Capital optimization through Variable Annuities product features

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Abstract

After an initial introduction to the American market in the 80s and to the Japanese market in 1999, the Variable Annuities (VA) product achieved a great success between 2000 and 2008. This is a product that meets the needs of the customers because it can offer much more benefit compared to the classical bond-type contracts.

VA is a type of unit-linked life insurance product. It is at the frontier of the finance and the insurance and aims to provide varied types of guarantees to the beneficiary. Thus, this is a hybrid derivative dependent on both the financial and actuarial components and sensitive to the two categories of risk.

Considering the specificity of the Variable Annuities product and the complexity of its risk exposure, an internal model is used when determining the required capital for this kind of product. In addition, the *Solvency Capital Requirement* (SCR), which represents the target capital to absorb the shock produced by some major risks, is named the *Short Term Economic Capital* (STEC) in the internal model of *AXA Life Invest*.

This thesis will present the various aspects of the Variable Annuities product and analyze the impacts of the important product features of VA on STEC under the Solvency II directives. For this, we present the status of the global market for this product and the description of all product features. Then, the pricing method, the hedging strategies and the risk management will be concerned as well.

Key words: Variable Annuities, Solvency Capital Requirement, internal model, optimization, pricing, hedging, risk management.

Résumé

Après une première introduction sur le marché américain dans les années 80 et japonais en 1999, le produit Variable Aanuities (VA) atteint un grand succès entre 2000 et 2008. C'est un produit correspondant bien aux besoins des clients car il peut offrir une espérance de gain plus importante comparée avec les contrats classique de type obligataire.

VA est un produit de type assurance vie en unités de compte. Il se trouve à la frontière de la finance et de l'assurance et vise à proposer de nombreux types de garanties aux bénéficiaires. Ainsi, c'est un produit dérivé hybride dépendant de composantes financières et actuarielles et sensible aux deux catégories de risques.

Considérant de la spécificité du produit VA et la complexité de ses risques exposés, un modèle interne est utilisé dans la détermination du capital requis. En plus, le *Solvency Capital Requirement* (SCR), qui représente le capital cible nécessaire pour absorber le choc provoqué par certains risques majeurs, est nommé *Short Term Economic Capital* (STEC) dans le modèle interne d'*AXA Life Invest*.

Ce mémoire présentera les différents aspects du produit VA et visera à analyser les impacts des paramètres et des composantes du produit VA sur le STEC sous la directive Solvabilité II. Pour cela, nous présenterons le marché global pour ce produit et la description de tous les paramètres du produit VA. Ensuite, la méthode de tarification utilisée, les risques exposés de ce produit et les stratégies pour les couvrir seront introduits.

Mots clés: Variable Annuities, Solvency Capital Requirement, modèle interne, optimisation, tarification, couverture, gestion des risques.

Synthesis

This thesis will present various aspects of Variable Annuities (VA) and will analyze the impact of the product features on the *Short Term Economic Capital* (STEC: *Solvency Capital Requirement* in the internal model of *AXA Life Invest*) under the Solvency II directives. In fact, the problematic relates to the optimization of the required capital through the change of the product features.

To provide a comprehensive analysis of the impacts on STEC, we will look at both the external factors such as the financial parameters, and the most important components of the product like the structure of some parameters and the financial and actuarial assumptions in the process of the product modelling. There are several steps to achieve it:

• We apply shocks to the financial parameters to quantify their impact on the STEC

⋆ We make a +10bp parallel shift in the yield curve.

⋆ The swaption volatility is raised by 1%.

• We modify the most important product parameters to analyze the impacts on the STEC

⋆ The RRC rate is increased by 50bp.

⋆ The income rate is increased by 20bp.

⋆ The roll-up rate is increased by 50bp.

• Study the impact of some more structuring elements of the product on the STEC

⋆ We change the ratchet frequency from 1 year to 5 years and then, we remove it to see its impact on the STEC.

⋆ We change the income rate structure. In order to do this, we fix the income rate for a policyholder aged 65 and then, we modify respectively the increment of the income rate per year from 10bp to 5bp and 20bp.

⋆ We study the variation of the STEC in the case where the Death Benefit structure is changed.

⋆ We add a new rider reward in the initial design.

• Verify the existence of the financial or actuarial assumptions improvements to be made for the product modelling to optimize the required capital and analyze the impact of these changes on the STEC

⋆ We use the new Capped Volatility Funds (CVF) modelling method to reduce the capital requirement caused by the volatility risk.

⋆ We change the mortality and longevity table inspired by the modelling assumptions for the Japanese product.

In each stage of the studies, the decomposition of the required capital is made and each component of the final level of the capital is analyzed before and after the change of the product features mentioned above.

There are six chapters in this thesis.

The chapter 1 begins with a general description of *AXA Life Invest* where I did my internship. In addition, the development of the Variable Annuities product in the global market, such as the American market, the European market and the Asian market is presented.

The chapter 2 provides a full description of all the components of the Variable Annuities product. For instance, some specific notations for the VA product: Account Value (AV) and Benefit Base (BB); the different types of guarantees: Guaranteed Minimum Death Benefit (GMDB) and Guaranteed Lifetime Withdrawal Benefit (GLWB); some different types of charges and profitability indicators.

The chapter 3 presents the methodology used for pricing the Variable Annuities product. More precisely, we utilize the Hull and White model to generate the rate and the Black and Scholes model to generate the equities. Moreover, the method to generate the random variables needed in the process is given. The calibration process and a description of the martingale test are also introduced.

The chapter 4 describes the strategy to cover the financial risks encountered by the Variable Annuities product. In this chapter, the formulas for calculating the Greeks are listed and the methods to reduce these types of risks are presented.

The chapter 5 relates to the Solvency II directives and provides a summary of the methodology to calculate the *Solvency Capital Requirement* using the internal model of *AXA Life Invest*. Then, the financial and actuarial risks to which the Variable Annuities product is exposed are presented. In addition, for each risk, the level of shock applied and the way to determine the capital requirement are given.

The chapter 6 is the most important part of this thesis. The analysis of the variations in each component of the STEC due to the modification of the product features is presented precisely in this chapter.

Synthèse

Ce mémoire présentera les différents aspects des Variable Annuities (VA) et visera à analyser les impacts des paramètres et des composantes du produit VA sur le *Short Term Economic Capital* (STEC : *Solvency Capital Requirement* dans le modèle interne d'*AXA Life Invest*) sous la directive Solvabilité II. La problématique est liée à l'optimisation du capital requis par la modification des composantes diverses du produit VA.

Pour pouvoir faire une analyse exhaustive des impacts sur le STEC, nous regarderons à la fois les facteurs externes, et les composantes les plus importantes du produit comme la structure des différents paramètres et les hypothèses actuarielles et financières pour la modélisation du produit. Il existe plusieurs étapes pour la réaliser :

- Nous appliquons les chocs sur les paramètres financiers pour quantifier son impact sur le STEC
  - ⋆ Nous faisons un déplacement parallèle de +10bp sur la courbe de taux.
  - ⋆ La volatilité de swaption est élevée de 1%.

- Nous modifions les paramètres les plus importants du produit pour regarder les impacts sur le STEC
  - ⋆ Le taux de RRC est augmenté de 50bp.
  - ⋆ Le taux de coupon est augmenté de 20bp.
  - ⋆ Le taux de roll-up est augmenté de 50bp.

- Etude de l'impact des éléments plus structurants du produit sur le STEC
  - ⋆ Nous changeons la fréquence de l'effet ratchet de 1 an en 5 ans et après nous le supprimons pour voir leurs impacts sur le STEC.
  - ⋆ Nous changeons la structure du taux de coupon. Pour ce faire, nous fixons sa valeur pour l'âge de 65 ans et faisons varier sa vitesse d'incrémentation de 10bp à 5bp et après à 20bp.
  - ⋆ Nous étudions le changement du STEC dans le cas où la structure de Death Benefit est modifiée.
  - ⋆ Nous ajoutons un nouveau rider reward dans le design initial.

- Vérification de la possibilité d'améliorer les hypothèses financières ou actuarielles dans la modélisation du produit pour optimiser le capital requis et analyse de l'impact de ces changements sur le STEC

⋆ Nous utilisons la nouvelle Capped Volatility Funds (CVF) modélisation pour diminuer le capital requis provoqué par le risque de volatilité.

⋆ Nous changeons la table de mortalité et de longévité inspirée par des hypothèses de modélisation du produit Japonais.

Dans chaque étape des études, une décomposition du capital requis est faite et chaque composante du niveau final de capital est analysée avant et après les changements mentionnés ci-dessus.

Ce mémoire contient six chapitres.

Le chapitre 1 commence par une description générale d'*AXA Life Invest* où j'ai réalisé mon stage de fin d'études. Ensuite, nous présentons le développement du produit Variable Annuities dans le marché global comme le marché américain, européen et asiatique.

Le chapitre 2 donne une description complète de tous les composantes et paramètres du produit Variable Annuities. Par exemple, les notations spécifiques du produit Variable Annuities : Account Value (AV) et Benefit Base (BB) ; les différent type de garanties : Guaranteed Minimum Death Benefit (GMDB) et Guaranteed Lifetime Withdrawal Benefit (GLWB) ; les charges diverses de ce type de produit et quelques indicateurs de profitabilité.

Le chapitre 3 présente la méthode utilisée pour la tarification du produit Variables Annuities. Nous utilisons le modèle de Hull and White pour générer le taux et le modèle de Black and Scholes pour générer les actions. De plus, la façon pour générer des variables aléatoires nécessaires dans le processus de tarification est donnée. La méthode de calibration et le descriptif du test martingale sont aussi introduits.

Le chapitre 4 décrit la stratégie pour couvrir les risques financiers auxquels le produit est exposé du produit Variable Annuities. Pour ce faire, les formules pour calculer les grecques sont listées et les méthodes pour diminuer ces types de risques sont introduites.

Le chapitre 5 concerne à la directive Solvabilité II et donne une présentation de la méthode pour calculer le *Solvency Capital Requirement* en utilisant le modèle interne d'*AXA Life Invest*. Les risques financiers et actuariels auxquels le produit Variable Annuities est exposé sont présentés. En plus, pour chaque risque, le niveau de choc appliqué et la façon pour déterminer le capital requis sont présentés.

Le chapitre 6 est la partie la plus importante de ce mémoire. L'analyse des variations de chaque composante du capital requis déterminéespar la modification des éléments du produit est présentée précisément dans ce chapitre.

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Chapter 1

AXA Life Invest and the Variable Annuities market

1.1 AXA Life Invest

The *AXA Group* was created through the merger and acquisition of several insurance companies, the oldest of which was named *La Compagnie d'Assurance Mutuelle contre l'Incendie* founded in 1817 and based in the city of Rouen. The name of AXA was adopted in 1985 for the reasons that the group needs a shorter and more upbeat name to make itself international. "AXA" was chosen because it starts with the letter 'A' so that it will be the first in lists and directories and it is easy to pronounce in every language. Nowadays, the *AXA Group* has become the first insurer in the world and the third in terms of the asset management. Present in 56 countries, especially in Europe, North America and the Asia-Pacific region, the 157,000 employees and distributors of AXA serve the needs of 102 million clients, both individuals and companies, across three major business lines: property-casualty insurance, life & savings, and asset management.

In order to manage the very complicated hedging of Variable Annuities products sold by partners companies among European countries, *AXA Group* created *AXA Life Europe Hedging Services* in 2006, which became *AXA Hedging Services* in 2008 with its extension to the Asian market. In parallel, created in 2008, *AXA Global Distributors*, another subsidiary, was an insurance intermediary dedicated to the commercialization for third parties networks of Variable Annuities, and regulated by the Central Bank of Ireland. *AXA Life Invest* (ALI) is born in 2012 after the merging of *AXA Hedging Services* and *AXA Global Distributors*. This is an independent entity of the *AXA Group* taking charge of all the aspects of the Variable Annuities products, including the hedging, the pricing, the creation and the distribution. With the expertise in providing investment and retirement protection solutions to customers, the total number of policies in force has now grown to over 298,969 with assets under management of 2.6€ billion as at 31st December 2013. *AXA Life Invest* is currently divided into two parts: one in Paris managing the European portfolios (Germany, UK, France, Italy, Spain, Switzerland, Portugal and Belgium), and one in Singapore handling Asian activities (Japan, Hong Kong and Australia).

The Parisian pole comprises various teams, each one playing a specific role in the Variable Annuities development.

The team where I made the internship is the *pricing* team. The main activities in the pricing team consist of creating new versions of the Variable Annuities product, pricing them and monitoring the profitability indicators of them.

The *structuring* team is responsible for the creation of the new model in the financial line of the Variable Annuities product. In addition, they take charge of the maintenance and the improvement of the modelling process.

The *hedging* team needs to compute the varied sensitivities to the market risk factors based on the daily data from the financial market and provide the hedging strategies of the Variable Annuities product.

The *life reporting* team is responsible for providing all report documentation and for the calculation of the P & L data.

The *risk management* team manages all sorts of risk confronted in the Variable Annuities business and creates the internal model under the Solvency II directives.

The *tool management and support* team takes charge of the upgrading process of the programs for pricing and for hedging the Variable Annuities product by reducing the running time and improving the quality of a certain program.

There exists the *actuarial* team in Dublin which concentrates on the actuarial assumption setting and supports other teams for the capital management.

There are also some important departments in *AXA Life Invest*. For example, the marketing team, the distribution team, and the new business development team.

1.2 The Variable Annuities market

Insurance companies used to offer to their clients products with constant annuities and to invest their revenues in bonds. Nevertheless, they became less attractive during the last twenty years due to the remarkable variation of interest rate and a better access to investment products. Investors became then more demanding regarding the yield of the products, which compelled insurers to develop a new range of products: Variable Annuities.

Variable Annuities products are mutual fund investments that have certain insurance-related guarantees, such as living benefits and death benefits. (Mutual funds are bundles of stocks or bonds or a mixture of both. They make it easy for small investors to diversify their holdings and invest with less risk.) They were launched in the United States in 1995 and appeared in France in 2006. Variable Annuities are attractive products especially with the retirement reforms and may represent 75 billion of Euros in 2015.

With a Variable Annuities product, a policyholder places his savings with an insurance company and chooses how his money will be invested. A policyholder chooses also his placement from a pre-selected list of funds which can range from aggressive stock funds to conservative bond funds. Therefore, the returns will vary depending on the underlying performance of the investments. This is why it is called a variable annuity. In comparison, with a fixed annuity, a policyholder does not choose the way of investments. Instead, the insurance company invests the savings and provides the customers with a stated guaranteed return.

1.2.1 Attractiveness of the Variable Annuities products

The Variable Annuities products are attractive to the policyholders as they offer the opportunity for them to participate in investment markets with the safety net of guarantees even in the event of the breakdown of the financial market. They can be regarded as a perfect alternative to the traditional fixed annuities products because they can offer different kinds of guarantees for the insured and the policyholders pay an explicit charge for the guarantees they choose that are valuable to them.

In the traditional fixed annuities, the amount paid for the insured does not change with inflation, which creates a problem even with a relatively low level of inflation rate, that the first payments of annuity have much more value than the final ones. In contract, Variable Annuities are different in the amount paid out for the policyholders, as the actual payout is linked to the price of the underlying assets invested in the deferral phase. If these assets rise in price, the annuity payment rises as well; conversely, if the assets fall in price, the annuity payment falls as well. The difference between the traditional fixed annuities and the Variable Annuities products leads to the rapid growth of the latter.

Further more, the fundamental demographics of the countries all over the world point to a major opportunity for retirement products because there will be a substantial increase in the number of retired people over the next decades as the large generation that was born after World War II reach retirement. Many major insurers in the world are aware of the fact that more and more people begin to make a plan for their retirement life with a serious consideration, which leads the insurers to set up new types of retirement products to attract potential customers.

1.2.2 The Variable Annuities products for the US market

Variable Annuities were first introduced into the United States by the *Teachers Insurance and Annuities Association-College Retirement Equity Fund* (TIAA-CREF) in 1952, to fund pension arrangements. This type of annuities solved one of the problems existing in the fixed annuities, namely, the inflation destroying the value of payments in the late annuities payment schedule. These new annuities were more complex and required regulatory approval from state insurance departments, which inhibited their rapid growth.

Variable Annuities came to be more attractive in the 1970's and 1980's in the United States, as the high level of inflation during that time leading to the policyholders' increasing fear of the fixed annuities product.

The market for these products increased dramatically during the 90s as more valuable and complicated guarantees are designed by the major insurers and offered to their customs to choose in order to boost the sales of products. Following the market downturn in the financial crisis taken place in the year 1997, the value of the guarantees was more fully appreciate as the policyholders realized that their wealth could be well protected even in the worst financial situation.

Nowadays, the Variable Annuities products continue to be popular, and many annuity designers believe that they can still be the hot products in the US financial markets in the future.

1.2.3 The Variable Annuities products for the European market

Until recently, the European life insurance market saw variable annuities as a potential blockbuster product. Major insurers and small competitors contemplated entering the market. *AXA Life Invest* issues the Variable Annuities products in France, Belgium, UK, Germany and Switzerland in the European continent.

The appeal of the Variable Annuities products for European insurers lies in the demographic changes taking place across the continent. Europe's over 50 populations is becoming bigger, wealthier and more varied in lifestyles. As consumers of financial products, over 50s have a wide range of needs depending on their current circumstances and their anticipated path through the various phases of pre and post retirement life. They require access to market returns in order to keep pace with a rising cost of living, but also look for protection of their assets and lifestyle given increased uncertainty from several sources: the volatility of asset returns, the level of state, how long they will live and in what state of health.

Once, the financial crisis revealed the risks of Variable Annuities and dimmed hopes for this new class of products. Following staggering losses and market exits, the doubt had been produced about the future development of the Variable Annuities products in the European market. But the reality proves that the Variable Annuities products can offer customers long-term guarantees and potentially high rates of return while giving insurers a new opportunity as the economy picks up.

1.2.4 The Variable Annuities products for the Asian market

Nowadays, *AXA Life Invest* issues the Variable Annuities products in China and Japan in the Asian continent. The introduction of such kind of product in Asian market can be traced back to the late 1990's. After the financial crisis taken place in 2007, Asia has emerged as a potential growth area for the Variable Annuities products and China is looked to be the top spot for this growth.

China's Insurance Regulatory Commission (CIRC) has piloted a program for Variable Annuities that gives insurers great opportunities in the business development. The potential for a 100% return with Variable Annuities makes them very popular with policyholders but a bit riskier for insurers. Chinese capital markets need to make sure that the insurer can hedge the Variable Annuity products with proper reinsurance strategies and continuous improvement of the product design, especially some of the guarantees offered to policyholders.

Chapter 2

Variable Annuities product features

2.1 General description of the Variable Annuities products (VA)

Variable Annuities is a type of life insurance contract. More specifically, it is a saving product created for the people who want to benefit of an amount of capital or annuity for their retirement lives. It is a kind of product at the frontier between the finance and the insurance, which has the objective to provide different types of guarantees for the policyholder. In this kind of contract, the initial premium is invested in varied funds (bonds, stocks, mutual funds) and the insurer promises to the insured a certain level of guaranteed income rate defined at the beginning of the contract. This kind of product corresponds to the needs of the clients because it can make more benefits for the insured compared to a classic bond-type saving product.

In insurance it is important to make the distinction between the following terms:

- Insurer: the physical or moral person who commits himself to pay the benefits foreseen in case of occurrence of the event on which the contract has been established;
- Subscriber: the physical or moral person who subscribed the contract;
- Insured: the physical or moral person on which relies the insured event;
- Beneficiary: the physical or moral person designated by the contract to receive the benefits described in the contract.

