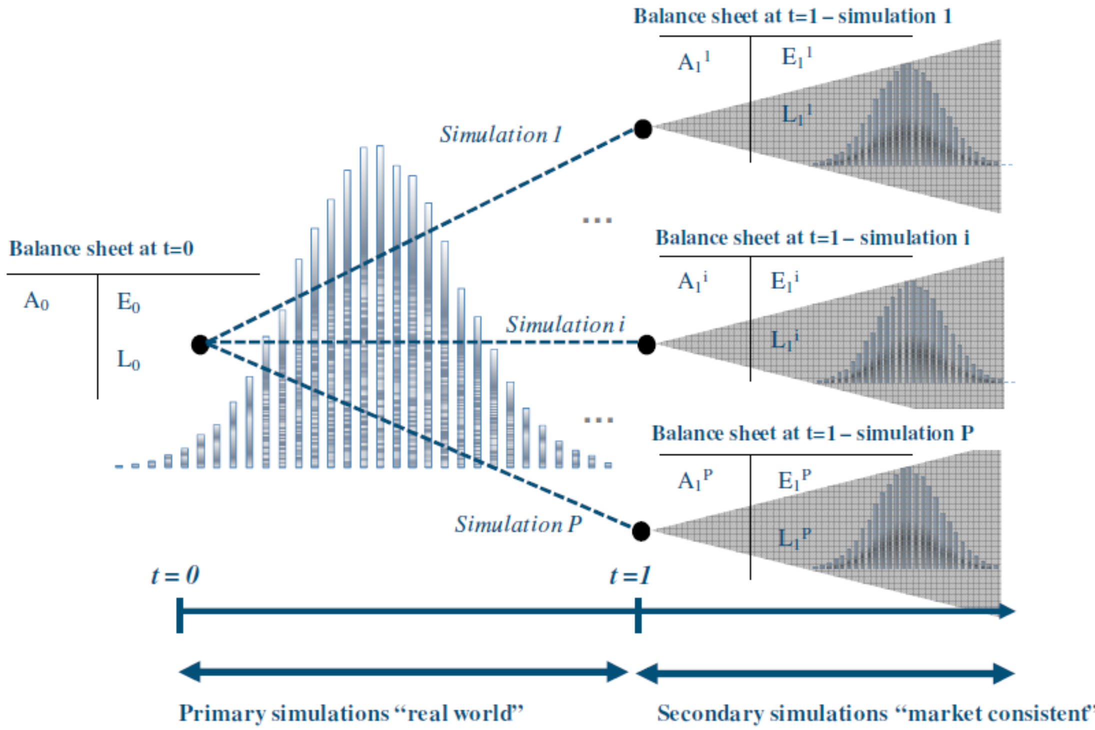
Estimating the distribution of own fund at t=1 is not an easy task. In the next part we present the methodology that we used to assess it: the nested simulations.

3.2 Capital calculation with the nested simulations method

In order to assess the solvency capital for equity and interest rate risk, we use the method of nested simulations. The concept of nested simulations becomes to be well-known in the insurance sector with the arrival of the Solvency II Directive as the calculation of the capital requirement naturally leads to this concept.

The following graph describes the method of nested simulations.

Figure 3.3: Nested Simulations[2]



Remark: We also use the terms "outer simulation" for primary simulation, and "inner simulation" for secondary simulation.

How does the nested simulations method work?

The concept of nested simulations to assess the solvency capital is based on two steps.

As the graph shows with the economic balance sheet at t=0, **the first step of the method consists in estimating the value of own capital at t=0**. Due to the complexity of this estimation, we use a Monte Carlo estimation by using risk-neutral world simulations to estimate the own capital at t=0.

The next step consists in estimating the distribution of own capital at t=1. This distribution is estimated by using a two-level simulation:

- We first simulate real world scenarios between t=0 and t=1. This real world simulation aims at diffusing the economic variables (equity returns, interest rates, inflation…) as closely as possible to the economic reality and this is done by

[2]Sources: Devineau et Loisel, "Construction of an acceleration algorithm of the nested simulations method for the calculation of the Solvency II economic capital"[10].

calibrating models using historical database that best represents their current and future evolutions. In this real world universe, assets offer a risk premium that is represented by an excess of return.

- At t=1, we have a collection of real world scenarios, but for each real world scenario we have to valuate the liabilities of the company. As we need to estimate the value of liabilities at t=1 to deduce the value of own capital at t=1, we need to carry out a new set of simulations at t=1. Thus at t=1, a Monte Carlo estimation with risk-neutral world simulations is required. Hence, we understand the term **Nested**, which describes the phenomenon of risk-neutral world simulations nested into real world simulations.

Why do we use the risk-neutral probability to valuate liabilities?

Under the real world, risky assets offer a risk premium and this makes difficult the realization of a valuation. Thus, when valuating we use the risk-neutral world. Under this probability the risk premium is equal to zero as investors are considered risk-neutral. It is important to note that risk-neutral probability is just a tool for valuation. The risk-neutral probability is based on two assumptions:

- **There is no arbitrage opportunity**: there is no financial strategy ensuring a payment at a future date with an initial investment equal to zero.
- **Completeness of the market**: a complete market is one in which the complete set of possible gambles on future states-of-the-world can be constructed with existing assets.

In practice, if the assumption of no arbitrage opportunity can be justified (when an opportunity arrives on the market it is quickly detected so that it disappears nearly instantaneously) the assumption of complete market is not verified in insurance as the options held depend on insurance risk like mortality risk for example. Thus these options cannot be replicated by assets and thus the market is not complete.

To summarize, estimating the distribution of own capital at t=1 with the nested simulations method means diffusing economic variables under real world between t=0 and t=1, and then valuating the economic balance sheet conditioned to the realization of these economic variables.

Inspired by the work performed by Devineau and Loisel on nested simulations (see [10]), we now describe the concept in a more theoretical way:

As we already said, Solvency II is based on the notion of economic balance sheet.

Economic Balance Sheet at t	
A_t	E_t L_t

with :

- A_t, Market value of assets at time t
- E_t, Equity at time t
- L_t, Fair value of the liabilities at time t.

As the balance sheet is in equilibrium, we have the relation : $E_t = A_t - L_t$.
The first step to calculate the SCR is to assess the economic balance sheet at t=0. Let's denote:

- $(F_t)_{t \geqslant 0}$ the filtration that characterizes the available information at t.
- Q the risk-neutral probability
- δ_u the discount factor in terms of a risk-free instantaneous rate r_u, $\delta_u = \exp(-\int_0^u r_h dh)$
- P_t, the liability cash-flows at t

The value A_0 is known: it is the market value of asset at t=0. In order to assess E_0, we use a Monte-Carlo estimation. Let's consider n simulations with $t \in [t;T]$. Then L_0 is calculated as follows:

$$L_0 = \mathbb{E}_Q\left[\sum_{u \geqslant 1} \delta_u P_u \mid F_0\right] = \frac{1}{n}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(\delta_t^i P_t^i \mid F_0\right)$$

Where δ_t^i and P_t^i are the values obtained in the simulation i at t.

Then, we deduce E_0 (also noted NAV_0 for Net Asset Value representing the difference between assets and liabilities) as follows:

$$E_0 = A_0 - L_0$$

Thus, we obtained the economic balance sheet at t=0.

The second step of the concept is to valuate the distribution of the own fund at t=1 (E_1 or NAV_1). The value of the liabilities is conditioned to the realization at t=1 that we found for each primary simulation. Let's describe this concept for the primary simulation p, and for S secondary simulations:

$$L_1^{(p)} = \mathbb{E}_Q\left[\sum_{t \geqslant 2}^{\infty} \frac{\delta_t}{\delta_1} P_t \mid F_1\right] = \frac{1}{s}\sum_{s=1}^{S}\sum_{t=2}^{T} \frac{\delta_t^{p,s}}{\delta_1^p} P_t^{p,s}$$

where:

- $L_1^{(p)}$ the fair value of liabilities for the first period for the primary simulation p

- δ_1^p the discount factor for the first period for the primary simulation p
- $\delta_t^{p,s}$ the discount factor at t, for the primary simulation p and the secondary simulation s.
- $P_t^{p,s}$ the liability cash-flow at t for the primary simulation p and the secondary simulation s

Thus we can determine the value of E_1 for each real world scenario and this gives us an estimation of the distribution of E_1.

The SCR is then calculated as:

$$SCR = E_0 - P(0,1) \ . \ q_{0.005}(E_1)$$

Where

- $P(0,1)$ is the price at t=0 of a Zero coupon bond that matures at t=1.
- $q_{0.005}(E_1)$ is the quantile of threshold 0.50% for the distribution of the E_1

<u>Proof:</u>
The SCR is defined as the capital needed in order to have:

$$P(E_1 < 0) \leqslant 0.5\%$$

Assume that the amount $S = -P(0,1).q_{0.005}(E_1)$ is invested in cash during the first year at risk-free return.
let E_1^{ajust},A_1^{ajust},L_1^{ajust} be the economic variables after incorporating S.

$$P\left(E_1^{ajust} < 0\right) = P\left(A_1^{ajust} - L_1^{ajust} < 0\right) \approx P\left(A_1 + \frac{S}{P(0,1)} - L_1 < 0\right)$$

The approximation is done because we assume that the impact of the capital S in the whole value of the first year results and in the conditional subsequent results is negligible.

$$P\left(E_1 + \frac{S}{P(0,1)} < 0\right) = P\left(E_1 - q_{0.005}(E_1) < 0\right)$$

so, we deduce:

$$P\left(E_1 < q_{0.005}(E_1)\right) = 0.50\%$$

In practice, the method of nested simulations is probably the most accurate method but it is heavy to implement and is time and storage consuming. There are the major limits of the concept.
The above description of the nested simulations shows that in order to implement the concept of nested simulations, we need to make some modeling assumptions. These assumptions have to be chosen with care.

3.3 Models and calibrations assumptions

Modeling assumptions used to estimate the capital requirement are supposed to be adapted to the risk of the company. Assumptions have to be made, and we have to be aware that different assumptions can lead to substantially different estimations of the capital requirement. Especially for the estimation of a quantile, models' structure and parameters' calibration will therefore have a direct impact on the estimation of the capital requirement. Models should be correctly chosen so as to produce adequately dispersed scenarios that give suitable calibration in the tail. For instance, the equity model must provide extreme scenarios with the appropriate probability, especially in the calculation of a quantile: choosing a model that under-estimates extreme scenarios would lead to the under-estimation of the solvency capital requirement. Also, calibrating a model on a very short historical database that does not include extreme scenarios would not lead to a satisfying calibration. The data used for calibration is important and should ensure that the real risk of the company is captured.

Insurers try to make the best assumptions about models and calibrations by using their knowledge about their portfolio. This subjectivity is supposed to lead to an estimation of the capital requirement adapted to the portfolio of the insurance company, and this is what an internal model aims at.

Chapter 4

Economic Scenario Generator

Introduction

The previous chapter introduced the concept of nested simulations that will be used to assess the capital for equity and interest rates risks for products of types GMAB and GMDB. In order to put this method into practice, we need to build an economic scenario generator (ESG). An ESG is a tool that produces forward-looking scenarios for multiple financial and economic variables. In our case, our economic scenario generator will provide economic scenarios for the evolution of equity returns and interest rates what will allow us to valuate a portfolio allocated in the monetary market, in bonds and in equities.
As presented in the concept of nested simulations, our economic scenarios generator must produce real world scenarios to time t=1, and risk-neutral scenarios after t=1. So in this chapter, the first section will present the equity modeling: the Black and Scholes equity model will be compared to the Merton equity model which is more appropriate in a quantile calculation as it provides heavier tails of distribution and this in the aim of avoiding the under-estimation of the capital requirement. We need to diffuse real world scenarios and risk neutral scenarios, thus we will present two methods used to calibrate the Merton model under risk real world and we will also present the change in the drift that allows us to produce risk-neutral equity scenarios.
The next section presents the modeling of interest rates. We will use the one-factor Cox-Ingersoll-Ross model motivated by the fact that this model overcomes the problem of negative interest rates. In fact we will show that this model produces strictly positive interest rates under certain constraints in its continuous form. But when discretizing the process with the explicit Euler scheme, the positivity of the simulated interest rates is not ensured anymore. To overcome this problem we will present another discretization scheme which is the implicit Euler Scheme. Similarly to the equity model, we need to diffuse risk real and then risk neutral interest rates scenarios. In this aim we will differentiate the risk-neutral world calibration from the risk real world calibration.
As it is observed in the reality, the evolutions of interest rates and equity returns are not independent and this has to be considered in our ESG. Thus, a section is dedicated

to the presentation of the dependence structure implemented: a linear dependence will be used using correlated Brownian motions. The regulator mandates some tests on the economic scenario generator, so last but not least, we give the results obtained when performing the test of the martingale. A test of convergence is also required by the regulator but this test will be presented later in the application.

4.1 Equity modeling

4.1.1 Useful theory: Ito Processes

Let X be the solution of the following stochastic differential equation:

$$X(t) = X(0) + \int_0^t m(X(s), s)ds + \int_0^t \sigma(X(s), S)dW(s)$$

Then X is a Ito process that can also be written as:

$$\mathbf{dX(t)} = m(X(t), t)\mathbf{dt} + \sigma(X(t), t)\mathbf{dW(t)}$$

Where :

- **W** is a Brownian Motion
- m the drift
- σ is the volatility

4.1.2 The Black and Scholes model

Many equity models exist, but probably the most famous model in finance, the Black and Scholes model or the Black Scholes Merton model is often used to diffuse the equities over a period of time.
Under the real world probability $\mathbb{P}$, the Black-Scholes differential equation is:

$$\boxed{\frac{dS_t}{S_t} = \mu dt + \sigma dW_t}$$

where:
W is a standard Brownian motion under historical probability
σ denotes the constant volatility of the risky asset
μ the constant drift

Even if the Black and Scholes model is very used for its simplicity to put into practice, it is based on quite heavy assumptions:

- There is no arbitrage opportunity: there is no financial strategy ensuring a payment at a future date with an initial investment equal to zero.
- It is possible to buy or sell any fraction of asset.
- The market is complete and continuous.
- Short selling is possible without any restriction.
- Constant volatility: probably the heaviest assumption, the measure of how much can a stock be expected to move in the near term is constant over time.
- Interest rates are constant and known : the Black-Scholes model uses a constant rate as the risk-free rate.
- Efficient markets: this assumption of the Black-Scholes model suggests that people cannot consistently predict the direction of the market or an individual stock.
- No dividends: he stock does not pay dividends.
- No commission and transaction costs: there are no fees for buying and selling stocks.
- Liquidity: it assumes that it is possible to purchase or sell any amount of stock at any given time.

By using the Ito Lemma:
Let X be a Ito process and let $f(t,x)$ be a $C^{1,2}$ function, then:

$$\boxed{\mathrm{d}f(X(t),t) = \left(\frac{\partial f}{\partial t} + m(X(t),t)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma\left(X(t),t\right)^2\frac{\partial f^2}{\partial x^2}\right)\mathrm{d}t + \sigma(X(t),t)\frac{\partial f}{\partial x}\mathrm{d}W(t)}$$

Let use the Ito lemma with $\mathrm{d}\ln(S_t)$:

$$\mathrm{d}\ln(S_t) = \left(\frac{1}{S_t}m(S_t,t) + \tfrac{1}{2}\left(\frac{-1}{S_t^2}\right)\sigma^2(S_t,t)\right)dt + \sigma\frac{1}{S_t}dW_t$$

$$\mathrm{d}\ln(S_t) = (\mu - \tfrac{1}{2}\sigma^2)dt + \sigma dW_t$$

$$\ln S_t = \ln S_0 + \textstyle\int_0^t \left(\mu - \tfrac{1}{2}\sigma^2\right)dt + \int_0^t \sigma dW_t$$

Finally we find, $\boxed{S_t = S_0 \exp\left((\mu - \tfrac{1}{2}\sigma^2)t + \sigma W_t\right)}$

It is interesting to note that the properties of a Brownian motion allow us to write: $\ln\left(\frac{S_t}{S_0}\right) \sim \mathcal{N}\left((\mu - \tfrac{1}{2}\sigma^2)t, \sigma^2 t\right)$

4.1.2.1 Risk-neutral world

Under risk neutral probability, we show by using the Girsanov theorem that the price of equities follows the following stochastic differential equation:

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t^{\mathbb{Q}}$$

where:

- r is the risk-free interest rate
- $W_t^{\mathbb{Q}}$ is a Standard Brownian motion under risk neutral probability

4.1.2.2 Model simulation

Models are usually expressed in a continuous form which permitted to discover some famous closed-form solutions. But in practice, we need to discretize the process in order to simulate trajectories (discrete data). Since the Black and Scholes model provides a closed-form solution for the price of equity as previously shown, we can simulate it without discretization error, with an exact discretization.

$$S_{t+\Delta t} = S_t \exp\left((\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma\left(W_{t+\Delta t} - W_t\right)\right)$$

The following properties of the Brownian motion,

- Independence in the increments
- $W_{t+\Delta t} - W_t \sim \mathbb{N}\left(0, \Delta t\right)$

allow to simulate $(W_{t+\Delta t} - W_t)$ by $\sqrt{\Delta t}.Z$ where $Z \sim \mathbb{N}\left(0, 1\right)$. Thus the exact discretization formula to simulate the Black and Scholes model is:

$$S_{t+\Delta t} = S_t \exp\left((\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}\ Z\right)$$

where $Z \sim \mathbb{N}\left(0, 1\right)$

4.1.2.3 Calibration

The calibration of an internal model is an important part. We have used a historical data of 5 years which is quite short. Please note that here we just want to present a method used for calibration, but in the context of an internal model it would be better to use larger database.
The **Maximum Likelihood** approach will be used to the problem of estimation of the parameters $(\mu, \sigma) = \Theta$ from the data. This method can be used for both continuous and

discrete random variables. The concept of this method is to estimate the parameter Θ for the assumed density function that will maximize the probability of having observed the given data sample. The probability of observing a particular data sample, is called the **Likelihood Function**, and is denoted l(Θ).