When entering this type of contract, the subscriber pays one or more initial amounts to the insurer in order to benefit from the guarantee defined by the contract. In the following paragraphs we will first introduce the vocabulary necessary to the understanding of this type of product. We will then explain the guarantees constituting the Benefit Base before detailing the characteristics of the products on which we will focus.

2.2 Fundamental definitions

2.2.1 Account Value (AV)

The Account Value is the total amount of savings existing in the policyholder's account at a specific date. This account is managed by the insurer and is invested in various funds. The evolution of its value depends on: the initial premiums paid by the subscriber, the market performance and the charges taken from this account.

2.2.2 Benefit Base (BB)

The Benefit Base is a reference amount that generates the future cash flow of Account Value, whose value can be regarded as the minimum amount to be refunded to the beneficiary when an event defined in the contract occurs. The value of Benefit Base can be revaluated by different kinds of mechanisms which will be introduced in the following paragraph.

2.2.3 Coupons

The coupons exist in the Guaranteed Minimum Withdrawal Benefit (GMWB) contract. They are the amount of money paid by the insurer each year during the payment period, which is computed as a percentage of the Benefit Base.

$$Coupon = Withdrawal\ Benefit\ Base \times Guaranteed\ Income\ Rate$$

2.2.4 Claims

The Claims are paid by the insurer to the beneficiary when the guaranteed amount is superior to the Account Value during the payment period. Claims can be null when the saving value of the contract covers all guaranteed amounts.

For a Guaranteed Minimum Withdrawal Benefit (GMWB) contract, if T

is the maturity of the contract, then the claims for a certain anniversary date $t \in \{0, 1, \cdots, T\}$, are defined as:

$$Claims_t = (Coupon - AV_t)_+$$

2.2.5 Real Rider Charge (RRC)

This is the amount effectively taken from the Account Value each year to cover the cost of the guarantee plus a certain margin. The Real Rider Charge allows the insurance to make benefit with issued Variable Annuities contracts. It is a fraction, fixed at the contract subscription, of the Account Value.

2.2.6 Economic Hedging Cost (EHC)

This is the annual cost for the guarantee. It reflects the amount of money that has to be taken annually from the Account Value to cover the present value of the claims. This amount depends on the Real Rider Charge.

$$EHC = \frac{Present\ Value(Claims)}{Present\ Value(Charges)} \times RRC$$

with $Present\ Value(Claims) = \sum_{t=0}^{T} e^{-rt} \times Claims_t$

and $Present\ Value(Charges) = \sum_{t=0}^{T} e^{-rt} \times Charges_t = \sum_{t=0}^{T} e^{-rt} \times AV_t \times RRC$

here T is the duration of the contract.

2.3 Mechanisms of revaluation of the Benefit Base

There exist different kinds of mechanisms in the process of revaluating the Benefit Base.

- Roll-up mechanism
- Ratchet mechanism
- Combo mechanism
- Roll-up on ratchet mechanism

2.3.1 Roll-up

This mechanism consists to guarantee the insured a minimum rate. The guaranteed amount is equal to the initial saving of the contract capitalized with a fixed rate determined at the beginning of the contract, which is called the roll-up rate.

Let BB_0 be the value of the initial guarantee, T the duration of the deferral period and r the roll-up rate. So, the BB with roll-up at an anniversary date $t \in \{0, 1, \cdots, T\}$ is:

$$\begin{aligned} BB_t^{roll} &= BB_0^{roll} \times (1 + rt) \quad \textit{linear rate} \\ BB_t^{roll} &= BB_0^{roll} \times (1 + r)^t \quad \textit{compound rate} \\ BB_t^{roll} &= BB_0^{roll} \times e^{rt} \end{aligned}$$

and we suppose that $BB_0^{roll} = BB_0 = AV_0$

2.3.2 Annual Ratchet

This mechanism makes it possible to revalue the Benefit Base each year.Let AV_t be the Account Value of the insured at time t. So, for each evaluation date $t \geq 1$, the value of Benefit Base with this mechanism can be written recursively:

$$BB_t^{rat} = max(BB_{t-1}^{rat}; AV_t)$$

and we suppose that $BB_0^{rat} = BB_0 = AV_0$

Furthermore, if T is the duration of the contract, the client who has chosen this guarantee will have at time $t \in \{0, 1, \cdots, T\}$:

$$BB_t^{rat} = \max_{s \in \{0,1,\cdots,t\}} AV_s$$

In fact, the guaranteed value equals the maximum among all the Account Value of each year's evaluation date of the contract.

There exists an enhanced version of the ratchet mechanism, which can be defined by the formula below:

$$BB_t^{rat} = max(BB_{t-1}^{rat}; AV_t + paid\ coupons)$$

Remark: In general, the difference between the two different evaluation dates of the contract is one year, but other frequency can be applied in our product (for example, half of a year). The ratchet mechanism can take place during the deferral period and the payment period.

2.3.3 Combo

This guarantee combines the two mechanisms roll-up and ratchet and is equal to the maximum between the roll-up base and the ratchet base.

The value of this guarantee can be written as:

$$BB_t^{comb} = max(BB_t^{roll}; BB_t^{rat})$$

with t the evaluation date of the contract.

Therefore, this guarantee can ensure the beneficiary both a minimum rate on his savings and the maximum value reached by this savings within the duration of this guarantee.

2.3.4 Roll-up on ratchet

This guarantee is also the combination of the two mechanisms mentioned before, but the bases do not evolve independently. In fact, the roll-up is applied directly on the ratchet.

$$\begin{array}{lcl} BB_t^{rat} & = & max(BB_{t-1}^{rat}; AV_t) \\ BB_t^{roll} & = & max(BB_{t-1}^{roll}; BB_{t-1}^{rat}) \times e^r \\ BB_t & = & BB_t^{roll} \end{array}$$

So,

$$BB_t = max(BB_{t-1}; \max_{s \in \{0,1,\cdots,t-1\}} AV_s) \times e^r$$

here t is the evaluation date of the contract.

2.4 Different types of guarantees

2.4.1 GMAB (Guaranteed Minimum Accumulation Benefit)

The policyholder gives an initial premium at time 0. This premium is invested on some financial products during a deferral period whose duration is fixed in advance. At the end of this period, the beneficiary receives the maximum between the Accumulation Benefit Base and the Account Value.

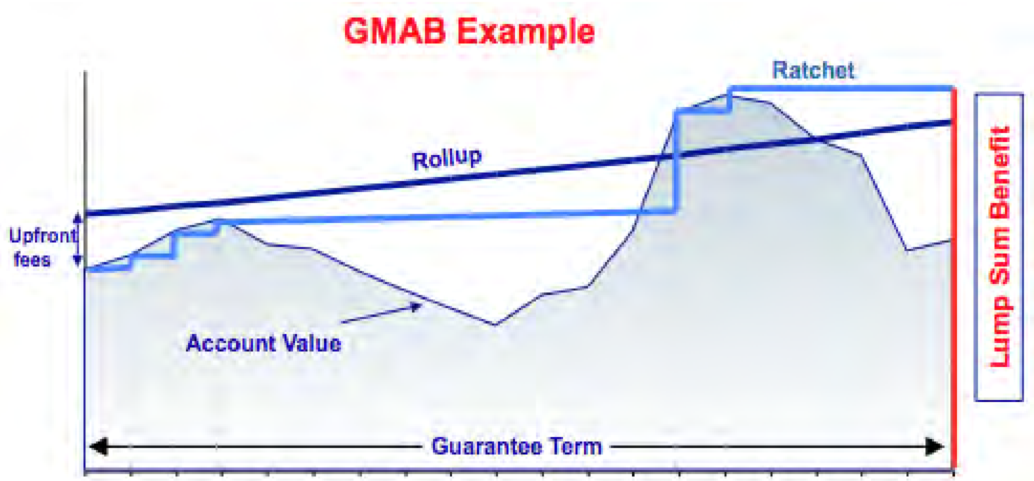
If T is the duration of the deferral period, the insurer gives to the beneficiary:

$$max(AV_T; AB_T)$$

which is equivalent to:

$$AV_T + (AB_T - AV_T)_+$$

A GMAB product can be seen as an European put option whose underlying is the Account Value and the strike is the amount of Accumulation Benefit Base reached at the end of the Deferral Period. This option cannot be exercised before the end of the deferral Period. The figure below shows an example of a GMAB product:



2.4.2 GMDB (Guaranteed Minimum Death Benefit)

The GMDB is comparable to the GMAB excepted it is paid to the successors when the policyholder dies. A certain amount is available for the beneficiary at the death of the insured person. This amount is equal to the maximum of the Death Benefit Base predefined and the Account Value.

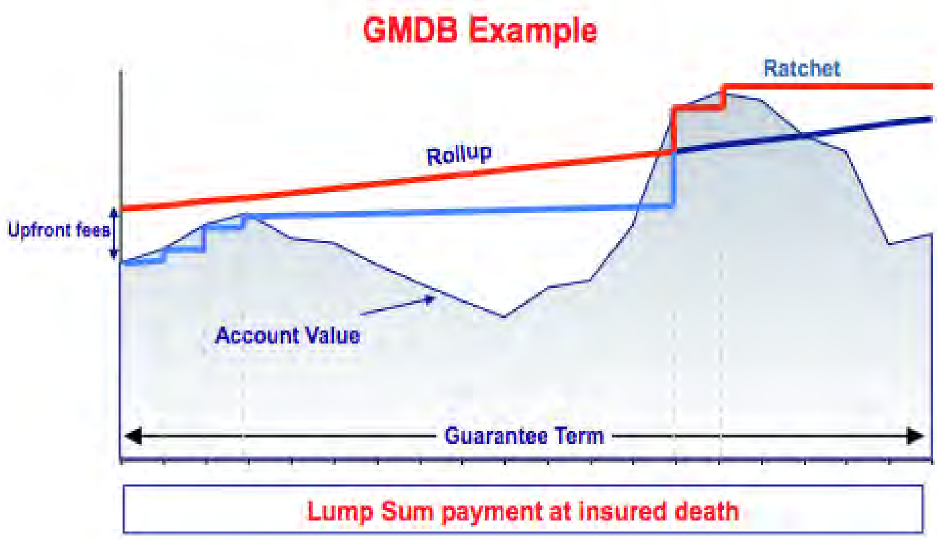
If τ is the death date of the insured, the insurer gives to the beneficiary:

$$max(AV_\tau; DB_\tau)$$

which is equivalent to:

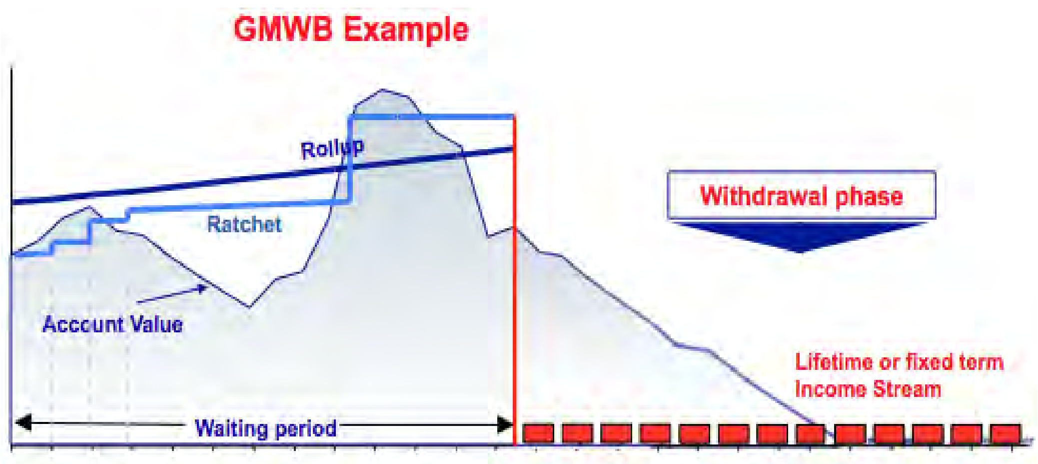
$$AV_\tau + (DB_\tau - AV_\tau)_+$$

The figure below shows an example of a GMDB product:



2.4.3 GMWB (Guaranteed Minimum Withdrawal Benefit) and GLWB (Guaranteed Lifetime Withdrawal Benefit)

GMWB is a guarantee on savings that offers the beneficiary the possibility to withdraw fixed coupons at specific dates. At the end of the Deferral period the payment period begins: the subscriber can withdraw constant coupons from the Account Value which are a percentage of the Withdrawal Benefit Base obtained at the end of the waiting period. If the Account Value reaches 0, it remains null and the insurer continues to pay the rest of the coupons with his own funds. However if the Account Value does not reach 0, the cost of this warranty is void for the insurer and the beneficiary takes back the value of the remaining savings at the end of the contract. The following figure shows an example of a GMWB product:



2.4.4 GMIB (Guaranteed Minimum Income Benefit)

This type of guarantee is really similar to the GMWB contract except that the policyholder cannot repurchase his contract. The annuities start after the accumulation period computed as a percentage of the Income Benefit Base.

2.4.5 GMSB (Guaranteed Minimum Surrender Benefit)

This guarantee offers the policyholder the possibility to surrender (without paying additional costs) after a surrender period. At the end of this period, the insured can choose to surrender and obtain the maximum between the Surrender Benefit Base and the Account Value.

This type of contract is similar to an American put option, which can be repurchased at any time before the maturity. The major difference between an American put option and a GMSB contract is that the policyholder should pay the ongoing charges (during the entire lifetime of the contract) whereas for an American put option, the charges are upfront (at the initial time of the contract).

2.5 Three phases in the regular GLWB product

2.5.1 The accumulation phase/deferral period

At the beginning of this period, the policyholder must give an initial premium. During this period, the initial investment is revalued with different mechanisms: roll-up, ratchet or the combination of these two. In most cases, the duration of this period is fixed at the beginning of the contract and selected by the insured, which is generally based on his retirement date. But for some products, its duration is not determined and can be changed during the lifetime of the contract.

2.5.2 The distribution phase/payment period

The payment period is the period during which the guaranteed benefit are obtained by withdrawing some coupons from the Account Value and distributing them to the policyholder. This period begins after the end of the accumulation phase and these payments are subtracted from the Account Value. The evolution of the Benefit Base can continue to take place (for example, the ratchet mechanism) in this period.

2.5.3 The activation phase

The guarantee of our product takes effect during this period. When the Account Value reaches 0, the insurer takes the responsibility to fulfil their engagement, which means the insurer needs to pay continually the cash flows owed to the beneficiary until the death of the latter.

2.6 Different types of fees

2.6.1 Management and Expense fees

This is a typical type of fee taken from the Account Value, which pays for the commissions, distribution, and administrative expenses of the contract. In general, these fees in the contract will be charged as a percentage of the Account Value.

2.6.2 Fund management fees

This type of fees are almost the same as an investment manager's fees in a mutual fund. These fees will vary depending on the various subaccount options within the Variable Annuities products. In general, they will be somewhat less than those charged by a managed mutual fund within the same investment category.

2.6.3 Surrender charges

Many Variable Annuities products will impose a surrender charge to reduce the loss caused by the diminution of the acquisition fees when it is cashed in before a specific period of time. The charge is made against the value of the investment when the contract is surrendered, and its purpose is to discourage a short-term investment by the purchaser.

2.7 Some profitability indicators

2.7.1 Value of in-force (VIF)

It represents the present value of the profits that will emerge from a block of life insurance policies over time. It can be calculated as the present value of the expected future earnings on the in-force business less the present value of the cost of holding required capital to support the in-force business.

2.7.2 New Business Value (NBV)

It is the sum of discounted cash flows received by the insurer divided by the Initial premium paid by the policyholder, which can be assimilated to an earnings/price ratio. From the insurer's point of view, it is a percentage of the Account Value that a contract will earn. It is one of the main profitability indicators.

$$NBV = \frac{\sum_{t=0}^{end\ of\ contract} e^{-rt} \times Cash\ flow(t)}{Initial\ premium}$$

2.7.3 Internal Rate of Return (IRR)

It is a measure of the rate of return on any investment, which can be considered as an indicator of capital efficiency. It can be understood as the discount rate at which the initial invested capital equals the net present value of future income. An investment is generally considered worthwhile if the IRR is greater than the return of a similar investment, or greater than the cost of the investment itself.

The IRR is the solution to the following formula:

$$Initial\ investment = Cash\ flow(0) = \sum_{t=1}^{end\ of\ contract} \frac{Cash\ flow(t)}{(1+IRR)^t}$$

2.8 Evolution of the Account Value

As defined in the previous paragraph, the AV is the total amount of savings existing in the insured's account.

Let us denote AV_t the Account Value at the time t. AV_0 is therefore, the initial investment of the insured, which is also the initial premium decremented by the acquisition fees. There exists a recurrence formula which allows passing the Account Value at the time t to the time $t+1$.

For a GLMB product, the evolution of the Account Value can be described as follows:

• The insurer pays the coupons only after the deferral period. Each coupon is withdrawn from the Account Value and is calculated as a certain percentage of the Benefit Base achieved at the end of the accumulation phase. The rest of AV is equal to: $(AV_t - coupon)_+$

• Then, the administrative fees and the hedging fees are taken from the remaining of the AV. The value of these fees is fixed at the beginning of the contract and is expressed as a percentage of the AV. After this step, the AV equals:

$$(AV_t - coupon)_+ \times (1 - rate\ of\ admin\ fees) \times (1 - rate\ of\ hedging\ fees)$$

This sum can be considered as a compensation for the insurer to cover the claims and expenses and constitutes the benefits of the insured.

• After the withdrawal of the possible coupons and the fees from the Account Value, the performance of the invested assets is applied. Let us note by r_t the profitability rate between the time t and $t+1$. This indicator can increase or decrease with the evolution of the financial market. The AV is now equal to:

$$(AV_t - coupon)_+ \times (1 - rate\ of\ admin\ fees) \times (1 - rate\ of\ hedging\ fees) \times (1 + r_t)$$

It is also called the savings before decrement and

$$\begin{aligned} AV_{t+1}^{before\ decrement} &= (AV_t - coupon)_+ \times (1 - rate\ of\ admin\ fees) \\ &\quad \times (1 - rate\ of\ hedging\ fees) \times (1 + r_t) \end{aligned}$$

In fact, this quantity is used to calculate the different types of Benefit Base.

• Finally, we decrement the Account Value of the insured by the rate of the lapse and the rate of the mortality. If τ_t is the rate of the lapse and q_t is the death probability of a person whose age is x during the time t and the time $t+1$, then, the Account Value at the time $t+1$ is:

$$AV_{t+1} = AV_{t+1}^{before\ decrement} \times (1-\tau_t) \times (1-q_t)$$

2.9 Evolution of the Benefit Base

As described in the previous paragraph, the BB is a reference amount that generates the future cash flow of the Account Value. Each GMXB contract guarantees to the beneficiary a minimum amount BB to be received at the end of the contract (at the death time of the policyholder if a GMDB is chosen), which can be evaluated with the mechanisms of roll-up, ratchet, or the combination of the two.

In our product GLWB, the roll-up base with the linear rate and the ratchet base are applied to calculate the Benefit Base.

$$\begin{aligned} BB_{t+1}^{roll} &= BB_t^{roll} \times (1+r) \\ BB_{t+1}^{rat} &= max(AV_{t+1}^{before\ decrement}; BB_t^{rat}) \end{aligned}$$

Then, the guarantee before decrement is represented by the maximum between the roll-up base and the ratchet base.

$$BB_{t+1}^{before\ decrement} = max(BB_{t+1}^{roll}; BB_{t+1}^{rat})$$

We can observe that this guarantee before decrement coincides with the combined mechanism if the roll-up base and the ratchet base are chosen in the contract.

Similarly to the evolution of the Account Value, the Benefit Base will decrease if we consider the effect of the lapse and the mortality.

So, the BB at the time $t+1$ is:

$$BB_{t+1} = BB_{t+1}^{before\ decrement} \times (1-\tau_t) \times (1-q_t)$$

Chapter 3

Pricing process of Variable Annuities

The price of the Variable Annuities is determined through a Monte Carlo methodology. To be precise, we generate n paths under the risk neutral probability and calculate the value of a policy as the average of all the values along each path. Generally, the path number that we take is 2000 or 5000.

In order to be able to price a policy, we need to simulate the following random variables at each time step:

- The short rate
- Nominal rates
- The Beta
- Equity returns
- Bond returns

The short rate r_t is the instantaneous rate of return of money at time t. Nominal rates represent the underlying rate corresponding to a zero coupon bond of maturity T at time t. The Beta corresponds to the amount by which money placed at the instantaneous rate will grow: $\beta(t) = exp(\int_0^t r_s ds)$. Equity returns and Bond returns represent respectively the returns of equities S_t and the returns of bonds $B(t, T)$ during the last month. We utilize the Hull and White model to simulate the rates and the Black and Scholes model for the simulation of the equities.

3.1 Hull and White model

The Hull and White model assumes that the short rate r_t has the following dynamic:

$$dr_t = (\theta(t) - ar_t)dt + \sigma(t)dW_t$$

where W_t is a Brownian motion, and a, $\sigma(t)$ and $\theta(t)$ are parameters to be determined.

A swap curve is given as a first input to calibrate this model. By interpolation and bootstrapping, it is transformed into a zero coupon curve where a zero coupon value is known for every maturity by time step of one month. In this model, $\theta(t)$ represents the long term rate. It is a function of a and $\sigma(t)$ and is determined by using the zero coupon curve and the forward rate curve deduced from it.

The parameter a represents the speed of the mean reversion and the parameter $\sigma(t)$ represents the volatility of the process. To determine the values of a and $\sigma(t)$, we need more market information. This information will come from the second input used to calibrate this model: swaption volatilities. a and $\sigma(t)$ will be chosen to allow the swaption model price fit as well as possible the swaption market price. More precisely, a and $\sigma(t)$ will be chosen to minimize the sum of the squares of the difference between the swaption model prices and the swaption market prices, which can be represented by the formula below:

$$\sum(price_{model} - price_{market})^2$$

In this way, the short rate, nominal rates, the Beta and bond returns can be simulated for each path and at each time step.

3.1.1 HJM model

The Hull and White model is one of the HJM model. HJM model is a class of models containing all the models diffusing zero coupon bonds and assuming the following dynamic under the risk neutral probabilit:

$$\frac{dB(t,T)}{B(t,T)} = r_t dt + \tau(t,T)dW_t$$

where $B(t,T)$ is the value of a zero coupon bond at time t whose maturity is T, r_t is the short rate, $\tau(t,T)$ is the volatility function and W_t is a Brownian motion. We notice that the price of a zero coupon bond at time T is 1 ($B(T,T) = 1$), so $\tau(T,T) = 0$.

The solution of the stochastic equation is given below:

$$B(t,T) = B(0,T) \times exp[\int_0^t r_s ds + \int_0^t \tau(s,T)dW_s - \frac{1}{2}\int_0^t \tau(s,T)^2 ds]$$

Taking $T = t$ in the previous equation and utilizing the fact $B(t,t) = 1$, we have:

$$1 = B(0,t) \times exp[\int_0^t r_s ds + \int_0^t \tau(s,t)dW_s - \frac{1}{2}\int_0^t \tau(s,t)^2]$$

Dividing the first equation by the second we can eliminate the short rate and we have:

$$B(t,T) = \frac{B(0,T)}{B(0,t)} \times exp[\int_0^t (\tau(s,T) - \tau(s,t))dW_s - \frac{1}{2}\int_0^t (\tau(s,T)^2 - \tau(s,t)^2)ds]$$

The forward continuous nominal rate $R_t(T,\theta)$ between T and $T+\theta$, fixed at t, satisfies the equation below:

$$exp(\theta R_t(T,\theta)) = \frac{1}{B_t(T,T+\theta)}$$

where $B_t(T,T+\theta)$ is the forward value of a zero coupon bond which satisfies, under the hypothesis of absence of arbitrage:

$$B_t(T,T+\theta) = \frac{B(t,T+\theta)}{B(t,T)}$$

This means that the forward continuous nominal rate $R_t(T,\theta)$ satisfies:

$$\begin{aligned} R_t(T,\theta) &= -\frac{1}{\theta}(ln(B_t(T,T+\theta)) \\ &= \frac{1}{\theta}[ln(B(t,T)) - ln(B(t,T+\theta))] \end{aligned}$$

We can deduce from the previous definition of the forward continuous nominal rate and the value of the zero coupon bond that:

$$\begin{aligned} R_t(T,\theta) &= \frac{1}{\theta}[ln(B(0,T)) - ln(B(0,T+\theta)) \\ &\quad + \int_0^t (\tau(s,T) - \tau(s,T+\theta))dW_s - \frac{1}{2}\int_0^t (\tau(s,T)^2 - \tau(s,T+\theta)^2)ds] \\ &= R_0(T,\theta) + \int_0^t \frac{\tau(s,T) - \tau(s,T+\theta)}{\theta}dW_s - \int_0^t \frac{\tau(s,T)^2 - \tau(s,T+\theta)^2}{2\theta}ds \end{aligned}$$

The forward short rate $f(t,T)$ is defined as:

$$f(t,T) = \lim_{\theta \to 0} R_t(T,\theta)$$

If we note $\gamma(s,T) = \dfrac{\partial \tau(s,T)}{\partial T}$, then the forward short rate $f(t,T)$ follows the equation below:

$$\begin{aligned}
f(t,T) &= \lim_{\theta\to 0}[R_0(T,\theta) + \int_0^t \frac{\tau(s,T)-\tau(s,T+\theta)}{\theta} dW_s \\
&\quad - \int_0^t \frac{\tau(s,T)^2 - \tau(s,T+\theta)^2}{2\theta} ds] \\
&= f(0,T) - \int_0^t \gamma(s,T) dW_s + \lim_{\theta\to 0} \int_0^t \frac{\tau(s,T+\theta)+\tau(s,T)}{2} \gamma(s,T) ds \\
&= f(0,T) - \int_0^t \gamma(s,T) dW_s + \int_0^t \tau(s,T)\gamma(s,T) ds
\end{aligned}$$

The instantaneous short rate $r_t = f(t,t)$ satisfies the equation:

$$r_t = f(0,t) - \int_0^t \gamma(s,t) dW_s + \int_0^t \gamma(s,t)\tau(s,t) ds$$

3.1.2 Link between HJM model and Hull and White model

The HJM model is equivalent to the Hull and White model when we take $\gamma(s,t) = -\sigma(s)exp(-a(t-s))$, or similarly we take $\tau(s,t) = -\sigma(s)\dfrac{1-exp(-a(t-s))}{a}$. In fact, the instantaneous short rate r_t given by the HJM model satisfies:

$$\begin{aligned}
r_t &= f(0,t) + \int_0^t \sigma(s) e^{-a(t-s)} dW_s + \frac{1}{a}\int_0^t \sigma(s)^2 e^{-a(t-s)}(1-e^{-a(t-s)}) ds \\
&= f(0,t) + e^{-at}\int_0^t \sigma(s) e^{as} dW_s + \frac{1}{a} e^{-at} \int_0^t \sigma(s)^2 e^{as}(1-e^{-a(t-s)}) ds \\
&= f(0,t) + e^{-at}\int_0^t \sigma(s) e^{as} dW_s + \frac{1}{a} e^{-at} \int_0^t \sigma(s)^2 e^{as} ds - \frac{1}{a} e^{-2at}\int_0^t \sigma(s)^2 e^{2as} ds
\end{aligned}$$

Differentiating the equation above we get:

$$\begin{aligned}
dr_t &= \frac{\partial f(0,t)}{\partial t} + e^{-at}(\sigma(t)e^{at}dW_t) - ae^{-at}(\int_0^t \sigma(s)e^{as}dW_s)dt + \frac{1}{a}e^{-at}(\sigma(t)^2 e^{at}dt) \\
&\quad - e^{-at}(\int_0^t \sigma(s)^2 e^{as} ds)dt - \frac{1}{a}e^{-2at}(\sigma(t)^2 e^{2at}dt) + 2e^{-2at}(\int_0^t \sigma(s)^2 e^{2as} ds)dt \\
&= \frac{\partial f(0,t)}{\partial t} + \sigma(t)dW_t - ae^{-at}(\int_0^t \sigma(s)e^{as}dW_s)dt \\
&\quad - e^{-at}(\int_0^t \sigma(s)^2 e^{as} ds)dt + 2e^{-2at}(\int_0^t \sigma(s)^2 e^{2as} ds)dt
\end{aligned}$$

This means that

$$dr_t + ar_t dt = af(0,t)dt + \frac{\partial f(0,t)}{\partial t}dt + (\int_0^t \gamma(s,t)^2 ds)dt + \sigma(t)dt$$

Therefore the Hull and White model is equivalent to the HJM model while we take:

$$\theta(t) = af(0,t) + \frac{\partial f(0,t)}{\partial t} + \int_0^t \gamma(s,t)^2 ds$$

Remark: In the following pricing process, we utilize the Hull and White model with the constant volatility parameter. That is to say, the short rate r_t has the dynamic given as below:

$$dr_t = (\theta(t) - ar_t)dt + \sigma dW_t$$

3.1.3 Diffusion process

- Short rate and forward rate

According to the previous analysis and calculation, the result of the Hull and White model is:

$$\begin{aligned} r_t &= f(0,t) + \frac{\sigma^2}{2a^2}(1-e^{-at})^2 + \sigma e^{-at}\int_0^t e^{as}dW_s \\ f(t,T) &= f(0,T) + \frac{\sigma^2}{a^2}e^{-aT}(e^{at}-1) - \frac{\sigma^2}{a^2}e^{-2aT}(e^{2at}-1) + \sigma e^{-aT}\int_0^t e^{as}dW_s \end{aligned}$$

The first equation is used to generate the short rate r_t and the second equation is used to generate the zero coupon bonds $B(t,T)$, which are expressed as a function of $f(t,T)$.

- Beta

Beta represents the amount of money that one would get by placing one unit of currency at the risk neutral rate. More precisely, Beta can be represented by the following formula:

$$\beta(t) = exp(\int_0^t r_s ds)$$

We can deduce the value of $\beta(t)$ by utilizing the above equation for the short rate r_t:

$$\begin{aligned} ln\beta(t) &= \int_0^t r_s ds \\ &= \int_0^t f(0,s)ds + \int_0^t \frac{\sigma^2}{2a^2}(1-e^{-as})^2 ds + \int_0^t \sigma e^{-as}(\int_0^s e^{au}dW_u)ds \\ &= -lnB(0,t) + \frac{\sigma^2}{2a^2}[t - \frac{2}{a}(e^{-at}-1) + \frac{1}{2a}(e^{-2at}-1)] \\ &\quad +\sigma\int_0^t e^{au}\int_u^t e^{-as}ds)dW_u \end{aligned}$$

So

$$\begin{aligned} \beta(t) &= \frac{1}{B(0,t)} exp[\frac{\sigma^2}{2a^2}t - \frac{\sigma^2}{a^3}(1-e^{-at}) + \frac{\sigma^2}{4a^3}(1-e^{-2at})] \\ &\quad exp(\frac{\sigma}{a}W_t)exp(-\frac{\sigma}{a}e^{-at}\int_u^t e^{-au}dW_u) \end{aligned}$$

This equation is used to generate the Beta in the Economic Scenario generator.

- Zero coupon bonds

The value of zero coupon bonds can at t can be calculated using the following formula:

$$B(t,T) = exp(-\int_t^T f(t,s)ds)$$

We can deduce the value of zero coupon bonds $B(t,T)$ by utilizing the above equation for the forward short rate $f(t,T)$:

$$\begin{aligned} -lnB(t,T) &= \int_t^T f(t,s)ds \\ &= \int_t^T f(0,s)ds + \int_t^T \frac{\sigma^2}{a^2}e^{-as}(e^{at}-1)ds \\ &\quad - \int_t^T \frac{\sigma^2}{2a^2}e^{-2as}(e^{2at}-1)ds + \int_t^T \sigma e^{-as}(\int_0^t e^{au}dW_u)ds \\ &= (\int_0^T f(0,s)ds - \int_0^t f(0,s)ds) + \frac{\sigma^2}{a^3}(e^{at}-1)(e^{-at}-e^{-aT}) \\ &\quad - \frac{\sigma^2}{4a^3}(e^{2at}-1)(e^{-2at}-e^{-2aT}) + \frac{\sigma}{a}(e^{-at}-e^{-aT})\int_0^t e^{au}dW_u \end{aligned}$$

and

$$\int_0^T f(0,s)ds - \int_0^t f(0,s)ds = ln(B(0,t)) - ln(B(0,T)) = ln\frac{B(0,t)}{B(0,T)}$$

So

$$\begin{aligned} B(t,T) &= \frac{B(0,T)}{B(0,t)} exp[-\frac{\sigma^2}{a^3}(e^{at}-1)(e^{-at}-e^{-aT}) + \frac{\sigma^2}{4a^3}(e^{2at}-1)(e^{-2at}-e^{-2aT})] \\ & exp[-\frac{\sigma}{a}(e^{-at}-e^{-aT}) \int_0^t e^{au} dW_u] \end{aligned}$$

This formula can be rewritten as below:

$$B(t,T) = X(t,T) exp(-Y(t,T)) r_t$$

with

$$X(t,T) = \frac{B(0,T)}{B(0,t)} exp(Y(t,T) f(0,t) - \frac{\sigma^2}{4a}(1-e^{-2at)}) Y(t,T))^2$$

and

$$Y(t,T) = \frac{1}{a}(1-e^{-a(T-t)})$$

3.2 Black and Sholes models

The Black and Sholes model assumes that the equities S_t follow the equations below:

$$\begin{cases} \frac{dS_t^i}{S_t^i} &= r_t dt + \sigma_t^i dW_t^i \\ dr_t &= (\theta_t - a r_t) dt + \sigma_t^0 dW_t^0 \\ \langle dW_t^i, dW_t^j \rangle &= \rho_{ij} dt \end{cases}$$

where σ_t^i are the deterministic volatilities of the equities, σ_t^0 is the deterministic volatility of the rate and ρ_{ij} represents the correlation between the different Brownian motions.

The solution of the first equation is:

$$\begin{aligned} S_t^i &= exp(\int_0^t r_s ds - \frac{1}{2}\int_0^t \sigma_s^2 ds + \int_0^t \sigma_s dW_s^i) \\ &= \beta_t exp(-\frac{1}{2}\int_0^t \sigma_s^2 ds + \int_0^t \sigma_s dW_s^i) \end{aligned}$$

The values of the equities will be simulated by discretizing and using a finite number of time steps. Moreover, the volatility σ_s is assumed to be constant over the time step. So, at each time step, we will utilize the following formula to generate the equities in the Economic Scenario Generator:

$$\frac{S_{t+1}^i}{S_t^i} = \frac{\beta^{t+1}}{\beta^t} exp(-\frac{1}{2}\sigma^2 \Delta t + \sigma \int_t^{t+\Delta t} dW_s^i)$$

Once the rates and β_t and β_{t+1} have been generated, all that remains to do is to simulate $_displaystyle \int_t^{t+\Delta t} dW_s^i$ which is a Gaussian variable of variance Δt and with a correlation equal to ρ_{ij} with the other similar integrals we need to simulate.

3.3 Random variables generation

We utilize the Monte Carlo method to price the variable annuities product. Therefore, we need to generate a large number of random variables in the process of pricing. In fact, the generation of random variables can be separated into three steps: the first one is to generate the independent uniform random variables, the second one is to generate the independent Gaussian variables and the last step is to transform the independent Gaussian variables into correlated Gaussian variables using the Cholesky decomposition.

Suppose that we have generate a series of independent Gaussian variables $X = (G_1, \ldots, G_n)'$, and the covariance matrix of this vector is $E(XX^t) = Id$. Now, we want to generate a vector of correlated variables following a specified correlation matrix A. This means that $E(YY^t) = A$. Since A is a correlation matrix, it is symmetric positive, and we can assume that is also definite (otherwise the random variables to generate are linearly linked and there is no need to simulate all of them). So we can use the Cholesky decomposition to write $A = TT^t$, where T is a lower triangular matrix. If $Y = TX$, then we have $E(YY^t) = E(TXX^tT^t) = TE(XX^t)T^t = TIdT^t = A$. Moreover, Y is obtained by linear combinations of independent Gaussian variables.

3.4 Calibration

As mentioned before, the parameters a and σ of the Hull and White model are determined by the process of calibration. These parameters are chosen to make the model price of the swaption is as close as the market price of the swaption. More precisely, we minimize the error function:

$$F(a, \sigma) = \sum_{i=1}^{n} (swaption_{model}(a, \sigma) - swaption_{market})^2$$

For a swaption of maturity T, tenor t_f, and with the payment exchanged every Δt, the swap rate S is given by:

$$S = \frac{B(t,T) - B(t, T + t_f)}{\Delta t \sum_{0 < k\Delta t \leq t_f} B(t, T + k\Delta t)}$$

The market price of a swaption is deduced from the implied volatilities using the Black Scholes formula. For a payer swaption of maturity T, tenor t_f, strike K and with the payment exchanged every Δt, the market price is given by:

$$swaption_{market} = \Delta t \sum_{0<k\Delta t \leq t_f} B(t, T + k\Delta t)[SN(d_1) - KN(d_2)]$$

with $d_1 = \dfrac{ln(\frac{S}{K}) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$ and $d_2 = d_1 - \sigma\sqrt{T-t}$

Particularly, for an at the money payer swaption, we have $K = S$, so the market price is given by:

$$swaption_{market} = (B(t,T) - B(t, T + t_f))[N(d) - N(-d)]$$

with $d = \dfrac{1}{2}\sigma\sqrt{T-t}$

The model price of a swaption is determined by the Hull and White model. Then, we utilize the Levenberg Marquardt algorithm to minimize the error function $F(a, \sigma)$.

3.5 Variance reduction

The pricing of our products is done through Monte Carlo simulation. The aim of this scenario generator is to generate n independent scenarios which will be used for this pricing. The law of great numbers implies that the mean of the prices along each path P_k will converge towards the true price of the product P. More precisely, the central limit theorem states that:

$$\sqrt{n}(\frac{1}{n}\sum_{k=1}^{n} P_k - P) \longrightarrow N(0, \sigma^2)$$

where σ is the standard deviation of P_k

This means that writing $\overline{P} = \dfrac{1}{n}\sum_{k=1}^{n} P_k$ a 95% confidence interval for the price P can be given:

$$P \in [\overline{P} - \frac{1.96\sigma}{\sqrt{n}}; \overline{P} + \frac{1.96\sigma}{\sqrt{n}}]$$

We can see that the precision of the pricing depends on two parameters: the number of paths n and the standard error of a pricing σ. The number of paths should therefore be as big as possible, but due to runtime reasons caused by the complexity of our products and the large number of products we need to price, we usually take $n = 5000$, which are relatively small.We therefore need to work on reducing the standard deviation σ.