By defining the log-returns of the data sample $(x_1, ..., x_n)$ as:

$$X_i = \ln\left(\frac{S_{i+1}}{S_i}\right)$$

The likelihood Function for independent and identically distributed (iid) variables is:

$$l(\Theta) = f_\Theta(x_1, ..., x_n) = \prod_{i=1}^{n} f_\Theta(x_i)$$

For numerical reasons, we usually convert the Likelihood Function to the **Log-Likelihood Function** $L(\Theta)$:

$$L(\Theta) = \sum_{i=1}^{n} \ln\left(f_\Theta(x_1, ..., x_n)\right)$$

The Maximum Likelihood Estimator (MLE) $\widehat{\Theta}$, is found by maximizing the Likelihood or Log-Likelihood function. In the Black and Scholes model, the log-returns increments form normal iid random variables, each with a known density determined by mean (m) and variance (v) $f_\Theta(x) = f(x; m; v)$ with:

$$m = \left[\widehat{\mu} - \frac{1}{2}\widehat{\sigma^2}\right]\Delta t \quad ; \quad v = \widehat{\sigma^2}\Delta t$$

By differentiating the Gaussian density function with respect to each parameter and setting the derivative to zero, the MLE method provides closed-form solutions for the mean (m) and variance (v) of the log-returns:

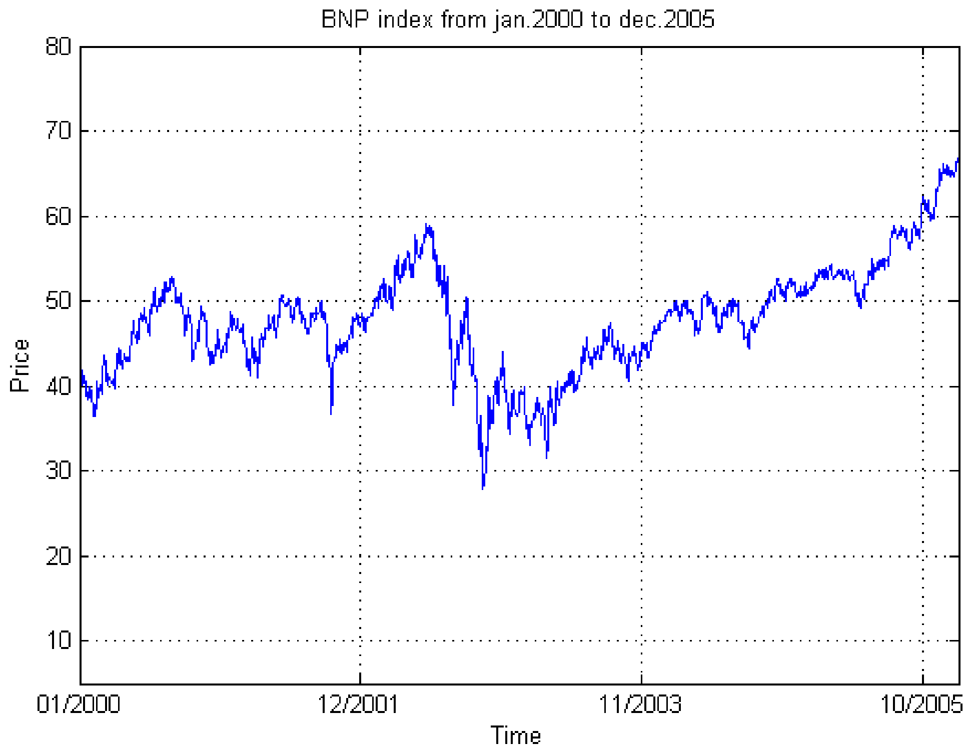
$$\widehat{m} = \frac{1}{n}\sum_{i=1}^{n} x_i \quad ; \quad \widehat{v} = \frac{1}{n}\sum_{i=1}^{n} (x_i - \widehat{m})^2$$

The estimator of the variance presented above is biased and in practice we usually use the following unbiased estimator:

$$\widehat{v} = \frac{1}{n-1}\sum_{i=1}^{n} (x_i - \widehat{m})^2$$

In order to illustrate this model, we have used the data sample of BNP Paribas from the 01.04.2000 to the 12.30.2005 (length of the sample is 1531). The average price over the period is about 47.8€ with a standard deviation of 6.46. The average return is 0.000273 with a standard deviation of 0.019741 ($\widehat{\sigma}$).

Figure 4.1: BNP PARIBAS Index from 01/2000 to 12/2005

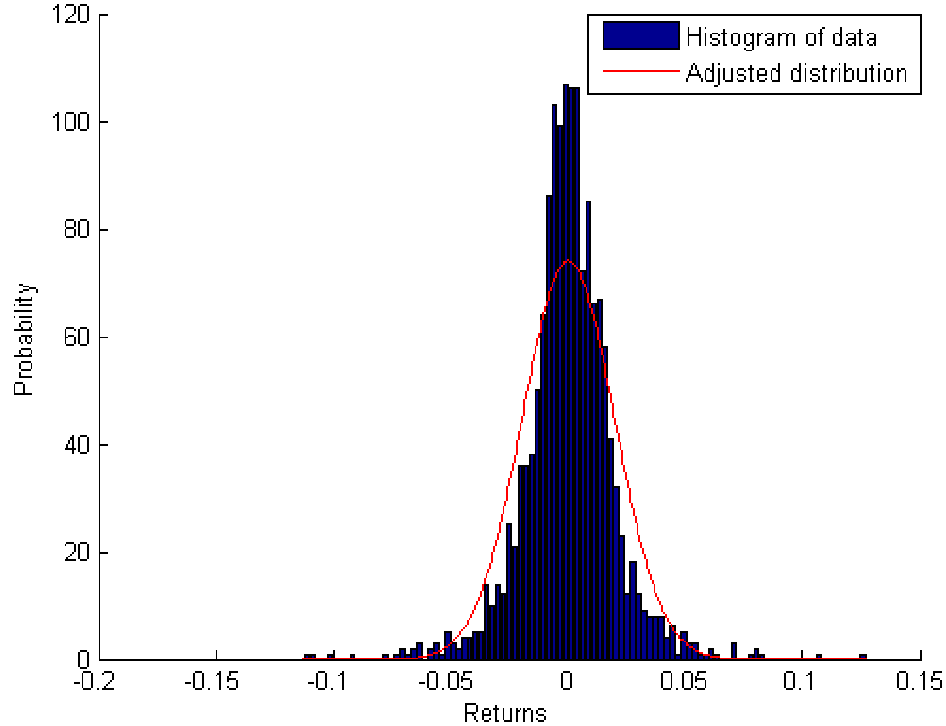


We find the following estimators:

Calibration	
$\widehat{\mu}$	0.0004684
$\widehat{\sigma}$	0.0197417

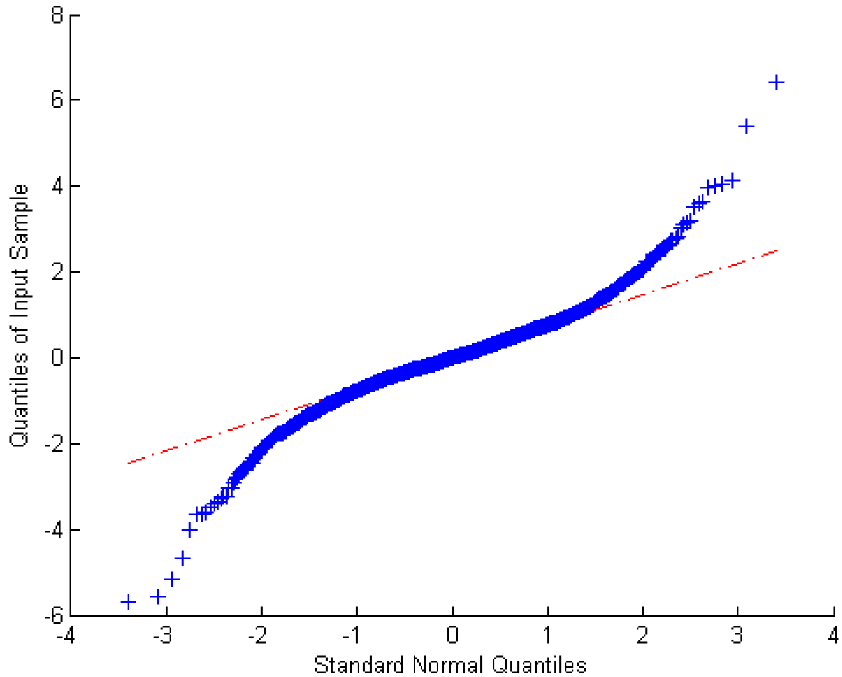
Using these estimations, it is interesting to compare the estimated distribution to the empirical distribution.

Figure 4.2: Fitted distribution of returns



The above picture shows that the adjusted distribution does not really fit the empirical distribution. We use a quantile-quantile plot in order to criticize the replication of the tails of distribution. The quantile-quantile plot (qqplot) is a graphical technique used for determining if two data sets come from a common distribution. The following picture presents the obtained quantile-quantile plot.

Figure 4.3: QQ-Plot of Sample Data versus Standard Normal

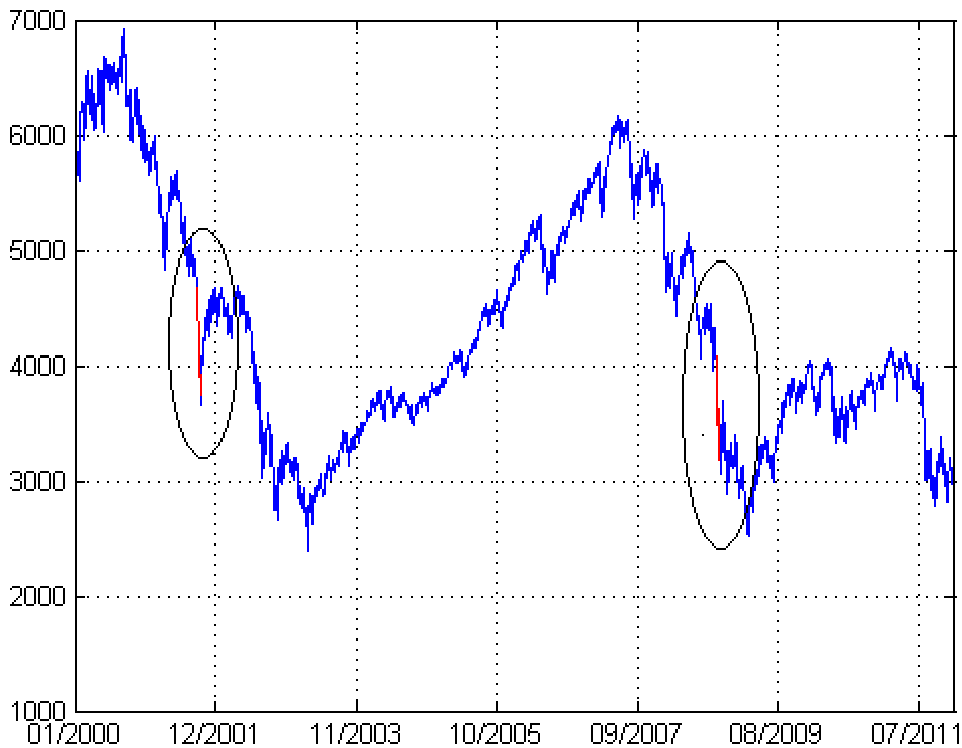


The above qqplot shows that the Gaussian assumption seems not to fit the returns distribution, and yet the Black Scholes model is very used in the professional world. Indeed, when putting theory into practice, the closed-form solution provided by this model allows for a relative simplicity. The Black and Scholes model is based on quite heavy assumptions that are not all verified on the markets. More particularly, some assumptions are particularly wrong when the market is perturbed or when the market knows some discontinuities. Thus, when using the Black and Scholes model, professionals must be careful about the assumptions taken especially in the calculation of the SCR which is a quantile. Thus, we present another model, the Merton model.

4.1.3 The Merton Model

The Black and Scholes model is often used for its simplicity but the tails of distribution do not fit the fat tails observed in the reality. In order to improve this lacuna, we introduce the Merton model. The publication of some economical results or other extreme events can affect the brutal changes in the financial world. The empirical observation below introduces this jump phenomenon.

Figure 4.4: CAC40 Index from 01/2000 to 12/2011



These jump observations can be modeled through **Poisson Jump processes**. In this model, we add a source of risk (Poisson jumps) without adding assets:

- The Brownian component represents the global fluctuations of the market due to the constant arrival of information concerning the overall market.
- The jump component represents information specific to the financial investment (good or bad news about the company,...).

Under the historical probability $\mathbb{P}$, the Merton model differential equation is:

$$\boxed{\frac{dS_t}{S_t} = \mu \mathbf{dt} + \sigma \mathbf{dW}_t + \mathbf{dJ}_t}$$

where the compounded Poisson process J is defined as :

$$J_t = \sum_{i=1}^{N_t} Z_i$$

with

$$\begin{cases} N_t \text{ follows a homogeneous Poisson Process with intensity } \lambda t \\ Z_i \sim \mathbb{N}\left(0;\sigma_z\right) \end{cases}$$

- W is a Brownian Motion
- σ the constant volatility of the risky asset
- μ the constant trend
- W, N and Z are mutually independent processes

The Merton model provides a closed-form solution:

$$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t + \sum_{k=1}^{N_t} Z_k\right)$$

We note that when $N_t = 0$ then the jump component is equal to zero. The stock return R(t) is equal to the following:

$$R(t) = \ln\left(\frac{S_t}{S_{t-1}}\right) = \left(\mu - \frac{\sigma^2}{2}\right) + \sigma\left(W(t) - W(t-1)\right) + \left(\sum_{k=1}^{N(t)} Z_k - \sum_{k=1}^{N(t-1)} Z_k\right)$$

Hence, the return still follows a log-normal distribution, but with the Merton model, at some date point, we can observe discontinuities in the curve and this is empirically observed. We note that the above stock return formula also shows that returns are independent and identically distributed. This means that the value of the return does not depend on the position of the time interval of the returns, but it depends on the step of time.

4.1.3.1 The martingale property

Under the condition of no arbitrage opportunity, a pricing rule implies that the present value of the price of the underlying asset is a martingale under the probability $\mathbb{Q}$.
For the Merton model, we show that the process $\widehat{S}_t = e^{rt} S_t$ is a martingale if and only if:

$$\mu = r - \lambda\left(\exp\left(\frac{\sigma_z^2}{2}\right) - 1\right)$$

This condition can be verified by calculating the expected value of the exponential of the Poisson jump part.

4.1.3.2 Model simulation

Similarly to the Black and Scholes model, the exact discretization can be used to simulate a Merton trajectory.

$$S_t = S_{t-\Delta t} \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}\epsilon_t + \sum_{j=1}^{N_{\Delta t}} Y_j\right)$$

Where

- $N_t \sim \mathrm{P}(\lambda\Delta t)$
- $Y_j \sim \mathbb{N}(0; 1)$
- $\epsilon_j \sim \mathbb{N}(0; 1)$

4.1.3.3 Calibration

In order to calibrate the Merton model, we first use the moment method estimation (MME). Then, we use the maximum likelihood approach.

The density [1] of returns is given by the following formula:

$$f(x) = \frac{\exp(-\lambda)}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \left[\frac{\lambda^n}{n!\sqrt{\sigma^2 + n\sigma_z^2}} \exp\left[-\frac{\left(x - \mu + \frac{\sigma^2}{2}\right)^2}{2\left(\sigma^2 + n\sigma_z^2\right)} \right]\right]$$

Using this density formula we can determine the central moments[2](moment about the mean), the idea of calculating the central moments and not just the simple moments comes from the coefficient $(x - \mu)$ in the density function of returns):

- Because of the symmetry of the distribution, odd central moments are equal to zero.

- Even central moments are given by the following formula:

$$\mathbb{E}\left[\left(R - \mathbb{E}\left(R\right)\right)^{2k}\right] = \frac{(2k)!}{2^k k!} \sum_{n=0}^{\infty} \frac{\lambda^n \exp\left(-\lambda\right)}{n!} \left(\sigma^2 + n\sigma_z^2\right)^k$$

[1] The mathematical proof is given in appendix

[2] The mathematical proof is given in appendix

Four linearly independent equations are necessary to assess the four unknown parameters $(\mu, \sigma, \lambda, \sigma_z)$. With the sample of observed returns $(r_1, ..., r_n)$, we deduce the following system of equations:

$$\begin{cases} \bar{r} = \frac{1}{n}\sum_{i=1}^{n} r_i = \mu - \frac{\sigma^2}{2} \\ \mathbb{E}\left[(R - \mathbb{E}(R))^2\right] = \frac{1}{n}\sum_{i=1}^{n}(r_i - \bar{r})^2 = \sigma^2 + \lambda\sigma_z^2 \\ \frac{1}{n}\sum_{i=1}^{n}(r_i - \bar{r})^4 = 3\exp(-\lambda)\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\left(\sigma^2 + n\sigma_z^2\right)^2 \\ \frac{1}{n}\sum_{i=1}^{n}(r_i - \bar{r})^6 = 15\exp(-\lambda)\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\left(\sigma^2 + n\sigma_z^2\right)^3 \end{cases}$$

Where R is the log-return, and n the number of observations. To solve numerically this system, we implement the Newton-Raphson algorithm. The sum in the above system is approximated by the sum of the first 20 terms (the convergence is fast).

Another method of calibration is to use the maximum likelihood estimator. The Log-likelihood function is:

$$L\left(x_1, ..., x_N, \mu, \sigma^2, \lambda, \sigma_z^2\right) = \prod_{i=1}^{N}\frac{\exp(-\lambda)}{\sqrt{2\pi}}\left[\sum_{n=0}^{\infty}\left[\frac{\lambda^n}{n!\sqrt{\sigma^2 + n\sigma_z^2}} \times \exp\left[-\frac{\left(x_i - \mu + \frac{\sigma^2}{2}\right)^2}{2\left(\sigma^2 + n\sigma_z^2\right)}\right]\right]\right]$$

In order to illustrate with a practical case, we apply the method of moments and the maximum likelihood method to the data set used for the Black and Scholes illustration (BNP Paribas Index from the 01.04.2000 to the 12.30.2005).

Parameters	Method of Moments	Maximum Likelihood
$\widehat{\mu}$	3.52622985e-004	5.0771594e-004
$\widehat{\sigma}$	0.01257	0.01215
$\widehat{\lambda}$	0.19976	0.28096
$\widehat{\sigma_z}$	0.0340298	0.0290436

Table 4.1: Parameters Estimation of the Merton model

Using two methods allows us to criticize the consistency of the results obtained. Indeed, we note that the two methods give the same order of magnitude even if the maximum likelihood method gives a larger value for $\widehat{\mu}$.

The variance of the Merton model is given by the following formula:

$$V = \overbrace{\sigma^2}^{\text{Brownian Motion Variance}} + \underbrace{\lambda\sigma_z^2}_{\text{Poisson Jump Variance}}$$

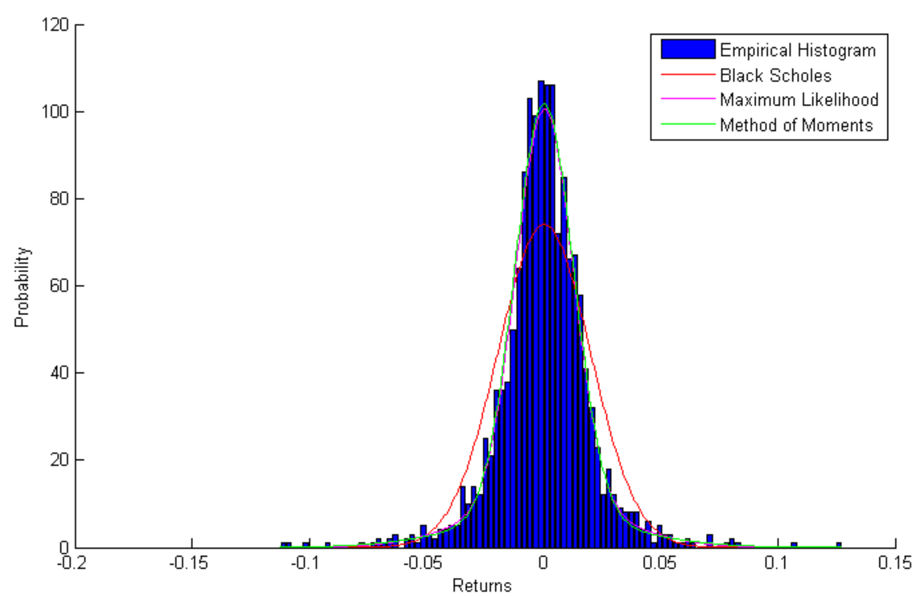
Thus, we can compare the variance part explained by the Poisson jump and by the Brownian motion with both methods of calibration:

Variance	Method of Moments	Maximum Likelihood
Annualized Variance[3]	9.73705%	9.62124%
Explained by the Poisson jump	≈ 59.397%	≈61.584%
Explained by the Brownian Motion	≈40.603%	≈38.416%

Table 4.2: Variance decomposition of the Merton model

As shown in the above table, the variance part explained by the Poisson jump component is a little more important with the maximum likelihood method than with the method of moments. Let's compare the different adjusted densities:

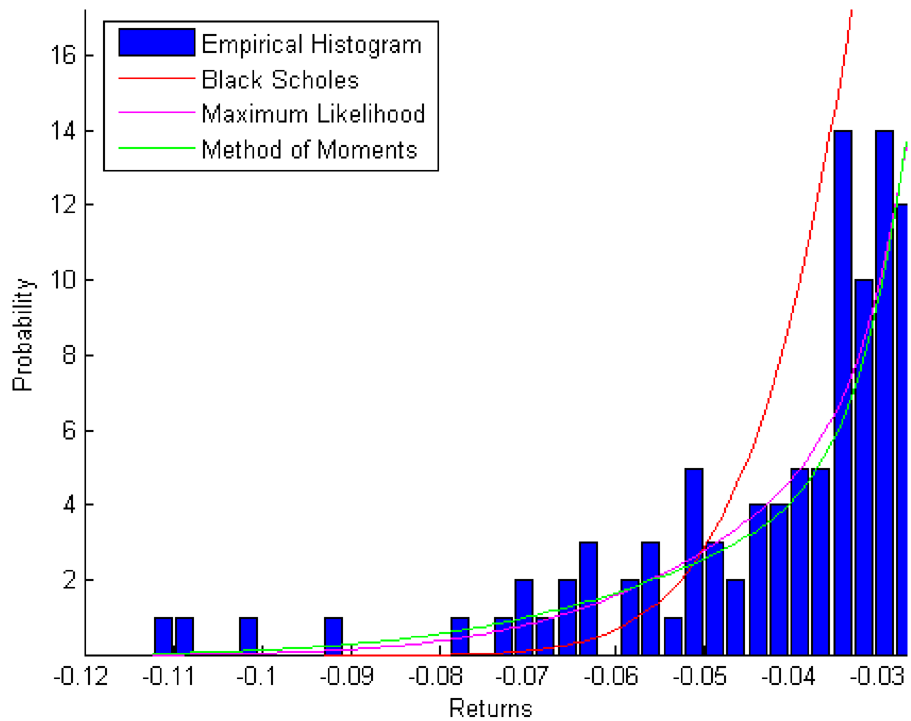
Figure 4.5: Comparison of the Merton model and the Black and Scholes model



[3]On the basis of 250 days a year

The comparison of the adjustments shows that the Merton model better fits the empirical distribution. We also see that the Merton adjustment presents larger tails of distribution than the Black and Scholes model as the following graph shows:

Figure 4.6: Zoom on the tail of distribution



The above figure shows that the Merton model better fits the empirical distribution of returns than the Black and Scholes model does. It is important to note, especially for the calculation of a quantile, that the tails of distribution are better adjusted with the Merton model than with the Black and Scholes model.

4.2 Interest rate modeling

In this section we present two one-factor models: the Vasicek model and the Cox-Ingersoll-Ross model (CIR). One-factor models are based on a quite heavy assumption: the short rate is correlated with every maturity rate. This is generally not observed in practice, however, one-factor models such as the Vasicek and the CIR model are very convenient to use as they provide a closed-form solution for zero-coupon prices. For more detail on interest rate model other than those presented here, the reader can refer to BRIGO and MERCURIO [4].

4.2.1 The Vasicek model

The Vasicek model (Vasicek, 1977) is one of the earliest stochastic short term interest rate. In this model, the short rate follows an Ornstein-Uhlenbeck process with constant positive coefficients. Under historical probability, the stochastic differential equation of the model is:

$$dr_t = a\,(b - r_t)\,dt + \sigma\ \mathbf{dW}_t$$

Where:

- r_t is the short rate at time t
- a is the speed of reversion
- b is the long term mean level
- σ is the instantaneous volatility
- W is a Brownian motion

The particularity of this model is that it features mean reversion: if the interest rate is above the long term mean ($r_t > b$) then $a\,(b - r_t)$ becomes negative so that the short rate will be pushed closer to the long term mean level b. Likewise, if the rate is inferior to b ($r < b$) then $a\,(b - r_t)$ is positive so that the rate will be pushed to be closer to the long run mean b. Thus, the idea of this model comes from the observed economic phenomenon that interest rates are pulled back to a long run average value over time. Nevertheless, with the Vasicek model there is a positive probability that r becomes negative. This phenomenon is economically non acceptable[4] and some models like the Cox-Ingersoll-Ross model have overcome this problem.

4.2.2 The Cox-Ingersoll-Ross model (CIR)

The CIR model was suggested in 1985. It follows a square-root process. Under historical probability, the CIR model is:

$$dr_t = a\,(b - r_t)\,dt + \sigma\sqrt{r_t}\ \mathbf{dW}_t$$

Where:

- r_t is the short rate at time t
- a is the speed of reversion

[4] We consider negative rates as non acceptable although we have recently observed negative interest rates, indeed the German treasury has just sold sovereign debt at a negative interest rate.

- b is the long term mean level
- σ is a constant parameter
- W is a Brownian motion

This model is an example of conditional volatility: when the short rate is high, the volatility of interest rate changes is also high, and vice versa. Another difference is that this model only produces strictly positive rates under the condition:

$$2ab > \sigma^2$$

Thus, when calibrating, it is important to make sure that this constraint is met.

4.2.2.1 Zero-coupon valuation with a CIR model

The zero-coupon rate at t maturing at T is the actuarial rate of a bond with a coupon rate equal to zero. At time t, the price of a zero-coupon bond maturing at T is given by:

$$P(t,T) = A(t,T)\exp\left(-B(t,T)r_t\right)$$

with:

$$A(t,T) = \left(\frac{2\gamma\exp\left(\dfrac{(\gamma+a)(t-T)}{2}\right)}{(\gamma+a)(\exp(\gamma(T-t))-1)+2\gamma}\right)^{\dfrac{2ab}{\sigma^2}}$$

$$B(t,T) = \frac{2(\exp(\gamma(T-t))-1)}{(\gamma+a)(\exp(\gamma(T-t))-1)+2\gamma}$$

$$\gamma = \sqrt{a^2+2\sigma^2} \quad ; \quad r_t : \text{the short rate}$$

Let $R(t,T)$ be the continued interest rate at t for maturity T-t, then the relation

$$R(t,T) = -\frac{1}{T-t}\ln(P(t,T))$$

leads to the following formula:

$$R(t,T) = \frac{r_t \times B(t,T) - \ln(A(t,T)}{T-t}$$

4.2.2.2 CIR simulation

In order to simulate with the Cox-Ingersoll-Ross model, we can use the explicit Euler scheme:

$$r_{t+\Delta t} = ab.\Delta t + r_t(1 - a\Delta t) + \sigma\sqrt{r_t.\Delta t}.Z$$

Where $Z \sim \mathbb{N}(0,1)$.

However, it is important to note that even when the condition $2ab > \sigma^2$ is met, it is possible to obtain negative interest rates with the explicit Euler scheme. If we consider that $r_{t+\Delta t}$ is positive then:

$$ab.\Delta t + r_t(1 - a\Delta t) + \sigma\sqrt{r_t.\Delta t}.Z > 0$$

which involves

$$Z > \frac{ab.\Delta t + r_t(1 - a\Delta t)}{-\sigma\sqrt{r_t.\Delta t}}$$

but $Z \sim \mathbb{N}(0,1)$ is not bounded from below, so whatever the condition on (a, b, σ) is, the probability to simulate a negative interest rate cannot be equal to zero. Thus, if we use the explicit Euler scheme, it can lead to negative short rate. Indeed, the condition necessary for positive interest rates is a condition that works only for the continuous form of the equation.

$$dr_t \approx r_{t+\delta t} - r_t$$

To solve this problem let's consider the process r_t as the limit of the explicit Euler scheme (see Alfonsi paper on the discretization schemes of the CIR process[2]):

$$r_t = \lim_{n\to\infty} \sum_{t_i<t} \left[\frac{a}{n}(b - r_{t_i}) + \sigma\sqrt{r_{t_i}}\left(W_{t_{i+1}} - W_{t_i}\right)\right]$$

Which leads to:

$$r_t = \lim_{n\to\infty} \sum_{t_i<t} \left[\frac{a}{n}\left(b - r_{t_{i+1}}\right) + \sigma\sqrt{r_{t_{i+1}}}\left(W_{t_{i+1}} - W_{t_i}\right) - \sigma\left(\sqrt{r_{t_{i+1}}} - \sqrt{r_{t_i}}\right)\left(W_{t_{i+1}} - W_{t_i}\right)\right]$$

Then we need to calculate $\langle\sqrt{r_t}, W_t\rangle$ by using the Ito lemma with the function $\sqrt{r_t}$ (twice differentiable function on $\mathbb{R}^{*+}$)

We find: $d\langle\sqrt{r_t}, W_t\rangle = \frac{\sigma}{2}dt$

then,

$$r_t = \lim_{n\to\infty} \sum_{t_i<t} \left[\frac{1}{n}\left(ab - \frac{\sigma^2}{2} - ar_{t_{i+1}}\right) + \sigma\sqrt{r_{t_{i+1}}}\left(W_{t_{i+1}} - W_{t_i}\right)\right]$$

We find the implicit Euler scheme:

$$r_{t_{i+1}} - r_{t_i} = \frac{1}{n}\left(ab - \frac{\sigma^2}{2} - ar_{t_{i+1}}\right) + \sigma\sqrt{r_{t_{i+1}}}\left(W_{t_{i+1}} - W_{t_i}\right)$$

If we consider that $\sqrt{r_{t_{i+1}}}$ is positive, let's assume $\sqrt{r_{t_{i+1}}} = X$:

$$X^2(1+\frac{a}{n}) - \sigma X\left(W_{t_{i+1}} - W_{t_i}\right) - r_{t_i} - \frac{1}{n}\left(ab - \frac{\sigma^2}{2}\right) = 0$$

By resolving this polynomial equation of the second degree, we prove that two solutions exist: $\forall(a,b,\sigma) \in \mathbb{R}^3_+$

$$\begin{cases} X_1 = \dfrac{\sigma\left(W_{t_{i+1}} - W_{t_i}\right) + \sqrt{\sigma^2\left(W_{t_{i+1}} - W_{t_i}\right)^2 + 4\left(1+\frac{a}{n}\right)\left(r_{t_i} + \frac{1}{n}\left(ab - \frac{\sigma^2}{2}\right)\right)}}{2\left(1+\frac{a}{n}\right)} \geqslant 0 \\[2ex] X_2 = \dfrac{\sigma\left(W_{t_{i+1}} - W_{t_i}\right) - \sqrt{\sigma^2\left(W_{t_{i+1}} - W_{t_i}\right)^2 + 4\left(1+\frac{a}{n}\right)\left(r_{t_i} + \frac{1}{n}\left(ab - \frac{\sigma^2}{2}\right)\right)}}{2\left(1+\frac{a}{n}\right)} \leqslant 0 \end{cases}$$

We have proved that a unique positive solution exists to the equation given by the implicit Euler scheme. We deduce:

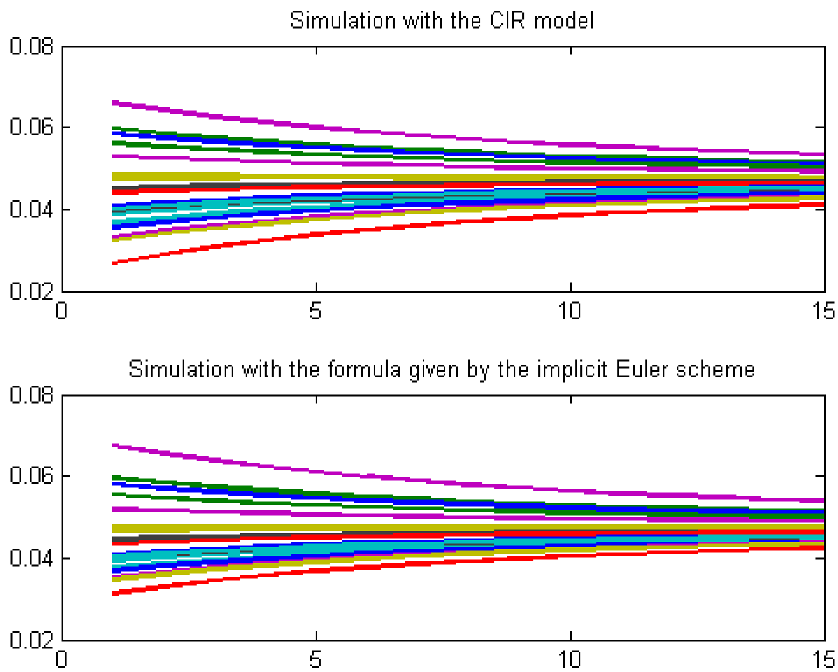
$$r_{t_{i+1}} = \left[\frac{\sigma\left(W_{t_{i+1}} - W_{t_i}\right) + \sqrt{\sigma^2\left(W_{t_{i+1}} - W_{t_i}\right)^2 + 4\left(1+\frac{a}{n}\right)\left(r_{t_i} + \frac{1}{n}\left(ab - \frac{\sigma^2}{2}\right)\right)}}{2\left(1+\frac{a}{n}\right)}\right]^2$$

This scheme preserves the monotonicity property (it is still an increasing function of r_t) satisfied by the CIR process:

$$\underbrace{r_0 < r_0^{'}}_{\text{Two initial conditions}} \qquad \Longrightarrow \quad r_t < r_t^{'}$$

It is interesting to compare the results given by the CIR model and the result obtained with the implicit Euler scheme. For a good comparison, we use the same Brownian motion for the two different discretizations, and we compare the obtained interest rate curves in one year ($dt = 1 year$).