We use the martingale test in order to reduce the variance. The process is to reject a set of paths which does not satisfy the martingale test and to recommence the generation until the condition of the test verifies.

- Martingale test on rates

In practice, we test the condition if $|(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\beta_{tk}}) \times \frac{1}{B(0,k)} - 1| < \varepsilon$ where n is the number of paths and ε the martingale level.

- Martingale test on equities

In practice, we test the condition if $|\frac{1}{n}\sum_{k=1}^{n}\frac{S_{tk}}{S_0\beta_{tk}} - 1| < \varepsilon$ where n is the number of paths and ε the martingale level.

Chapter 4

Hedging strategies

4.1 Option Value

To estimate the evolution of the liability on the Variable Annuities products, the Liability Option Value, or more simply the Option Value is introduced. The Option Value represents the Variables Annuities contract value from the insurer's point of view. This value measures the difference between what the insurance receives from the policyholder through the charges and what the insurance gives to the policyholder through the claims.

The sum of the discounted claims (cash flows received by the policyholder) is given by:

$$PV(Claims) = \sum_{t=1}^{T} \beta_t \times Claims_t$$

The sum of the discounted charges (cash flows given to the insurance) is given by:

$$PV(Charges) = \sum_{t=1}^{T} \beta_t \times Charges_t$$

where β_t is the discount factor and $PV(.)$ is a function giving the sum of the discounted future cash flows.

Finally, for the insurer, the Option Value is worth:

$$OV = PV(Charges) \times \frac{EHC}{RRC} - PV(Claims)$$

where the EHC (Economic Hedging Cost) is the real cost of claims for the insurer. $RRC - EHC$ represents the insurance realized margin. The EHC is computed such that the Option Value is null at the contract subscription.

4.2 Some formulas to calculate the sensitivities

• Delta is a measurement of how the option value is affected when change is occurs in the underlying price.

$$Delta(x\%)_t = \frac{OV(AV_t \times (1 + x\%)) - OV(AV_t \times (1 - x\%))}{2 \times x\% \times AV_t}$$

Generally $x = 1$.

• Gamma is used to measure the rate of change in the delta with respect to the change in the underlying price.

$$Gamma(x\%)_t = \frac{OV(AV_t \times (1 + x\%) + OV(AV_t \times (1 + x\%)) - 2 \times OV(AV_t)}{(AV_t \times x\%)^2}$$

• Rho is a measurement of how the option value is affected when change occurs in the interest rate.

$$Rho_t = \frac{OV(Courbe_{taux} + \Delta Courbe_{taux}) - OV(Courbe_{taux} - \Delta Courbe_{taux})}{2 \times \Delta Courbe_{taux}}$$

Generally $\Delta Courbe_{taux} = 10bp$.

• Rho convexity is used to measure the rate of change in the rho with respect to the change in the interest rate.

$$\begin{aligned} Rho\ convexity_t &= \frac{1}{(\Delta Courbe_{taux})^2} \times [OV(Courbe_{taux} + \Delta Courbe_{taux}) \\ &\quad + OV(Courbe_{taux} - \Delta Courbe_{taux}) - 2 \times OV(Courbe_{taux})] \end{aligned}$$

4.3 Some hedging strategies

4.3.1 Delta neutral strategy

Delta neutral strategy is defined as keeping a portfolio's value neutral to small changes in the underlying stock's price. Delta is the sensitivity of an option's value to the stock price while all other variables remain unchanged. Because the option pricing equations are partial differential equations, Delta is mathematically represented as: $\delta = \frac{\partial V}{\partial S}$. This can be read as the partial derivative of the options value with respect to changes in the underlying stock's price.

A delta hedging is a simple type of hedging strategy that is widely used by derivative dealers to reduce or eliminate a portfolio's exposure to an underlying asset. The dealer calculates the portfolio's delta with respect to the underlier and then adds an offsetting position in the underlier to make the portfolio's delta zero. The offsetting position may take various forms, but a spot,or future position in the underlier is typical. All that is really required is that the position's delta offset that of the original portfolio.

In some cases, while the position's delta is hedged, it still has negative gamma, and likely negative vega as well. Such residual gamma and vega exposures are inevitable when options positions are delta hedged. One solution is to use the delta-gamma hedging strategy, in which options are added to a portfolio to achieve both a zero delta and zero gamma. Not only will this eliminate gamma exposure, but it will largely address vega exposure as well. Because options can be expensive, dealers rarely employ delta-gamma hedging.

Another problem with delta hedging an options position is the fact that the position's delta will change with movements in the underlying asset, thereby throwing off the delta hedge. The inevitable solution to this problem is to constantly adjust the delta hedge as the underlier moves. This technique is called dynamic hedging.

4.3.2 Gamma neutral strategy

Using gamma neutral hedging strategies involves creating options positions that have an overall gamma value that is zero, or very close to zero. The principle is to ensure that the delta value of such positions stays stable regardless of how the underlying security moves.

Gamma neutral options strategies can be used to create new positions or to adjust an existing one. The goal is to use a combination of options that will make the overall gamma value as close to zero as possible. A zero value will mean that the delta value shouldn't move when the price of the underlying security moves.

By creating a position that is gamma neutral, but delta positive, one can benefit from predictable profits without being exposed to exponential losses if things don't turn out as the dealer predicted. This is useful if the trader wish to hold a long term position on a security expected to increase in value over time, but wish to reduce the effect of any unexpected moves.

It's possible to create an options position that isn't affected by any moves in the price of an underlying security, but that will benefit from changes in the implied volatility. To do this a trader has to make sure a position is both gamma neutral and delta neutral. Doing so will effectively make the trader profit when implied volatility rises.

Chapter 5

Risk management

Since the initial Solvency I Directive 73/239/EEC was introduced in 1973, more elaborate risk management systems have been developed. While the Solvency I Directive was aimed at revising and updating the current European solvency regime, Solvency II has a much wider scope.

The Solvency II framework has three main pillars:

• Pillar 1 covers all the quantitative requirements. This pillar aims to ensure firms are adequately capitalized with risk-based capital. All valuations in this pillar are to be done in a prudent and market-consistent manner. Companies may use either the standard formula approach or an internal model approach. The use of internal models will be subject to stringent standards and prior supervisory approval to enable a firm to calculate its regulatory capital requirements using its own internal model.

• Pillar 2 imposes higher standards of risk management and governance within a firm's organization. This pillar also gives supervisors greater powers to challenge their firms on risk management issues. It includes the Own Risk and Solvency Assessment (ORSA), which requires a firm to undertake its own forward-looking self-assessment of its risks, corresponding capital requirements, and adequacy of capital resources.

• Pillar 3 aims for greater levels of transparency for supervisors and the public. There is a private annual report to supervisors, and a public solvency and financial condition report that increases the level of disclosure required by firms. Any current returns will be completely replaced by reports containing core information that firms will have to make to the regulator on a quarterly and annual basis. This ensures that a firm's overall financial position is better represented and includes more up-to-date information.

The Solvency II directive is based on the concept of market consistent valuation of the assets and liabilities. And European insurers are required to be in full compliance with this framework. It requires the companies to have enough capital to withstand the adverse changes to the fair value of liabilities (FVL) over one year with a 99.5th percentile confidence level.

One of the most important parts of this framework is to reflect the new risk management practices to define required capital. A solvency capital requirement may have the following purposes:

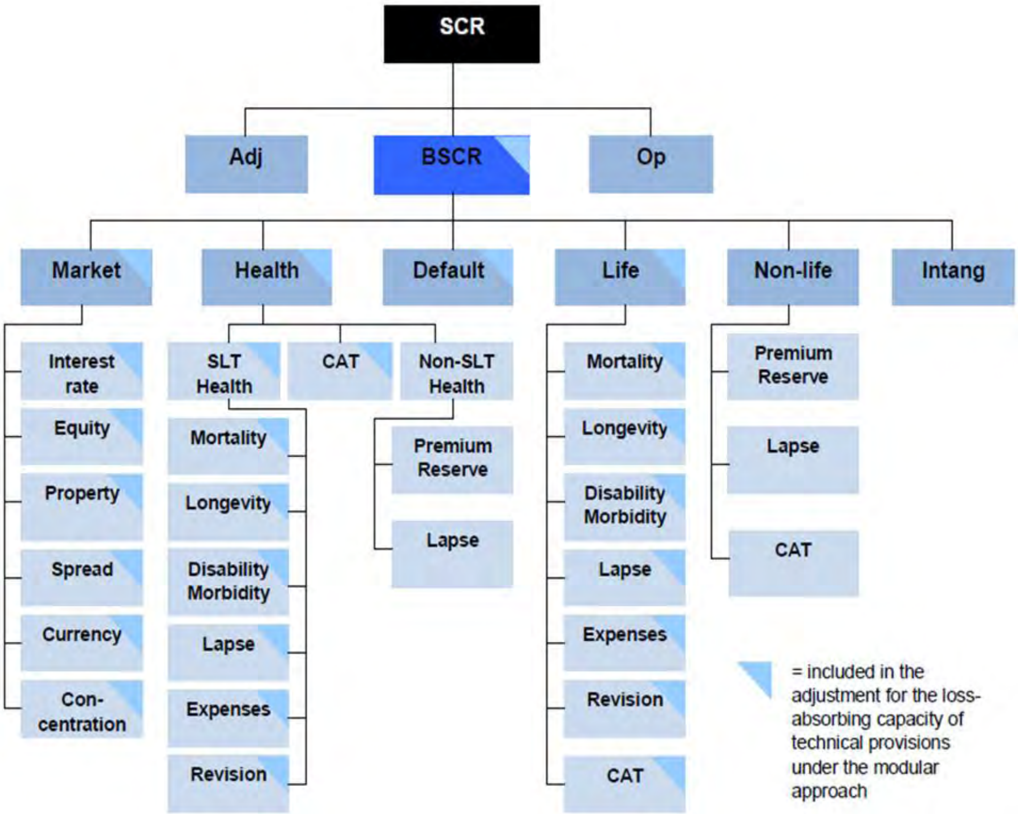
- To reduce the risk that an insurer would be unable to meet claims;

- To reduce the losses suffered by policyholders in the event that a firm is unable to meet all claims fully;

- To provide early warning to supervisors so that they can intervene promptly if capital falls below the required level;

- To promote confidence in the financial stability of the insurance sector.

The overview of the SCR calculation with the standard formula is given below:



SCR figures are calculated including and excluding the loss-absorbing capacity of technical provisions (LAC of TP), which means the calculation of both gross and net SCR figures is demanded.

In the internal model of AXA Life Invest, the solvency capital requirement is named as the Short Term Economic Capital (STEC), defined as a measure of the required capital level to withstand a once over 200 years shock over a one-year period. There are three components of STEC existed in the internal model: the Life STEC, the Market STEC and the Operational STEC. It is a value at risk measure and it is determined by the difference between the baseline VIF and the shocked VIF.

5.1 Life risk

5.1.1 Catastrophe risk

The catastrophe risk stems from extreme or irregular events whose effects are not sufficiently captured in other sub-risks (for example, the pandemic and the nuclear explosion). The aim of this sub-risk is to quantify the risk of a sudden extreme event, which would lead to a significant temporary increase in the mortality as well as in the medical expenses and disability claims. The catastrophe risk is contingent on mortality, and in the internal model of AXA Life Invest, the mortality has been identified as the key component of this risk, which is noted as Cat_Mortality.

An absolute stress of 1.2‰ is to be applied as an additional increase on the base mortality rates in the first year of the projection. That is to say, the capital requirement should be equal to the loss in AFR (Available Financial Resources) that results from an instantaneous increase of the mortality rate in the following 12 months.

$SCR_{cat} = \Delta AFR_{cat}$ the variation of AFR in case of a combination of the catastrophe shocks, excluding the loss-absorbing capacity of technical provision.

5.1.2 Mortality risk

The mortality risk is the risk of loss, or of adverse change in the value of liabilities, resulting from changes in the level, trend or volatility of mortality rate, where an increase in the mortality rate leads to an increase in the value of insurance liabilities. The aim of this sub-risk is to quantify the risk of underestimating mortality assumption for the business adversely sensitive to an increase in mortality. The risk of adverse deviation of the mortality is captured through two components: trend and level. And, in the internal model of AXA Life Invest, the trend component of mortality risk, noted as Mortality_Trend, has been considered as the key component of this risk, which captures the risk of overestimating the improvements of mortality assumption for the business adversely sensitive to an increase in mortality.

The capital requirement should be equal to the loss in AFR that would result from an instantaneous permanent and proportional increase of 15% in the mortality rates used for the calculation of technical provisions.

$SCR_{mort} = \Delta AFR_{mort}$ the variation of AFR in case of a +15% mortality rates shock, excluding the loss-absorbing capacity of technical provision.

5.1.3 Longevity risk

The longevity risk is the risk of loss, or of adverse change in the value of liabilities, resulting from changes in the level, trend or volatility of mortality rates, where a decrease in the mortality rates leads to an increase in the value of insurance liabilities. The aim is to quantify the risk of overestimating mortality assumption for the business adversely sensitive to an increase in mortality.

The scenarios are calibrated based on the general population data (Human Mortality Database) and updated with the most recently available population data. These scenarios represent the 99.5% percentile level in the sense of the most adverse scenario for the longevity deviation.

5.1.4 Lapse risk

The lapse risk is the risk of loss, or of adverse change in the value of liabilities, resulting from changes in the expected exercise rates of policyholder options. Scope of the stress is full lapses, cancellations, transfers (to other insurer for example), surrenders, proportional partial withdrawals, paid-up status (meaning reduced guarantees due to the suspension of regular premium payment), renewal premiums, flexible premiums and single premiums when included.

- Lapse up risk

This sub-risk captures the risk of higher than expected lapses on a permanent basis. The by-default stress scenarios defined by the group are: a 35% permanent proportional increase in the best estimate lapse rate for business that is adversely exposed to an increase in lapse.

$$SCR_{lapse\_up} = \Delta AFR_{R(up)}$$

where $R_{up}(R) = 135\%R$ is the shocked rate to be applied in case of the increase of lapse rate.

- Lapse down risk

This sub-risk captures the risk of lower than expected lapses on a permanent basis. The by-default stress scenarios defined by the group are: a -35% permanent proportional increase in the best estimate lapse rate for business that is adversely exposed to an increase in lapse.

$$SCR_{lapse\_down} = \Delta AFR_{R(down)}$$

where $R_{down}(R) = 65\%R$ is the shocked rate to be applied in case of the decrease of lapse rate.

- Mass lapse risk

This sub-risk aims to capture the risk of adverse unanticipated lapses in the coming year as a result of a mass phenomenon due to an additional lapse driver which is only observable in extreme scenario (for example, fear of bankruptcy and reputation issue).

The stress factors are different for individual business and group business. For individual business, a stress of an additional increase of 22% on the lapse rates in the first year of the projection.

$$SCR_{lapse\_mass} = \Delta AFR_{R(mass)}$$

where $R_{mass}(R) = 22\%$ is the shocked rate to be applied in case of mass lapse.

5.1.5 Expense risk

The expense risk arises from the variation in the expenses incurred in servicing insurance and reinsurance contracts. The aim is to quantify the risk that the level of expenses is higher than expected.

The by-default stress scenarios defined by the group are: a permanent proportional increase of 10% in expected expense.

$SCR_{exp} = \Delta AFR_{exp}$ the variation of AFR in case of the combination of the expense shocks, excluding the loss-absorbing capacity of technical provision.

5.1.6 Life risk aggregation

The capital requirement for life risk is derived by combining the capital requirements for the life sub-risks using a correlation matrix as follows:

$$SCR_{life} = \sqrt{\sum_{r,c} CorrLife_{r,c} \times SCR_r \times SCR_c}$$

where $CorrLife_{r,c}$ is the entries of the correlation matrix CorrLife. SCR_r, SCR_c are the capital requirements for individual life sub-risks according to the rows and columns of correlation matrix CorrLife and where the correlation matrix CorrLife is defined as follows:

	Cat	Mortality	Longevity	Lapse up	Lapse down	Expense	Mass lapse
Cat	100%	0%	0%	0%	0%	25%	0%
Mortality	0%	100%	-50%	25%	0%	25%	0%
Longevity	0%	-50%	100%	0%	25%	0%	0%
Lapse up	0%	25%	0%	100%	50%	25%	0%
Lapse down	0%	0%	25%	50%	100%	0%	0%
Expense	25%	25%	0%	25%	0%	100%	50%
Mass lapse	0%	0%	0%	0%	0%	50%	100%

5.2 Market risk

Market risk arises from uncertainty in:

- the level of interest rates, exchange rates, stock prices and property prices;
- the risky components of interest sensitive financial instruments;
- the concentration of risks in portfolios.

It consists of the following sub-risk categories in the internal model of AXA Life Invest: equity, interest rate, currency, spread, volatility, inflation, market and concentration. Interest rate, equity, property and currency risks are calculated using a scenario approach whereas spread and concentration risks are derived from a factor model.

5.2.1 Equity risk

Equity risk arises from the level or volatility of market prices for equities. Exposures to equity risk refer to all assets and liabilities whose value is sensitive to changes in equity prices.

In order to calculate the equity SCR, the existing equity will be separated into two types:
Type 1: the equities listed in regulated markets in countries which are member of the EEA (European Economic Area) or the OECD (Organisation for European Economic Co-operation);
Type 2: the equities listed in stock exchanges in countries which are not member of the EEA or OECD; the equities not listed; the private equities; the hedge funds; the commodities and other alternative investments.

The Equity SCR (both gross and net of LAC of TP) is calculated as:

$$SCR_{equity} = \sqrt{SCR_{type1}^2 + 2 \times 0.75 \times SCR_{type1} \times SCR_{type2} + SCR_{type2}^2}$$

where the SCR_{type1} (respectively SCR_{type2}) is the simple sum of the capital requirements for: standard equities, standard equities of strategic nature and duration based equities.

The capital requirements for type 1 and type 2 equities are calculated as follows:

	Type 1	Type 2
Standard equity	39% + symmetric adjustment	49% + symmetric adjustment
Standard equity of strategic nature	22%	22%
Duration based equity	22%	22%

where the symmetric adjustment is calculated as below:

$$\begin{aligned} SA &= min[10\%; max[-10\%; \frac{1}{2}(\frac{CI - AI}{AI} - 8\%)]] \\ CI &= current\ level\ of\ MSCI\ Europe\ Index \\ AI &= weighted\ average\ over\ the\ last\ 36\ months\ of\ daily\ MSCI\ Europe\ Index \end{aligned}$$

5.2.2 Interest Rate risk

The SCR for the interest rate risk equals the loss in the own funds that results from an instantaneous increase or decrease in basic risk-free interest rate at different maturities in accordance with the specific shocks listed in the table below.

Maturity	≤1	2	3	4	5	6	7	8	9	10	
Interest Rate Up	70%	70%	64%	59%	55%	52%	49%	47%	44%	42%	
Interest Rate Down	-75%	-65%	-56%	-50%	-46%	-42%	-39%	-36%	-33%	-31%	
Maturity	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**≥90**
Interest Rate Up	39%	37%	35%	34%	33%	31%	30%	29%	27%	26%	20%
Interest Rate Down	-30%	-29%	-28%	-28%	-27%	-28%	-28%	-28%	-29%	-29%	-20%

For missing maturities the shocks are interpolated linearly. For maturities greater than 90 years the stresses of +/-20% are maintained. The same shocks have to be applied simultaneously to all currencies.

The value of SCR for the interest rate risk is equal to the larger capital requirement for either the sum for each currency of the risk of an increase in interest rates or the risk of a decrease in interest rates.

$$SCR_{int} = max(0; \sum_{all\ fx} IR_{up}; \sum_{all\ fx} IR_{down})$$

When applying the stress factors in this risk, the absolute change in both the upward and downward side, should be at least equal to 1%.

Example: Suppose that there exists a yield curve where 1 year maturity is 0.6%, the 3 year maturity is 1.2% and the 5 year maturity is 2.2%. In this case, the shocked yield curves are as follows:

Initial Rate	Up shock	Down shock
1year: 0.6%	1.6% =0.6% + max(1%, 0.6%*70%)	0% =max(0%, 0.6%+min(-1%, 0.6%*-75%))
3year: 1.2%	2.2% =1.2% + max(1%, 1.2%*64%)	0.2% =max(0%, 1.2%+min(-1%, 1.2%*-56%))
5year: 2.2%	3.41% =2.2% + max(1%, 2.2%*55%)	1.19% =max(0%, 2.2%+min(-1%, 2.2%*-46%))

5.2.3 Concentration risk

Concentration risk is calculated using a factor-based formula. It is defined as the risk regarding the accumulation of exposures with the same counterparty, both direct and indirect.

The gross Concentration SCR is calculated based on the net exposures E_i, ratings and the amount of total Assets.

$$SCR_{conc} = \sum_i \sqrt{Conc_i^2}$$

where $Conc_i = g_i \times XS_i$ and $XS_i = max(0; E_i - CT_i \times Asset_i)$ with E_i is the exposure to Asset i and CT_i is the concentration threshold for Asset i.

The risk factor g_i and the threshold CT_i are given as below:

Ratings	AAA-AA	A	BBB	BB	B	CCC or lower	Unrated
Credit Quality Step	0-1	2	3	4	5	6	7
Excess Exposure Threshold CT_i	3%	3%	1.50%	1.50%	1.50%	1.50%	1.50%
Risk Factor g_i	12%	21%	27%	73%	73%	73%	73%

5.2.4 Volatility risk

Volatility risk is the risk that the change of price of a portfolio will be determined by the changes in the volatility of a risk factor. It usually applies to the portfolio of derivatives instruments, where the volatility of its underlying is a major influence of price.

A measure for the sensitivity of a price of a portfolio (or asset) to changes in volatility is vega, the rate of change of the value of the portfolio with respect to the volatility of the underlying asset.

5.2.5 Inflation risk

Inflation risk, which is also called the purchasing power risk, is the risk that the cash flows from an investment will not be as worthy as in the future because of the changes in the purchasing power due to the inflation.

As we know, the duration of a VA contract is generally very long. In this way, we need to consider the inflation risk during the life of the VA product. And it is important to realize that the inflation risk is not the risk that there exists inflation in the future, but the risk that the inflation level is higher than expected.

5.2.6 Market risk aggregation

The aggregation to total market risk figures is done using a correlation matrix. The correlation matrix differs depending whether the interest rate SCR is derived from the up shock or from the down shock of interest rates. In the first case the correlation is 0, else 50%.

The gross Market SCR (excluding LAC of TP) calculation is performed as follows:

$$SCR_{market} = \sqrt{\sum_{i,j} CorrMKT_{i,j} \times SCR_i \times SCR_j}$$

5.3 Operational risk

Operational risk is the risk of a change in value caused by the fact that actual losses, incurred for inadequate or failed internal processes, people and systems, or from external events (including legal risk), differ from the expected losses.

In the internal model of AXA Life Invest, the operational risk is calculated as an aggregation of the market STEC and the life STEC, and then multiplied by a certain coefficient, whose value equals 15%.

Chapter 6

Impact of the product features on the STEC

The core mission of my internship is the analysis of the Variable Annuities product features, the objective being the capital requirement optimization. More precisely, it is a research project of *AXA Life Invest* with the purpose of enhancing the company's overall economic strength, which is on the top of the internal research agenda.

Along with the more prudential developments of the *Solvency II directives*, more specific requests on insurance company's capital requirement have been made in order to protect both the insurer and the policyholder's interest as well as to strengthen the stability of the insurance sector as a whole. However, it is essential for an insurance company to reduce its capital for various reasons. For instance, the excessive capital might induce some negative impacts on company's business development.

On one hand, if the capital cannot be efficiently invested into the financial market, it would be suboptimal for the insurance company to maximize its own profitability, which in turn will reduce the social welfare in the economy. Furthermore, in short-run perspective, it will bring down the company's competitiveness in the sector; in a long-run basis, it is harmful for the company's business development.

On the other hand, it is very costly for a financial company to hold excess capital buffer, since the opportunity cost in the financial market is huge. In order to balance such kind of loss, the charges caused by the higher required capital might be transferred to the policyholders in the form of higher prices of the insurance product.

As a consequence, the initial purpose of providing protection to both the policyholder and the insurance company's interest has not been fulfilled successfully. In contrast, if the capital requirement can be optimized, the profitability of an insurance company will be largely improved and it will contribute to the business development of the company in the future. Due to all these reasons, we need to concentrate on how to optimize the capital requirement.

In order to analyze the impact of the product features on the capital requirement, we utilize an initial product design throughout the study and we modify in the further step the important components of the product and the modelling assumptions.

The initial design combines two guarantees at the same time: the Guaranteed Lifetime Withdrawal Benefit (GLWB) and the Guaranteed Minimum Death Benefit (GMDB). It has the important product features as follows:

- The minimum age at subscription of the contract is 45 whereas the maximum is 80. In addition, a policyholder has the right to receive his coupons only after 60.

- The minimum duration for the accumulation period is 2 years.

- The RRC rate differs for the three Capped Volatility Funds: for the CVF30, 1.05%; for the CVF40, 1.2% and for the CVF50, 1.35%.

- For the single life contracts, the income rate depends on the age when the policyholder receives his coupon for the first time. More precisely, the income rate for a policyholder aged 60 is 3% and the increment is 10bp per year, growing until the age of 75. The table below shows the evolution of the income rate:

Age	60	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75+
%	3	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9	4	4.1	4.2	4.3	4.4	4.5

For the joint life contracts, the income rate is 0.5% below the ones for the single life contracts, and the applicable income rate is based on the age of the younger life.

• The roll-up rate is 2.5% (simple rate) and this effect is applied until the first withdrawal of the coupon and for a maximum of 15 years. In addition, the roll-up effect is combined with an annual ratchet effect. Moreover, the annual ratchet is applied during the accumulation and payment period.

• For the GLWB guarantee, the initial benefit base equals to the initial net premium (the initial premium reduced by the upfront charges).

Before the end of the accumulation period, the Withdrawal Benefit is calculated as below:

$$WB_t = max(ratchet\,base_t; 1+min(t,15)\times roll-up\,rate)\times initial\,net\,premium$$

After the accumulation period the roll-up effect is not applied, and the Withdrawal Benefit is calculated as below:

$$WB_t = max(WB_{t-1}; AV_t)$$

For the GMDB guarantee, the beneficiary will receive an amount of money when the policyholder is dead. If the policyholder dies before the end of the accumulation period, the amount is equal to the maximum between the Account Value and the initial net premium. If not, the amount is equal to the rest of the Account Value.

6.1 Impact of some financial parameters

This is the first step of our study. We apply some financial shocks, such as making a +10bp parallel shift in the yield curve and increasing the swaption volatility by 1%, to see the variation of the Market STEC.

The reason of our consideration of these two financial parameters results from the fact that they are used to calibrate the model. More precisely, the parameters in the model are determined by using taking them as two important inputs.

A swap curve is given as a first input to calibrate this model. It can be transformed into a zero coupon curve by interpolation and bootstrapping, where a zero coupon value is known for every maturity by time steps of 1 month. This allows us to compute the forward rate as the derivative of this zero coupon curve. Then $\theta(t)$ in the Hull and White model can be determined by absence of arbitrage using this zero coupon curve and the forward rates deduced from this curve.

To determine the values of a and σ in the Hull and White model, we need more market information. The information will come from the second input used to calibrate the model, swaption volatilities. a and σ will be chosen in order to allow the swaption model prices to fit as well as possible the swaption market prices. More precisely a and σ are chosen in order to minimize the sum of the squares of the difference between the swaption market prices and the swaption model prices.

As a consequence, we can conclude that these two financial parameters are so important in our product design. They attract our attention to study their impact on the Market STEC.

6.1.1 Impact of the swap rate

A swap is a derivative in which two counterparties agree to exchange periodically their cash flows with each other within a period of time in the future. The most common transaction of a swap is the interest rate swap. In this type of contract, a nominal principle is fixed at the beginning. We suppose that there are two counterparties A and B. Then within the duration of the contract, A will pay B with a fixed rate while B will pay A with a floating rate. The swap rate is the fixed rate in the interest rate swap contract.

If we apply the shock of improving the swap rate by 10bp, the market STEC will decreased by 3.4% from 5.23% to 5.05% of the initial premium.

The Greeks in the guarantee component have the variation as follows:

	Delta	Gamma	Rho	Rho_convexity	Equity_vol	IR_vol
modified	23.2852	-1.2864	4.5035	-0.0379	-49.8722	-21.767
initial	24.3898	-1.3366	4.8943	-0.0403	-50.507	-20.7857

The variation of the Greeks in the base product component is given below:

	Delta	Gamma	Rho	Rho_convexity	Equity_vol	IR_vol
modified	7.6133	-0.1336	-0.3187	-0.003	-4.7156	-1.9165
initial	7.8268	-0.1385	-0.2887	-0.003	-4.4863	-1.7645

The Greeks in the base product component have not changed significantly. So, we can conclude that the variation of market STEC results from the variation of the Greeks in the guarantee component. In addition, we can notice that the Delta, Rho and the absolute value of equity volatility will decrease while the absolute value of interest rate volatility will increase. Then, we will choose a certain model point to comprehend the variation of the Greeks in order to simplify the analysis.

For a policyholder who has invested 10000 euros in the Variable Annuities contract, the tables below demonstrate the present value of the EHC charges and the claims in different baseline case and shocked scenario case.

initial	PV(EHC Charges)	PV(Claims)	modified	PV(EHC Charges)	PV(Claims)
AV(1-1%)	1887	1906	AV(1-1%)	1846	1806
Baseline case	1885	1885	Baseline case	1844	1787
AV(1+1%)	1884	1866	AV(1+1%)	1843	1768

The Delta at $t = 0$ for this model point is computed using the formula mentioned before. So, the Delta with the initial swap assumption equals 18.5% ($[(1884 - 1866) - (1887 - 1906)]/(2\% * 10000) = 18.5\%$) and the Delta with the new swap assumption equals 17.5% ($[(1843 - 1768) - (1846 - 1806)]/(2 * 1\% * 10000) = 17.5\%$). In this way, the value of Delta decreases.

initial	PV(EHC Charges)	PV(Claims)	modified	PV(EHC Charges)	PV(Claims)
rate(1-0.1%)	1926	1990	rate(1-0.1%)	1885	1885
Baseline case	1885	1885	Baseline case	1844	1787
rate(1+0.1%)	1844	1787	rate(1+0.1%)	1804	1695

The Rho at $t = 0$ for this model point is computed using the formula mentioned before. So, the Rho with the initial assumption equals 6.05% and the Rho with the new assumption equals 5.45%. The value of Rho decreases.

initial	PV(EHC Charges)	PV(Claims)	modified	PV(EHC Charges)	PV(Claims)
IR_vol(1-0.1%)	1892	1851	IR_vol(1-0.1%)	1851	1752
Baseline case	1885	1885	Baseline case	1844	1787
IR_vol(1+0.1%)	1878	1919	IR_vol(1+0.1%)	1837	1822

The IR_vol at $t = 0$ for this model point is computed using the formula mentioned before. According to the formula, the IR_vol with the initial assumption equals -41% and the new IR_vol is equal to -42%. So, the absolute value of IR_vol increases.

initial	PV(EHC Charges)	PV(Claims)	modified	PV(EHC Charges)	PV(Claims)
Equity_vol(1-0.1%)	1885	1852	**Equity_vol(1-0.1%)**	1844	1755
Baseline case	1885	1885	**Baseline case**	1844	1787
Equity_vol(1+0.1%)	1887	1926	**Equity_vol(1+0.1%)**	1846	1826

The Equity_vol at $t = 0$ for this model point is computed using the formula mentioned before. According to the formula, the Equity_vol with the initial assumption equals -36% and the new Equity_vol is equal to -34.5%. So, the absolute value of Equity_vol decreases.

6.1.2 Impact of the swaption volatility

A swaption is an option granting its owner the right but not the obligation to enter into an underlying swap. There are two types of swaption contracts: one is the payer swaption and the other one is the receiver swaption. The former gives the owner of the contract the right to enter into a swap where he pays with a fixed rate and receives with a floating rate. The latter gives the owner of the contract the right to enter into a swap where he pays with a floating rate and receives with a fixed rate.

If we apply the shock of improving the swaption volatility by 1%, the market STEC will not change greatly compared to the result with the initial swaption volatility assumption.

The Greeks in the guarantee component has the variation as follows:

	Delta	Gamma	Rho	Rho_convexity	Equity_vol	IR_vol
modified	24.1616	-1.3232	4.7943	-0.04	-50.326	-21.0026
initial	24.3898	-1.3366	4.8943	-0.0403	-50.507	-20.7857

The variation of the Greeks in the base product component is given below:

	Delta	Gamma	Rho	Rho_convexity	Equity_vol	IR_vol
modified	7.7649	-0.1352	-0.3039	-0.003	-4.5454	-1.8003
initial	7.8268	-0.1385	-0.2887	-0.003	-4.4863	-1.7645

We notice that the Greeks in both the guarantee component and the base product component have not changed significantly. That is to say, the shock of improving the swaption volatility by 1% will not make an important impact on the market STEC.

6.2 Impact of some important parameters

This is the second step of our study. We modify some important product features, such as the roll-up rate, the RRC rate, the ratchet frequency and the guaranteed income rate, to see the influence of these variations on the Life STEC.

6.2.1 Impact of the roll-up rate

As introduced in the previous chapter, the roll-up rate is defined as a fixed rate used to revalue the initial premium in order to calculate the Benefit Base. In the initial design, the roll-up rate is fixed at 2.5% and in the process of changing this product feature, its value is improved by 50bp to 3%.

The Life STEC for each sub-risk has the values as follows:

	Cat Mortality	Mortality Trend	Longevity	Lapse up	Lapse down	Expenses	Mass lapse
modified	0.8113	0	329.9981	6.7149	0	24.9109	96.2521
initial	0.9004	0	318.1769	14.5107	0	24.9734	113.5487

Compared to the initial design, the longevity risk is augmented while the lapse up risk and the mass lapse risk is decreased. And, after the aggregation of these sub-risks, the Life STEC is altered from 3.44% to 3.48% of the initial premium.

In order to explain the changes of these three risks, we look at their values in both the guarantee part and the base product part.

In the guarantee part, the results are as follows:

	Cat Mortality	Mortality Trend	Longevity	Lapse up	Lapse down	Expenses	Mass lapse
modified	-0.2209	103.3456	-331.6314	31.8157	-32.3908	0	32.8641
initial	-0.2628	100.6654	-325.043	26.2106	-25.4028	0	24.5974

In the base product part, the results are as follows:

	Cat Mortality	Mortality Trend	Longevity	Lapse up	Lapse down	Expenses	Mass lapse
modified	-0.5903	-3.3795	1.6334	-38.5306	44.5265	-24.9109	-129.1161
initial	-0.5955	-3.4447	1.7083	-39.8511	46.3558	-24.9381	-130.06

We can learn from these data that the changes in the longevity risk, lapse up risk and mass lapse risk are caused by their variations in the guarantee part, since there aren't much differences for the base product part between the situation where the roll-up rate is equal to 3%, and where the roll-up rate is equal to 2.5%. Therefore, we analyse the variations in the guarantee part for these three risks to explain their changes in the Life STEC.

If the roll-up rate is raised from 2.5% to 3%, the Benefit Base calculated with the new roll-up rate will be increased. That is to say, the insurer offers a rising level of guaranteed Benefit Base to the policyholder. As a result, the coupon withdrawn from the Account Value by policyholders will be augmented as well since its value is computed as the guaranteed Benefit Base multiplied by the income rate.

The Life STEC for the guarantee part is calculated by the difference between the option value of the shock scenario and that of the baseline case. As described before, the option value is defined as the present value of EHC charges minus claims. Since the amount of EHC charges does not vary as much as the claims if the shock is applied to the baseline, the value of life STEC for each sub-risk can be obtained approximately by the present value of claims of the baseline minus that of the shock scenario in our analysis.

For the longevity risk, we can notice that the value of Life STEC for this sub-risk is negative, which results from the increase of the present value of claims if the longevity shock is applied. To analyse the impact caused by the increase of roll-up rate, we compare the absolute value of Life STEC for this part of risk.

In order to simplify the analysis, we can suppose that in the shock scenario, a policyholder can live n years longer than in the baseline. As a result, the numbers of coupon received by the policyholders will be augmented because they are still alive, which leads to the increase of claims paid by the insurer. Since the claims are paid in the last few years of the contract when the rest of Account Value cannot cover the coupons needed by the policyholders, the value of each annual claim is equal to that of a coupon. Combined with the previous analysis, the absolute value of life STEC for the longevity risk will be approximately equal to the value of n coupons. As the value of each coupon is augmented with the modified roll-up rate, the absolute value of Life STEC for this sub-risk increases as well.

For the lapse up risk and the mass lapse risk, as their evolutions are quite the same, we explain the movement of mass lapse risk in order to simplify the process. We can suppose that in the shock scenario where the mass lapse risk is applied, $x\%$ of policyholders choose to terminate their contracts at the first year, which leads to the case where the present value of claims of the shock scenario is turned to be $(1 - x\%) \times PV(Claims)_{baseline}$. As a result, the Life STEC for the mass lapse risk is approximately equal to $(1-(1-x\%)) \times PV(Claims)_{baseline}$, which is $x\% \times PV(Claims)_{baseline}$. The lapse rate $x\%$ in the baseline and in the shock scenario is very close, which is about 22.6%. And, as the value of a single coupon goes up in our new hypothesis, the present value of claims in the baseline augments as well, which leads to the increase of the Life STEC for the mass lapse risk.

6.2.2 Impact of the RRC rate

As explained before, the RRC (Real Rider Charge) is the guarantee charge. This charge is the amount that is withdrawn periodically from the Account Value. It is calculated as a percentage of the Account Value fixed at the beginning of the contract and named the RRC rate in our analysis. In our initial design, the RRC rate is fixed at 1.05% for the CVF30, 1.2% for the CVF40, and 1.35% for the CVF50 since it is obvious that an insured needs to pay more charges when he invests in a more risky fund. Then, we improve the RRC rate by 50bp for each fund in the new case.

The Life STEC for this hypothesis has the values as follows:

	Cat Mortality	Mortality Trend	Longevity	Lapse up	Lapse down	Expenses	Mass lapse
modified	1.1309	0	340.6946	22.0344	0	23.5788	157.157
initial	0.9004	0	318.1769	14.5107	0	24.9734	113.5487

There are significant variations in the STEC for longevity risk, lapse up risk and mass lapse risk compared to the initial design. The augmentation of STEC in these sub-risks results in the increase of Life STEC as a whole, whose value is changed from 3.44% to 3.82% of the initial premium.