Figure 4.7: Implicit Euler Scheme vs Explicit Euler Scheme



Graphically, we see that the implicit Euler scheme gives larger values than the explicit Euler scheme.
Other solutions exist to avoid negative interest rates. For example, it is usually considered that:

$$\widehat{r}_{t+\delta_t} = \max\left(\widehat{r}_t\left(1 - a\delta_t\right) + ab\delta_t + \sigma\sqrt{\delta_t\widehat{r}_t}Z; 0\right)$$

This solution puts negative interest rate to zero as there are very few negative simulations. Another solution consists in taking the module of the result as the square root of negative rate introduces complex numbers:

$$\widehat{r}_{t+\delta_t} = |\widehat{r}_t\left(1 - a\delta_t\right) + ab\delta_t + \sigma\sqrt{\delta_t\widehat{r}_t}Z|$$

These two last solutions are simple solutions, and they are not really consistent. This is the reason why we will prefer the solution given by the implicit Euler scheme to simulate.

4.2.2.3 Link between the historical and risk-neutral parameters

The historical parameters and the risk-neutral parameters can be chosen independently. In general, the risk neutral parameters (a_1, b_1, σ_1) differ from the historical parameters (a, b, σ). But, in practice, it is often used to constrain these parameters, using a risk premium parameter λ. These constraints are used to get a simple interpretation of the change of parameters, in other word in the change of probability. Thus it is usually assumed that $\sigma_1 = \sigma$. We present the general framework of this method in **appendix**. In our case, we will use two independent calibrations. We will present a method to estimate the parameters independently using interest rate instruments (zero-coupon prices) for the risk-neutral world and historical data for the historical world.

4.2.2.4 CIR calibration

It is important to insist on the different methods used to calibrate the CIR process. Under the historical probability, we use historical data of interest rates, whereas in risk-neutral world, we replicate the price of interest rate instruments.

Risk-neutral world calibration

In order to calibrate the CIR model, we use the zero-coupon curve published[5] (12.31.2012) by the french institute of Actuaries, "Institut des Actuaires". Please note that normally, in the context of an internal model, the curve published by the EIOPA should be used. We will estimate the parameters (σ, a, b) by minimizing the distance (the Least Squares approach) between the observed zero-coupon prices and the zero-coupon prices given by the formula of the CIR model (we can also do it with zero-coupon rates):

$$\min_{\sigma,a,b} \left[\sum_{i=1}^{n} \left[P_i - P_{ob,i}\right]^2 \right] \quad or \quad \min_{\sigma,a,b} \left[\sum_{i=1}^{n} \left[R_i - R_{ob,i}\right]^2 \right]$$

Where:

- P_i is the theoretical zero-coupon price of maturity i
- $P_{ob,i}$ is the observed zero-coupon price of maturity i
- R_i is the theoretical interest rate of maturity i
- $R_{ob,i}$ is the observed interest rate of maturity i

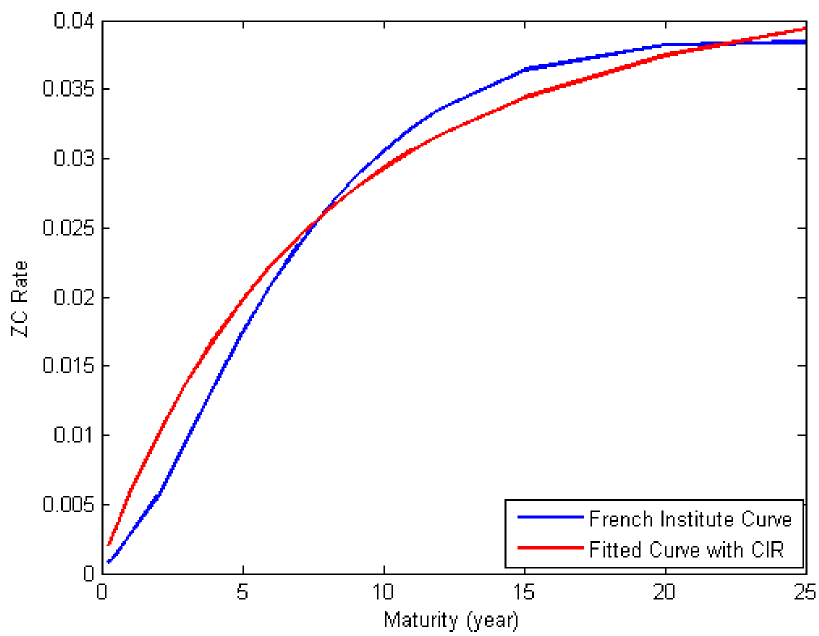
As we have to make an assumption for the short rate r_t, we assume it is equal to the 3-month Euro-Libor rates. We decide to estimate the parameters (a,b) for fixed values of σ under the constraint $2\% < \sigma < 8\%$. We choose the value of σ that best fits the curve and minimizes the above formula. With this calibration method we obtain the following results:

[5] This curve is given in appendix

Parameters	Estimators
$\hat{\sigma}$	0.05
$\hat{a}$	0.224
$\hat{b}$	0.0485

We note that the condition $2ab > \sigma^2$ is met. Using these estimators, we can now plot the fitted interest rates curve and compare it to the observed zero-coupon interest rate curve.

Figure 4.8: CIR Adjustment



We can see that it is hard to explain the full interest rates curve with only the short rate, and this is a limit of our model. Please note that in theory, we should recalibrate the parameters of the model for each outer scenario but as this work is difficult in practice, we will use the same parameters in our application.

Real world calibration

The calibration under historical world requires to use historical data, thus we use the Euro-Libor (3 months) database. On a first approach, we use the maximum likelihood approach method based on the assumption that the residuals follow a normal distribution. However, in practice, by plotting the distribution of conditional residuals, it appears that the normal assumption is not verified.

Thus, we have tested another method based on the Quasi-Maximum Likelihood approach. It consists in optimizing the log-likelihood function computed as if the normal assumption on residuals was empirically verified. It is interesting to use the observed error terms to simulate instead of simulating Brownian motions. In other words, the error terms simulation would consist in randomly selecting in the observed residuals.

Using the implicit Euler scheme, the assumption on residuals is:

$$\epsilon_{t+1} = \frac{(1+a)\, r_{t+1} - r_t - ab + \dfrac{\sigma^2}{2}}{\sigma\sqrt{r_{t+1}}} \sim \mathbb{N}(0,1)$$

After maximizing the likelihood function, we obtain 3 estimators $(\widehat{a}, \widehat{b}, \widehat{\sigma})$. Given these three estimators, we need to back-test them. The test consists in validating the calibration method.

Back-testing explanation:

For the back-test, the database is cut into several parts with the same length. For instance, we calibrate the model using elements 1 to 100 of the database, thus we obtain estimators, and we check if the (observed) element 101 is in the following interval defined with the percentile observed in the element 1 to 100 (this interval corresponds to the thresholds 1% and 99%):

$$r_{101} \in \left[\left[\frac{\sigma p_{0.01} + \sqrt{\sigma^2 p_{0.01}^2 + R_t}}{2(1+a)} \right]^2 ; \left[\frac{\sigma p_{0.99} + \sqrt{\sigma^2 p_{0.99}^2 + R_t}}{2(1+a)} \right]^2 \right]$$

with

- $R_t = 4(1+a)\left(r_t + ab - \dfrac{\sigma^2}{2} \right)$
- $p_{.99}$ and $p_{0.01}$ are respectively the 99^{th} and the 1^{st} percentile observed in the data used for the calibration.

We repeat this method by using elements 2 to 101, and we check if r_{102} is in the confidence interval. Using a large data base (length 2000) it is possible to verify if the calibration method is appropriate.
We have performed this test, and we have observed that by calibrating 1000 times, the predicted value is outside the interval 9 times. This shows that the model is well adjusted to the threshold.
However, in the context of nested simulations, implementing this method in an ESG is not an easy task, this is the reason why we will use for the application of this thesis the parameters given by the maximum likelihood approach.

4.2.3 Discount Factors

In order to calculate the Best Estimate, we have to determine which discount factors (DF_t) to use to calculate the present value of future cash-flows.
For insurance products, the values of the cash-flows (F_t) used in the calculation of the Best Estimate depend on two sources of risk: the financial risk and the insurance risk. We can represent these two sources with two filtered probability spaces: $(\Omega^f, (F_t^i)_{t\geq 0}, P^f)$ for the financial risk and $(\Omega^i, (F_t^i)_{t\geq 0}, P^i)$ for the insurance risk. Then the Best Estimate is calculated as:

$$BE = \mathbb{E}^{P^f \otimes P^i}\left[DF_t \times F_t\right]$$

Given that the value of the cash-flows only depend on the insurance risk (for example only the mortality):

$$BE = \mathbb{E}^{P^f}\left[DF_t\right] \times \mathbb{E}^{P^i}\left[F_t\right] = P(0,T) \times \mathbb{E}^{P^i}\left[F_t\right]$$

Then the initial zero-coupon curve can be used to calculate the Best Estimate if the cash-flows do not depend on the financial world.
However, in our case, the value of the cash-flow also depends on the evolution of the underlying assets, and this is the reason why we cannot use the initial zero-coupon curve. Especially when we consider interactions between assets and liabilities, we have to remain consistent and use the simulated short rate of each trajectory to calculate the discount factors.

4.3 Correlation

In order to remain consistent with the empirical observations, it is important to introduce a dependence between interest rates evolution and stock returns evolution. To do this, we will use correlated Brownian motions. The Cholesky decomposition is used to build correlated Brownian motions.

4.3.1 The Cholesky decomposition

Every symmetric, positive definite matrix A can be decomposed into a product of a unique lower triangular matrix L and its transpose ${}^T L$:

$$A = L.{}^T L$$

where L is called the Cholesky factor of A, and can be interpreted as a generalized square root of A.

4.3.2 Correlated Brownian Motions

We use correlated Brownian motions as the dependence structure between interest rates and equities.

$$\mathbf{dW}_t^1.\mathbf{dW}_t^2 = \rho.dt$$

$\mathbf{W}_t^1$ and $\mathbf{W}_t^2$ are respectively the Brownian motion used in the equity model and the Brownian motion used in the CIR model. $\mathbf{W}_t^1$ and $\mathbf{W}_t^2$ are correlated , M is the correlation matrix of $W_t = (\mathbf{W}_t^1, \mathbf{W}_t^2)$.

$$M = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

Remark: We note that for $\rho = 1$ ou $\rho = -1$, the correlation matrix is not positive definite.
By using the Cholesky decomposition, there exists a unique matrix L such as $M = L^T L$.
So we need to build two correlated Brownian motions by simulating an independent Brownian motion $B_t = (\mathbf{B}_t^1, \mathbf{B}_t^2)$.
For independence reasons, we show that:

$$cov(B_t) = \begin{pmatrix} var(B_t^1) & 0 \\ 0 & var(B_t^2) \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} = t. \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

By stating $W_t = {}^T LB_t$ comes:

$$cov(W_t, W_t) = cov({}^T LB_t, {}^T LB_t) = {}^T L. \underbrace{cov(B_t)}_{tI_2}.L = t({}^T L.L) = t.M$$

So in order to simulate correlated Brownian motions we have to simulate independent Brownian motions $B_t = (\mathbf{B}_t^1, \mathbf{B}_t^2)$ and the correlated Brownian motions, $W_t = (\mathbf{W}_t^1, \mathbf{W}_t^2)$ are given by the following formula:

$$W_t = {}^T L.B_t$$

4.3.3 Correlation estimation

The estimation of the coefficient ρ depends on the composition of the portfolio. For instance, if the portfolio is composed of European equities, then we can estimate ρ as the historical correlation between the returns of the Euro Stoxx 50 index and the EONIA (Euro OverNight Index Average) for the last ten years.
It is important to note that the estimation of this parameter depends on the choice of historic data. Between 2000 and 2006, we observe a larger correlation between the Euro Stoxx 50 and the EONIA compared to the 2006-2012 data set.
The data set from 2000 to 2012 gives the following result:

Estimation	
ρ	0,07
QIS5 (upward Shock)	0,50

We observe that our estimation is very different from the correlation given in the Fifth Quantitative Impact Study.

4.4 Tests

The regulation mandates some tests on the economic scenario generator.

Test of the martingale: This test is used to ensure that the asset, under risk neutral world, provides an average return equal to the risk-free rate. We have realized this test with 6000 simulations, and we find a relative error of about 0,30%.

Test on the parameters: Parameters of the CIR model are supposed to reproduce the risk-free interest rates term structure. In our case, the test consists in simulating the short rate a lot of times, and comparing the average curve obtained with the risk-free rate curve. Due to the assumptions taken when using a one-factor model, this test was not satisfied. Two factors models would have been more appropriate but the choice of the CIR model is also motivated by its simplicity to put into practice.

Chapter 5

Asset and liability model

Introduction

The previous chapter presented the economic scenario generator. The next step is to present how we use these scenarios to evaluate the situation of the company (cash-in and-out flows) at each step. Thus these scenarios are integrated in an asset and liability model. So in this chapter we first present the method used to evaluate the assets at each step. Then we present the modeling assumptions of liabilities with a deterministic model taken from the mortality table TH0002. After that we detail the steps in the asset and liability model to evaluate the economic situation of the company at each step. The last part of this thesis is dedicated to the implementation of the tool. Indeed, as the method of nested simulations is very time and storage consuming, we have tried to avoid loops in the computer code and we present the matrix calculation method used to implement the model and optimize the run time.

5.1 Asset and liability management

Asset and Liability Management (ALM) reflects the relationship between the liabilities and the assets covering them, taking into account interactions between them.

5.1.1 Asset in ALM

The asset is composed of bonds, equities and a monetary part. Bonds are supposed to be at par. The coupon is also assumed to be fixed. The simulation of an interest rate curve at each step, will allow us to valuate bonds in the portfolio.
The market value of bonds is given by:

$$MV_t = \sum_{k>t}^{T} cashflows(k) \times \exp\left(-R(t,k)\right)$$

R(t,k) is the zero-coupon rate at t of maturity the date of the removal of the coupon k. The market value, MV_t, is used to assess the value of the fund, and also to rebalance the fund each year.
As bonds are considered to be at par, the value at t=0, of the yearly coupon for a bond maturing in t=T, is calculated as:

$$C = \frac{1 - \exp(-R(T))}{\sum_{k=1}^{T} \exp(-R(t,k))}$$

Thus, the value of bonds changes over time, depending on the evolution of interest rates.
As under risk-neutral world the average return is equal to the risk-free rate, we have to remain consistent and **use the one-year interest rate simulated with the CIR model at each step as the risk-free interest rate used in the equity model.** At each step, the portfolio is rebalanced with the same initial risk profile.

5.1.2 Liability in ALM

Guarantee payments are made at the end of each year after deducing the fees. We consider that the premium is a unique premium. In our application we will not study the cover of the guarantees with the risk premium (naked product).

We will study a closed portfolio of policies (there is no new policyholders) starting at t=0, maturing at t=10. The portfolio is studied as a unique contract with a model point. The model point is to represent the average contract, based on the following characteristics:

- age of the policyholder (it can be the average of the group)
- sex of the policyholder
- GMAB guarantee at maturity T
- GMDB guarantee
- Fees charged on the value of the contract

As we do not consider lapse other than mortality, we retrieve at each step, a proportion of q_x of the fund and consider that the proportion of q_x dies. Thus, mortality is taken deterministic.

5.1.3 Asset and liability interactions

Conditioned by the scenario given by the ESG, the ALM model has to make the company one year older. The ALM model proceeds as follows (example to go from t=t to t=t+1):

- Step 1: Valuation of the fund at t=t+1 using the ESG
- Step 2: Calculation of the cash-flows which are fees and the death proportion. It is important to note that the value of the guarantee is not retrieved from the fund, but from the own funds.
- Step 3: Calculation of the cash-flows used in the calculation of the Best Estimate.
- Step 4: Fund rebalancing, after deducing the cash-flows, the fund is rebalanced to its original asset allocation. To rebalance the bonds part, we sell the bonds in our portfolio by calculating their market value and we buy again bonds of the same initial maturity.

5.2 Best Estimate

In this part we describe how we calculate the Best Estimate of a product including a GMAB and a GMDB. The Best Estimate is defined as the expected present value of future cash-flows. It takes into account all the cash in and out flows, required to meet the insurer's obligations over time.

Best Estimate =Insurer's Obligations - policyholders' Obligations

Realistic probability assumptions are essential for the calculation of the Best Estimate and it is calculated under the risk-neutral probability. It is important to describe the cash-flows taken into account in the calculation of the Best Estimate.

Chargings:

As all financial products have fees, the amount depends on the investment fund. The expense charged on the contract are intended to cover the insurer's fees. The difference between real expense and expense received by the insurer is what the insurer expects to earn. Fees are generally separated into:

- An investment management fee: this fee is charged for the management of the fund. It may vary depending on the type of investment.
- Insurance charges: they include administrative and distribution charges. They are generally an annual amount.
- Surrender fees: these charges apply if the policyholder surrenders before the maturity of the contract.

It is important to note that those fees cannot only be seen as cash-in flows for the company, because the company has to pay other actors like a broker for instance. So the only flows that can be seen as a cash-in-flows in the calculation of the Best Estimate is the difference between the fees charged on the contract and the operating costs. The amount of operating costs remains a cash-out-flow.

We consider that the fees received at t are paid by the proportion of the portfolio remaining at (t-1).