The following step is to look at the changes in both the guarantee part and the base product part.

In the guarantee part,

	Cat Mortality	Mortality Trend	Longevity	Lapse up	Lapse down	Expenses	Mass lapse
modified	0.0938	111.7035	-344.0501	58.435	-63.017	0	103.8354
initial	-0.2628	100.6654	-325.043	26.2106	-25.4028	0	24.5974

We can notice that the absolute value of the longevity risk is augmented, and the value of the lapse up risk and the mass lapse risk goes up as well.

Since the RRC rate is enhanced, the insurer will receive more charges taken from policyholder's account. Therefore, the Account Value will be decreased in a more rapid way, which results in the diminution of its present value. As a consequence, the amount of claims given to the policyholder augments. In this way, the reasons for variations in these three sub-risks are quite the same with the ones in the change of roll-up rate.

In the base product part,

	Cat Mortality	Mortality Trend	Longevity	Lapse up	Lapse down	Expenses	Mass lapse
modified	-1.2247	-6.903	3.3555	-80.4694	91.898	-23.5788	-260.9924
initial	-0.5955	-3.4447	1.7083	-39.8511	46.3558	-24.9381	-130.06

Compared to the initial design, the absolute value of the lapse up risk and the mass lapse risk are increased. We'll try to explain the evolution of these two sub-risks.

When calculating the STEC for the base product part, the value of VIF is mainly driven by the value of the present value of RRC charges, noted $PV(RRC\ Charges)$. If the shock of lapse up or the mass lapse is applied to the baseline, the $PV(RRC\ Charges)$ will be decreased, which results from the hypothesis that more people choose to terminate their contracts than previewed. Therefore, the values of lapse up risk and the mass lapse risk are negative. To analyse the impact of RRC rate on these sub-risks, we compare the absolute value of STEC for these sub-risks.

As explained before, the present value of RRC charges plays an important role in the evolution of the value of VIF. The present value of RRC Charges in the shock scenario can be regarded as the one in the baseline multiplied by a certain number x, which is inferior to 1. And, the present value of RRC Charges in the baseline is increased as we improve the RRC rate by 50bp. In conclusion, the STEC for these two risks will be augmented in absolute value.

6.2.3 Impact of the ratchet frequency

In the initial design, the frequency of the ratchet mechanism applied in the deferral period for the valuation of the Benefit Base is one year. As the ratchet frequency can make a difference in the STEC, we change this product feature in order to see the impacts. The first step is to increase the frequency from one year to five years. The second step is to remove this mechanism in the process of evaluating the Benefit Base, which can be considered as the case where the ratchet frequency is infinite.

The STEC for each sub-module risk are presented below:

	Cat Mortality	Mortality Trend	Longevity	Lapse up	Lapse down	Expenses	Mass lapse
1 year	0.8583	0	323.3347	13.6405	0	24.9381	105.4625
5 years	0.8609	0	318.3684	15.7541	0	24.7696	105.587
No ratchet	0.8545	0	317.4	14.1838	0	24.6819	104.1682

In the guarantee part,

	Cat Mortality	Mortality Trend	Longevity	Lapse up	Lapse down	Expenses	Mass lapse
1 year	-0.2628	100.6654	-325.043	26.2106	-25.4028	0	24.5974
5 years	-0.2686	98.9867	-320.068	24.7131	-22.9702	0	23.6752
No ratchet	-0.2642	98.7651	-319.0644	25.9003	-24.6663	0	24.6375

In the base product part,

	Cat Mortality	Mortality Trend	Longevity	Lapse up	Lapse down	Expenses	Mass lapse
1 year	-0.5955	-3.4447	1.7083	-39.8511	46.3558	-24.9381	-130.06
5 years	-0.5923	-3.4198	1.6996	-40.4672	47.6901	-24.7696	-129.2622
No ratchet	-0.5903	-3.3859	1.6643	-40.0842	47.5501	-24.6819	-128.8057

The Life STEC in these three cases does not vary a lot. The impact is from 3.45% to 3.41%, then to 3.39% of the initial premium. Since the second change of the product design can be regarded as the ratchet frequency is altered to infinity, we can conclude that the more frequently the ratchet mechanism takes effect, the more Life STEC an insurer should hold.

By comparing the data in these three cases, we can notice that the variation of the Life STEC results mainly from the variation of the longevity risk, especially in the guarantee component. In addition, we can see that the absolute value of the longevity risk in the guarantee component will be declined as the gap between two valuation dates of the ratchet mechanism is longer. So, the following analysis is concentrated on explaining the reasons for this kind of change.

The STEC for each sub-risk is calculated as the difference between the option value in the shock scenario and the one in the baseline. And the option value for a certain scenario is defined as the present value of the EHC charges minus the one of the claims. So, we can write it together as follows:

$$\begin{aligned} STEC_{guarantee} &= OV_{shock} - OV_{baseline} \\ &= (PV(EHCCharges)_{shock} - PV(Claims)_{shock}) \\ &\quad -(PV(EHCCharges)_{baseline} - PV(Claims)_{baseline}) \end{aligned}$$

Since the $PV(Claims)$ is the main driver compared to the $PV(EHCCharges)$ in the evolution of the option value, we can simplify the initial formula by:

$$STEC_{guarantee} = PV(Claims)_{baseline} - PV(Claims)_{shock}$$

If the longevity shock is applied, the present value of the claims will be augmented because of the higher life expectancy of a policyholder. As a result, the present value of the claims in the shock scenario can be considered as the baseline option value multiplied by a certain number larger than 1. In this way, the guarantee component of STEC is calculated as: $PV(Claims)_{baseline} \times (1 - x)$ with $x > 1$. That's why the value of STEC in the guarantee part for the longevity risk is negative.

Then, we turn to view the variation of present value of the claims in the baseline. In the process of revaluating the Benefit Base, two types of mechanisms have been applied: one is the roll-up, the other one is the ratchet. The final Benefit Base is determined by the maximum of these two mechanisms.

The value of guaranteed Benefit Base with the annual ratchet mechanism equals the maximum among all the Account Value of each year's valuation, which can be represented by the formula below:

$$BB_t^{rat} = max_{0 \leqslant s \leqslant t} AV_s$$

If the ratchet frequency is changed from one year to five years, the Benefit Base revaluated with this mechanism will be dropped as it equals the maximum of the Account Value of five years' valuation. And, if no ratchet mechanism is applied in the revaluation of the Benefit Base, the value of Benefit Base will fall down as well. To sum up, the guaranteed Benefit Base provided by the insurer will be reduced as the ratchet mechanism takes effect less frequently. As a result, the amount of coupon withdrawn from the Account Value will be decreased since the coupon is calculated as: Benefit Base $\times$ Income Rate. Therefore, the claims paid by the insurer will be decreased. That is to say, the present value of the claims in the baseline will be decreased compared to the initial design.

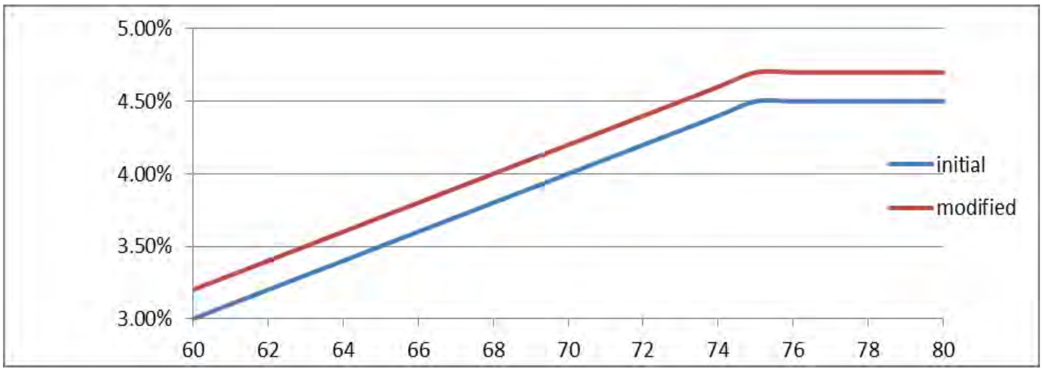
In conclusion, combining with the previous analysis, the value of guarantee component of STEC for the longevity risk will be declined.

6.3 Impact of the income rate

In this part of essay, we try to find a way to change the income rate structure in order to optimize the required capital and make some analysis on the impact of the modification.

6.3.1 Impact of the Income rate+20bp

First step is to improve the income rate by 20bp. The evolution of income rate structure in these two situations can be described by the following graph.



The table below presents the value of Life STEC in each sub-risk.

	Cat Mortality	Mortality Trend	Longevity	Lapse up	Lapse down	Expenses	Mass lapse
Modified	0.6624	0	343.1636	0	3.1546	24.6687	65.7294
Initial	0.9004	0	318.1769	14.5107	0	24.9734	113.5487

Great changes take place in the longevity risk, lapse up risk and mass lapse risk. Then, we look at the values of STEC for each sub-risk in both the guarantee part and the base product part.

In the guarantee part,

	Cat Mortality	Mortality Trend	Longevity	Lapse up	Lapse down	Expenses	Mass lapse
Modified	-0.0763	109.0805	-344.7328	44.7485	-47.2567	0	62.2887
Initial	-0.2628	100.6654	-325.043	26.2106	-25.4028	0	24.5974

In the base product part,

	Cat Mortality	Mortality Trend	Longevity	Lapse up	Lapse down	Expenses	Mass lapse
Modified	-0.5861	-3.2828	1.5692	-38.182	44.1021	-24.6687	-128.0182
Initial	-0.5955	-3.4447	1.7083	-39.8511	46.3558	-24.9381	-130.06

When comparing the variations of these sub-risks separately in the guarantee part and in the base product part, we can notice that the changes of all these risks in the Life STEC are caused by their changes in the guarantee part.

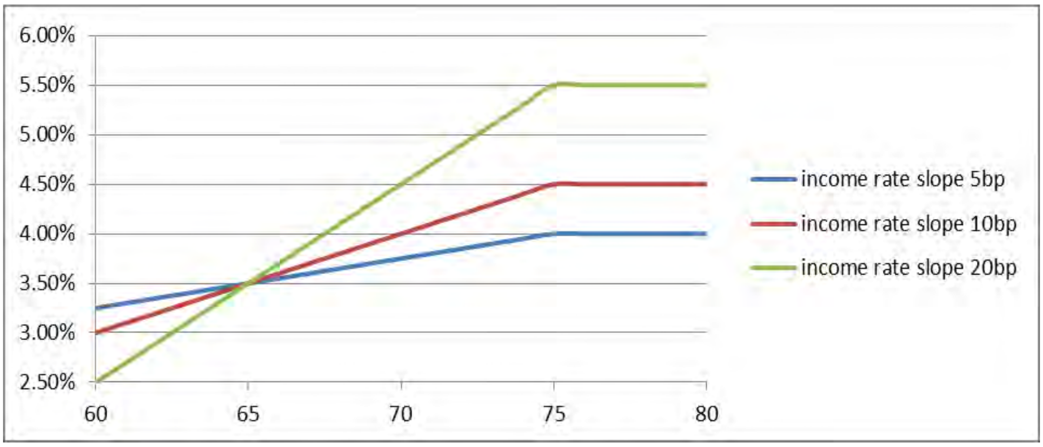
The longevity risk is increased in absolute value and the lapse up risk and mass lapse risk is augmented as well. The longevity risk has more effect in absolute value when the initial income rate is enlarged by 20bp. Since the income rate is augmented by 20bp, the coupon is increased as well and in this way, the claims paid by the insurer will be augmented as well, which leads to the increase of the present value of claims. As the longevity shock takes effect in the initial design, the present value of claims in the shock scenario will be enlarged, considered as the present value of claims in the baseline multiplied by x, and x>1 in this situation. Following the same reasons in the previous process of changing the product feature, we can explain the reasons for the change.

6.3.2 Impact of the income rate structure

In the initial design of the product, the income rate structure can be described as follows. The guaranteed income rate varies with the insured's age at the beginning of the payment phase and is fixed at 3% for an insured aged 60, which is the minimum age allowed at the beginning of the payment period. This value will be increased if the policyholder decides to receive the coupons later. The increment equals 0.1% per year, and it continues until the age of 75. After this age, the income rate for each insured who commences to withdraw his coupons remains at a stable level, which is equal to 4.5% $(3\% + 0.1\% \times (75 - 60) = 4.5\%)$.

Via previous analysis of the product features impact on STEC, we find that the longevity risk plays an important role in the life STEC. Therefore, we try to find some methods to reduce this value in order to decrease the value of STEC.

Considering that most of the French people begin their retirement at the age of 65, in the processes of changing this product feature, the level of income rate is fixed for this age. In addition, the increment speed for the guaranteed income rate is modified from 0.1% to 0.2%, respectively to 0.05%. The evolution of income rate structure in these two situations can be described by the following graph.



- Income rate slope = 20bp

The tables below present the STEC for each risk in the guarantee part and the base product part at $t = 0$ for income rate slope=20bp and 10bp.

In the guarantee part,

	Cat Mortality	Mortality Trend	Longevity	Lapse up	Lapse down	Expenses	Mass lapse
20bp	-0.2329	106.0956	-337.0669	25.3324	-23.8201	0	28.4281
10bp	-0.2628	100.6654	-325.043	26.2106	-25.4028	0	24.5974

In the base product part,

	Cat Mortality	Mortality Trend	Longevity	Lapse up	Lapse down	Expenses	Mass lapse
20bp	-0.5898	-3.1264	1.2728	-39.7854	46.4469	-24.9128	-129.3432
10bp	-0.5955	-3.4447	1.7083	-39.8511	46.3558	-24.9381	-130.06

The results for the Life STEC are as follows:

	Cat Mortality	Mortality Trend	Longevity	Lapse up	Lapse down	Expenses	Mass lapse
20bp	0.8228	0	335.7941	14.453	0	24.9128	100.9151
10bp	0.8583	0	323.3347	13.6405	0	24.9381	105.4625

In this case, we can notice that the most important impacts are those of the longevity risk and the mass lapse risk. More exactly, the longevity risk is increased by 3.8% compared to the initial design and the mass lapse risk is decreased by 4.3%, which makes the Life STEC rising from 3.45% to 3.56% of the initial premium. In the following, we analyze what determines the evolution of the longevity risk and the mass lapse risk both in the guarantee part and the base product part.

If the annual increment of the income rate is larger (while fixing the income rate at the age of 65), the average income rate will be greater than the one of the initial design. In this way, the coupon paid out for the insured will be higher because the annual amount of payment is calculated by multiplying the income rate by the guaranteed benefit base.

For the guarantee part, the impact of income rate structure can be described as the difference of STEC in these two situations. More specifically, it's the difference of the STEC for each sub-risk in the case where the income rate slope is 20bp minus the one of 10bp.

$$Impact = STEC^{20bp}_{guarantee} - STEC^{10bp}_{guarantee}$$

The guarantee component of the STEC for each sub-risk is defined as the difference between the option value of shock scenario and the option value of baseline.

$$Impact = (OV^{20bp}_{shock} - OV^{20bp}_{baseline}) + (OV^{10bp}_{shock} - OV^{10bp}_{baseline})$$

When the option value is calculated, it is equal to the present value of EHC charges minus that of claims.

$$OV = PV(EHCCharges - Claims)$$

When comparing the impact of the EHC charges and claims while changing the income rate structure, we can conclude that the EHC charge plays a less important role than the claims in the change of option value. As the option value impact is mainly driven by the claims change, we can analyze the impact of income rate structure on the option value by looking at the effect on the claims.

$$\begin{aligned}
Impact &= -Impact_{claims} \\
&= -Impact^{20bp}_{claims} + Impact^{10bp}_{claims} \\
&= -(PV(Claims)^{20bp}_{shock} - PV(Claims)^{20bp}_{baseline}) \\
&\quad +(PV(Claims)^{10bp}_{shock} - PV(Claims)^{10bp}_{baseline}) \\
&= -(PV(Claims)^{20bp}_{shock} - PV(Claims)^{10bp}_{shock}) \\
&\quad +(PV(Claims)^{20bp}_{baseline} - PV(Claims)^{10bp}_{baseline})
\end{aligned}$$

As the value of coupon augments in average, the claims paid by the insurer will be increased when the income rate slope increases from 10bp to 20bp.

When the longevity shock is applied to the initial scenario, the policyholder will have a higher life expectancy compared to the baseline, which leads to the increase of claims. As a result, we can consider the claims of shock scenario as that of baseline multiplied by a certain number which is superior to 1, noted x_1 for the situation of 20bp and x_2 for the situation of 10bp.

$$Impact = (1 - x_1) \times PV(Claims)^{20bp}_{baseline} - (1 - x_2) \times PV(Claims)^{10bp}_{baseline}$$

Via the observation of empirical data, we find that x_1 and x_2 are very close. So, the previous formula can be written as:

$$Impact = (1 - x) \times (PV(Claims)^{20bp}_{baseline} - PV(Claims)^{10bp}_{baseline})$$

According to the previous analysis, $1 - x < 0$ and $PV(Claims)^{20bp}_{baseline} - PV(Claims)^{10bp}_{baseline} > 0$. In this way, we can explain why the STEC for longevity risk falls down in the situation where the increment speed of income rate is increased.

The reasons for the increase of mass lapse risk in this case can be explained almost in the same way. If the mass lapse shock is applied to the initial scenario, it means that more lapses will take place during the first year, which leads to the decrease of the claims paid by the insurer. As a result, the claims of shock scenario can be regarded as that of the baseline multiplied by a certain number which is inferior to 1, noted x_1 for the situation of 20bp and x_2

for the situation of 10bp. As x_1 and x_2 are very close, we can conclude by the following formula that the STEC for mass lapse risk goes up in this case.

$$Impact = (1 - x) \times (PV(Claims)^{20bp}_{baseline} - PV(Claims)^{10bp}_{baseline})$$

For the base product part, we can also analyse the impact of income rate structure by the following formula:

$$Impact = STEC^{20bp}_{baseproduct} - STEC^{10bp}_{baseproduct}$$

The STEC for each sub-risk is defined as the difference between the VIF of shock scenario and the VIF of baseline.

$$Impact = (VIF^{20bp}_{shock} - VIF^{20bp}_{baseline}) - (VIF^{10bp}_{shock} - VIF^{10bp}_{baseline})$$

VIF base product is calculated as follows:

$$VIF_{baseproduct} = PV(RRCcharges - EHCcharges + income - expenses - commissions)$$

Since all the charges and fees are calculated as a percentage of the present value of Account Value, we can compare the present value of the underlying instead of the VIF in each baseline and shock scenario in order to simplify the analysis.

$$\begin{aligned} Impact &= (PV(AV)^{20bp}_{shock} - PV(AV)^{20bp}_{baseline}) - (PV(AV)^{10bp}_{shock} - PV(AV)^{10bp}_{baseline}) \\ &= (PV(AV)^{20bp}_{shock} - PV(AV)^{10bp}_{shock}) - (PV(AV)^{20bp}_{baseline} - PV(AV)^{10bp}_{baseline}) \end{aligned}$$

Since the coupon is augmented, the Account Value will decrease faster, which leads to the decrease of the present value of Account Value (noted $PV(AV)$).