Guarantee payment:

Guarantees are cash-out-flows for the company. Under the notations:

- G_t^D the death guarantee at t
- G^A the accumulation guaranteed amount at the maturity of the contract
- f the fixed percentage of fees charged on the value of the fund
- F_t is the fund value at t for the number of remaining living policyholders. At each step, the surrender value of the units for the dead policyholders and charged fees are retrieved from the fund whereas the value of the guarantee is not retrieved from the fund but from the own funds of the company.

Let's consider that the GMDB is characterized by a payment at the end of the year of $max(G_t^D, (1-f).F_t)$ if the policyholder dies between t and t-1 (we consider that the amount value of the fund taken in the calculation of the cash-flow is the value of the unit after deducing the fees), and the GMAB is characterized by a payment of $max(G^A, (1-f).F_t)$ at the maturity of the contract.

Under the following assumptions:

- Payments are done at the end of the year
- Policyholders, aged x, die at the ends of years
- fees are a percentage of the fund value at each end of year and are separated into f_{real} and f_c, respectively the real fees (paid by the insurance) and the charged fees. In fact real fees depend on the number of policies but in this formula to simplify we consider that real fees are a percentage of the fund value.

Thus, the Best Estimate at t=1 is calculated as (The cash-flows at t=1 has already been deduced from the asset of the company):

$$
\begin{aligned}
\text{BE} &= \frac{1}{S}\sum_{i=1}^{S}\sum_{t=2}^{T} DF_t^i \left[{}_{1/t-1}q_x.\max\left(G_t^D; F_t^i.(1-f_c)\right) + \left(f_{real}-f_c\right).F_t^i\right] \\
&+ DF_T^i.\left[{}_T p_x.max\left(G^A;(1-f_c).F_T^i\right)\right]
\end{aligned}
$$

with:

- F_t^i is the value of the fund at time t for the inner scenario i and S is the number of inner scenarios.

- DF_t^i is the discount factor at time t for the inner scenario i.

- ${}_{1/i-1}q_x$ the probability that a person aged x dies between i-1 and i.

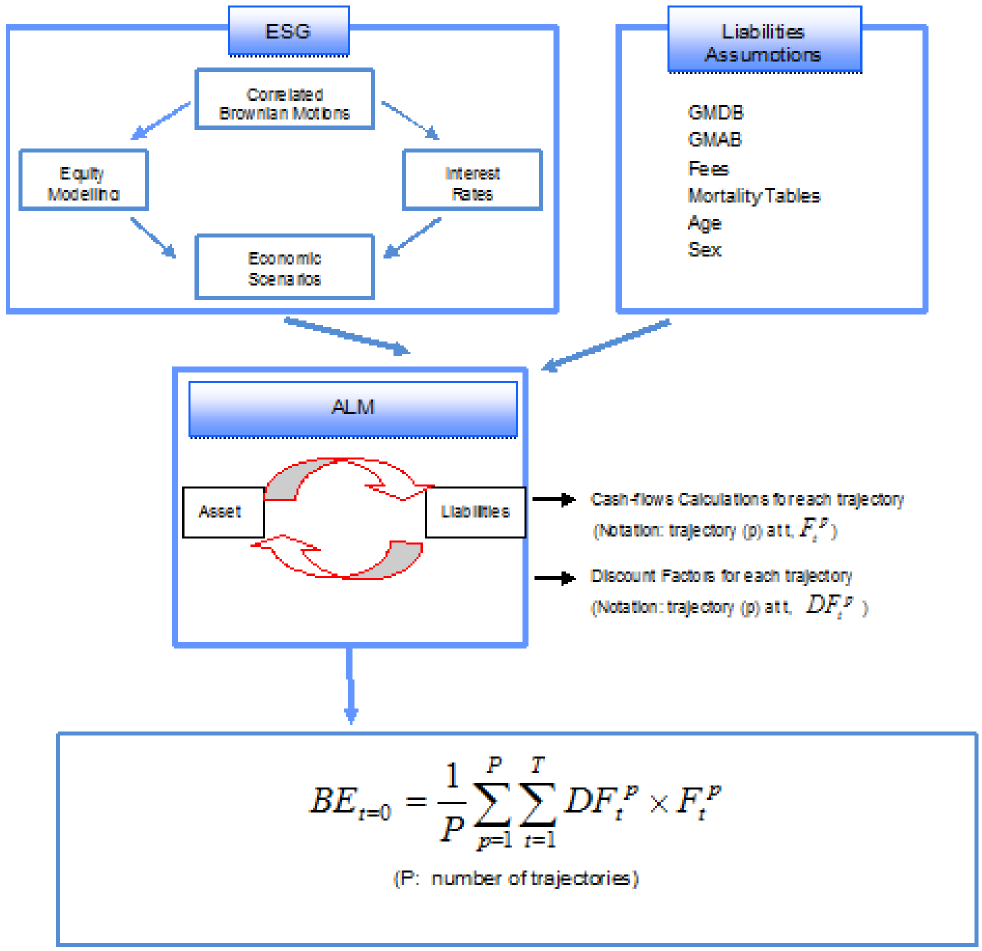
- ${}_T p_x$ the probability that a person aged x is still alive after T years.

It is important to note that when we use ${}_{1/i-1}q_x$ and ${}_T p_x$ from a mortality table, we assume that the mortality rates do not change in the future (risk of mortality trend).

5.3 Global structure

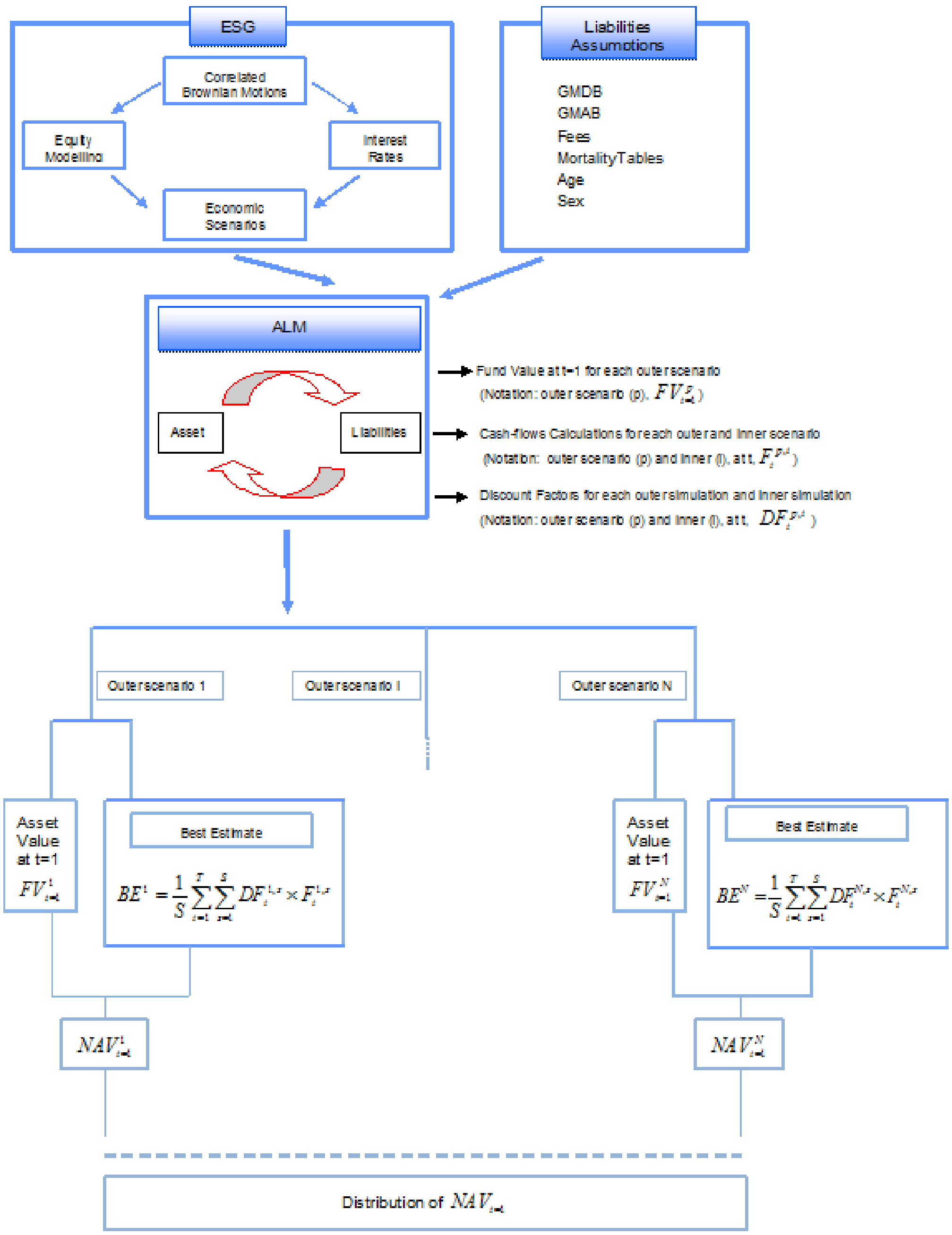
In order to calculate the net asset value at t=0, we need to calculate the Best Estimate at t=0. The following picture presents the structure of the estimation of the Best Estimate at t=0 ($BE_{t=0}$).

Figure 5.1: Best Estimate calculation at t=0



The following picture presents the architecture of the estimation of the distribution of the net asset value at t=1 used for the market risk module calculation.

Figure 5.2: $NAV_{t=1}$ calculation

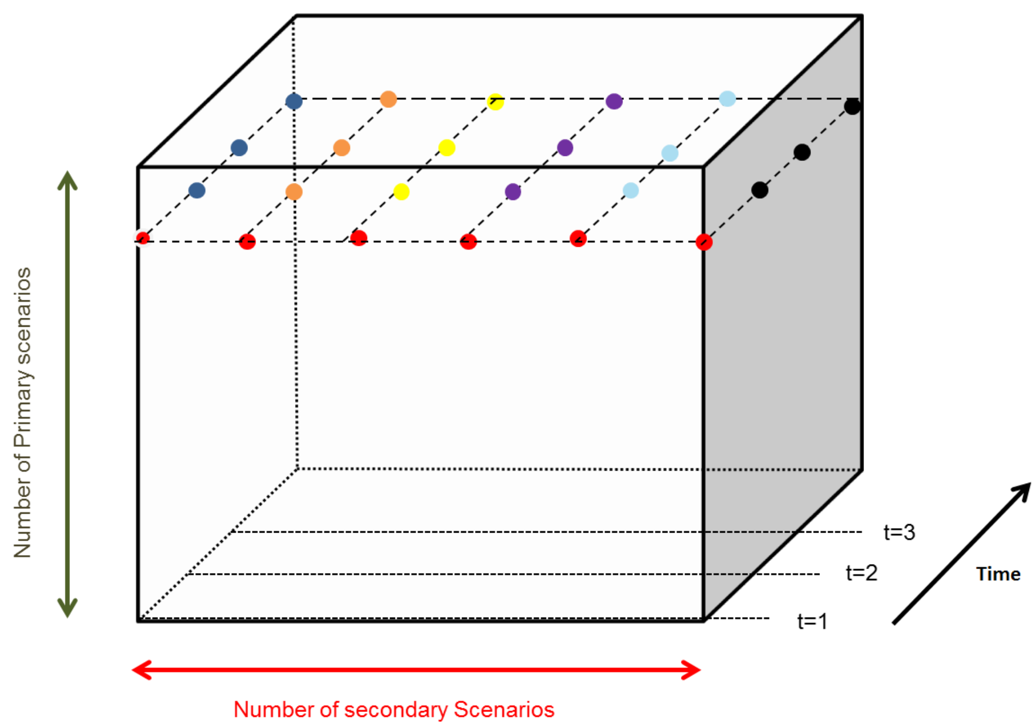


5.4 Implementation

The implementation was made using MATLAB (Matrix Laboratory). MATLAB is a programming environment for algorithm development, data analysis, visualization, and numerical computation. During the implementation several difficulties had to be faced. The major difficulty to face was the run time and the storage necessary when using the nested simulation approach. In order to take up this calculation challenge we have used matrix calculus. Indeed, MATLAB is particularly powerful in the manipulation of matrix calculus. This is the reason why, the implementation has followed one main idea: **using matrix operations instead of loops**.

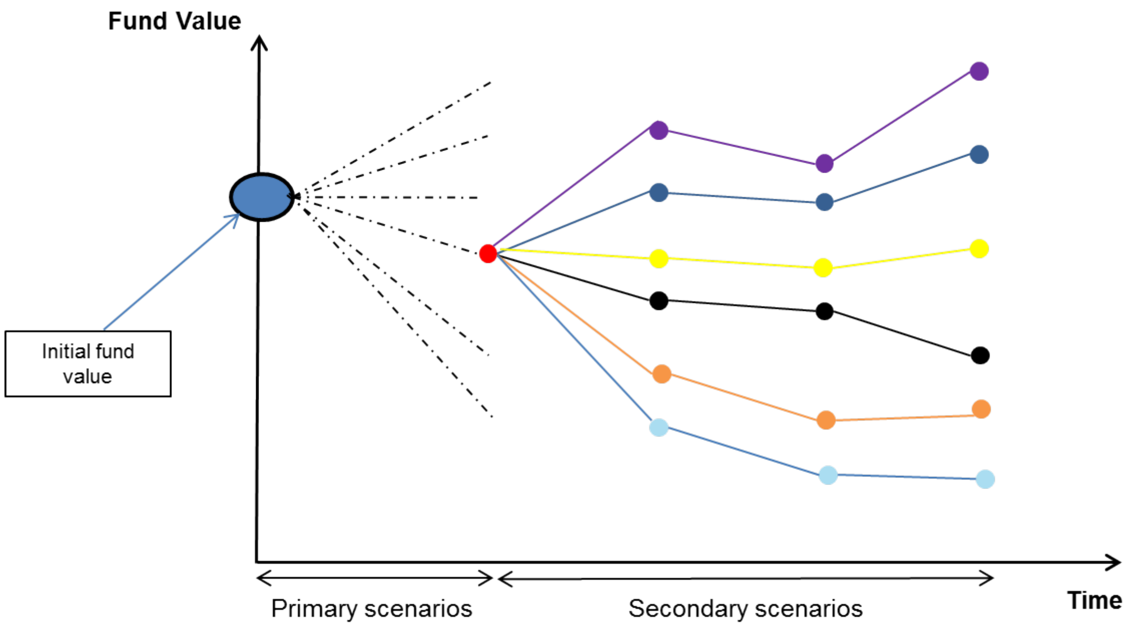
In our implementation, we use three dimensional matrices. The following picture illustrates the ESG implementation.

Figure 5.3: Three dimensional matrix for the ESG



On the above picture, the red points of the cube represent a realization of the fund at t=1 (an outer scenario). As we have to make a second level of simulation, we repeat the value of the first realization on the line of the matrix, this is the reason why there are several red points. The other points ($t > 1$) represent the secondary scenarios conditioned to the realization of the primary scenario (red point). Each floor of the matrix represents an outer scenario and the inner scenarios conditioned to it. The following picture illustrates the mechanism of the three dimensional matrix presented above.

Figure 5.4: Illustration of the implementation



Using three dimensional matrices allows to calculate the three dimensional cash-flows matrix for the Best Estimate. We also use three dimensional matrices to store realizations of the CIR model, and this simplifies the calculation of the present value of future cash-flows. Once we have the matrix of cash-flows, we can use basic matrix operations with MATLAB. It is important to note that the time step of the simulation was one year, if we had chosen a shorter time step the run time and the storage would have significantly increased this is to say that even if the nested simulations concept is a very accurate method this method is limited by the capacity of computers.
Implementing with three dimensional matrices has reduced time computing, but it demands large matrices storage and the capacity of storage is limited. However using matrix calculation instead of loops is a good way to optimize the implementation.

Chapter 6

Application

Introduction

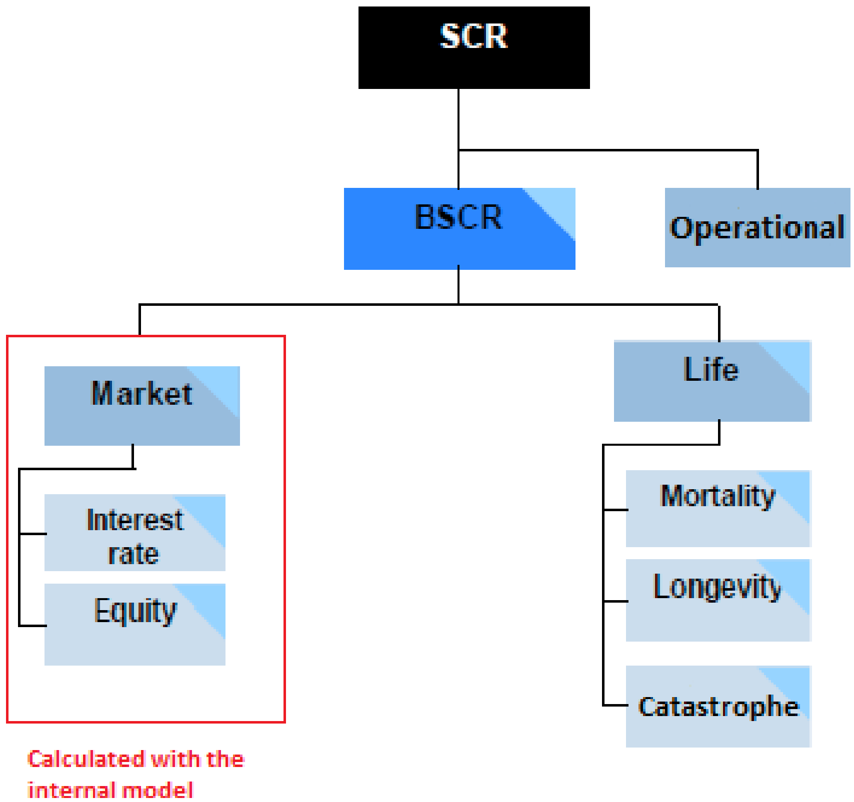
This chapter presents an application of the implementation performed so far. The aim of this chapter is to present the capital calculation for products of types GMAB and GMDB. The first section is dedicated to the presentation of the context of the application. Then, we illustrate the calculation of the capital requirement for interest rates and equity risks with the nested simulations approach through a sensitivity study. Please note that in our application we approximate the market SCR by the aggregation of the equity required capital and the interest rates required capital. Normally to talk about market SCR we should add the other risks given in the standard formula. To complete the application, we have included our study in the context of a simplified partial internal model approach by estimating the life SCR with the shocks given by the standard formula. Thus this chapter illustrates a sensitivity study of the required capital for interest rates and equity risks but it also exemplifies the calculation of the life SCR using the shocks of the standard formula.

6.1 Context of our application

We consider a company who sells contracts maturing in ten years. To improve the visibility of the study of sensibility of the market risk, we will study three cases: in the first one the company sells contracts including just a GMAB (there is no GMDB), in the second one the company sells contracts including just a GMDB (there is no GMAB), and in the third case the company sells contracts including both guarantees. The third case will be used to illustrate the study of the convergence in the next section.

In the following application we will make some simplifying assumptions in order to have a first approximation of the solvency capital requirement. The following graph illustrates the structure adopted:

Figure 6.1: Simplified structure of the SCR



As our contract only includes in-case-of survival and in case-of-death guarantees, we do not have to consider the modules Health and Non-Life.
Moreover the following assumptions are also made:

- There is no loss-absorbency of technical provisions and deferred tax compensating the loss of the insurer. *Adjustment* = 0.

- The risk arising from intangible assets is equal to zero.

- The counterparty default risk is equal to zero. This assumption is a rather strong assumption in the current economic context but it remains consistent if we consider a portfolio composed of sovereign bonds with high rating.

Concerning the market risk module, we simplify by doing the following assumptions:

- As there is no investment in property: *Property*=0.

- The spread risk is equal to zero: it is a strong assumption especially in the current context (recent downgrades and default of governments).

- Assets, liabilities and all the cash-flows of the company are in the same currency. Thus the currency risk is equal to zero: $Currency = 0$.

- The portfolio is well-diversified: the concentration risk is equal to zero. Thus, $concentration$=0.

- The liquidity is supposed sufficient.Thus, $Illiquidity$ =0.

For the life module, as products contain life guarantees we will consider:

- Management expenses remains stable, thus the expense sub-module is equal to zero. This assumption is consistent if the product has been sold for a long time, management fees are known.

- The lapse has not been modeled, so we will not study the lapse and the lapse sub-module is equal to zero.

- There is no disability or morbidity guarantee.

On the liability side:

The portfolio of the company is composed of a group of policyholders. We consider that the group is homogeneous. This assumption is rather consistent, as in practice policyholders are generally grouped with a model point which consists in "calculating the average characteristics of the group". The group shares the following characteristics:

- Male, aged 60

- The death probability is taken from table TH0002

- They all own the same kind of policies:

  - We consider that policyholders pay a single premium at the beginning of the contract.

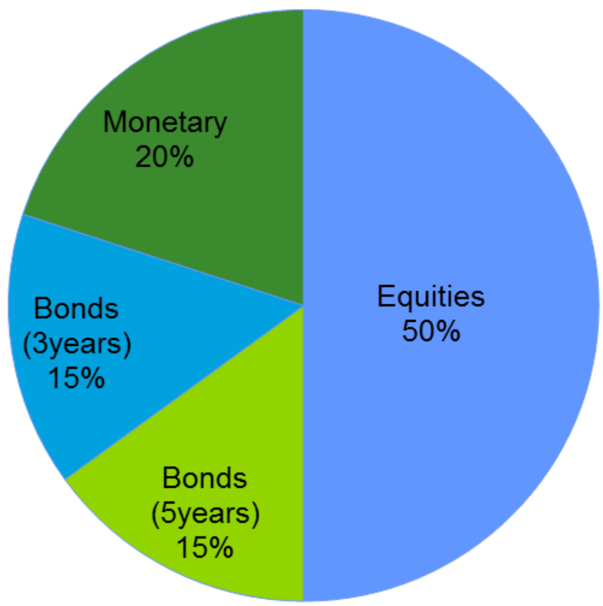
  - We first consider that the death guarantee (GMDB) is a 2% roll-up guarantee.

  - We also first consider that the accumulation guarantee pays the initial amount invested.

  - 0.50% are charged each year on the value of the contracts. We consider that company's costs are equal to charged fees.

- In our application, the lapse is not modeled, and only the q_x proportion is retrieved from the fund.

The investment of the unit-linked is allocated as follows:



The above pie-chart shows that the portfolio includes three-year maturity bonds and five year-maturity bonds. As the maturity of the policies is ten years, the duration of the bonds' part can appear inconsistent from an ALM point of view but our implementation re-balances the fund each year as the initial defined risk profile. Thus, bonds are sold every year and new bonds of maturity 3 and 5 years are bought every year. The coupons are calculated depending on the interest rate curve. Equities have a volatility of 20% (σ) and an historical return of 5%. The equity model used is the Merton model, and consequently to the result of the section about the calibration of the Merton model, we have adjusted the parameters so that the Poisson component explains 60% of the total variance and the Brownian part 40%. The own funds are invested safely, on a monetary fund. We will consider that the risk premium for the guarantees is included in the own funds. The hedging strategy of the guarantees is not studied in this application.

Thus the accounting balance sheet of the company at t=0 is:

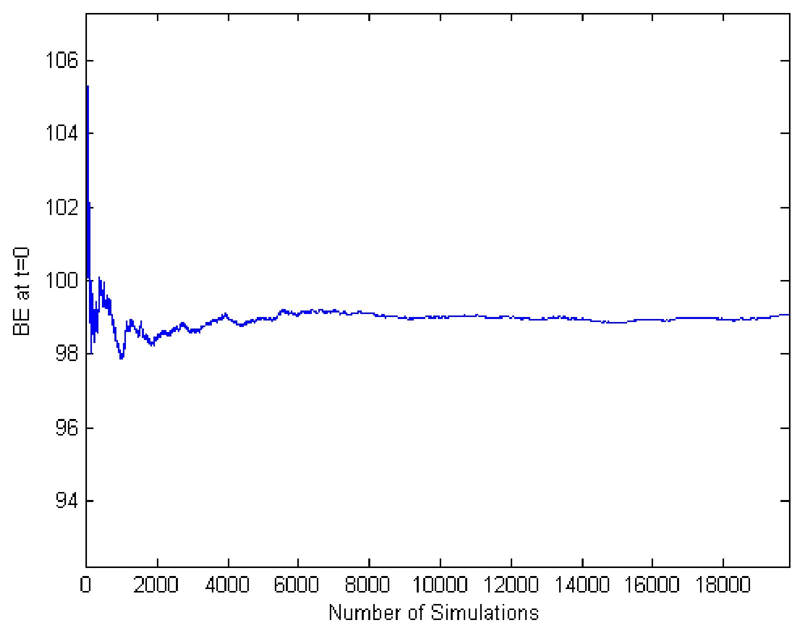
Accounting Balance sheet at t=0			
Asset		Liabilities	
Equities	50 000 000	Own Capital	20 000 000
Bonds	30 000 000	Mathematical provisions	100 000 000
Monetary	40 000 000		
Total	120 000 000	Total	120 000 000

We will make vary the characteristics of the contracts (a contract including just a GMAB, or just a GMDB, or including both). We consider that the accounting value of assets is equal to the market value of assets at t=0. Thus, there is no unrealized gain or loss of the assets portfolio.
Before calculating the economic balance sheet, we need to test the convergence of our tool and determine the numbers of simulations to use.

6.2 Convergence

Studying the convergence is important as we need to determine a sufficient number of simulations, and we need to control the accuracy of the estimation by calculating a confidence interval. The following graph presents the convergence of the Best Estimate at t=0 depending on the number of simulations.

Figure 6.2: Convergence of the Best Estimate at t=0



The convergence is quite slow, given the above picture, we will take 6000 simulations. The study of the convergence of the Best Estimate at t=1 is similar to the previous convergence study, thus we will take 6000 inner scenarios.

As we previously mentioned, the Best Estimate at t=0 is calculated as:

$$BE_{t=0} = \frac{1}{P}\sum_{p=1}^{P}\sum_{t=1}^{T} DF_t^p \times F_t^p$$

We consider that our trajectories are independent and follow the same distribution, (the theory also precises that variables are square-integrable) then we can built a confidence interval (level 95%). As the calculation of the cash-flows implies a maximum, the distribution is not a Gaussian distribution. However using the Strong Law of Large Numbers and the Central Limit Theorem, we can determine a confidence interval:

$$\begin{cases} \left[\widehat{BE}_{t=0} - 1.96 \times \frac{\widehat{\sigma}}{\sqrt{P}} \; ; \; \widehat{BE}_{t=0} + 1.96 \times \frac{\widehat{S_p}}{\sqrt{P}}\right] \\ \widehat{S_p} = \sqrt{\frac{1}{P}\sum_{p=1}^{P}\left(\widehat{BE}_{t=0}^{p} - \widehat{BE}_{t=0}\right)^2} \end{cases}$$

Numerically, we find a confidence interval for the Best Estimate at t=0, for P=6000 simulations:

$$BE_{t=0} = 98\ 918\ 200€ \pm 0.21 \text{ million}$$

If we take $BE_{t=0}$ =98 918 200€ for the value of the estimation (as we have to choose one), then the economic balance sheet is presented as follows:

Economic Balance sheet at t=0			
Asset		Liabilities	
		$NAV_{t=0}$	21 081 800
Market Value of Assets	120 000 000	Best Estimate	98 918 200
Total	120 000 000	Total	120 000 000

We also need to determine the number of outer scenarios for a sufficient convergence of the quantile of the net asset value at t=1. As it is more complicated to determine a confidence interval for the estimation of the quantile of the $NAV_{t=1}^{mkt}$, and as we are limited by the run time, we will limit the number of outer simulations to 5000. In order to study the convergence of the calculation of $NAV_{t=1}^{mkt}$, we will calculate the relative error of the estimator of the $NAV_{t=1}^{mkt}$ compared to the result estimated with 5000 outer simulations and 6000 inner simulations. The following table summarizes the results:

Number of outer simulations	Relative Error
1000	4.1%
2000	2.6%
3000	0.74%
4000	0.16%

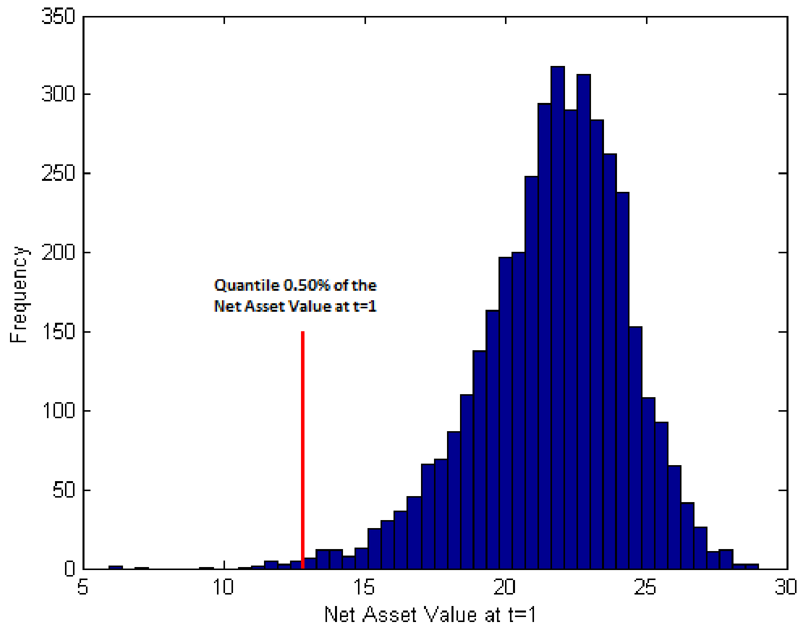
Given the above results, we will use 6000 inner scenarios and 4000 outer scenarios. Indeed, the implementation of this application with three dimensional matrices demands a lot of storage but our simplified case, the storage is not so large and this allows us to have results rather quickly.

Thus, using 4000 outer simulations and 6000 inner simulations, we estimate in our example the value of the quantile 0.50% of the net asset value at t=1 (for market risk):

$$q_{0.005} NAV_{t=1}^{mkt} = 13\ 139\ 664\ €$$

This value was extracted from the following distribution of $NAV_{t=1}^{mkt}$:

Figure 6.3: Distribution of the Net Asset Value at t=1



We can deduce from this study, the market SCR of our example:

$$SCR_{market} = 8\ 046\ 202€$$

6.3 Sensitivity study

In this section, we will study the sensitivity of capital requirement for market risk. In our model, this capital requirement is calculated as:

$$SCR_{Market} = NAV_{t=0} - P(0,1)q_{0.005}NAV_{t=1}^{mkt}$$

We will study the sensitivity of SCR_{Market} to:

- The Guarantees
- The allocation of the unit-linked
- The volatility of equities
- The age of policyholders

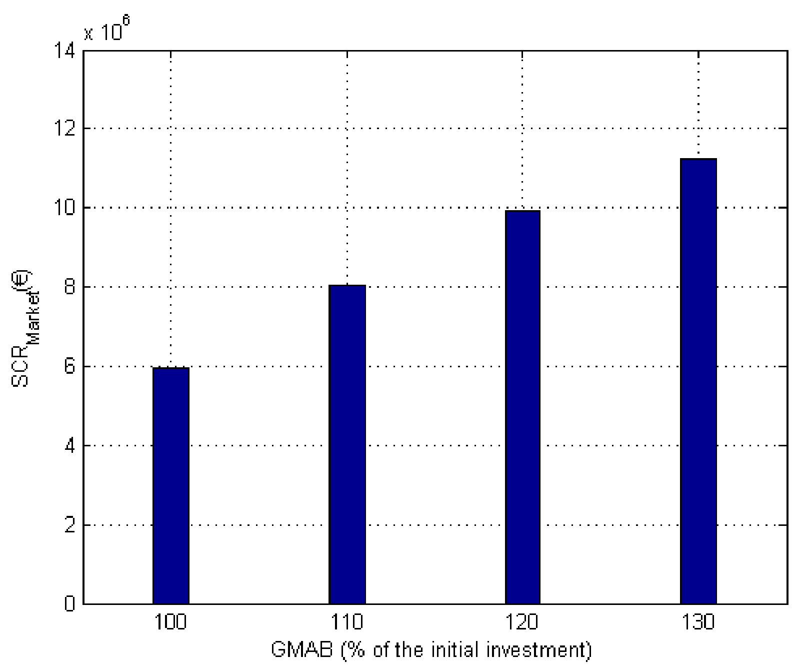
6.3.1 Sensitivity to the guarantees

In this section we want to study the sensitivity of SCR_{Market} to the guarantees. Let's first consider that the company sells policies including just a GMAB whose characteristics are given in the table below:

Table 6.1: Sensitivity of the SCR_{Market} to the GMAB guarantee

GMAB Characteristics (% of the initial investment)	SCR_{Market} **(€)**
100%	5 962 996
110%	8 053 923
120%	9 935 567
130%	11 229 981

Figure 6.4: Impact of the GMAB guarantee on the SCR_{Market}

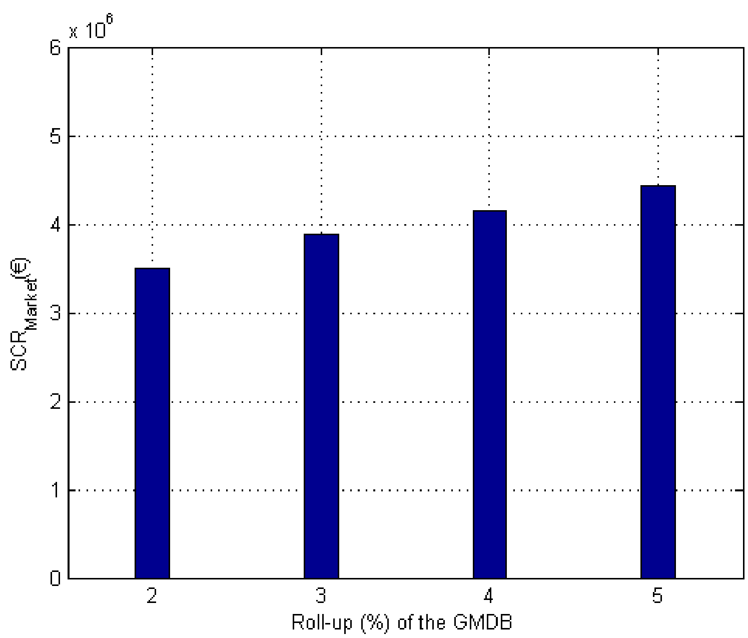


The above table shows that if we increase the guaranteed amount of the GMAB, the risk increases for the company. When the percentage of the guarantee goes from 100% to 130%, the SCR_{Market} increases of about 90%. Indeed, when increasing the guarantee of the GMAB, the number of unfavorable scenarios increases (scenarios where the fund has less performed than the guarantee), so the company must have more capital to meet its obligations with the probability of 99.5%. The increase of SCR_{Market} is quite fast, and this is the reason why a company must be careful to the characteristics of the guarantee. Let's now study the sensitivity of the SCR_{Market} to the death guarantee. For more visibility, in this sensitivity study we consider that the contract includes only a GMDB. The following table presents the results:

Table 6.2: Sensitivity of the SCR_{Market} to the GMDB guarantee

GMDB Characteristics (Roll-up rate)	SCR_{Market} **(€)**
2%	3 502 124
3%	3 884 298
4%	4 156 061
5%	4 437 428

Figure 6.5: Impact of the GMDB guarantee on the SCR_{Market}



Similarly to the accumulation guarantee, when the GMDB roll-up rates increases, the SCR_{Market} also increases as expected (the number of unfavorable scenarios increases). When the roll-up rate goes from 2% to 5% the SCR_{Market} increases of about 26%. This shows that an increase in the roll-up rate has a significant impact on the SCR_{Mkt}.