If the longevity risk is applied to the baseline, the $PV(AV)$ will be increased. Therefore, the $PV(AV)$ can be written as $x \times PV(AV)_{baseline}$, where $x > 1$ and depends on the income rate structure. The impact becomes:

$$Impact = (x_1 - 1) \times PV(AV)^{20bp}_{baseline} - (x_2 - 1) \times PV(AV)^{10bp}_{baseline}$$

As x_1 and x_2 are very close, we can rewrite the previous formula as:

$$Impact = (x - 1) \times (PV(AV)^{20bp}_{baseline} - PV(AV)^{10bp}_{baseline})$$

According to the previous analysis, $x - 1 > 0$ and

$$PV(AV)^{20bp}_{baseline} - PV(AV)^{10bp}_{baseline} < 0$$

In this way, we can explain why the STEC for the longevity risk falls down in the situation where the increment speed of income rate is increased.

If the mass lapse shock is applied to the initial scenario, the $PV(AV)$ will be decreased. Following the same reasoning as above, we can rewrite the impact as:

$$Impact = (x - 1) \times (PV(AV)^{20bp}_{baseline} - PV(AV)^{10bp}_{baseline})$$

In this way, we can conclude that the STEC for mass lapse risk goes up in this case.

- Income rate slope = 5bp

In the guarantee part,

	Cat Mortality	Mortality Trend	Longevity	Lapse up	Lapse down	Expenses	Mass lapse
5bp	-0.2679	98.1606	-319.0967	27.1683	-26.7002	0	24.8787
10bp	-0.2628	100.6654	-325.043	26.2106	-25.4028	0	24.5974

In the base product part,

	Cat Mortality	Mortality Trend	Longevity	Lapse up	Lapse down	Expenses	Mass lapse
5bp	-0.5988	-3.6319	1.9688	-39.9977	46.4754	-24.9652	-130.5195
10bp	-0.5955	-3.4447	1.7083	-39.8511	46.3558	-24.9381	-130.06

The results for the Life STEC are as follows:

	Cat Mortality	Mortality Trend	Longevity	Lapse up	Lapse down	Expenses	Mass lapse
5bp	0.8666	0	317.1279	12.8295	0	24.9652	105.6408
10bp	0.8583	0	323.3347	13.6405	0	24.9381	105.4625

In this case, we can notice that the most important impact is the one in the longevity risk, especially in the guarantee part. The absolute value of the longevity risk is decreased by about 1.8%, which makes the Life STEC reducing from 3.45% to 3.40% of the initial premium. This is coherent with the previous result because we change the income rate structure in opposite direction and it leads to the variation of STEC for the longevity risk in different way.

From this data, we can see clearly that the change of income rate slope from 10bp to 5bp is lower than the change from 10bp to 20bp. If we consider an average of the income rate for all the model points, the result is as follows:

5bp	3.54%
10bp	3.66%
20bp	3.90%

When comparing the average income rate for each situation, we can note that the income rate slope change from 10bp to 5bp leads to a lower average coupon than in the situation where the income rate slope is changed from 10bp to 20bp. This helps us to comprehend the less important impact of income rate when the increment speed is lower.

6.4 Impact of the DB guarantee

In the existing model, our product provides two kinds of guarantees to the clients: one is the GLWB; the other one is the GMDB.

To calculate the Withdrawal Benefit Base, two types of revalorization methods are applied, the roll-up effect and the ratchet effect. The rollup effect is applied during the period, $t = 0$ and $t = min(\textit{deferral period}; 15)$ whereas the ratchet effect is active during the whole life of the product. So, during the deferral period, the rollup base and the ratchet base are combined and form the rollup on ratchet base.

The Death Benefit equals only to the initial net premium paid by the insured and the duration of this type of guarantee is just equal to the deferral period, which is not rich enough to compensate the Withdrawal Benefit payments. For this reason, we need to make some changes on the initial design to improve the Life STEC for this product.

To sum up, in the initial design, the Death Benefit given to the beneficiary is represented by the formula below:

If T is the duration of the deferral period and t is the death date of the insured, then

$$Death\ Benefit = \begin{cases} max(AV_t; Initial\ net\ premium) & t < T \\ AV_t & t \geq T \end{cases}$$

6.4.1 DB with the roll-up and ratchet in the deferral phase

The first step is to add a roll-up and a ratchet effect on the Death Benefit in the deferral period in order to make it equal to the guaranteed Withdrawal Benefit. At the same time, the duration of the Death Benefit remains the same which is equal to the deferral period.

This kind of change in the Death Benefit structure can be represented by the formula below:

If T is the duration of the deferral period and t is the death date of the insured, then

$$Death\ Benefit = \begin{cases} max(AV_t; WB_t) & t < T \\ AV_t & t \geq T \end{cases}$$

In the guarantee part,

	Cat Mortality	Mortality Trend	Longevity	Lapse up	Lapse down	Expenses	Mass lapse
Modified	-0.4535	98.3469	-323.7978	27.0741	-26.3354	0	28.0278
Initial	-0.2628	100.6654	-325.043	26.2106	-25.4028	0	24.5974

In the base product part,

	Cat Mortality	Mortality Trend	Longevity	Lapse up	Lapse down	Expenses	Mass lapse
Modified	-0.5955	-3.4447	1.7083	-39.8511	46.3558	-24.9381	-130.06
Initial	-0.5955	-3.4447	1.7083	-39.8511	46.3558	-24.9381	-130.06

In total, the Life STEC is presented as follows:

	Cat Mortality	Mortality Trend	Longevity	Lapse up	Lapse down	Expenses	Mass lapse
Modified	1.049	0	322.0895	12.777	0	24.9381	102.0322
Initial	0.8583	0	323.3347	13.6405	0	24.9381	105.4625

We can notice that in the base product part, the STEC for each sub-risk remains unchanged. This is because of the invariability of all the items utilized to compute this component since each one is calculated as a percentage of the present value of Account Value. More exactly, the Account Value is linked to the performance of the market in the deferral period while in the payment phase the Account Value is affected by the present value of the charges and the coupons withdrawn from the account of insured. In this way, the changed Death Benefit structure does not have influence in the present value of the Account Value when comparing with the initial design. Therefore, the data for the base product stays the same.

As a result, the variation of life STEC is due to the change in the guarantee component. However, we can find that the value of the guarantee part does not have much variation. This is resulted from the slight variation of the present value of the claims. The claims in our product can be separated in two parts: one for the Death Benefit guarantee, the other one for the Withdrawal Benefit guarantee. After adding the two mechanisms to improve the level of the Death Benefit, the present value of the Death claims is raised in a non-significant way while the present value of the Withdrawal claims remains unchanged. So, there isn't much change in the present value of the claims. Therefore, the Life STEC has not much variation compared to the initial design.

6.4.2 DB for life

The second step is to extend the duration of the Death Benefit to the policy maturity, which means during all period of the product.

This kind of modification on the Death Benefit structure can be represented by the formula below:

$$Death\ Benefit = \begin{cases} max(AV_t; Initial\ net\ premium) & t < T \\ max(AV_t; Initial\ net\ premium - paid\ coupons) & t \geq T \end{cases}$$

If T is the duration of the deferral period and t is the death date of the insured, then

In the guarantee part,

	Cat Mortality	Mortality Trend	Longevity	Lapse up	Lapse down	Expenses	Mass lapse
Modified	0.0636	59.3518	-268.171	54.6227	-56.6989	0	104.4101
Initial	-0.2628	100.6654	-325.043	26.2106	-25.4028	0	24.5974

In the base product part,

	Cat Mortality	Mortality Trend	Longevity	Lapse up	Lapse down	Expenses	Mass lapse
Modified	-0.5955	-3.4447	1.7083	-39.8511	46.3558	-24.9381	-130.06
Initial	-0.5955	-3.4447	1.7083	-39.8511	46.3558	-24.9381	-130.06

In total, the Life STEC is presented as follows:

	Cat Mortality	Mortality Trend	Longevity	Lapse up	Lapse down	Expenses	Mass lapse
Modified	0.5319	0	266.4628	0	10.343	24.9381	25.6498
Initial	0.8583	0	323.3347	13.6405	0	24.9381	105.4625

As explained before, the present value of the Account Value stays the same in the process of changing the Death Benefit structure, which leads to the unchanged state of the base product component of Life STEC. Therefore, in the following analysis, we concentrate on the guarantee part to study the impact of this modified product feature.

In the Life STEC, great changes take place in the longevity risk and the mass lapse risk. For the longevity risk, it is calculated as the option value in the applying longevity risk scenario minus the option value in the baseline case.

$$\begin{aligned} STEC_{longevity} &= OV_{longevity} - OV_{baseline} \\ &= (PV(EHC\ Charges)_{longevity} - PV(Claims)_{longevity}) \\ &\quad -(PV(EHC\ Charges)_{baseline} - PV(Claims)_{baseline}) \end{aligned}$$

Since the variation level of the present value of the EHC charges is less than the one of the claims, we pay little attention to the impact of the EHC charges on this part of study.

$$STEC_{longevity} = PV(Claims)_{baseline} - PV(Claims)_{longevity}$$

As we can divide the total claims into the death claims portion and the withdrawal claims portion, the STEC for the longevity risk can be represented below:

$$\begin{aligned} & STEC_{longevity} \\ = \; & (PV(Death\ Claims)_{baseline} + PV(Withdrawal\ Claims)_{baseline}) \\ & -((PV(Death\ Claims)_{longevity} + PV(WithdrawalClaims)_{longevity}) \\ = \; & (PV(DeathClaims)_{baseline} - PV(DeathClaims)_{longevity}) \\ & -(PV(WithdrawalClaims)_{baseline} + PV(WithdrawalClaims)_{longevity}) \end{aligned}$$

When the longevity shock is applied, the present value of the death claims will be decreased in comparison with the baseline case and the present value of the withdrawal claims will be increased. In this way, the first part of the above formula is positive while the second part is negative. Since the negative part plays a more important role, the STEC for the longevity risk is eventually negative.

Now, we take into account the initial design to explain the reasons for the decrease of the death claims while the longevity shock takes effect. The beneficiary will receive an amount of compensation in the payment period and the value of this payment equals the Account Value at the death date of the insured. If the life expectancy of the insured is improved, the beneficiary will receive the indemnity later in the payment period compared to the baseline situation. The Account Value will be decreased through the time. Therefore, the death benefit given by the insurer will drop as well.

The increase of the withdrawal benefit results from the longer life of the policyholder. More exactly, the rest money in the policyholder's account cannot cover all the coupons guaranteed for the beneficiary and the exceeded part will be taken charge by the insurer. The longer life the policyholder will have, the more claims the insurer will be responsible for. In this way, the withdrawal benefit rises.

In addition, to compare the impact of the modified death benefit structure with the initial one, we make the difference of STEC for the longevity risk in these two cases. Since the present value of the withdrawal claims is linked to the withdrawal benefit and the income rate which remains unchanged in these two cases, the distinction of the STEC can be written as follows:

$$\begin{aligned} & STEC_{modified} - STEC_{initial} \\ = \ & (PV(DeathClaims)_{baseline}^{modified} - PV(DeathClaims)_{longevity}^{modified}) \\ & -(PV(DeathClaims)_{baseline}^{initial} - PV(DeathClaims)_{longevity}^{initial}) \end{aligned}$$

That is to say, to compare the impact of the new death benefit structure with the initial one is to compare the influence in the death claims part.

The initial death benefit in the payment period is the Account Value at the death date of the insured while the modified design is the maximum between the Account Value and the initial premium minus the coupons paid before. Therefore, in the baseline case, the modified death benefit is larger than the one in the initial design. Then, the longevity shock takes effect, so the death claims will be reduced. The modified value of the death claims can be viewed as the initial one multiplied with a certain number inferior to 1, noted as x. In this way, the distinction of STEC can be written as: $(1-(1-x)) \times PV(DeathClaims)_{baseline}$ with $x < 1$. This value is positive, which means the longevity risk makes more difference in the modified design than the initial one. In this way, the STEC for the longevity risk is augmented.

For the mass lapse risk, we can commence the study by directly using the formula below:

$$\begin{aligned} & STEC_{modified} - STEC_{initial} \\ = \; & (PV(DeathClaims)^{modified}_{baseline} - PV(DeathClaims)^{modified}_{masslapse}) \\ & -(PV(DeathClaims)^{initial}_{baseline} - PV(DeathClaims)^{initial}_{masslapse}) \end{aligned}$$

We know that in the baseline case, the death benefit in the shocked scenario is larger than the initial one. And, with the application of the mass lapse risk, the death claims will be decreased and its value can be regarded as the initial death claims multiplied by a certain number inferior to 1. In this way, the distinction of STEC for the mass lapse risk is positive. That is to say, the impact of the mass lapse risk makes more influence in the modified design.

6.4.3 DB for life with the roll-up and ratchet effect in the deferral phase

The third step is to extend the duration of the Death Benefit Base to the policy maturity and add a roll-up and a ratchet effect on the Death Benefit in the deferral period in order to make it equal to the guaranteed Withdrawal Benefit.

This method of changing the Death Benefit structure can be represented by the formula below:

If T is the duration of the deferral period and t is the death date of the insured, then

$$Death\ Benefit = \begin{cases} max(AV_t; WB_t) & t < T \\ max(AV_t; WB_t - paid\ coupons) & t \geq T \end{cases}$$

In the guarantee part,

	Cat Mortality	Mortality Trend	Longevity	Lapse up	Lapse down	Expenses	Mass lapse
Modified	0.0863	34.9964	-224.1371	77.2244	-82.5453	0	155.9958
Initial	-0.2628	100.6654	-325.043	26.2106	-25.4028	0	24.5974

In the base product part,

	Cat Mortality	Mortality Trend	Longevity	Lapse up	Lapse down	Expenses	Mass lapse
Modified	-0.5955	-3.4447	1.7083	-39.8511	46.3558	-24.9381	-130.06
Initial	-0.5955	-3.4447	1.7083	-39.8511	46.3558	-24.9381	-130.06

In total, the Life STEC is presented as follows:

	Cat Mortality	Mortality Trend	Longevity	Lapse up	Lapse down	Expenses	Mass lapse
Modified	0.5092	0	222.4289	0	36.1895	24.9381	0
Initial	0.8583	0	323.3347	13.6405	0	24.9381	105.4625

The life STEC changes significantly in the longevity risk and in the mass lapse risk in this modified death benefit structure. As explained before, we divide the claims into two parts: the death claims and the withdrawal claims. In addition, we compare the distinction of the STEC for each sub risk by studying the influence in the death claims part.

In the initial design, the death benefit in the payment period equals the Account Value while in the modified design, its value is equal to the maximum between the Account Value and the Withdrawal Benefit subtracted the paid coupons. Therefore, the death claims will be improved in the baseline case.

Then, if the longevity shock is applied, the value of death claims declines. And, this can be seen as the death claims in the baseline case multiplied by a number inferior to 1. Therefore, the distinction of the STEC for the longevity risk in the modified design and in the initial design is positive. The value of the STEC will be augmented.

The reason for the effect of the mass lapse risk can be described in the same way since the evolution of the death claims is similar when applying these two risks.

6.4.4 Comparison of the three modifications

The table below presents the important capital and profitability data for the above three cases:

Capital and Profitability	Initial design	DB+roll-up+ratchet	DB for life	DB+roll-up+ratchet for life
STEC Market	5.23%	5.90%	5.90%	6.16%
STEC Life	3.45%	3.43%	2.73%	2.35%
STEC Op	1.04%	1.13%	1.06%	1.07%
Total STEC	7.34%	7.95%	7.48%	7.51%
VIF	6.54%	6.38%	3.01%	1.17%
MVM	1.95%	1.95%	1.71%	1.44%
Capital (GRM)	6.43%	7.49%	9.92%	11.54%

According to the data, the decrease in the Life STEC meets our prediction for the reasons that a richer Death Benefit Base will compensate better the impact of the longevity shock on the Withdrawal Benefit payments. However, the increase in the Market STEC needs our more consideration.

We consider that the value of EHC is another feature that we need to take into account. In this way, we recalculate the value of EHC to see the changes in the Market STEC.

The results of the important capital and profitability data are as follows:

Capital and Profitability	Initial design	DB+roll-up+ratchet	DB for life	DB+roll-up+ratchet for life
STEC Market	5.23%	5.24%	6.08%	6.64%
STEC Life	3.45%	3.43%	2.73%	2.35%
STEC Op	1.04%	1.04%	1.09%	1.14%
Total STEC	7.34%	7.34%	7.66%	7.99%
VIF	6.54%	6.38%	3.01%	1.17%
MVM	1.95%	2.19%	2.19%	2.19%
Capital (GRM)	6.43%	6.81%	10.66%	13.00%

The recalculation of the EHC will not make the decrease in the Market STEC as we expected. And the Life STEC will remain the same because the recalculation of the EHC will not make any impact on the Life STEC. The reasons are as follows:

$$Life\ STEC = -min(Life\ STEC_{base\ product} + Life\ STEC_{guarantee}; 0)$$

with

$$Life\ STEC_{base\ product} = OV_{shock} - OV_{baseline}$$

and

$$OV = PV(EHC\ Charges - Claims)$$

In addition,

$$Life\ STEC_{guarantee} = VIF\ BP_{shock} - VIF\ BP_{baseline}$$

and

$$VIF\ BP = PV(RRC\ Charges - EHC\ Charges + Incomes - Expenses - Commisions)$$

If we make the sum of these two parts, the EHC Charges will be eliminated and

$$Life\ STEC = -min(VIF_{shock} - VIF_{baseline}; 0)$$

So, we can explain the unchanged value of the Life STEC.

We can also notice that in the third modification of the Death Benefit structure, the VIF drops to a great degree. In this way, we augment the level of the RRC rate to compensate the great reduction of the VIF in order to see if there exists any improvement on the capital requirement.

Capital and Profitability	DB+roll-up+ratchet for life	DB+roll-up+ratchet for life+RRC+50bp	DB+roll-up+ratchet for life+RRC+80bp
STEC Market	6.64%	6.52%	6.47%
STEC Life	2.35%	2.42%	2.47%
STEC Op	1.14%	1.12%	1.12%
Total STEC	7.99%	7.90%	7.89%
VIF	1.17%	2.89%	3.94%
MVM	2.19%	2.19%	2.19%
Capital (GRM)	13.00%	11.15%	10.08%

The improved RRC rate does not make significant difference on the Total STEC. In fact, the decrease of the Market STEC and the increase of the Life STEC balance the total amount. At the same time, the improvement of the VIF does occur since we have augmented the charge rate to enforce the benefit of the product. When taking all these factors into account, the decrease of the capital requirement can be explained.

6.5 Impact of the CVF modeling

The *Capped Volatility Fund* (CVF) strategy aims at maintaining a stable level of volatility for the whole portfolio. Different from the traditional fixed-allocation strategy, a CVF strategy moves money from the risky assets to the safer assets in order to achieve the right level of volatility for the investment.

In the present product for the French market, three possible funds are provided for the policyholder to invest. That is, CVF30, CVF40 and CVF50, where 30, 40 and 50 represent the percentage of the average equity exposure in each fund. As expected, the target volatility is different for each fund. More precisely, for the CVF30 fund, the target volatility is fixed at 5% while 5.5% for the CVF40 fund and 6.5% for the CVF50 fund. If the volatility of a certain fund is inferior to the target, the money invested in the safer baskets will be transferred to the risky one. In this way, more benefit will be achieved since the investment structure is more dynamic than before.

In the initial modelling methodology, the *Capped Volatility Fund* is simulated simply as a equity whose volatility is equal to the target one. This method makes the pricing of the CVF very conservative and lessens the expected level of return. As a result, we utilize a more accurate method to simulate the CVF and calculate the average percentage of the equities in the risky basket. Then, we simulate a new basket with the allocation of the equities and bonds obtained from the previous step.