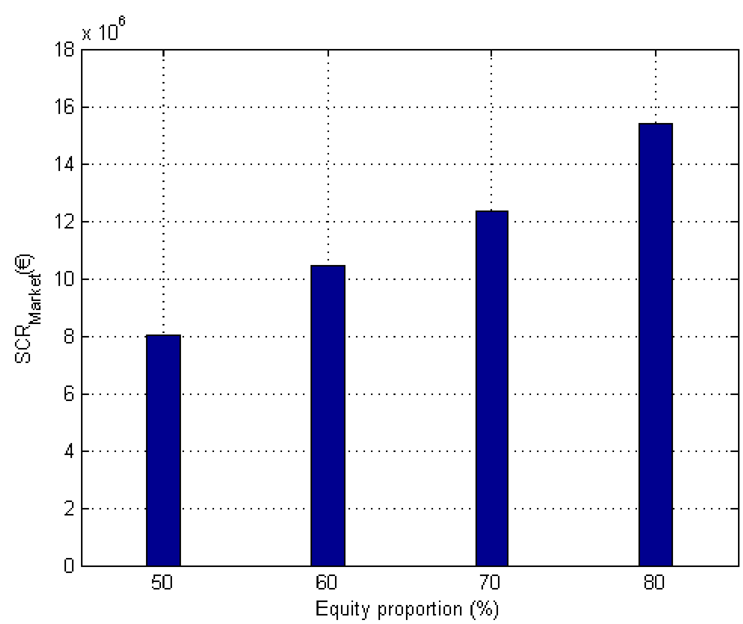
6.3.2 Sensitivity to the allocation

In this section, we present the sensitivity of the SCR_{Market} to the risk profile of the investment. We study the sensitivity for contracts including both a GMAB (100% of the initial investment) and a GMDB (2% roll-up). In order to study the allocation of the investment, we make vary the equity and bonds parts without changing the monetary part. The following table presents the results:

Table 6.3: Sensitivity of the SCR_{Market} to the allocation

Allocation	Equities	50%	60%	70%	80%
	Bonds	30%	20%	10%	0%
	Monetary	20%	20%	20%	20%
SCR_{Market} (€)		8 046 201	10 470 101	12 349 461	15 390 331

Figure 6.6: Impact of the equity proportion on the SCR_{Market}



The results show that when the equity part of the fund increases, the SCR_{Market} increases (nearly linearly). Thus, it means that requirements increase when the equity part increases. This is an expected result, indeed, as the volatility of the fund increases, the SCR_{Market} also increases. As the volatility of equity returns is larger than the volatility of bond returns, it appears consistent that the SCR_{Market} increases when the equity part increases. This shows that if the insurer wants to reduce its exposure to market risk, a possibility is to reduce the part of the fund invested in equities.

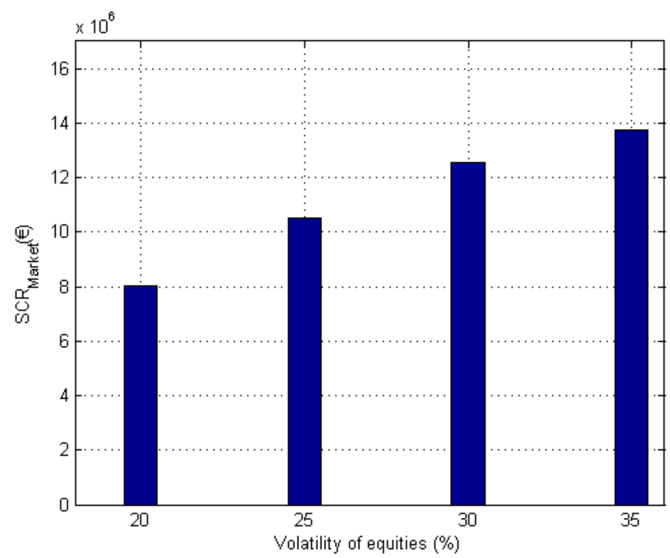
6.3.3 Sensitivity to the volatility of equities

In this part, we test the sensitivity of the SCR_{Market} to the volatility of equities in the fund. We illustrate this example with the same initial allocation (which is 50% equity, 20% monetary and 30% bonds) and same guarantees (GMAB 100%, and GMDB 2% roll-up). The following table presents the results:

Table 6.4: Sensitivity of the SCR_{Market} to volatility of equities

Volatility of equities	SCR_{Market}**(€)**
20%	8 046 201
25%	10 488 521
30%	12 556 212
35%	13 757 942

Figure 6.7: Impact of the volatility of equities on the SCR_{Market}



The results show that the SCR_{Market} is very sensible to the volatility of equities. Thus if the volatility of equities goes from 20 to 35%, the SCR_{Market} increases by about 70%, which is a considerable increase. Thus if the insurer wants to reduce its exposure to market risk, a possibility is to invest in less volatile equities and as we can see, the decrease is quite fast.

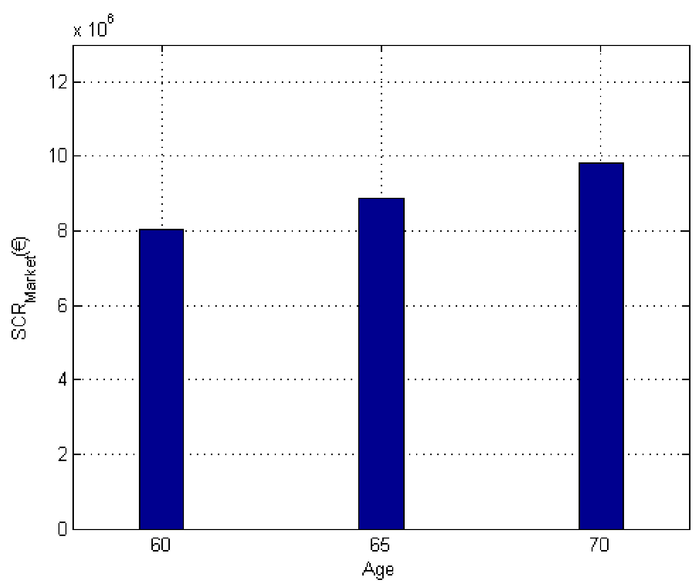
6.3.4 Sensitivity to the age of policyholders

In this part, we study the sensitivity of the SCR_{Market} to the age of the policyholders at t=0. Longevity and mortality are not market risks, however this sensitivity study shows that our model is sensible to the characteristics of the portfolio. The following table summarizes the results:

Table 6.5: Sensitivity of the SCR_{Market} to the age of policyholders

Age	SCR_{Market}(€)
60	8 046 202
65	8 884 205
70	9 815 256

Figure 6.8: Impact of the age of policyholders on the SCR_{Market}



The results given in the above table have been tested on the basic contract which includes a 2% roll-up GMDB and a 100% GMAB. The results show that the SCR_{Market}

increases when policyholders are older. Indeed, as contracts include a GMAB and a GMDB, both longevity risk (the probability that a person is still alive at a date is under-estimated) and mortality risk (people die prematurely) exist for these contracts. However, in our example, the value guaranteed by the GMDB (2% roll-up) is higher than the value guaranteed by the GMAB (100%), and moreover, the GMDB guarantee is generally applied before the GMAB guarantee, so if policyholders are older, it means that it is more likely that more policyholders die and use the death guarantee, what means that the company pays a larger amount (and earlier) when the death guarantee is applied (compared to the GMAB).

6.4 Solvency Capital Requirement

In this section we illustrate the calculation of the Solvency Capital Requirement with our internal model. We remind that we consider that the company sells contracts including a GMAB (100% of the initial investment) and a GMDB (2% roll-up guarantee).

Market risk module:
The SCR_{Market} has already been calculated in the previous section with our internal model.

$$SCR_{Market} = 8\ 046\ 202 \text{ €}$$

Life risk module:
We have to calculate the life risk module. In our case, we have shown that we have to calculate the longevity sub-module, the mortality risk sub-module and the catastrophe risk sub-module.

Mortality risk sub-module:
The mortality risk is associated to the death guarantee. Article SCR.7.16 in the QIS5 stipulates for the mortality shock, "*A permanent 15% increase in mortality rates for each age and each policy where the payment of benefits (either lump sum or multiple payments) is contingent on mortality risk*". By applying this shock we find a capital requirement for the mortality risk sub-module of:

$$Life_{Mort} = 162\ 923\text{€}$$

Catastrophe risk sub-module:
The catastrophe risk sub-module in the QIS5 aims to capture extreme mortality event. Article SCR.7.82 in the QIS5 stipulates for the life catastrophe shock, "*Absolute increase*

in the rate of policyholders dying over the following year of 1.5 per mille (only applicable to policies which are contingent on mortality)". By applying this shock we found a capital requirement for the catastrophe sub-module of:

$$Life_{Cat} = 1\ 055\ 921€$$

Longevity risk sub-module:
The longevity risk sub-module in the QIS5 is associated to the GMAB guarantee. Article SCR.7.28 in the QIS5 stipulates for the longevity shock,"*a (permanent) 20% decrease in mortality rates for each age and each policy where the payment of benefits (either lump sum or multiple payments) is contingent on longevity risk*".By applying this shock we find:

$$Life_{Lg} = 0$$

This result is consistent with the results of the sensitivity study as we have observed that when the contracts includes a GMAB and a GMDB the company is exposed to the longevity risk or to the mortality risk. In our case, if policyholders live longer, then less death guarantees will be applied but more accumulation (GMAB) will be applied. But the death guarantee amount is larger than the accumulation guarantee and the death guarantee is also paid earlier. Thus it is expected to find a risk capital for longevity equal to zero.

Aggregation of the life module:
In order to aggregate the life risk sub-modules, the following correlation matrix is given in the QIS5 (Article SCR.7.7):

Table 6.6: Correlation coefficients for the life risk module

	Mortality	Longevity	Catastrophe
Mortality	1	-0.25	0.25
Longevity	-0.25	1	0
Catastrophe	0.25	0	1

Using this matrix we find the capital requirements for the life module:

$$SCR_{Life} = \sqrt{Life_{Mort}^2 + Life_{Lg}^2 + Life_{Cat}^2 + 0.5 \times Life_{Mort}\,(Life_{Cat} - Life_{Lg})}$$

Numerically, we found:

$$SCR_{Life} = 1\ 107\ 939€$$

Basic Solvency Capital Requirement:
The EIOPA precises that "*In order to fully integrate the partial internal model into the standard formula for the purpose of calculating the Solvency Capital Requirement, insurance and reinsurance undertakings should use the correlation matrices of the standard formula set out in these technical specifications*". Thus, after calculating the capital requirement for the life risk module, we can now use the correlation coefficient between the market risk module and the life risk module given in the QIS5 which is 0.25 (Article SCR.1.32) to calculate the Basic Solvency Capital Requirement.
The BSCR is calculated as:

$$BSCR = \sqrt{SCR_{Market}^2 + 0.50 \times SCR_{Market} \times SCR_{Life} + SCR_{Life}^2}$$

We found:

$$BSCR = 8\ 392\ 034€$$

Operational module:
The operational risk module represents the risk arising from inadequate actions or events. The definition of this risk and the calculation of its module is given in the QIS5 (Article SCR.3.6):

$$SCR_{op} = min\left(0.3 \times BSCR; Op\right) + 0.25 \times Exp_{ul}$$

Where:

- Exp_{ul} is the amount of annual expenses incurred during the previous 12 months in respect life insurance where the investment risk is borne by the policyholders.
- Op is the basic operational risk charge for all business other than life insurance where the investment risk is borne by the policyholders.

As we do not consider the previous twelve months,$Exp_{ul} = 0$.
Elements before t=0 are taken equal to zero. We calculate Op with the standard formula in the QIS 5 and we find Op=0. Thus, the operational risk module is equal to zero.

$$SCR_{Op} = 0$$

Solvency Capital Requirement calculation:
Finally, in our case, the SCR is calculated as:

$$SCR = BSCR + SCR_{Op} = BSCR$$

Numerically,

$$SCR = 8\ 392\ 034€$$

Risk Margin:
Several simplification are proposed in the QIS5 in order to estimate the risk margin. We have chosen the method using the modified duration. Article TP.5.49. in the QIS5 stipulates:

$$CoCM = (\frac{CoC}{(1+r_1)}) \times Dur_{mod}(0) \times SCR_{t=0}$$

Where:

- $CoCM$ the risk margin.
- $SCR_{t=0}$ is the solvency capital requirement at t=0.
- $Dur_{mod}(0)$ is the modified duration.
- CoC is the cost-of-capital rate given in the QIS 5 (equal to 6%).

To calculate the modified duration we first calculate the duration:

$$Dur(0) = \frac{\sum_t \frac{tF_t}{(1+r_t)^t}}{\sum_t \frac{F_t}{(1+r_t)^t}}$$

We calculate this duration using the cash-flows of the central scenario.
We find:

$$Dur(0) = 9.2059$$

And then, we deduce the modified duration by:

$$Dur_{mod}(0) = \frac{Dur(0)}{(1+r_a)}$$

Where r_a is the actuarial rate equal to the constant rate that gives the same duration.
We find:

$$r_a = 3.88\%$$

$$Dur_{mod}(0) = 8.8621$$

Thus we can calculate the risk margin:

$$CoCM = 4\ 450\ 291\ €$$

Solvency II Balance sheet:
For our application, we obtained the following Solvency II balance sheet (in millions of euros):

Figure 6.9: Solvency II balance sheet

Solvency Ratio:
The solvency ratio reflects the solvency of the company. It is defined as the ratio between the own funds and the SCR: numerically we find **1.982**. This value describes a relatively satisfying solvency of the company.

Chapter 7

Perspective

In this thesis, we have presented a partial internal model using nested simulations. A perspective of the project is to extend the partial internal model to a full internal model. However, we have seen that using the method of nested simulations involves large storage and run time, and we are conscious that even a good optimization of the calculation with three dimensional matrices will not be sufficient enough to model a whole company. The nested simulations method is probably the most accurate method, but other methods exist to take up this calculation challenge. Thus in this part, we present one of them, the Least Squares Monte Carlo.

7.1 The Least Squares Monte Carlo approach

The Least Squares Monte Carlo approach is similar to the full nested simulations: the LSMC approach consists in using the same numbers of outer scenarios but we reduce the number of inner scenarios (risk-neutral scenario) to one per outer scenario. Using only one inner scenario leads to very inaccurate liability valuations, but the LSMC consists in doing a regression through these very inaccurate valuations, hence the name Least Squares. We then use the fitted regression curve instead of the inaccurate valuations obtained with a single inner scenario. The following graph explains the LSMC approach:

Figure 7.1: Least Squares Monte Carlo approach

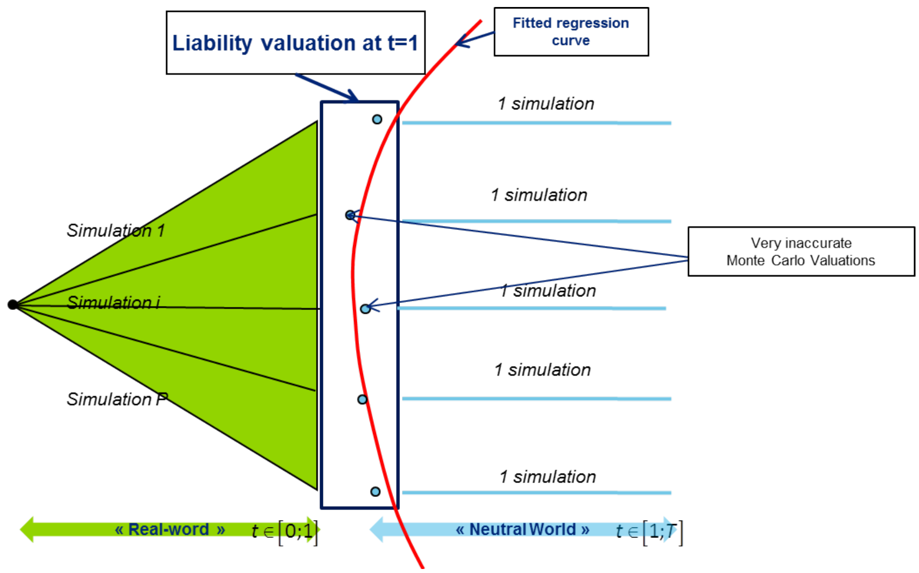
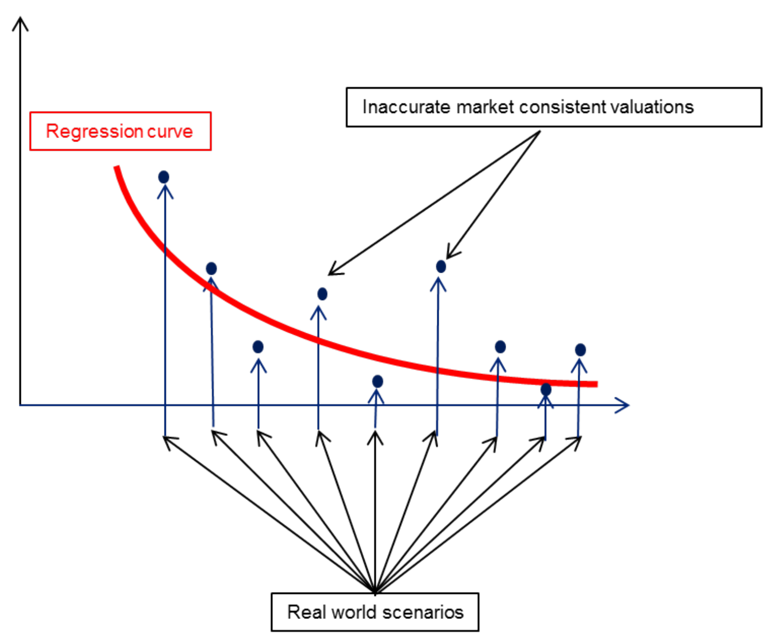


Figure 7.2: Least Squares Monte Carlo regression curve



The Blue points on the first graph represent the net asset value at t=1 obtained by a single inner scenario. The red curve represents the fitted regression curve that will be used instead of simulating a lot of inner scenarios. Thus, on the regression, the liability

valuation at t=1 is the response variable and the explanatory variables are the input risk drivers of the economic scenario generator. The regression function gives us an approximation of the liability value at t=1. This concept can appear quite unlikely, but it is used and it works.
The perspective of the project is to create a full internal model using the Least Squares Monte Carlo method. Even if this method can appear simple, finding the right regression function is a real challenge.