We compare the results of the Market STEC in the two modelling methods. In addition, we focus only on the CVF50 to simplify the analysis.

The Greeks in the guarantee component has the variation as follows:

	Delta	Gamma	Rho	Rho_convexity	Equity_vol	IR_vol
modified	25.5204	-1.5483	5.881	-0.0427	-14.7494	-13.0635
initial	24.3872	-1.2836	4.9725	-0.0396	-55.9433	-20.9273

The variation of the Greeks in the base product component is given below:

	Delta	Gamma	Rho	Rho_convexity	Equity_vol	IR_vol
modified	8.2189	-0.1604	-0.1432	-0.0023	-1.1853	-0.8856
initial	7.6661	-0.1259	-0.3173	-0.0028	-5.0731	-1.8403

We notice that the application of the new modelling methodology leads to the decrease of the Market STEC from 5.21% to 5.09% of the initial premium. The variation of the Market STEC is not as significant as expected, because although the absolute value of the equity volatility and the swaption volatility decreases both in the guarantee component and base product component, the augmentation of the absolute value of Gamma compensates their effects. However, the diminution of the Vega meets our expectation since the objective of the new modelling method is to lower the impact of the volatility to the underlying assets.

6.6 Impact of the mortality and longevity structure

In this part of thesis, we try to modify the mortality and longevity structure by using the modelling assumptions of the Japanese product and make analysis on the effect of such kind of change in the modelling process.

As the STEC for the longevity risk is very small for the Japanese product compared to others, and the main cause of this result might be the mortality structure of Japanese product, we decide to use the modelling methodology for the mortality and longevity structure in our product design.

Nowadays, the mortality function of Japanese product during the payment period is given below:

$$mortality\ rate = Base\ mortality\ table \times Base\ mortality\ improvement$$

where $Base\ mortality\ improvement = (1-r(valuation\ age))^{valuation\ age-2000}$ with r the improvement rate applied to the base mortality rate dependent on the age of insured in the valuation year.

For this part of analysis, we focus only on a female policyholder born in the year 1962 and aged 50 at the valuation year 2012. We consider three possible funds structures (CVF30, CVF40 and CVF50) as the fund performance has an impact on the amount of required capital as well.

In the process of changing the mortality table, we fix the French data for an individual born in 1950. Then, the formula mentioned before for the Japanese product is applied while reconstructing the remaining mortality table used by calculating the longevity risk for this specified insured. We expect that the application of this new mortality rate will lead to a considerable decrease of the shock of longevity.

For the part of changing the longevity table, we can utilize the following formula to calculate:

$$longevity_{new} = longevity_{old} \times \frac{(1-r(age_0))^{12}}{(1-r(age_t))^{t+12}}$$

After the change of mortality and longevity structure, the STEC for each sub-risk has the value as follows:

	Cat Mortality	Mortality Trend	Longevity	Lapse up	Lapse down	Expenses	Mass lapse
modified	0.3873	0	13.0646	0	43.562	28.8173	0
initial	0.8468	0	306.5914	13.6405	0	28.6621	64.0211

In the guarantee part,

	Cat Mortality	Mortality Trend	Longevity	Lapse up	Lapse down	Expenses	Mass lapse
modified	0.1049	71.4555	-13.4325	78.3108	-82.4068	0	115.7248
initial	-0.3549	81.1811	-306.6827	35.6447	-32.7753	0	29.0306

In the base product part,

	Cat Mortality	Mortality Trend	Longevity	Lapse up	Lapse down	Expenses	Mass lapse
modified	-0.4923	-0.3285	0.3679	-31.7095	38.8447	-28.8173	-93.158
initial	-0.4919	-0.4105	0.0913	-31.5283	38.4973	-28.6621	-93.0517

Comparing the data of the initial design and of the modified mortality table and longevity shock structure, we can find that the life STEC decreases a lot, which is changed from 3.17% to 0.56%. The value of each sub-risk of life STEC in the base product part does not have much changes and the main evolution of the life STEC is caused by the variation in the guarantee part.

We can notice that in the guarantee part, the value of each sub-risks are of the same sign with the one under the initial hypothesis except for the catastrophic mortality risk, which is negative in the previous and turns to be positive in the modified mortality structure.

When the value of catastrophic risk is negative, it means that the shock option value is inferior to the baseline option value. On the contrary, when its value turns to be positive, it means that the shock option value is superior to the baseline option value.

We write the formula for calculation the STEC in the guarantee part as follows:

$$\begin{aligned} STEC_{guarantee} &= OV_{shock} - OV_{baseline} \\ &= (PV(EHCCharges)_{shock} - PV(Claims)_{shock}) \\ &\quad -(PV(EHCCharges)_{baseline} - PV(Claims)_{baseline}) \\ &= (PV(EHCCharges)_{shock} - PV(EHCCharges)_{baseline}) \\ &\quad +(PV(Claims)_{baseline} - PV(Claims)_{shock}) \end{aligned}$$

In general, if the catastrophe shock is applied, the mortality rate will be augmented at the first year of the contract and it leads to the decrease of both the EHC Charges and the Claims. So, in the formula presented above, the first part will be inferior to 0 and for the second part, the value will be positive. By analysing the result in the initial and modified design, we find that the evolution of EHC charges plays a more important role in the initial design while the evolution of claims has much more influence on this risk in the modified design. In this way, the different sign of this sub risk can be explained.

As expected, the value of STEC for the longevity risk will be decreased by the amelioration of the initial mortality table and the modified longevity risk structure. The reason for this change is that the product of the mortality rate and the longevity shock goes down when calculated using the improvement rate of the Japanese product.

The value of the mortality trend risk decreases as well. When this risk is applied, it means that greater mortality rate will be employed to calculate the life STEC. In the initial design, the baseline mortality rate is the data from the French population, while in the modified design the baseline mortality rate is changed to be the one using the improvement rate for the Japanese product, which is less than the previous one. As a result, the variation from the baseline to the shock scenario under the modified hypothesis is less than the one in the initial design. That is to say, the capital required to withstand this risk will be decreased.

In order to simplify the analysis of the impact of the changed mortality rate and the longevity risk structure for the lapse risk, we concentrate on the evolution of the lapse up risk. Since the mortality rate has been decreased by the modified structure, the claims provided by the insurer will be increased. And, the claims can be considered as the main driver of the evolution of this risk. In this way, the capital required to hold for this risk will be approximately equal to the present value of the baseline claims minus the claims in the shock scenario. That is to say, $STEC_{guarantee} \approx$

$PV(Claims)_{baseline} - PV(Claims)_{shock}$.

Since the shocked claims can be viewed as the baseline claims multiplied by a certain number inferior to 1, $STEC_{guarantee} \approx PV(Claims)_{baseline} \times (1-x)$ with $x < 1$, and according to the previous analysis, the present value of the charges in the baseline increases under the new hypothesis, the STEC for this risk will be augmented.

6.7 Impact of the reward rider

In this part of study, we try to add a new rider reward in the initial design to see its impact on the STEC.

The reward rider gives the policyholder the possibility to receive the coupons with a higher income rate in the early years of the payment period. This new rider is attractive to the policyholder because they can obtain more money during the first few years of the retirement when they want to enjoy the active phase of their retirement. Once the new rider takes effect, we will add on respectively an additional 1% and 2% to the initial guaranteed income rate for the first few years of the payment period. The duration of this income rate improving period equals the duration of the deferral period. For example, if the policyholder defers taking the coupons for 7 years, then the value of each coupon will be higher for the first 7 years of the payment period. At the same time, the roll-up mechanism taking place in the deferral period will not be applied as the combined rider would be too expensive. However, the annual ratchet still applies to the benefit base.

6.7.1 Analysis of the variation in the Life STEC

We listed below the results of each component in the Life STEC in the situation where 1% of the reward rider is applied to the initial design.

In total, the Life STEC is presented as follows:

	Cat Mortality	Mortality Trend	Longevity	Lapse up	Lapse down	Expenses	Mass lapse
Modified	1.0338	0	276.8341	28.8189	0	24.5374	138.1526
Initial	0.8583	0	323.3347	13.6405	0	24.9381	105.4625

The significant variations take place in the longevity risk and the mass lapse risk. Then we look into the variation for these two risks both in the guarantee part and in the base product part.

In the guarantee part,

	Cat Mortality	Mortality Trend	Longevity	Lapse up	Lapse down	Expenses	Mass lapse
Modified	-0.2854	85.736	-279.2632	25.6307	-25.5282	0	22.3894
Initial	-0.2628	100.6654	-325.043	26.2106	-25.4028	0	24.5974

In the base product part,

	Cat Mortality	Mortality Trend	Longevity	Lapse up	Lapse down	Expenses	Mass lapse
Modified	-0.7484	-4.3866	2.4291	-54.4495	64.2122	-24.5374	-160.5419
Initial	-0.5955	-3.4447	1.7083	-39.8511	46.3558	-24.9381	-130.06

We can notice that in the guarantee part, the absolute value of the STEC in the case where the longevity risk is applied will be decreased. According to the analysis, the STEC for the longevity risk is approximately equal to the difference between the present value of the claims in the baseline and the one in the shocked scenario. Since the application of the longevity risk is to extend the payment period of the coupons, we can suppose that the duration of the extended payment period is n years. Therefore, the increment of the claims is approximately equal to the value of n coupons. While the roll-up mechanism is not applied in the deferral phase, the Withdrawal Benefit Base will be decreased. As a result, the value of each coupon calculated as the income rate times the Withdrawal Benefit Base will be decreased as well. In this way, we can conclude that the impact of the longevity shock will be weakened.

The significant changes take place in the base product part are the increase of the absolute value of the STEC for the lapse up risk and the mass lapse risk. As introduced before, the STEC for the base product part is calculated as the difference of the $VIF_{base\ product}$ between the shocked scenario and the baseline case. In addition to this, each item in the $VIF_{base\ product}$ is a certain percentage of the present value of the Account Value. In this way, we can compare the $PV(AV)$ in order to simplify the analysis. If the lapse up shock or the mass lapse shock is applied in the baseline case, the $PV(AV)$ will be decreased since a large number of policyholders will end their contracts in advance. And, this can be seen as the present value of the Account Value in the baseline case multiplied by a certain number inferior to 1. In addition, in the baseline case, the present value of the Account Value will be decreased when the reward rider is applied to the initial design. Such kind of variation is due to the improvement of the coupons at the first few years of the payment period, leading to the diminution of the present value of the Account Value more rapidly. Therefore, we can explain the more important impact of the lapse up risk and the mass lapse risk in the base product component.

6.7.2 Comparison of the two modifications

Capital and Profitability	Initial design	Reward 1%	Reward 2%
STEC Market	5.23%	4.49%	4.82%
STEC Life	3.45%	3.18%	3.05%
STEC Op	1.04%	0.92%	0.95%
Total STEC	7.34%	6.46%	6.67%
VIF	6.54%	7.99%	5.98%
MVM	1.95%	1.90%	1.59%
Capital (GRM)	6.43%	3.60%	5.61%

We can find that this is the best result that we have obtained because this change of the product feature makes the best improvement in the process of optimization of the capital requirement compared to other modifications.

The decrease of the two important component of the Total STEC, both the Life STEC and the Market STEC, lead to the reduction of the Total STEC.

For the variation in the Life STEC, we have discussed it before. For the variation around the Market STEC, the increment of the Market STEC results from the higher market sensitivity of the product since the reward rate has been improved from 1% to 2%.

For the capital requirement fixed by the GRM, it is calculated as $1.5 \times Total\ STEC - VIF + MVM$. The MVM is calculated as $6\% \times PV(STEC_{non\ headgeable})$, with $STEC_{non\ headgeable}$ represents the STEC except for the market part, which can be hedged in our hypothesis. Since the Life STEC and the Operational STEC decrease at the same time, the value of MVM is reduced as well. In addition, as we have mentioned before, the VIF is an indicator showing the profitability of the company. We notice that in the second case where 2% of reward rate is applied, the VIF will be reduced since there will be more claims paid by the insurer. However, even with the reduction of the VIF, the decrease in the capital requirement is still significant. This corresponds to our objective of the project, which aims to diminish the capital requirement.

Since the profitability indicator VIF in the second modification is less than the one presented in the initial design, we try to look into the result when the reward rate is equal to 1.5%. We can imagine that the result of the VIF is positioned between the two cases above, which can also lead to the decrease of the Total STEC and the capital requirement compared to the initial design.

The results are as follows:

Capital and Profitability	Initial design	Reward 1.5%
STEC Market	5.23%	4.65%
STEC Life	3.45%	3.11%
STEC Op	1.04%	0.93%
Total STEC	7.34%	6.56%
VIF	6.54%	7.01%
MVM	1.95%	1.59%
Capital (GRM)	6.43%	4.42%

These results coincide well with our expectation and the entire indicators for the capital requirement and the profitability are better than the data in the initial design. And the reward rate equal to 1.5% is rather attractive to the customers since the additional 1.5% of the coupon will be paid in their active phase when they commence their retirement life.

Conclusion

In the study of the capital requirement optimization, we have changed various kinds of product features to reduce the level of capital needed to cover the shocks taking place both in the financial and actuarial circumstances.

In the first step of our study, we modify the most important parameters of the product. We have the results as follows:

Capital and Profitability	Initial design	IR+10bp	IR_vol+1%	RRC+50bp	Income rate+20bp	Roll-up+50bp
STEC Market	5.23%	5.05%	5.25%	5.38%	5.35%	5.32%
STEC Life	3.45%	3.34%	3.45%	3.82%	3.53%	3.48%
STEC Op	1.04%	1.01%	1.05%	1.10%	1.07%	1.06%
Total STEC	7.34%	7.10%	7.36%	7.75%	7.52%	7.44%
VIF	6.54%	6.98%	6.31%	9.31%	4.78%	6.13%
MVM	1.95%	1.95%	1.96%	1.93%	1.72%	1.88%
Capital (GRM)	6.43%	5.62%	6.69%	4.26%	8.21%	6.91%

- When we make a +10bp parallel shift in the yield curve, the present value of the claims will be decreased since it is influenced by the discount factor which has a direct relation with the interest rate. Therefore, both the Market and Life STEC are slightly lower and the VIF is a little higher, which leads to the lower level of the capital requirement.

- When the volatility of the interest rate is improved by 1%, only the Market STEC is influenced and is augmented slightly, while the Life STEC remains unchanged. In addition, the profitability level is lessened by the increased volatility of the interest rate. As a result, the overall capital requirement is higher.

- The increase of the RRC rate means a higher level of the charges, which leads to the increase of the claims since at the initial time, the present value of the EHC charges should be equal to the present value of the claims. Therefore, the higher market sensitivity and more STEC for the longevity risk, which leads to the increase in both the Market and Life STEC. However, the improvement of the profitability level is significant, which reduces the required capital.

- The increase of the income rate leads to higher claims that should be given to the insured. Therefore, the lower profitability and the slightly higher STEC in both the Market and Life STEC take place. As a result, the two combined components lead to an important augmentation of the capital requirement.

- The evolution of each component of the capital requirement when the roll-up rate is increased is nearly the same as for the income rate. The only difference is that the variation of the capital requirement is lower since each component is less affected when this product feature is changed.

Capital and Profitability	Initial design	Ratchet 5 years	Without Ratchet
STEC Market	5.23%	5.18%	5.16%
STEC Life	3.45%	3.41%	3.39%
STEC Op	1.04%	1.03%	1.03%
Total STEC	7.34%	7.26%	7.24%
VIF	6.54%	6.55%	6.49%
MVM	1.95%	1.94%	1.93%
Capital (GRM)	6.43%	6.29%	6.30%

The second step of our study concerns in the modification of some more structuring components of the product. We begin by the modification of the ratchet frequency:

We notice that the ratchet frequency does not have an important impact on the Total STEC since both the Life and Market STEC have not varied greatly. We can explain it by a lower ratchet realization probability when the roll-up is applied. That is to say, the impact on the Benefit Base is mainly produced by the roll-up mechanism. As a consequence, the impacts on the Total STEC and the capital requirement are negligible.

Then, we change the structure of the income rate:

Capital and Profitability	Initial design	Income rate slope 5bp	Income rate slope 20bp
STEC Market	5.23%	5.24%	5.17%
STEC Life	3.45%	3.40%	3.56%
STEC Op	1.04%	1.04%	1.04%
Total STEC	7.34%	7.31%	7.36%
VIF	6.54%	6.55%	6.41%
MVM	1.95%	1.91%	2.07%
Capital (GRM)	6.43%	6.33%	6.70%

We can see that the modification of the structure of the income rate does not have an important impact on the Total STEC, VIF or Capital requirement. This is due to the slight variation of the average income rate caused by such a modification.

The results when the Death Benefit structure is changed and the EHC is recalculated are given as below:

Capital and Profitability	Initial design	DB+roll-up+ratchet	DB for life	DB+roll-up+ratchet for life
STEC Market	5.23%	5.24%	6.08%	6.64%
STEC Life	3.45%	3.43%	2.73%	2.35%
STEC Op	1.04%	1.04%	1.09%	1.14%
Total STEC	7.34%	7.34%	7.66%	7.99%
VIF	6.54%	6.38%	3.01%	1.17%
MVM	1.95%	2.19%	2.19%	2.19%
Capital (GRM)	6.43%	6.81%	10.66%	13.00%

A richer Death Benefit by adding a roll-up and a ratchet or by extending its duration or both of them does reduce the longevity impact, as expected. In addition, the main effect seems to be given by the extension of the duration of the Death Benefit, but not by the application of the revalorization methods. This results form the fact that the mortality is augmented when an insured becomes older, compensating the impact caused by the longevity risk. On the other hand, even when the EHC is recalculated, the Market STEC is still higher than in the initial design. At the same time, the profitability decreases significantly. Therefore, the capital requirement is augmented.

The third step, we try to find some amelioration of the product modelling assumptions. We use respectively the new Capped Volatility Funds modelling methods and the modified mortality and longevity table. The results are as follows:

Capital and Profitability	Initial design	New CVF modelling	New mortality and longevity modelling
STEC Market	7.97%	5.09%	9.34%
STEC Life	3.17%	3.61%	0.56%
STEC Op	1.39%	1.04%	1.42%
Total STEC	9.79%	7.33%	9.96%
VIF	4.71%	8.28%	0.89%
MVM	1.95%	1.33%	1.95%
Capital (GRM)	11.93%	4.04%	16.01%

• When the financial assumption is changed, the Market STEC does decrease as expected due to the significant diminution of the STEC for the volatility risk. Therefore, with a more important effect made by the Market STEC, the Total STEC decreases even in the case where the Life STEC is augmented. Then, with the enhanced profitability level, the Capital requirement in this case falls down significantly.

• When the new actuarial assumption is applied, the STEC for the longevity risk is almost reduced to zero, leading to a drastic reduction of the Life STEC as expected. But the higher level of the Market STEC leads to the improvement of the Total STEC. Then, the remarkable lower level of the VIF makes the Capital requirement augment significantly.

The best result takes place when the new rider reward is added to the initial design. It has made both the Total STEC and the Capital requirement decrease while the profitability level is improved. In this way, we have made both the capital optimization and the profitability improvement.

Capital and Profitability	Initial design	Reward 1.5%
STEC Market	5.23%	4.65%
STEC Life	3.45%	3.11%
STEC Op	1.04%	0.93%
Total STEC	7.34%	6.56%
VIF	6.54%	7.01%
MVM	1.95%	1.59%
Capital (GRM)	6.43%	4.42%

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