Conclusion

With the arrival of the Solvency II Directive, insurers must prepare themselves to have a more accurate view of the risks they bear, but also to have an economic view of their balance sheet. The directive requires the calculation of the solvency capital requirement and as we described in this thesis, this capital can be estimated by using a standard formula or a (full or partial) internal model. However sellers of variable annuities will not be allowed to use the standard formula. Thus, the aim of this thesis was to present a method to estimate a required capital with an internal model but also with the standard formula for those products, and more precisely for GMAB and GMDB products.

We have presented a simplified partial internal model that calculates the SCR in two steps: first the required capital for the market risk (simplified to the interest rates and equity risks) is estimated through our internal model, and then the shocks of the standard formula are used to calculate the required capital associated to the life risk module. As in the first step we need to estimate the distribution of own funds at the end of the first year, we have used the method of nested simulations. The implementation of this method requires two levels of simulations. The first step to develop it was to model the assets. We chose the Merton model for the equity part as it takes into account discontinuities due to the brutal arrival of positive or negative information and as it presents fat tails of distribution. The one-factor interest rate model Cox-Ingersoll-Ross was used, the choice of this model was motivated by its simplicity to put into practice but also because it does not produce negative interest rates. The choices of the models are important, but we have shown that insurers will also have to focus on their calibrations. Also in this thesis, we have distinguished the risk-neutral calibration from the real world calibration. After modeling the asset, we have determined the cash-flows for the Best Estimate calculation. And finally, we have taken into account the interactions between the assets and the liabilities with an ALM model.

The implementation of this tool with MATLAB was the major part of this thesis. It is important to highlight the means to put in place in order to implement the model. This is also the reason why the model's complexity will also be limited by the means available to the insurer.

Thanks to the tool developed, we could test on a simplified case the sensitivity of the required capital for market risk to the characteristics of the guarantees and to the structure of the investment. Thus, the results of our sensitivity study show that insurers can reduce their exposure to market risk by decreasing the proportion of the fund invested in equities, or by investing in less volatile equities. By varying the age of policyholders, we highlighted the sensitivity of our tool to the characteristics of policyholders. Insurers could realize this kind of study in order to estimate and monitor their solvency ratio.

Using our partial internal model, we have illustrated a SCR calculation on a simplified case. This example has highlighted the importance of the market risk for GMAB and GMDB products as it represents the major part of the solvency capital requirement. It also justifies the use of a partial internal model to better assess the market risk for these products. After calculating the risk margin, we have presented the economic balance sheet according to the Solvency II Directive.

A perspective of our project is to extend our partial internal model to a full internal model that includes all the risks borne by the insurer. In order to decrease the run time and the storage, the full nested simulations could be replaced by the Least Squares Monte Carlo concept that limits the number of inner scenarios to a single one.

Means to comply with the Solvency II regulatory framework are heavy and the effective date of the directive is still being discussed. Solvency II is a long run project and it marks the arrival of a new culture of risk.

List of Figures

List of Tables

Theoretical elements on the Merton model

In the Merton model, the return is defined as:

$$\begin{aligned}
\text{R(t)} &= \ln\left(\frac{S_t}{S_{t-1}}\right) \\
&= \ln\left(S_t\right) - \left(S_{t-1}\right) \\
&= \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W(t) + \textstyle\sum_{k=1}^{N_t} Z_k - \left[\left(\mu - \frac{\sigma^2}{2}\right)(t-1) + \sigma W(t-1) + \textstyle\sum_{k=1}^{N_{t-1}} Z_k\right] \\
&= \left(\mu - \frac{\sigma^2}{2}\right) + \sigma\left(W(t) - W(t-1)\right) + \left(\textstyle\sum_{k=1}^{N_t} Z_k - \sum_{k=1}^{N_{t-1}} Z_k\right) \\
&= \left(\mu - \frac{\sigma^2}{2}\right) + \sigma W(1) + \textstyle\sum_{k=1}^{N(1)} Z_k
\end{aligned}$$

Let's now calculate the density of return:

$$P\left[R \leq x\right] = P\left[R \leq x \mid N(1) = 0\right] \times P\left[N(1) = 0\right] + \sum_{n=1}^{\infty} P\left[R \leq x \mid N(1) = n\right] \times \left[N(1) = n\right]$$

$N(1)$ is a homogeneous Poisson process, then $\forall n \geq 0$:

$$P\left[N(1) = n\right] = \exp(-\lambda)\frac{\lambda^n}{n!}$$

We can deduce form the independence of the variable in the model that:

$$\left(\mu - \frac{\sigma^2}{2}\right) + \sigma\left(W(t) - W(t-1)\right) + \left(\sum_{k=N_{t-1}}^{N_t} Z_k\right) \sim \mathbb{N}\left(\mu - \frac{\sigma^2}{2}, \sqrt{\sigma^2 + n\sigma_z^2}\right)$$

Then, we can write:

$$P\left[R \leq x \mid N(1) = n\right] = P\left[\left(\mu - \frac{\sigma^2}{2}\right) + \sigma W(1) + \sum_{k=1}^{N(1)} Z_k \leq x\right]$$

$$\frac{\partial}{\partial x} P\left[R \leq x \mid N(1)=n\right]=\frac{1}{\sqrt{2\pi\left(\sigma^2+n\sigma_z^2\right)}} \exp\left(-\frac{\left(x-\mu+\frac{\sigma^2}{2}\right)^2}{2\left(\sigma^2+n\sigma_z^2\right)}\right)$$

Then we deduce the density of return:

$$f(x)=\frac{\exp(-\lambda)}{\sqrt{2\pi}} \sum_{n=0}^{\infty}\left[\frac{\lambda^n}{n!\sqrt{\sigma^2+n\sigma_z^2}} \exp\left[-\frac{\left(x-\mu+\frac{\sigma^2}{2}\right)^2}{2\left(\sigma^2+n\sigma_z^2\right)}\right]\right]$$

Given this density function, we can use it to determine the formula of central moments. For symmetric reasons, uneven central moments are equal to zero. For even central moments, we use the density of returns to determine the central moments of order 2k:

$$E[(R-E[R])^{2k}]=\sum_{n=0}^{\infty} \frac{\lambda^n \exp(-\lambda)}{\sqrt{2\pi}} \int_{-\infty}^{\infty}\left[\frac{x^{2k}}{n!\sqrt{\sigma^2+n\sigma_z^2}} \exp\left[-\frac{\left(x-\mu+\frac{\sigma^2}{2}\right)^2}{2\left(\sigma^2+n\sigma_z^2\right)}\right]\right] dx$$

then we apply the change of variable, $u=\frac{\left(x-\mu+\frac{\sigma^2}{2}\right)}{\left(\sigma^2+n\sigma_z^2\right)}$

we deduce that,

$$\int_{-\infty}^{\infty}\left[\frac{x^{2k}}{n!\sqrt{2\pi\left(\sigma^2+n\sigma_z^2\right)}} \exp\left[-\frac{\left(x-\mu+\frac{\sigma^2}{2}\right)^2}{2\left(\sigma^2+n\sigma_z^2\right)}\right]\right] dx=\left(\sigma^2+n\sigma_z^2\right)^k c_{2k}$$

Where $c_{2k}=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}u^{2k}exp(-\frac{u^2}{2})du=\frac{(2k)!}{2^k k!}$
We deduce the result,

$$\mathbb{E}\left[\left(R-\mathbb{E}\left(R\right)\right)^{2k}\right]=\frac{(2k)!}{2^k k!}\sum_{n=0}^{\infty}\frac{\lambda^n \exp\left(-\lambda\right)}{n!}\left(\sigma^2+n\sigma_z^2\right)^k$$

Curve of zero-coupon rates, March 2012

A zero-coupon curve is published monthly by the French Insitute of Actuaries.

Maturity	ZC Rate
0.25	0.081%
0.33	0.094%
0.42	0.11%
0.50	0.13%
1	0.27%
2	0.55%
3	0.95%
4	1.363%
5	1.739%
6	2.07%
7	2.37%
8	2.63%
9	2.86%
10	3.051%
11	3.22%
12	3.35%
15	3.64%
20	3.82%
25	3.84%

Risk premium in the CIR model

Both the Vasicek and the Cox-Ingersoll-Ross Models provide closed-form solutions for zero-coupon bond prices and this is very appreciated, this is also the reason why the form of the risk premium is usually chosen so as to preserve the general form of the model. In theory, the volatility of the model is the same under both historical and risk-neutral world. The general form of interest rate models can be expressed both under historical and risk-neutral probabilities:

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t$$
$$dr_t = (\mu(t, r_t) - \lambda(t)\sigma(t, r_t))dt + \sigma(t, r_t)dW_t^{\mathbb{Q}}$$

$\lambda(t)$ is the risk premium, it represents the excess premium of a risky asset compared to a non-risky asset.

By applying the above formula, we find the Vasicek equations under both historical ($\mathbb{P}$) and risk-neutral probabilities ($\mathbb{Q}$):

$$dr_t = a(b - r_t)dt + \sigma dW_t$$

$$dr_t = a(b_\lambda - r_t)dt + \sigma dW_t^{\mathbb{Q}}$$

With $\begin{cases} b_\lambda = b - \dfrac{\lambda\sigma}{a} \\ \lambda = \dfrac{\mu - r}{\sigma} \end{cases}$ The zero-coupon price formula remains the same, but only the parameters are changed.

For the Cox-Ingersoll-Ross model, we use a similar reasoning, the risk premium is given by:

$$\lambda(t, r) = \frac{\lambda}{\sigma}\sqrt{r}$$

Under risk neutral probability, the Cox-Ingersoll-Ross equation becomes:

$$dr_t = a_\lambda(b_\lambda - r_t)dt + \sigma\sqrt{r_t}dW_t^{\mathbb{Q}}$$

$$\text{With} \begin{cases} a_\lambda = a\ (1+\lambda) \\ b_\lambda = \dfrac{b}{1+\lambda} \end{cases}$$

Glossary

ALM: Asset and Liability Management (ALM) reflects the relationship between the liabilities and the assets covering them, taking into account interactions between them.

CEIOPS: Commitee of European Insurance and Occupational Pensions Supervisors , became the EIOPA in 2011.

CIR model: The Cox-Ingersoll-Ross model model was suggested in 1985. It follows a square-root process that permits to produce strictly positive interest rates.

Directive: A directive is a legislative act of the European Union, which requires member states to achieve a particular result without dictating the means of achieving that result. It can be distinguished from regulations which are self-executing and do not require any implementing measures. Directives normally leave member states with a certain amount of leeway as to the exact rules to be adopted. Directives can be adopted by means of a variety of legislative procedures depending on their subject matter.

EIOPA: the European Insurance and Occupational Pensions Authority (EIOPA), composed of high-level representatives from the insurance and occupational pensions supervisory authorities of the European Union's Member States, advises the Commission in particular with technical aspects, and with the consistency implementation of Solvency II.

ESG: An Economic Scenario generator is a tool that produces forward-looking scenarios for multiple financial and economic variables.

GMxB: The Guaranteed Minimum x Benefit refer to the guaranteed living and death benefits associated with variable annuity business. In the terminology GMxB, the letter x refers to the type of guarantee.

Nested Simulations: the term Nested Simulations is used to feature the two levels of simulations used in this thesis, it is also called stochastic into stochastic or simulations into simulations. Thus in this thesis we talk about a neutral-risk simulations nested into real-risk simulations.

LSMC concept: The Least Squares Monte Carlo concept is an alternative method to the full nested simulations. It reduces the number of secondary simulations to a single one, and uses a regression function to determine the value of liabilities.

QIS: a Quantitative Impact Study (QIS) is a field-testing exercise, run to assess the practicability, implications and possible impact of specified approaches to insurers' capital setting under Solvency II.

QIS5: the Fifth Quantitative Impact Study refers to the quantitative impact study conducted between August and October 2010.

Ratchet: It is a possible method to define the amount guaranteed. The guaranteed amount is equal to the greater of the contract value at guarantee application time, or premium payments, or the contract value on a specified date.

Roll-up: It is a possible method to define the amount guaranteed. It is the greater between the initial investment value increased at a specified rate of interest (the roll-up rate) and the value of the fund when the guarantee is applicable.

SCR: the Solvency Capital Requirement is the value-at-risk of the basic own funds of an insurance or reinsurance undertakings subject to a confidence level of 99.5% over a one year period.

Unit-linked product: It is a type of life insurance where the cash value of a policy varies according to the current value of the net asset value of the underlying investment assets. The investment risk is borne exclusively by the policyholder.

Variable Annuities: it is the US term to describe unit-linked products with secondary guarantees. Variable annuities are basically unit-linked contracts with additional guarantees.

Bibliography

[1] AIDAN E., MIONE G., (2011): Développement d'un outil de simulation dans le cadre de Solvency II. Actuarial thesis, ENSAE.

[2] ALFONSI A., (2005): On the discretization schemes for the CIR (and Bessel squared) processes. 1-5.
http://www.cmap.polytechnique.fr/~mazari/NoufelFrikha/Alfonsi_2005.pdf

[3] BLAMONT D., SAGOO P., (2009): Pricing and hedging of variable annuities.
http://www.risk.net/data/lifepensions/pdf/cutting_edge_0209.pdf

[4] BRIGO D., MERCURIO F., (2009): Interest rate models - Theory and Practice. Springer, 58-68.

[5] CEIOPS,(2010): Draft Report on Variable Annuities. Consultation Paper no. 83
https://eiopa.europa.eu/fileadmin/tx_dam/files/consultations/consultationpapers/CP%20No.%2083%20-%20Draft%20Report%20on%20Variable%20Annuities.pdf

[6] CEIOPS, (2010): CEIOPS' Advice for Level 2 Implementing Measures on Solvency II: Partial internal models. Consultation Paper no. 65
https://eiopa.europa.eu/fileadmin/tx_dam/files/consultations/consultationpapers/CP65/CEIOPS-L2-Advice-Partial-Internal-Models.pdf

[7] CEIOPS (2010), QIS Technical Specifications.

[8] CORRIGAN J., FINKELSTEIN G., (2009), Implications of Solvency II for product Development.
http://uk.milliman.com/perspective/solvency-II/pdfs/implications-solvency-II-product.pdf

[9] DELOITTE INTERNAL TRAINING (2011), Internal model with the Least Squares Monte Carlo.

[10] DEVINEAU L., LOISEL S. (2009):Construction d'un algorithme d'accélération de la méthode des ≪ simulations dans les simulations ≫ pour le calcul du capital

économique Solvabilité II.
http://hal.archives-ouvertes.fr/docs/00/36/96/50/PDF/Devineau-Loisel-ISFA-WP2101-2009-v2.pdf

[11] DEVOLDER P. (1993): Finance Stochastique.
Editions de l'université de Bruxelles.

[12] EIOPA, (2011), Report on Variable Annuities.
https://eiopa.europa.eu/fileadmin/tx_dam/files/publications/reports/Report-on-Variable-Annuities.pdf

[13] GENEST M., (2010), Etude de rentabilité d'un contrat de type Variable Annuities. Actuarial Thesis (EURIA).

[14] KAMEGA A. (2011), Générateurs de scénarios économiques en assurance. Course EURIA

[15] NTEUKAM O.T, Least Squares Monte Carlo, course EURIA

[16] LE MER S., (2010) Capital économique en assurance vie. Actuarial Thesis (DAUPHINE)

[17] PLANCHET F., THEROND P., JACQUEMIN J. (2005): Modèles financiers en assurance. Economica

[18] PLANCHET F., THEROND P.: L'impact de la prise en compte des sauts boursiers dans les problématiques d'assurance.
http://www.ressources-actuarielles.net/__c1256cfc001e6549.nsf/0/300dd2604e5b29c6c1256fe6002a0e68?OpenDocument

[19] PLANCHET F., IMBEAULT M., LAGRAS W., LOLJEEH R., ZHAO S., (2011), Performance d'un contrat variable annuities, Survey ISFA.

[20] PLANCHET F., THEROND P., Simulation de trajectoires de processus continus.
http://therond.pierre.free.fr/Simulation%20de%20trajectoires%20de%20processus%20continus.pdf

Websites:
www.ressources-actuarielles.net
https://eiopa.europa.eu/